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Super-Exponential Extinction Time of the Contact Process on Random Geometric Graphs

VAN HAO CAN

Aix Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France
Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, 10307 Ha Noi, Vietnam
(e-mail: cvhao89@gmail.com)

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In this paper we prove lower and upper bounds for the extinction time of the contact process on random geometric graphs with connection radius tending to infinity. We obtain that for any infection rate $\lambda > 0$, the contact process on these graphs survives a time super-exponential in the number of vertices.

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1. Introduction

We will study the contact process on random geometric graphs (RGGs) in $d \geqslant 2$ dimensions with intensity function $g = g_n(x)$ and connection radius R, denoted by $G(n, R, g)$.

An RGG is constructed as follows. The vertex set is composed of the atoms of a Poisson point process with intensity g on $[0, \sqrt[d]{n}]^d$. Then for any two vertices $v \neq w$, we draw an edge between them if $\|v - w\| \leqslant R$, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$. We will assume throughout this paper that there are positive constants b and B, such that

$$0 < b \leqslant g(x) \leqslant B < +\infty \quad \text{for all } x. \tag{1.1}$$

Here and below, we remove the subscript n in the function g_n for simplicity.

The contact process is one of the most studied interacting particle systems and is also often interpreted as a model to describe the spread of a virus in a network (see for instance [14]). Mathematically, it can be defined as follows: given a locally finite graph $G = (V, E)$ and $\lambda > 0$, the contact process on G with infection rate λ is a pure jump Markov process $(\xi_t)_{t \geqslant 0}$ on $\{0,1\}^V$. Vertices of V (also called sites) are regarded as individuals which are either infected (state 1) or

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healthy (state 0). By considering ξ_t as a subset of V via $\xi_t \equiv \{v : \xi_t(v) = 1\}$, the transition rates are given by

$$\xi_t \to \xi_t \setminus \{v\} \text{ for } v \in \xi_t \text{ at rate } 1, \quad \text{and}$$
$$\xi_t \to \xi_t \cup \{v\} \text{ for } v \notin \xi_t \text{ at rate } \lambda |\{w \in \xi_t : \{v,w\} \in E\}|,$$

where $|A|$ is the cardinality of a set A.

Originally the contact process was studied on integer lattices or homogeneous trees. More recently, probabilists started investigating this process on some families of random networks like configuration models, or preferential attachment graphs; see for instance [1, 2, 4, 5, 17, 18].

Random geometric graphs have been extensively studied for a long time by many authors; see in particular Penrose's book [21]. Recently, these graphs have also been considered as models of wireless networks (see *e.g.* [13]). Therefore, there has been interest in processes occurring on them, including the contact process in both theoretical and practical approaches; see for example [9, 10, 22].

In this paper, we are in particular interested in the extinction time of the contact process,

$$\tau_n = \inf\{t : \xi_t^1 = \varnothing\},$$

where (ξ_t^1) is the contact process on $G(n,R,g)$ starting with all nodes infected.

It has been shown that the contact process on finite graphs dies out almost surely, thus $\tau_n < \infty$ a.s. Now, it is interesting to determine the order of magnitude of τ_n. For sparse graphs, that is, graphs in which the number of edges is of the order of the number of vertices, we will show that the extinction time is at most exponential in the number of vertices (see Lemma 5.1). On the other hand, we will show in Section 2.1 that the extinction time of the contact process on a complete graph is super-exponential in the number of vertices. For the random geometric graph $G(n,g,R)$, we observe that w.h.p. the number of vertices is $\Theta(n)$ and the graph locally looks like a complete graph (all vertices in a ball of radius $R/2$ form a clique). Hence, we can expect that $\log \tau_n$ is super-linear in n as in the case of the complete graph. (Note that there are graphs which are not sparse but for which $\log \tau_n = O(n)$, for example the configuration model with infinite mean degree; see Theorem 1.2(ii) in [4].)

Ganesan [10, Theorem 1.2] considers the contact process on an equivalent model of $G(n,R,g)$ in two dimensions. Translating to our model, Ganesan proves that if $R \to \infty$ and $R^2 = O(\log n)$, then there exist positive constants $c = c(\lambda)$ and $C = C(\lambda)$, such that w.h.p. $Cn\log n \geqslant \log \tau_n \geqslant cnR^2/\log n$.

In our main result, we will prove that in all dimensions larger than or equal to 2, for any $\lambda > 0$ and for all R large enough, w.h.p. $\log \tau_n = \Theta(n\log(\lambda R^d))$.

Theorem 1.1. *Let $d \geqslant 2$ and τ_n be the extinction time of the contact process on the graph $G(n,R,g)$ with g satisfying* (1.1) *starting from full occupancy. Then there exist positive constants c, C and K depending only on d, b, B, such that the following statements hold.*

(i) *For any $R = R(n)$ and $\lambda = \lambda(n)$ satisfying $n \geqslant R^d \geqslant K/(\lambda \wedge 1)$, w.h.p.*

$$\tau_n \geqslant \exp(cn\log(\lambda R^d))$$

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and

$$\frac{\tau_n}{\mathbb{E}(\tau_n)} \xrightarrow[n\to\infty]{(\mathcal{L})} \mathcal{E}(1),$$

with $\mathcal{E}(1)$ *an exponential random variable with mean one.*

(ii) *For all* $R > 0$*, w.h.p.*

$$\tau_n \leqslant \exp(Cn\log(\lambda R^d)).$$

Part (i) implies that when R tends to infinity, the contact process survives a time super-exponential in n regardless of the value of λ. We usually say that in this case the critical value of the infection rate is zero. On the other hand, recently in [20] Ménard and Singh have shown that when R is *fixed*, there is a non-trivial phase transition of the contact process on *infinite* random geometric graphs (*i.e.* the vertices are atoms of a Poisson point process on the whole space $\mathbb{R}^d$). More precisely, they prove that there exists a constant $\lambda_c > 0$, such that if $\lambda < \lambda_c$, the contact process dies out a.s. whereas if $\lambda > \lambda_c$, it survives forever with positive probability. Moreover, in [3, Section 5.3.3], by slightly improving some details in the proof of Ménard and Singh, we show that $\lambda_c(R) = \Theta(R^{-d})$.

It has been observed in many examples that the contact process on a sequence of finite graphs, say (G_n), converging locally to some limiting graph, say G, exhibits a phase transition at the same critical value of infection rate as on the limit G: in the sub-critical regime, the contact process on G dies out a.s. (resp. the extinction time τ_n of the process on G_n is of order $\log(|G_n|)$), whereas in the super-critical regime, the contact process survives forever with positive probability (resp. $\log \tau_n$ is of order $|G_n|$); see for instance [2, 5, 6, 17, 19].

Our Theorem 1.1(i) implies that in the case of random geometric graphs, this phase transition also holds in a 'highly' super-critical phase, *i.e.* it holds when $\lambda > C\lambda_c$, with C a positive constant independent of R.

We now make some comments on the proof of Theorem 1.1. The proof of (i) consists of two main steps. First, we find in $G(n,R,g)$ a *key subgraph* composed of $\lceil cnR^{-d} \rceil$ adjacent complete graphs, each of size $\lfloor cR^d \rfloor$; see Lemma 3.2. The proof of this part is based on the existence of long paths in super-critical site percolation in $\mathbb{Z}^d$. Secondly, we study the extinction time of the contact process on this key subgraph. This part is based on a comparison between the contact process and a super-critical oriented percolation; see Section 4. The proof of (ii) follows from a quite general argument: the extinction time of the contact process on a graph $G = (V,E)$ is at most $\exp(C|V|\log(|E|/|V|))$, for some positive constant C.

The paper is organized as follows. In Section 2 we prove some preliminary results on the contact process on complete graphs and the oriented percolation in two dimensions. In Section 3 we prove the existence of a key subgraph mentioned above. In Section 4 we study the contact process on this key subgraph. In Section 5 we conclude the proof of Theorem 1.1 by using results in the previous sections. Finally, in Section 6, we study some extensions: the case $d = 1$ and the equivalent model considered in [10].

We now fix some notation. The size of a graph G will be the cardinality of its set of vertices and we denote it by $|G|$. For $\mu > 0$, we let $\mathrm{Poi}(\mu)$ denote a Poisson random variable with mean μ, and we let $\mathcal{E}(\mu)$ be an exponential random variable with mean $1/\mu$. For $x > 0$, we let $\lfloor x \rfloor$ (resp.

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$\lceil x \rceil$) denote the greatest (resp. least) integer less (resp. greater) than or equal to x. If f and g are two real functions, we write $f = O(g)$ if there exists a constant $C > 0$, such that $f(x) \leqslant Cg(x)$ for all x, $f = \Theta(g)$ if $f = O(g)$ and $g = O(f)$, and $f = o(g)$ if $g(x)/f(x) \to 0$ as $x \to \infty$. The term w.h.p. means 'with probability tending to 1'.

2. Preliminaries

2.1. Contact process on complete graphs

We let K_m denote the complete graph of size m. Similarly to the results for the contact process on star graphs in [5, 18], we prove the following.

