Math. Proc. Camb. Phil. Soc. (2011), **150**, 557 © *Cambridge Philosophical Society* 2011
doi:10.1017/S0305004111000107
First published online 9 March 2011

Separable Banach lattices on which every bounded linear operator is regular

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(*Received* 2 *September* 2010; *revised* 1 *December* 2010)

Abstract

We give a complete description of those separable Banach lattices E with the property that every bounded linear from E into itself is the difference of two positive operators.

1. *Introduction*

The question of when every bounded linear operator T mapping a Banach lattice E into another Banach lattice F must be *regular*, i.e. the difference of two positive operators, has been of interest since the early days of the study of Banach lattices, e.g. [**6**, **7**]. There are many cases in which it is known whether or not this is possible, but no sign as yet of knowing precisely when it is true. One case when we are much closer to a full answer is when $E = F$, when Abramovich in [**1**] showed that if all bounded linear operators from E into itself are regular then E must be isomorphic either to an AL-space or to an AM-space. Although equality holds for any AL-space (and therefore for isomorphic Banach lattices) there is equality for some AM-spaces (notably those that are Dedekind complete with a strong order unit, but also for others, see [**2**] and [**11**]) but not in many other cases (see [**11**] for many examples). Even in this special a setting it is not clear how to formulate a conjecture as to exactly which AM-spaces do have this property. In this note we give a complete solution in the rather special case of separable Banach lattices.

Our terminology and notation are standard, see e.g. [**8**]. In particular $\mathcal{L}(E)$ denotes the space of norm bounded linear operators on E whilst $\mathcal{L}^r(E)$ is the space of regular operators on E.

2. *The results*

Before proceeding to the main proof we need to prove a characterization of those separable AM-spaces which have a strong order unit, which may be of interest in its own right. As a preliminary to proving this, we need to show how to convert sequences with a minimal disjointness property into sequences which are genuinely disjoint. In the following statement, δ_{mn} denotes the Kronecker delta function, taking the value 1 if $m = n$ and 0 otherwise.

PROPOSITION 1. *Let E be a Banach lattice, (f_n) a sequence of lattice homomorphism functionals on E, (x_n) a sequence in E_+ with $f_m(x_n) = \delta_{mn}$ for $m, n \in \mathbb{N}$. Then there is a* disjoint *sequence (y_n) in E_+ with $y_n \leqslant x_n$ for all $n \in \mathbb{N}$ and with $f_m(y_n) = \delta_{mn}$.*

https://doi.org/10.1017/S0305004111000107 Published online by Cambridge University Press

Proof. Define

$$p = \sum_{n=2}^{\infty} \frac{x_n}{2^n \|x_n\|}$$

and set $y_1^1 = x_1 \vee p - p$, $z = x_1 \vee p - x_1$, so that $y_1^1 \perp z$, $0 \leqslant y_1^1 \leqslant x_1$ and $0 \leqslant z \leqslant p$. Now set $y_n^1 = 2^n \|x_n\| z \wedge x_n$ for $n \geqslant 2$ so again we have $y_n^1 \leqslant x_n$. If $m \neq n$ then $0 \leqslant f_m(y_n^1) \leqslant f_m(x_n) = 0$. As $f_1(x_n) = 0$ for $n \geqslant 2$ and f_1 is continuous we have $f_1(p) = 0$ so that

$$f_1(y_1^1) = f_1(x_1) \vee f_1(p) - f_1(p) = f_1(x_1) \vee 0 - 0 = f_1(x_1) = 1.$$

Note that

$$f_m(p) = \sum_{n=2}^{\infty} \frac{f_m(x_n)}{2^n \|x_n\|} = \frac{1}{2^m \|x_m\|}$$

and that if $m \geqslant 2$ then $f_m(z) = f_m(x_1) \vee f_m(p) - f_m(x_1) = 0 \vee f_m(p) - 0 = f_m(p)$, so that $f_m(y_m^1) = 2^m \|x_m\| f_m(z) \wedge f_m(y_m^1) = 1$. Thus $f_m(y_n^1) = \delta_{mn}$. As $y_1^1 \perp z$ we also have $y_1^1 \perp y_n^1$ for $n \geqslant 2$.

Apply this procedure now to the sequence $(y_n^1)_{n=2}^{\infty}$ and we obtain a new sequence $(y_n^2)_{n=2}^{\infty}$ with

$$\begin{aligned} f_m(y_n^2) &= \delta_{mn} \quad (n \geqslant 2) \\ y_n^2 &\leqslant y_n^1 \quad (n \geqslant 2) \\ y_2^2 &\perp y_n^2 \quad (n \geqslant 3). \end{aligned}$$

Note that if $n \geqslant 2$ then $0 \leqslant y_n^2 \leqslant y_n^1 \perp y_1^1$ so that $y_n^2 \perp y_1^1$.