Lemma 2.1. *Assume that $\lambda \leqslant 1$ and $m\lambda \geqslant 640$. Then the following assertions hold.*

(i) *Let (ξ_t) be the contact process on K_m. Then*

$$\mathbb{P}\left(\inf_{T_m/2 \leqslant t \leqslant T_m} |\xi_t| \geqslant m/4 \mid |\xi_0| \geqslant m/4 \right) \geqslant 1 - 2T_m^{-1},$$

with $T_m = \exp(m \log(\lambda m)/16)$.

(ii) *Let K_m^1 and K_m^2 be two disjoint complete graphs of size m, and let $K_{m\text{-}m}$ be the graph formed by adding an edge between these two graphs. Let (ξ_t) be the contact process on $K_{m\text{-}m}$. Then*

$$\mathbb{P}(|\xi_{T_m} \cap K_m^2| \geqslant m/4 \mid |\xi_0 \cap K_m^1| \geqslant m/4) \geqslant 1 - 5T_m^{-1}.$$

Proof. Part (i) follows from the claims

$$\mathbb{P}\left(\inf_{0 \leqslant t \leqslant T_m} |\xi_t| \geqslant m/4 \mid |\xi_0| \geqslant m/2 \right) \geqslant 1 - T_m^{-1}, \tag{2.1}$$

$$\mathbb{P}(\exists t \leqslant T_m/2 : |\xi_t| \geqslant m/2 \mid |\xi_0| \geqslant m/4) \geqslant 1 - T_m^{-1}. \tag{2.2}$$

First, we observe that $|\xi_t|$ increases by 1 with rate $\lambda |\xi_t|(m - |\xi_t|)$ and decreases by 1 with rate $|\xi_t|$. Therefore, the skeleton of $(|\xi_t|)$ is a random walk (U_r) trapped at 0, which satisfies $U_0 = |\xi_0|$ and

$$U_{r+1} = U_r + 1 \quad \text{with probability } p_1 = \frac{\lambda(m - U_r)}{\lambda(m - U_r) + 1},$$
$$U_{r+1} = U_r + 1 \quad \text{with probability } 1 - p_1.$$

We now prove (2.1). Assume that $|\xi_0| \geqslant m/2$. Then $U_0 \geqslant m/2$. Moreover, if $U_r \in (m/4, 3m/4)$ then $p_1 \geqslant \lambda m/(\lambda m + 4)$. Hence, when $U_r \in (m/4, 3m/4)$, it stochastically dominates a random walk (X_r) satisfying $X_0 = m/2$ and

$$X_{r+1} = X_r + 1 \quad \text{with probability } \frac{\lambda m}{\lambda m + 4},$$
$$X_{r+1} = X_r - 1 \quad \text{with probability } \frac{4}{\lambda m + 4}.$$

Then θ^{X_r} is a martingale, where

$$\theta = \frac{4}{\lambda m}.$$

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Let q be the probability that X_r goes below $m/4$ before hitting $3m/4$. It follows from the optional stopping theorem that

$$q\theta^{m/4} + (1-q)\theta^{3m/4} \leqslant \theta^{m/2}.$$

Therefore, using $\lambda m \geqslant 640$, we get

$$q \leqslant \theta^{m/4} = (4/\lambda m)^{m/4} \leqslant T_m^{-3}/(2m^2). \tag{2.3}$$

Hence, the random walk (X_r) (and thus $(|\xi_t|)$) makes at least $\lfloor m^2 T_m \rfloor$ upcrossings between $m/2$ and $3m/4$ before hitting $m/4$ with probability larger than

$$1 - \lfloor m^2 T_m \rfloor T_m^{-3}/(2m^2) \geqslant 1 - T_m^{-1}/2. \tag{2.4}$$

The law of the waiting time between two upcrossings of $(|\xi_t|)$ stochastically dominates $\mathcal{E}(L)$, with $L = \lambda \lfloor m/2 \rfloor (m - \lfloor m/2 \rfloor) + \lfloor m/2 \rfloor$, the waiting time when $|\xi_t| = \lfloor m/2 \rfloor$.

Suppose that $(|\xi_t|)$ makes more than $\lfloor m^2 T_m \rfloor$ consecutive upcrossings. Then the time that $(|\xi_t|)$ stays above $m/4$ stochastically dominates S, the sum of $\lfloor m^2 T_m \rfloor$ i.i.d. exponential random variables with mean $1/L$. By applying Chebyshev's inequality, we get

$$\mathbb{P}(S < \lfloor m^2 T_m \rfloor / 2L) \leqslant 4/(\lfloor m^2 T_m \rfloor) \leqslant T_m^{-1}/2. \tag{2.5}$$

Since $L \leqslant m^2/2$, we deduce (2.1) from (2.4) and (2.5).

We now prove (2.2). Assume that $|\xi_0| \geqslant m/4$. Using a similar argument as for (X_r), we get that when $U_r \in (m/8, m/2)$, it stochastically dominates a random walk (Y_r) satisfying $Y_0 = m/4$ and

$$\begin{aligned} Y_{r+1} &= Y_r + 1 \quad \text{with probability } p_2 = \frac{\lambda m}{\lambda m + 2}, \\ Y_{r+1} &= Y_r - 1 \quad \text{with probability } 1 - p_2. \end{aligned}$$

Let us define

$$\sigma_Y = \inf\{r : Y_r \geqslant m/2\} \quad \text{and} \quad \tilde{\sigma}_Y = \inf\{r : Y_r \leqslant m/8\}.$$

Then, similarly to (2.3), we have

$$\mathbb{P}(\tilde{\sigma}_Y < \sigma_Y) \leqslant (2/\lambda m)^{m/8} \leqslant T_m^{-1}/3. \tag{2.6}$$

Since $Y_r - (2p_2 - 1)r$ is a martingale, it follows from the optional stopping theorem that

$$m/4 = \mathbb{E}(Y_{\sigma_Y \wedge r}) - (2p_2 - 1)\mathbb{E}(\sigma_Y \wedge r) \leqslant m/2 - (2p_2 - 1)\mathbb{E}(\sigma_Y \wedge r).$$

Therefore, using $m\lambda \geqslant 640$, we get

$$\mathbb{E}(\sigma_Y \wedge r) \leqslant \frac{m}{4(2p_2 - 1)} \leqslant m/3.$$

Letting t go to infinity, we obtain

$$\mathbb{E}(\sigma_Y) \leqslant m/3.$$

Thus, using Markov's inequality, we have

$$\mathbb{P}(\sigma_Y \geqslant mT_m) \leqslant \mathbb{E}(\sigma_Y)/mT_m \leqslant T_m^{-1}/3. \tag{2.7}$$

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Now, let us define

$$\sigma = \inf\{t : |\xi_t| \geqslant m/2\} \quad \text{and} \quad \tilde{\sigma} = \inf\{t : |\xi_t| \leqslant m/8\}.$$

Then, by (2.6),

$$\mathbb{P}(\tilde{\sigma} < \sigma) \leqslant \mathbb{P}(\tilde{\sigma}_Y < \sigma_Y) \leqslant T_m^{-1}/3. \tag{2.8}$$

On the other hand, when $|\xi_t| \in (m/8, m/2)$ the waiting time at each stage is an exponential random variable with mean less than $1/M$, with $M = \lambda \lfloor m/8 \rfloor (m - \lfloor m/8 \rfloor) + \lfloor m/8 \rfloor$, the mean of the waiting time when $|\xi_t| = \lfloor m/8 \rfloor$. Therefore

$$\sigma 1(\sigma < \tilde{\sigma}) \preceq \sum_{i=1}^{\sigma_Y} E_i,$$

where (E_i) is a sequence of i.i.d. exponential random variables with mean $1/M$ and independent of σ_Y. Hence

$$\begin{aligned}\mathbb{P}(T_m/2 \leqslant \sigma < \tilde{\sigma}) &\leqslant \mathbb{P}(\sigma_Y \geqslant mT_m) + \mathbb{P}\left(\sum_{i=1}^{\lfloor mT_m \rfloor} E_i \geqslant T_m/2\right) \\ &\leqslant 2T_m^{-1}/3. \end{aligned} \tag{2.9}$$

Here, we have used (2.7) to bound the first term, and for the second one we note that

$$\mathbb{E}(E_i) = 1/M \leqslant 64/(7\lambda m^2) \leqslant 1/(70m);$$

thus, using a standard large deviation result, we get a bound for this term. Now, it follows from (2.8) and (2.9) that

$$\mathbb{P}(\sigma \geqslant T_m/2) \leqslant T_m^{-1},$$

which proves (2.2).

For (ii), let u and v be two vertices in K_m^1 and K_m^2 respectively, such that there is an edge between u and v. Let (ξ_t') (resp. (ξ_t'')) be the contact process on K_m^1 (resp. K_m^2). By (i), we have

$$\mathbb{P}(\xi'_{T_m} \neq \varnothing \mid |\xi_0'| \geqslant m/4) \geqslant 1 - 2T_m^{-1}. \tag{2.10}$$

We now claim that

$$\mathbb{P}(\exists t \leqslant m^2/2 : |\xi_t''| \geqslant m/4 \mid \xi'_{m^2/2} \neq \varnothing) \geqslant e^{-m/4}. \tag{2.11}$$

To prove (2.11), it amounts to showing that

$$\mathbb{P}(\exists t \leqslant m/2 : |\xi_t''| \geqslant m/4 \mid |\xi_0''| = 1) \geqslant 2e^{-m/4}, \tag{2.12}$$

and

$$\mathbb{P}(v \text{ gets infected before } m^2/4 \mid \xi'_{m^2/4} \neq \varnothing) \geqslant 1/2. \tag{2.13}$$

For (2.12), observe that when $|\xi_t''| \leqslant m/4$, it increases by 1 in the next stage with probability

$$\frac{\lambda(m - |\xi_t''|)}{\lambda(m - |\xi_t''|) + 1} \geqslant \frac{3\lambda m}{3\lambda m + 4} > 0.9,$$

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as $\lambda m \geqslant 640$. Moreover, the waiting time to the next stage is an exponential random variable with mean less than 1. Therefore, the probability that in all the $\lceil m/4 \rceil$ first stages, $|\xi_i''|$ increases and the waiting time is less than 1, is larger than

$$(0.9(1-e^{-1}))^{\lceil m/4 \rceil} \geqslant 2e^{-m/4},$$

which implies (2.12). For (2.13), we note that

$$\{\xi'_{m^2/4} \neq \varnothing\} \subset \bigcap_{i=0}^{\lfloor m^2/8 \rfloor - 1} \mathcal{E}_i,$$

where

$$\mathcal{E}_i = \{\exists v_i \in K_m^1 : \xi'_{2i}(v_i) = 1\}.$$

We define

$$\begin{aligned}\mathcal{I}_i = \mathcal{E}_i \cap \{&\text{there is no recovery at } u_i \text{ in } [2i, 2i+1] \text{ and there is an infection spread from} \\ &u_i \text{ to } u \text{ in } [2i, 2i+1], \text{ there is no recovery at } u \text{ in } [2i, 2i+2] \text{ and there is an} \\ &\text{infection spread from } u \text{ to } v \text{ in } [2i+1, 2i+2]\}.\end{aligned}$$

If $u_i \equiv u$, we only consider the recovery in u and the infection spread from u to v. We see that if one of $(\mathcal{I}_i)$ occurs, then v gets infected before $m^2/4$, and for any $i = 0, \ldots, \lfloor m^2/8 \rfloor - 1$,

$$\mathbb{P}(\mathcal{I}_i \mid \cap_{j=0}^{i} \mathcal{E}_j) \geqslant e^{-3}(1-e^{-\lambda})^2 \geqslant \lambda^2/(4e^3), \tag{2.14}$$

as $\lambda \leqslant 1$. Therefore, by using induction we have

$$\begin{aligned}&\mathbb{P}(v \text{ is not infected before } m^2/4) \\ &\leqslant \mathbb{P}\left(\left(\bigcup_{i=0}^{\lfloor m^2/8 \rfloor - 1} \mathcal{I}_i\right)^c \cap \left(\bigcap_{i=0}^{\lfloor m^2/8 \rfloor - 1} \mathcal{E}_i\right)\right) \\ &\leqslant (1-\lambda^2/(4e^3))^{\lfloor m^2/8 \rfloor} \\ &\leqslant 1/2,\end{aligned} \tag{2.15}$$

since $\lambda m \geqslant 640$. Thus (2.13) follows.

We now prove (ii) by using (2.11). Suppose that $\xi'_{T_m} \neq \varnothing$. We divide the time interval $[0, T_m/2]$ into $\lfloor T_m/m^2 \rfloor$ small intervals of length $m^2/2$. Then (2.11) implies that in each interval with probability larger than $e^{-m/4}$, there is a time s, such that that $|\xi_s''| \geqslant m/4$. Hence, similarly to (2.15), we have

$$\mathbb{P}(\exists s \leqslant T_m/2 : |\xi_s''| \geqslant m/4 \mid \xi'_{T_m/2} \neq \varnothing) \geqslant 1 - (1-e^{-m/4})^{\lfloor T_m/m^2 \rfloor} \geqslant 1 - T_m^{-1}. \tag{2.16}$$

Suppose that $|\xi_s''| \geqslant m/4$ with $s \leqslant T_m/2$. Then (i) implies that $|\xi''_{T_m}| \geqslant m/4$ with probability larger than $1 - 2T_m^{-1}$. Combining this with (2.10) and (2.16), we get (ii). □

2.2. Oriented percolation on finite sets

For any positive integer ℓ, we consider an oriented percolation process on $[0, \ell]$ with parameter q defined as follows. Let

$$\Gamma = \{(i,k) \in [0,\ell] \times \mathbb{N} : i+k \text{ is even}\}.$$

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For each pair of sites (i,k) and $(j,k+1)$ with $j=i\pm 1$, we draw an arrow from (i,k) to $(j,k+1)$ with probability p, all these events being independent. Given the initial configuration $A\subset[0,\ell]$, the oriented percolation $(\eta_t)_{t\geqslant 0}$ is defined by

$$\eta_t^A=\{i\in[0,\ell]:\exists\, j\in A \text{ s.t. } (j,0)\to(i,t)\} \text{ for } t\in\mathbb{N},$$

where the notation $(j,0)\to(i,t)$ means that there is an oriented path from $(j,0)$ to (i,t). If $A=\{x\}$, we simply write (η_t^x). We call (η_t) a *Bernoulli-oriented percolation with parameter q*.

The oriented percolation on $\mathbb{Z}$, denoted by $(\bar\eta_t)$, was investigated by Durrett in [7]. Using his results and techniques, we will prove the following.

Lemma 2.2. *Let (η_t) be the oriented percolation on $[0,\ell]$ with parameter q. Then there exist positive constants ε and c, independent of q and ℓ, such that if $q\geqslant 1-\varepsilon$ then the following statements hold.*

(i) *For any ℓ, and $x\in[0,\ell]$,*

$$\mathbb{P}(\exists r,s\leqslant 2\ell,\ \text{s.t. } \eta_r^x(0)=1,\eta_s^x(\ell)=1)\geqslant c.$$

(ii) *For any ℓ,*

$$\mathbb{P}(\eta_{t_\ell}^{\mathbf{1}}\neq\varnothing)\geqslant 1-1/t_\ell,$$

where $t_\ell=\lfloor(1-q)^{-c\ell}\rfloor$ and $(\eta_t^{\mathbf{1}})$ is the oriented percolation starting with $\eta_0^{\mathbf{1}}=[0,\ell]$.

(iii) *There exist a positive constant $\beta\in(0,1)$ and an integer $s_\ell\in[\exp(c\ell),2\exp(c\ell)]$, such that*

$$\mathbb{P}(|\eta_{s_\ell}^{\mathbf{1}}\cap[(1-\beta)\ell/2,(1+\beta)\ell/2]|\geqslant 3\beta\ell/4)\geqslant 1-\exp(-c\ell).$$

Proof. Part (i) is similar to Theorem B.24(a) in [14], and (ii) can be proved using a contour argument as in [7, Section 10].

We now prove (iii). Let $(\bar\eta_t)$ be the oriented percolation on $\mathbb{Z}$. Then

$$\alpha=\mathbb{P}(\bar\eta_t^{\,0}\neq\varnothing\,\forall t)\to 1\quad\text{as } q\to 1.$$

Hence we can assume that $\alpha>3/4$. Now we define

$$\ell_1=\lfloor(8-\alpha)\ell/16\rfloor,\quad \ell_2=\lfloor(8+\alpha)\ell/16\rfloor\quad\text{and}\quad \ell_3=\lfloor\ell/4\rfloor.$$

We claim that there exists a positive constant c, such that for any $A\subset[\ell_1,\ell_2]$ with $|A|\geqslant 3\alpha\ell/32$,

$$\mathbb{P}(|\eta_{\ell_3}^A\cap[\ell_1,\ell_2]|\geqslant 3\alpha\ell/32)\geqslant 1-\exp(-c\ell).\tag{2.17}$$

Suppose that (2.17) holds for a moment; we now prove (iii). Let A_0 be an arbitrary subset of $[\ell_1,\ell_2]$ satisfying $|A_0|\geqslant 3\alpha\ell/32$. Then we define

$$\mathcal{M}_1=\{|\eta_{\ell_3}^{A_0}\cap[\ell_1,\ell_2]|\geqslant 3\alpha\ell/32\}.$$

It follows from (2.17) that

$$\mathbb{P}(\mathcal{M}_1)\geqslant 1-\exp(-c\ell).\tag{2.18}$$

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Moreover, if $\mathcal{M}_1$ happens, $\eta^{A_0}_{\ell_3} \cap [\ell_1, \ell_2]$ contains a subset A_1 whose cardinality is larger than $3\alpha\ell/32$. Thus we can define

$$\mathcal{M}_2 = \mathcal{M}_1 \cap \{|\eta^{A_1,\ell_3}_{2\ell_3} \cap [\ell_1, \ell_2]| \geqslant 3\alpha\ell/32\},$$

where for all $0 \leqslant s \leqslant t$ and $A \subset [0, \ell]$,

$$\eta^{A,s}_t = \{i \in [0, \ell] : \exists\, j \in A \text{ s.t. } (j, s) \to (i, t)\}.$$

Similarly, for all $k \geqslant 2$ we define

$$\mathcal{M}_{k+1} = \mathcal{M}_k \cap \{|\eta^{A_k,k\ell_3}_{(k+1)\ell_3} \cap [\ell_1, \ell_2]| \geqslant 3\alpha\ell/32\}, \tag{2.19}$$

where A_k is a subset of $\eta^{A_{k-1},(k-1)\ell_3}_{k\ell_3} \cap [\ell_1, \ell_2]$ satisfying $|A_k| \geqslant 3\alpha\ell/32$.