Repeating this process inductively and taking $y_n = y_n^n$ gives the desired sequence.

A Banach lattice in which every norm bounded disjoint positive family has a supremum is known to be Dedekind complete and to have a strong order unit. It remains an open question as to whether or not every norm bounded disjoint positive family being order bounded suffices to prove the existence of a strong order unit. Even if we assume initially that we are dealing with an AM-space this is still unclear. We do however have a partial result that suffices for our purposes.

The proof makes use of results from the theory of compact Choquet simplexes. Recall that if K is a compact convex set in a locally convex space then $A(K)$ denotes the space of continuous real-valued affine functions on K. Equip this space with the supremum norm over K and the pointwise partial order. We will denote the set of extreme points of K by $\mathcal{E}(K)$. If $0 \in \mathcal{E}(K)$ then we will write $A_0(K)$ for those $f \in A(K)$ with $f(0) = 0$. If E is an AM-space then the set $K = \{f \in E_+^* : \|f\| \leqslant 1\}$ is a compact Choquet simplex under the weak* topology and E is isometrically order isomorphic to $A_0(K)$. The Choquet simplexes that arise in this way are rather special, one characterization being that the set $[0, 1] \times \mathcal{E}(K)$ is closed, [**5**, théorème 1·30]. For a general simplex it is not true that the closure of a face need be a face, see [**3**] for example, but for the simplexes arising from AM-spaces the closure of a face is again a face, see [**5**, théorème 1·30]. In particular, if $A \subset \mathcal{E}(K)$ then its convex hull, $\mathrm{co}(A)$ is a face and therefore $F = \overline{\mathrm{co}}(A)$ is also a face. Milman's converse of the Krein-Milman theorem, see [**9**, p. 9] or [**10**, Proposition 1·5], states that $\mathcal{E}(F) \subset \overline{A}$. It is simple, using Hahn–Banach separation for example, to see that E has a strong order unit if and only if the origin is *not* in the weak* closure of $\mathcal{E}(K) \setminus \{0\}$.

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PROPOSITION 2. *If E is a separable AM-space in which every norm bounded disjoint positive sequence is order bounded then E has a strong order unit.*

Proof. Suppose that E is a separable AM-space without a strong order unit. We identify E with $A_0(K)$ as above. As E is separable, the weak* topology on K is metrizable and because E does not have a strong order unit the origin is in the closure of $\mathcal{E}(K) \setminus \{0\}$. By metrizability we may find a sequence of distinct points $k_n \in \mathcal{E}(K)$ with $k_n \to 0$. For each $n \in \mathbb{N}$, set $F_n = \overline{\text{co}}(\{k_m : m \neq n\}$ which is a closed face of K. As $k_n \notin \overline{\{k_m : m \neq n\}}$ it is certainly true that $k_n \notin \mathcal{E}(F_n)$. As k_n is extreme in K if it did lie in F_n it would certainly be extreme, so that $k_n \notin F_n$. Thus $\{k_n\}$ and F_n are disjoint closed faces of K. It is a simple consequence of Edwards' Theorem, [**4**, Proposition II·3·13], that there is $f_n \in A(K)$ with $f_{n|F_n} \equiv 0$, $f_n(k_n) = 1$ and $0 \leqslant f_n(k) \leqslant 1$ for all $k \in K$. As $0 \in \overline{\{k_m : m \neq n\}} \subset F_n$, $f_n(0) = 0$ so that $f_n \in A_0(K)$, which we are identifying with E.

The maps $f \mapsto f(k_n)$ are lattice homomorphisms on $A_0(K)$ and we have $f_n(k_m) = \delta_{mn}$ so we may invoke Proposition 1 to produce a positive disjoint sequence (g_n) in $A_0(K)$ with $g_n \leqslant f_n$ and $g_n(k_m) = \delta_{mn}$. The sequence (g_n) is certainly norm bounded. If e were an upper bound for the sequence then we would have $e(k_n) \geqslant g_n(k_n) = 1$ for all $n \in \mathbb{N}$, so that $0 = e(0) = \lim_{n\to\infty} e(k_n) \geqslant 1$.

In the statement of the following theorem, c denotes the usual space of convergent real sequences, c_0 the subspace of null sequences and "isomorphic" means the existence of a linear operator that is both a Banach space and order isomorphism.