By (2.17), we have for all $k \geqslant 1$

$$\mathbb{P}(\mathcal{M}_{k+1} \mid \mathcal{M}_k) \geqslant 1 - \exp(-c\ell),$$

or equivalently

$$\frac{\mathbb{P}(\mathcal{M}_{k+1})}{\mathbb{P}(\mathcal{M}_k)} \geqslant 1 - e^{-c\ell}. \tag{2.20}$$

Using (2.18) and (2.20), we obtain that for $k_\ell = \lfloor e^{c\ell/2} \rfloor$,

$$\mathbb{P}(\mathcal{M}_{k_\ell}) \geqslant (1 - e^{-c\ell})^{k_\ell} \geqslant 1 - 1/k_\ell. \tag{2.21}$$

We have

$$\mathcal{M}_{k_\ell} \subset \{|\eta^{\mathbf{1}}_{s_\ell} \cap [\ell_1, \ell_2]| \geqslant 3\alpha\ell/32\}, \tag{2.22}$$

where $s_\ell = \ell_3 \times k_\ell$.

On the other hand, if $\beta = \alpha/8$, then $\ell_1 = \lfloor (1-\beta)\ell/2 \rfloor$ and $\ell_2 = \lfloor (1+\beta)\ell/2 \rfloor$. Hence, using (2.21) and (2.22), we get

$$\mathbb{P}(|\eta^{\mathbf{1}}_{s_\ell} \cap [(1-\beta)\ell/2, (1+\beta)\ell/2]| \geqslant 3\beta\ell/4) \geqslant 1 - \exp(-c\ell/2),$$

which implies that (iii) holds with $\beta = \alpha/8$.

Now it remains to prove (2.17). First, we observe that for all $t \leqslant \ell_3$,

$$\bar{\eta}^{[\ell_1,\ell_2]}_t \subset [\ell_1 - t, \ell_2 + t] \subset [\ell_1 - \ell_3, \ell_2 + \ell_3] \subset [0, \ell],$$

where $(\bar{\eta}_t)$ is the oriented percolation on $\mathbb{Z}$. Therefore, for any $A \subset [\ell_1, \ell_2]$,

$$(\bar{\eta}^A_t)_{0 \leqslant t \leqslant \ell_3} \equiv (\eta^A_t)_{0 \leqslant t \leqslant \ell_3}.$$

Hence, to simplify notation, we use (η_t) for both processes in the interval $[0, \ell_3]$. To prove (2.17), it suffices to show that there exists a positive constant c, such that

$$\mathbb{P}(|\eta^x_{\ell_3} \cap [\ell_1, \ell_2]| \geqslant 3\alpha\ell/32 \mid \eta^x_{\ell_3} \neq \varnothing) \geqslant 1 - \exp(-c\ell) \quad \text{for all } x \in [\ell_1, \ell_2], \tag{2.23}$$

$$\mathbb{P}(\eta^A_{\ell_3} \neq \varnothing) \geqslant 1 - \exp(-c\ell) \quad \text{for all } A \subset [\ell_1, \ell_2] \text{ with } |A| \geqslant 3\alpha\ell/32. \tag{2.24}$$

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To prove (2.23), we define for any $A \subset \mathbb{Z}$ and $t \geqslant 0$

$$r_t^A := \sup\{x : \exists y \in A, (y,0) \to (x,t)\},$$
$$l_t^A := \inf\{x : \exists y \in A, (y,0) \to (x,t)\}.$$

Then (2.23) is a consequence of the following claims.

(a) If $[\ell_1,\ell_2] \subset [l^x_{\ell_3}, r^x_{\ell_3}]$, then

$$\eta^{\mathbf{1}}_{\ell_3} \cap [\ell_1,\ell_2] \equiv \eta^x_{\ell_3} \cap [\ell_1,\ell_2].$$

(b) $\mathbb{P}(|\eta^{\mathbf{1}}_{\ell_3} \cap [\ell_1,\ell_2]| \geqslant 3(\ell_2-\ell_1)/4) \geqslant 1-\exp(-c\ell)$, as $3/4 < \alpha$.
(c) $\mathbb{P}([l^x_{\ell_3}, r^x_{\ell_3}] \supset [\ell_1,\ell_2] \mid \eta^x_{\ell_3} \neq \varnothing) \geqslant 1-\exp(-c\ell)$.

We start with claim (a). Suppose that $[\ell_1,\ell_2] \subset [l^x_{\ell_3}, r^x_{\ell_3}]$. Then there exists $y \leqslant \ell_1$ and $z \geqslant \ell_2$ together with two oriented paths: γ_1 from $(x,0)$ to (y,ℓ_3), and γ_2 from $(x,0)$ to (z,ℓ_3). Now, let u be any element of $\eta^{\mathbf{1}}_{\ell_3} \cap [\ell_1,\ell_2]$. Then $\ell_1 \leqslant u \leqslant \ell_2$ and there exists a vertex $v \in \mathbb{Z}$ and an oriented path γ' from $(v,0)$ to (u,ℓ_3). The path γ' is forced to intersect γ_1 or γ_2. In both cases, this implies the existence of an oriented path from $(x,0)$ to (u,ℓ_3). Hence $u \in \eta^x_{\ell_3} \cap [\ell_1,\ell_2]$. We have just proved that

$$\eta^{\mathbf{1}}_{\ell_3} \cap [\ell_1,\ell_2] \subset \eta^x_{\ell_3} \cap [\ell_1,\ell_2].$$

The reverse is trivial, hence we obtain (a).

Claim (b) follows from a result of Durrett and Schonmann [8, Theorem 1]. (Note that in [8] the result is proved for the contact process, but as mentioned by the authors the proof works just as well for oriented percolation.) In fact, it still holds if we replace $3/4$ by any $\alpha' < \alpha$. To prove (c), we observe that if $\eta^x_{\ell_3} \neq \varnothing$ then

$$r^x_{\ell_3} = r^{(-\infty,x]}_{\ell_3}.$$

Moreover, by the main result of Section 11 in [7], there is a positive constant c, such that for all integer x,

$$\mathbb{P}(r^{(-\infty,x]}_{\ell_3} \leqslant x + \alpha\ell_3/2) \leqslant \exp(-c\ell).$$

Therefore, if $x \in [\ell_1,\ell_2]$, then

$$\mathbb{P}(r^x_{\ell_3} \geqslant \ell_2 \mid \eta^x_{\ell_3} \neq \varnothing) \geqslant 1-\exp(-c\ell),$$

since $x + \alpha\ell_3/2 \geqslant \ell_1 + \alpha\ell_3/2 \geqslant \ell_2$. Similarly

$$\mathbb{P}(l^x_{\ell_3} \leqslant \ell_1 \mid \eta^x_{\ell_3} \neq \varnothing) \geqslant 1-\exp(-c\ell).$$

Then claim (c) follows from the last two estimates.

Now we prove (2.24) by using the same arguments as in Section 10 in [7]. We say that A is more spread out than B (and write $A \succ B$) if there is an increasing function φ from B into A such that $|\varphi(x) - \varphi(y)| \geqslant |x-y|$ for all $x,y \in B$. (Note that this implies $|A| \geqslant |B|$.) In [7], Durrett proves that there is a coupling such that if $A \succ B$ then

$$\eta^A_t \succ \eta^B_t \quad \text{for all } t \geqslant 0,$$

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and as a consequence $|\eta_t^A| \geqslant |\eta_t^B|$ for all t. Hence

$$\mathbb{P}(\eta_t^A = \varnothing) \leqslant \mathbb{P}(\eta_t^B = \varnothing).$$

On the other hand, by (ii),

$$\mathbb{P}(\eta_{\ell_3}^{[\ell_1,\ell_1+\ell_4]} = \varnothing) \leqslant \exp(-c\ell),$$

with $\ell_4 = \lfloor 3\alpha\ell/32 \rfloor - 1$. We observe that $A \succ [\ell_1, \ell_1 + \ell_4]$ for any A with $|A| \geqslant 3\alpha\ell/32$. Thus (2.24) follows from the last two inequalities. □

3. Existence of a key subgraph

In this section, by using a result on the existence of long paths in a super-critical site percolation, we will show that the random geometric graph contains a key subgraph composed of $\lceil cn/R^d \rceil$ complete graphs of size $\lfloor cR^d \rfloor$, with some $c > 0$.

The Bernoulli site percolation on $\mathbb{Z}^d$ with parameter p is defined as usual: designate each vertex in $\mathbb{Z}^d$ to be open independently with probability p and closed otherwise. A path in $\mathbb{Z}^d$ is called open if all its sites are open. Then there is a critical value $p_c^s(d) \in (0,1)$, such that if $p > p_c^s(d)$, then a.s. there exists an infinite open path (cluster), whereas if $p < p_c^s(d)$, a.s. there is no infinite cluster.

Lemma 3.1. *Consider the Bernoulli site percolation on $[0,n]^d$ with $d \geqslant 2$ and $p > p_c^s(2)$. Then there exists a positive constant $\rho = \rho(p,d)$, such that w.h.p. there is an open path whose length is larger than ρn^d.*

Now, we define the key subgraph. For $\ell, m \in \mathbb{N}$, we let $\mathcal{C}(\ell, m)$ denote the graph obtained by glueing a complete graph of size m to each vertex in a path of length ℓ.

Lemma 3.2. *Suppose that $d \geqslant 2$ and g satisfies* (1.1)*. Then there exist positive constants c and C, such that if $n \geqslant R^d \geqslant C$ then w.h.p. $G(n,R,g)$ contains a copy of $\mathcal{C}(\lceil cnR^{-d} \rceil, \lfloor cR^d \rfloor)$ as a subgraph.*

Proof. If n/R^d is bounded from above, then w.h.p. $G(n,R,g)$ contains a clique of size of order n and thus the result follows. Indeed, by definition the vertices in $A = [0, R/\sqrt{d}]^d$ form a complete graph. Moreover the number of vertices in A is a Poisson random variable with mean $\int_A g(x)dx = \Theta(R^d) = \Theta(n)$, and hence w.h.p. it is of order n.

We now assume that n/R^d tends to infinity. Let $\ell = \lfloor \sqrt[d]{n}/(R/2\sqrt{d}) \rfloor$. We divide the box $[0, \sqrt[d]{n}]^d$ into ℓ^d smaller boxes of equal size, enumerated by $(E_a)_{a \in [1,\ell]^d}$, whose side length is $R/(2\sqrt{d})$. We see that if v and w are in the same small box or in adjacent ones, then $\|v - w\| \leqslant R$, hence these two vertices are connected. This implies that the vertices on a small box form a clique and two adjacent cliques are connected.

For any $a \in [1,\ell]^d$, let

$$X_a = \#\{v : v \in E_a\}$$

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denote the number of vertices located in E_a. Then (X_a) are independent and X_a is a Poisson random variable with mean

$$\mu_a = \int_{E_a} g(x)dx \geqslant b\left(\frac{R}{2\sqrt{d}}\right)^d =: \mu, \tag{3.1}$$

since $g(x) \geqslant b$ for all x. For any a, we define

$$Y_a = 1(\{X_a \geqslant \mu/2\}).$$

Since $\mathbb{P}(\mathrm{Poi}(\mu) \geqslant \mu/2) \to 1$ as $\mu \to \infty$, it follows from (3.1) that $\mathbb{P}(Y_a = 1) \to 1$ as $R \to \infty$. Therefore there is a positive constant C, such that if $R^d \geqslant C$ then

$$\mathbb{P}(Y_a = 1) \geqslant p := (1 + p_c^s(2))/2.$$

We note that the Bernoulli random variables (Y_a) are independent. Hence, if we say the small box E_a is open when $Y_a = 1$ and closed otherwise, then we get a site percolation on $[1,\ell]^d$ which stochastically dominates the Bernoulli site percolation on $[1,\ell]^d$ with parameter $p > p_c^s(2)$. Then Lemma 3.1 gives that w.h.p. there is an open path of length $\rho\ell^d = \Theta(nR^{-d})$. On the other hand, in each open box, there is a clique of size $\mu/2\Theta(R^d)$ and these cliques in adjacent open boxes are connected. Hence, the result follows by taking c small enough. □

Proof of Lemma 3.1. We set $m = \lfloor n^{1/4} \rfloor$. For $n, d \geqslant 2$, we say that the box $[0,n]^d$ is ρ-*good* if the site percolation cluster on it satisfies the following.

There exist two vertices x in $\{m\} \times [0,n]^{d-1}$ and y in $\{n-m\} \times [0,n]^{d-1}$ and an open path composed of three parts: the first one included in $[0,m] \times [0,n]^{d-1}$ has length larger than m and ends at x; the second one included in $[m, n-m] \times [0,n]^{d-1}$ has length larger than ρn^d, starts at x and ends at y; the third one included in $[n-m,n] \times [0,n]^{d-1}$ starts at y and has length larger than m. □

We now prove by induction on d that if $p > p_c^s(2)$, there is a positive constant $\rho_d = \rho(p,d)$, such that w.h.p. the box $[0,n]^d$ is ρ_d-good. Then Lemma 3.1 immediately follows.

When $d = 2$, the statement is proved by Grimmett in [12, Theorem 1]. We will prove it for $d = 3$; the proof for $d \geqslant 4$ is exactly the same and will not be reproduced here.

For $1 \leqslant i \leqslant n$, let $\Lambda_i = \{i\} \times [2m, n-2m]^2$. We define

$$n_1 = n - 4m \quad \text{and} \quad m_1 = \lfloor n_1^{1/4} \rfloor.$$

We say that the ith plane is *nice* (or Λ_i is nice) if the site percolation on this plane satisfies the following: Λ_i is ρ_2-good (we consider Λ_i as a box in $\mathbb{Z}^2$), and in each of the rectangles $\{i\} \times [m, 2m+m_1] \times [0,n]$ and $\{i\} \times [n-2m-m_1, n-m] \times [0,n]$, there is a unique connected component of size larger than m_1. See Figure 1 for a sample of a nice plane.

The result for $d = 2$ implies that w.h.p. Λ_i is ρ_2-good. On the other hand, we know that w.h.p. in the percolation on a box of size n there is a unique open cluster having diameter larger than $C \log n$ for some C large enough (see for example Theorem 7.61 in [11]). Thus w.h.p. there is a unique open cluster of size larger $(C \log n)^d$. Hence Λ_i is nice w.h.p. for all $i = 1, \ldots, n$. Moreover, the events $\{\Lambda_i \text{ is nice}\}$ are independent since the planes are disjoint. Therefore $\mathcal{A}_n$ holds w.h.p.

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with

$$\mathcal{A}_n = \{\#\{i : m \leqslant i \leqslant n-m, \Lambda_i \text{ is nice}\} \geqslant n/2\}.$$

On $\mathcal{A}_n$, there are more than $n/2$ disjoint open paths (they are in disjoint planes), each of which has length larger than $\rho_2 n_1^2$. Thus, to obtain an open path of length of order n^3, we will glue these long paths using shorter paths in good boxes of nice planes. To do that, we define the following construction:

$\mathcal{B}_n = \{$for all $1 \leqslant i \leqslant \lfloor n/2 \rfloor$, there exist open paths:

$\ell_{2i-1} \subset \{2i-1\} \times [m, 2m] \times [0, n]$ whose end-vertices are u and v with third coordinates 0 and n respectively;

$\ell'_{2i-1} \subset \{2i-1\} \times [n-2m, n-m] \times [0, n]$ whose end-vertices are u' and v' with third coordinates 0 and n respectively;

$\ell_{2i} \subset \{2i\} \times [m, 2m] \times [0, n]$ whose end-vertices are z and t with second coordinates m and $2m$ respectively;

$\ell'_{2i} \subset \{2i\} \times [n-2m, n-m] \times [0, n]$ whose end-vertices are z' and t' with second coordinates $n-2m$ and $n-m$ respectively$\}$.

We observe that ℓ_{2i-1} is a bottom–top crossing and ℓ_{2i} is a left–right crossing in two consecutive rectangles. Then they intersect when we consider only the last two coordinates, and the same holds for ℓ'_{2i-1} and ℓ'_{2i}. Hence on $\mathcal{B}_n$, for all $1 \leqslant i \leqslant n-2$, there exist $a_i \in [m, 2m] \times [0, n]$ and $b_i \in [n-2m, n-m] \times [0, n]$, such that

$$\begin{aligned} (i, a_i) \in \ell_i \quad \text{and} \quad (i+1, a_i) \in \ell_{i+1}, \\ (i, b_i) \in \ell'_i \quad \text{and} \quad (i+1, b_i) \in \ell'_{i+1}. \end{aligned}$$

In other words, we can jump from the ith plane to the next one in two ways. Moreover, on $\mathcal{A}_n$ for all i such that the ith plane is nice, the first part of the long open path in Λ_i is connected to ℓ_i (as these paths are in the same rectangle $\{i\} \times [m, 2m + m_1] \times [0, n]$ and have length larger than m_1), and similarly the third part is connected to ℓ'_i; see Figure 1.