THEOREM 3. *A separable Banach lattice E has the property that $\mathcal{L}(E) = \mathcal{L}^r(E)$ if and only if one of the following three conditions hold*:

(i) *E is isomorphic to a (separable) AL-space*;
(ii) *E is isomorphic to a finite product of copies of c*;
(iii) *E is isomorphic to c_0.*

Proof. Clearly the property that $\mathcal{L}(E) = \mathcal{L}^r(E)$ is unaffected by change of norm. In case (1) a proof of equality may be found, for example, in [**8**, theorem 1·5·11], in case (2) it is a consequence of [**11**, theorem 2·12] and in case (3) it follows from [**11**, theorem 2·2].

If we have equality, then E is isomorphic to either an AL-space or to an AM-space by [**1**]. We need only address the case that E is isomorphic to an AM-space. As above, we identify E with $A_0(K)$, where K is a metrizable compact Choquet simplex with 0 as an extreme point. Consider the open subsets $P_{f,\epsilon} = \{k \in \mathcal{E}(K) : f(k) > \epsilon\}$, for $f \in A_0(K)_+$ and $\epsilon > 0$, of $\mathcal{E}(K) \setminus \{0\}$. Suppose first that all such sets are finite. We are dealing with a metric topology, so each point in $P_{f,\epsilon}$ must be isolated. Furthermore, the union of all $P_{f,\epsilon}$ is the whole of $\mathcal{E}(K) \setminus \{0\}$ as if $f(k) = 0$ for all positive f we also have $f(k) = 0$ for all f which means that $k = 0$. Thus $\mathcal{E}(K) \setminus \{0\}$ is a discrete space. If E does not have a strong order unit then 0 is a limit point and there can be no non-zero limit points ℓ as if there were we can pick $f \geqslant 0$ with $f(\ell) > 0$ and therefore $f(k) > 0$ for an infinite number of choices of $k \in \mathcal{E}(K) \setminus \{0\}$. It follows that $\mathcal{E}(K)$ may be identified with the one point compactification of $\mathbb{N}$, still assuming that E does not have a strong order unit. In particular $\mathcal{E}(K)$ is a closed subset of K so that K is a *Bauer* simplex and by Theorem II.4.1 of [**4**] $A(K)$ may be identified with the space of all continuous real valued functions on $\mathcal{E}(K)$, which is in turn isomorphic to c. Thus $A_0(K)$, and hence E, is isomorphic to c_0.

If E does not have a strong order unit and some set $P_{f,\epsilon}$ is infinite then by scaling we may suppose that there is $e \in A_0(K)$ such that $\{k \in \mathcal{E}(K) : e(k) \geqslant 1\}$ is an infinite set. Using

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the compactness and metrizability of K we may find a sequence of distinct points (k_n) in $\mathcal{E}(K)$ with $e(k_n) \geqslant 1$ for all $n \in \mathbb{N}$, $k_n \to k \in K$ and each $k_n \neq k$. By Proposition 2 there is a norm bounded sequence (x_n) in E_+ which is not bounded above. We may now define a bounded linear operator $T \in \mathcal{L}\big(A_0(K)\big)$ by the formula

$$Tf = \sum_{n=1}^{\infty} (f(k_n) - f(k))x_n,$$

as in the proof of Case I of the proof of [**11**, theorem 2·8]. Suppose that $U \in \mathcal{L}(A_0(K))$ with $U \geqslant T, 0$, then for each $n \in \mathbb{N}$ there is $f_n \in A_0(K)_+$ with $f_n(k_n) = 1$ and $f_n(k) = 0$ using Edwards' Theorem again. As lattice operations in $A_0(K)$ are pointwise on $\overline{\mathcal{E}(K)}$, if we set $g_n = f_n \wedge e$ then we still have $g_n(k_n) = 1$ and $g_n(k) = 0$. Thus, for all $n \in \mathbb{N}$, we have

$$Ue \geqslant Ug_n = \sum_{m=1}^{\infty} (g_n(k_m) - g_n(k))x_m \geqslant x_n$$

which contradicts the fact that (x_n) is not bounded above.

There remains to consider only the possibility that E has a strong order unit, when it is isomorphic to some space $C(X)$, where X is a compact metrizable Hausdorff space. If X is a finite set then E is finite dimensional, so certainly isomorphic to an AL-space. If X is infinite then it will contain a non-trivial convergent sequence and it then follows from [**11**, theorem 2·15] that we are in case (2).

Acknowledgements. At the International Conference of Young Scientists on "Mathematical Analysis and Mathematical Modeling" (Vladikavkaz, 12–19, July, 2010) I gave a talk titled *When are all bounded operators between classical Banach lattices regular?* At the end of that talk Prof. A. Kusraev asked about possible further advances in the area. That question, for which I would like to thank him, led directly to this paper.

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