On $\mathcal{A}_n \cap \mathcal{B}_n$, we can find in $[m, n-m] \times [0, n]^2$ a path of length larger than $\rho_2 n^3/3$. Indeed, let i be the first index, such that $i \geqslant m$ and Λ_i is nice. We start at an end-point – *e.g.* from the right – of the long path in Λ_i, then go along this long path towards the other end-point. Then we can go to ℓ_i and arrive at (i, a_i). Now we jump to $(i+1, a_i)$ (recall that it is a neighbour of (i, a_i)). If the $(i+1)$th plane is not nice, we go to $(i+1, a_{i+1})$ to jump to the next plane (note that both $(i+1, a_i)$ and $(i+1, a_{i+1})$ are in ℓ_{i+1}). If the $(i+1)$th plane is nice, we can now touch and then go along to the long path in this plane and arrive at $(i+1, b_{i+1})$ to jump to the next plane. By continuing this procedure, we can go through all the long paths of nice planes in the definition of $\mathcal{A}_n$. The resulting path is in $[m, n-m] \times [0, n]^2$ and has length larger than $\rho_2 n^3/3$.

Moreover, in the slabs $[0, m] \times [0, n]^2$ and $[n-m, n] \times [0, n]^2$, w.h.p. we can find two paths of length larger than m which are connected to the long path we have just found above. These paths form the required three-part long path. Therefore, on $\mathcal{A}_n \cap \mathcal{B}_n$, w.h.p. the box $[0, n]^3$ is ρ_3-good with $\rho_3 = \rho_2/3$.

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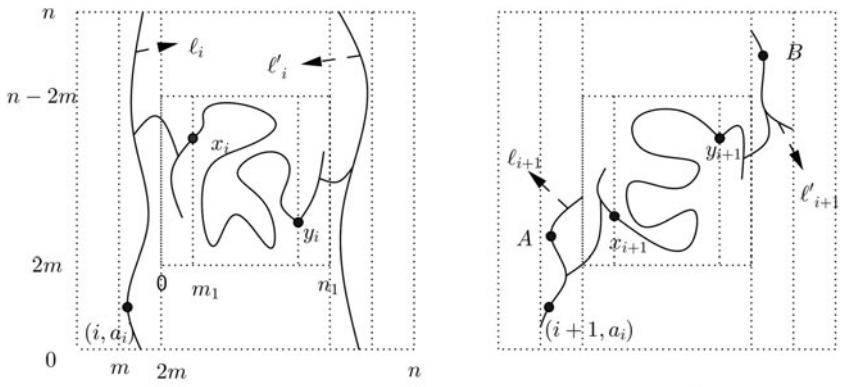


Figure 1. Glueing two long paths; $A = (i+1, a_{i+1})$, $B = (i+1, b_{i+1})$.

Now it remains to show that $\mathcal{B}_n$ holds w.h.p. We observe that the probability of the existence of such a path ℓ_i is larger than $1 - \exp(-cm)$ for some $c > 0$ (see for instance (7.70) in [11]). Thus $\mathcal{B}_n$ holds w.h.p.

Here we summarize the change in the induction step from $d-1$ to d when $d \geqslant 4$. First, in the definition of a *nice* box, we consider

$$\Lambda_i = \{i\} \times [2m, n-2m]^{d-1},$$

and the uniqueness of the connected component of size larger than m_1 in the slabs $\{i\} \times [m, 2m + m_1] \times [0,n]^{d-2}$ and $\{i\} \times [n-2m-m_1, n-m] \times [0,n]^{d-2}$. Second, in the definition of $\mathcal{B}_n$, we consider $\ell_i \subset \{i\} \times \{m\}^{d-3} \times [0,n]^2$ and $\ell'_i \subset \{i\} \times \{n-m\}^{d-3} \times [0,n]^2$, two bottom–top (resp. left–right) crossings in the last two coordinates when i is odd (resp. even). □

4. Contact process on the key subgraph

In this section we study the extinction time and the metastability of the contact process on key subgraphs defined in the previous section.

Lemma 4.1. *Let $\tau_{\ell,M}$ be the extinction time of the contact process on $\mathcal{C}(\ell, M)$ starting from full occupancy. Then there exist positive constants c and K independent of λ, such that if $\bar{\lambda} M \geqslant K$, then*

$$\mathbb{P}(\tau_{\ell,M} \geqslant \exp(c\ell M \log(\bar{\lambda} M))) \geqslant 1 - \exp(-c\ell M \log(\bar{\lambda} M)), \tag{4.1}$$

with $\bar{\lambda} = \lambda \wedge 1$.

Proof. Let (ξ_t) be the contact process on $\mathcal{C}(\ell, M)$ with parameter $\lambda > 0$. It is sufficient to consider the case $\lambda \leqslant 1$, since the contact process is monotone in λ. We assume also that $M\lambda \geqslant 640$.

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For $i \in [0,\ell]$, we say that i is *lit* at time t (the term is taken from [5]) if the number of infected vertices in its attached complete graph at time t is larger than $M/4$.

Let $T = \exp(M\log(\lambda M)/16)$. For $r \geqslant 0$ and $i, j \in [0,\ell]$ such that $|i-j| = 1$ and $i+r$ is even, we define

$$\begin{aligned} Z^r_{i,j} = {} & 1(\{i \text{ is not lit at time } rT\}) \\ & + 1(\{i \text{ is lit at time } rT \text{ and } i \text{ lights } j \text{ at time } (r+1)T\}), \end{aligned}$$

where 'i lights j at time $(r+1)T$' means that

$$|\{y \in C(j) : \exists x \in C(i) \cap \xi_{rT} \text{ s.t. } (x,rT) \longleftrightarrow (y,(r+1)T) \text{ inside } C(i) \cup C(j) \cup \{i,j\}\}| \geqslant M/4,$$

where $C(i)$ is the complete graph attached at i. Then $(Z^r_{i,j})$ naturally define an oriented percolation by identifying

$$\{Z^r_{i,j} = 1\} \Leftrightarrow \{(i,r) \to (j,r+1)\}.$$

It follows from Lemma 2.1(ii) that

$$\mathbb{P}(Z^r_{i,j} = 1 \mid \mathcal{F}_{rT}) \geqslant 1 - 5T^{-1} \quad \text{for all } r \geqslant 0 \text{ and } |i-j| = 1,$$

where $\mathcal{F}_t$ denotes the sigma-field generated by the contact process up to time t.

Moreover if $x \neq i$ and $y \neq j$, then $Z^r_{x,y}$ is independent of $Z^r_{i,j}$. Hence by a result of Liggett, Schonmann and Stacey [15] (see also Theorem B26 in [14]), the distribution of the family $(Z^r_{i,j})$ stochastically dominates the measure of a Bernoulli-oriented percolation with parameter

$$q \geqslant 1 - T^{-\gamma},$$

with $\gamma \in (0,1)$. Moreover, if λM is large enough, then $1 - T^{-\gamma} > 1 - \varepsilon$, with ε as in Lemma 2.2.

In summary, when λM is large enough, the distribution of $(Z^r_{i,j})$ stochastically dominates that of an oriented percolation on $[0,\ell]$ with density close to 1. On the other hand, it follows from Lemma 2.2(ii) that the oriented percolation process survives up to the step

$$\lfloor (1-q)^{-c\ell} \rfloor \geqslant \lfloor T^{c\gamma\ell} \rfloor \geqslant \exp(c\gamma\ell M \log(\lambda M)),$$

with probability larger than

$$1 - \exp(-c\gamma\ell M \log(\lambda M)),$$

for some constant $c > 0$. Hence the result follows. □

We now prove a metastablity result for connected graphs containing a copy of $\mathcal{C}(\ell, M)$.

Lemma 4.2. *Let (G^0_n) be a sequence of connected graphs, such that $|G^0_n| \leqslant n$, for all n. Let τ_n denote the extinction time of the contact process on G^0_n starting from full occupancy. Assume that G^0_n contains a subgraph H_n, which is isomorphic to $\mathcal{C}(\ell_n, M)$. Then there exists a positive constant K, such that if $M \geqslant K/(\lambda \wedge 1)$ and*

$$\frac{\ell_n}{d_n \vee \log n} \to \infty, \tag{4.2}$$

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where $d_n = \max_{v\in G_n^0} d(v,H_n)$, *then*

$$\frac{\tau_n}{\mathbb{E}(\tau_n)} \quad \xrightarrow[n\to\infty]{(\mathcal{L})} \quad \mathcal{E}(1).$$

Proof. According to a result of Mountford [16, Proposition 1.2], it suffices to show that there exists a sequence (a_n), such that $a_n = o(\mathbb{E}(\tau_n))$ and

$$\sup_{v\in V_n} \mathbb{P}(\xi^v_{a_n} \neq \xi_{a_n}, \xi^v_{a_n} \neq \varnothing) = o(1), \tag{4.3}$$

where $(\xi_t)_{t\geqslant 0}$ denotes the process starting from full occupancy.

Set $\bar{\lambda} = \lambda \wedge 1$. By Lemma 4.1, we get that if $\bar{\lambda}M$ is large enough, then

$$\mathbb{E}(\tau_n) \geqslant \exp(c\ell_n M \log(\bar{\lambda}M)), \tag{4.4}$$

with c as in this lemma. By (4.2), there is a sequence (φ_n) tending to infinity, such that

$$\frac{\ell_n}{k_n} \to \infty, \tag{4.5}$$

with

$$k_n = \lfloor (\log n \vee d_n)\varphi_n \rfloor.$$

Now define

$$b_n = s_{k_n} T \quad \text{and} \quad a_n = 2b_n + 1,$$

with s_{k_n} as in Lemma 2.2(iii) and $T = \exp(M\log(\bar{\lambda}M)/16)$.

Then (4.4) and (4.5) show that $a_n = o(\mathbb{E}(\tau_n))$, so it remains to prove (4.3) for this choice of (a_n). To this end it is convenient to introduce the dual contact process. Given some positive real t and A a subset of the vertex set V_n of G_n, the dual process $(\hat{\xi}^{A,t}_s)_{s\leqslant t}$ is defined by

$$\hat{\xi}^{A,t}_s = \{v \in V_n : (v, t-s) \longleftrightarrow A \times \{t\}\},$$

for all $s \leqslant t$. For any v, we have

$$\begin{aligned}
&\mathbb{P}(\xi^v_{a_n} \neq \xi_{a_n}, \xi^v_{a_n} \neq \varnothing) \\
&\quad = \mathbb{P}(\exists w \in V_n : \xi^v_{a_n}(w) = 0, \xi^v_{a_n} \neq \varnothing, \hat{\xi}^{w,a_n}_{a_n} \neq \varnothing) \\
&\quad \leqslant \sum_{w\in V_n} \mathbb{P}(\xi^v_{a_n} \neq \varnothing, \hat{\xi}^{w,a_n}_{a_n} \neq \varnothing, \text{ and } \hat{\xi}^{w,a_n}_{a_n-t} \cap \xi^v_t = \varnothing \text{ for all } t \leqslant a_n),
\end{aligned} \tag{4.6}$$

So let us now prove that the last sum above tends to 0 when $n \to \infty$.

By the hypothesis, G_n^0 contains a subgraph H_n which is isomorphic to $\mathcal{C}(k_n, M)$. Hence, H_n contains a chain of $k_n + 1$ vertices $x_0, \ldots, x_{k_n}$, such that x_i is connected to x_{i+1} for all $0 \leqslant i \leqslant k_n - 1$. Moreover, the vertex x_i is attached to a complete graph of size M, say $C(x_i)$, for all $0 \leqslant i \leqslant k_n$.

Now we slightly change the definition of a lit vertex, and say that x_i is lit if the number of its infected neighbours *in* $C(x_i)$ is larger than $M/4$ for $i = 0, \ldots, k_n$.

We first claim that for any v

$$\mathbb{P}(\mathcal{A}(v)^c, \xi^v_{b_n} \neq \varnothing) = o(1/n), \tag{4.7}$$

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where

$$\mathcal{A}(v) = \{\xi^v_{b_n} \neq \varnothing, |\{i \in [(1-\beta)k_n/2, (1+\beta)k_n/2] : x_i \text{ is lit at time } b_n\}| \geqslant 3\beta k_n/4\},$$

with β as in Lemma 2.2.

Suppose for a moment that (4.7) holds. Then we also have

$$\mathbb{P}(\hat{\mathcal{A}}(w)^c, \hat{\xi}^{w,2b_n+1}_{b_n} \neq \varnothing) = o(1/n), \tag{4.8}$$

with

$$\hat{\mathcal{A}}(w) = \{\hat{\xi}^{w,2b_n+1}_{b_n} \neq \varnothing, \exists S \subset [(1-\beta)k_n/2, (1+\beta)k_n/2] \text{ with } |S| \geqslant 3\beta k_n/4 \text{ and } W_i \subset C(x_i)x \text{ with } |W_i| \geqslant M/4 \,\forall i \in S : (x, b_n+1) \longleftrightarrow (w, 2b_n+1) \,\forall x \in \cup_{i\in S} W_i\}.$$

Note that $\mathcal{A}(v)$ and $\hat{\mathcal{A}}(w)$ are independent for all v and w. Moreover, on $\mathcal{A}(v) \cap \hat{\mathcal{A}}(w)$, there are more than $\beta k_n/2$ vertices which are lit in both the original and the dual processes. More precisely, there is a set $S \subset [(1-\beta)k_n/2, (1+\beta)k_n/2]$ with $|S| \geqslant \beta k_n/2$ and sets $U_i, W_i \subset C(x_i)$ with $|U_i|, |W_i| \geqslant M/4$ for all $i \in S$, such that

$$\begin{aligned} (v,0) &\longleftrightarrow (x, b_n) \quad \text{for all } x \in \cup_{i\in S} U_i, \\ (y, b_n+1) &\longleftrightarrow (w, 2b_n+1) \quad \text{for all } y \in \cup_{i\in S} W_i. \end{aligned}$$

It is not difficult to show that there is a positive constant c, such that for any non-empty sets $U_i, W_i \subset C(x_i)$,

$$\mathbb{P}(U_i \times \{b_n\} \overset{C(x_i)}{\longleftrightarrow} W_i \times \{b_n+1\}) \geqslant c,$$

where the notation

$$U_i \times \{b_n\} \overset{C(x_i)}{\longleftrightarrow} W_i \times \{b_n+1\}$$

means that there is an infection path inside $C(x_i)$ from a vertex in U_i at time b_n to a vertex in W_i at time b_n+1.

Moreover, conditionally on the sets U_i, W_i, these events are independent. Therefore,

$$\mathbb{P}(\exists i : U_i \times \{b_n\} \overset{C(x_i)}{\longleftrightarrow} W_i \times \{b_n+1\} \mid U_i, W_i) \geqslant 1 - (1-c)^{\beta k_n/2} = 1 - o(1/n),$$

by our choice of k_n. This implies that

$$\mathbb{P}(\mathcal{A}(v), \hat{\mathcal{A}}(w), \hat{\xi}^{w,a_n}_{a_n-t} \cap \xi^v_t = \varnothing \text{ for all } t \leqslant a_n) = o(1/n). \tag{4.9}$$

Combining (4.7), (4.8) and (4.9), we obtain (4.6). Hence, it now remains to prove (4.7).

Fix a vertex $v \in G^0_n$. We call i_* an index, such that

$$d(v, x_{i_*}) \leqslant d(v, H_n) + 1 \leqslant d_n + 1. \tag{4.10}$$

As in Lemma 4.1, we define an oriented percolation $(\tilde{\eta}_r)_{r\geqslant 0}$ on $[0, k_n]$ as follows. For $0 \leqslant i, j \leqslant k_n$ and $r \geqslant 0$, such that $|i-j| = 1$ and $i+r$ is even, we let $Z^r_{i,j} = 1$ (or equivalently $(i,r) \to (j, r+1)$) if either x_i is not lit at time rT or x_i is lit at time rT and x_i lights x_j at time $(r+1)T$.

As in Lemma 4.1, there exists a positive constant K, such that if $\bar{\lambda} M \geqslant K$, then $(\tilde{\eta}_r)$ stochastically dominates a Bernoulli-oriented percolation with parameter $1-\varepsilon$, with ε as in Lemma 2.2.

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Assume that d_n is even; if not, we just take the smallest even integer larger than d_n. Then we set

$$\tilde{d_n} = d_n + 2k_n.$$

Now define for $k \geqslant 0$,

$$C_k = \{\exists r,s \in [k\tilde{d_n} + d_n, (k+1)\tilde{d_n}] \text{ s.t. } \tilde{\eta}_r^{i_*,k\tilde{d_n}+d_n}(0) = 1, \tilde{\eta}_s^{i_*,k\tilde{d_n}+d_n}(k_n) = 1\},$$

where, for any $A \subset [0,k_n]$ and $t \geqslant s \geqslant 0$,

$$\tilde{\eta}_t^{A,s} = \{x \in [0,k_n] : \exists y \in A, (y,s) \to (x,t)\}.$$

Then, using Lemma 2.2(i), we get

$$\mathbb{P}(C_k \mid \mathcal{F}_{k\tilde{d_n}+d_n}) \geqslant c. \tag{4.11}$$

Using the same arguments as for claim (a) in Lemma 2.2, we observe that on C_k,

$$\tilde{\eta}_r^{\mathbf{1}} = \tilde{\eta}_r^{i_*,k\tilde{d_n}+d_n} \quad \text{for all } r \geqslant (k+1)\tilde{d_n}. \tag{4.12}$$

Define

$$\mathcal{E} = \{|\tilde{\eta}_{s_{k_n}}^{\mathbf{1}} \cap [(1-\beta)k_n/2, (1+\beta)k_n/2]| \geqslant 3\beta k_n/4\},$$

with s_{k_n} as in Lemma 2.2(iii). Using (4.12), we get that on $C_k \cap \mathcal{E}$, if $(k+1)\tilde{d_n} \leqslant s_{k_n}$ then

$$|\tilde{\eta}_{s_{k_n}}^{i_*,k\tilde{d_n}+d_n} \cap [(1-\beta)k_n/2, (1+\beta)k_n/2]| \geqslant 3\beta k_n/4. \tag{4.13}$$

Let $K_n = \lfloor s_{k_n}/\tilde{d_n} \rfloor$ and for any $0 \leqslant k \leqslant K_n - 1$, we define

$$A_k = \{\xi_{k\tilde{d_n}}^v \neq \varnothing\},$$

and

$$B_k = \{\xi_{k\tilde{d_n}}^v \times \{k\tilde{d_n}\} \to (x_{i_*}, (k\tilde{d_n} + d_n - 1)T)\} \cap \{x_{i_*} \text{ is lit at time } (k\tilde{d_n} + d_n)T\} \cap C_k.$$

We have

$$\{\xi_{b_n}^v \neq \varnothing\} \subset \bigcap_{k=0}^{K_n-1} A_k. \tag{4.14}$$

On the other hand, if x_{i_*} is lit at time rT and $\tilde{\eta}_s^{i_*,r}(i) = 1$ for $s > r$, then x_i is lit at time sT. Hence by (4.13) on $\mathcal{E}$, if one of the events $(A_k \cap B_k)$ happens, then $\mathcal{A}(v)$ occurs. Combining this with (4.14), we get

$$\{\xi_{b_n}^v \neq \varnothing\} \cap \mathcal{A}(v)^c \subset \mathcal{E}^c \cup \left(\bigcap_{k=0}^{K_n-1} A_k \cap B_k^c\right). \tag{4.15}$$

Using Lemma 2.2(iii), for the first term we obtain the bound

$$\mathbb{P}(\mathcal{E}^c) \leqslant \exp(-ck_n) = o(1/n), \tag{4.16}$$

by the choice of k_n. For the second term, by using (4.10) and a similar argument as for (2.14), we have

$$\mathbb{P}((v,t) \to (x_{i_*}, t + (d_n - 1)T)) \geqslant \exp(-C(d_n - 1)T) \quad \text{for any } t \geqslant 0,$$

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for some constant $C > 0$. On the other hand, if x_{i_*} is infected at time t then it is lit at time $t+T$ with probability larger than $\exp(-CT)$. Therefore combining with (4.11), we get that for any $k \leqslant K_n - 1$,

$$\mathbb{P}(B_k^c \mid \mathcal{G}_k)\mathbf{1}(A_k) \leqslant 1 - c\exp(-Cd_nT),$$

where $\mathcal{G}_k = \mathcal{F}_{kd_n}$. Iterating this, we get

$$\mathbb{P}\left(\bigcap_{k=0}^{K_n-1} A_k \cap B_k^c\right) \leqslant (1 - c\exp(-Cd_nT))^{K_n-1} = o(1/n), \tag{4.17}$$

where the last equality follows from the definition of s_{k_n}. Combining (4.15), (4.16) and (4.17), we get (4.7) and finish the proof. □

5. Proof of Theorem 1.1

5.1. Proof of (i)

We first prove the lower bound on τ_n. By Lemma 3.2, there are positive constants c and K such that if $R^d \geqslant K/(\lambda \wedge 1)$, then w.h.p. $G(n,R,g)$ contains a subgraph H_n which is isomorphic to $\mathcal{C}(\ell_n, M)$, with $\ell_n = \lceil cnR^{-d} \rceil$ and $M = \lfloor cR^d \rfloor$.

If ℓ_n is bounded (or $R^d = \Theta(n)$), then $\mathcal{C}(\ell_n, M)$ contains a complete graph of size of order n. Then Lemma 2.1(i) implies that w.h.p. the extinction time is larger than $\exp(cn\log(\lambda n))$, for some $c > 0$.

If ℓ_n tends to infinity, then the result follows from Lemma 4.1.

To prove the convergence in law of $\tau_n/\mathbb{E}(\tau_n)$, we recall some known results about the diameter of the giant component and the size of small components in RGGs. There is a positive constant R_0, such that if $R > R_0$, then w.h.p.

(a) the diameter of the largest component is $D_n = O(n^{1/d}/R)$,
(b) the size of the second largest component is $O((\log n)^{d/(d-1)})$.

The first claim is proved by Friedrich, Sauerwald and Stauffer in [9, Corollary 6] and the second one is proved in Penrose's book [21, Theorem 10.18] when $g \equiv 1$. It is not hard to generalize these results for our model with g bounded both from below and above.

The second claim together with Lemma 5.1 below show that w.h.p. the extinction time of the contact process on G_n and on G_n^0 – the largest component – are equal. We are now in a position to complete the proof of (i).

- If $R^d = o(n/\log n)$, then

$$\frac{nR^{-d}}{\log n} \to \infty \quad \text{and} \quad \frac{nR^{-d}}{D_n} \to \infty,$$

since by (a), $D_n = O((nR^{-d})^{1/d})$. On the other hand, $\ell_n = \Theta(nR^{-d})$ and

$$d_n = \max_{v \in G_n^0} d(v, H_n) \leqslant D_n.$$

Thus

$$\frac{\ell_n}{d_n \vee \log n} \to \infty.$$

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Therefore, Lemma 4.2 implies the convergence in law of $\tau_n/\mathbb{E}(\tau_n)$.

- If $n/\log n = O(R^d)$, then H_n contains a complete graph of size larger than $\sqrt{n}$. On the other hand, for all $k,\ell \in \mathbb{N}$ the complete graph of size $k\ell$ always contains a copy of $\mathcal{C}(k,\ell)$. Hence, G_n^0 contains a copy of $\mathcal{C}(\lfloor n^{1/4}\rfloor, \lfloor n^{1/4}\rfloor)$. We have

$$\frac{\lfloor n^{1/4}\rfloor}{d_n \vee \log n} \to \infty,$$

since

$$d_n \leqslant D_n = O((nR^{-d})^{1/d}) = O((\log n)^{1/d}).$$

Thus the result follows from Lemma 4.2. □

5.2. Proof of (ii)

We prove an upper bound on the extinction time of the contact process on an arbitrary graph.

Lemma 5.1. *Let τ_G be the extinction time of the contact process on a graph $G=(V,E)$ starting from full occupancy. Then*

(a) $\mathbb{P}(\tau_G \leqslant F(|V|,|E|)) \geqslant 1-\exp(-|V|)$,
(b) $\mathbb{E}(\tau_G) \leqslant 2F(|V|,|E|)$

with

$$F(|V|,|E|) = |V|\left(2+\frac{4\lambda|E|}{|V|}\right)^{|V|}.$$

Proof. Observe that (b) is a consequence of (a) and the following. For any $s>0$,

$$\mathbb{E}(\tau_G) \leqslant \frac{s}{\mathbb{P}(\tau_G \leqslant s)}.$$

This result is Lemma 4.5 in [17].

We now prove (a). Let us denote by (ξ_t) the contact process on G starting with full occupancy. By using Markov's property and the monotonicity of the contact process, it suffices to show that

$$\mathbb{P}(\xi_1 = \varnothing) \geqslant \exp(-|V|\log(2+4\lambda|E|/|V|)). \tag{5.1}$$

Observe that the process dies at time 1 if, for any vertex v, it heals before 1 and does not infect any neighbour. Let σ_v be the time of the first recovery at v; then $\sigma_v \sim \mathcal{E}(1)$. Let $\sigma_{v\to}$ be the time of the first infection spread from v to one of its neighbours. Then it is the minimum of $\deg(v)$ i.i.d. exponential random variables with mean λ and thus $\sigma_{v\to} \sim \mathcal{E}(\lambda\deg(v))$. Moreover, σ_v and $\sigma_{v\to}$ are independent. Therefore

$$\mathbb{P}(\sigma_v < \min\{\sigma_{v\to}, 1\}) = \frac{1-e^{-(1+\lambda\deg(v))}}{1+\lambda\deg(v)} \geqslant \frac{1}{2(1+\lambda\deg(v))}.$$

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On the other hand, these events $\{\sigma_v < \min\{\sigma_{v\rightarrow}, 1\}\}_v$ are independent. Then, using Cauchy's inequality, we get that

$$\begin{aligned}\mathbb{P}(\xi_1 = \varnothing) &\geqslant \prod_{v\in V}(2+2\lambda \deg(v))^{-1} \\ &\geqslant \left(\frac{2|V| + 2\lambda \sum_{v\in V}\deg(v)}{|V|}\right)^{-|V|} \\ &= \left(2+\frac{4\lambda|E|}{|V|}\right)^{-|V|},\end{aligned}$$

which implies (5.1). □

Proof of Theorem 1.1(ii). The upper bound on τ_n follows from Lemma 5.1 and the following: w.h.p. $G(n,R,g) = (V_n, E_n)$ with

- $|V_n| \leqslant 2Bn$
- $|E_n| \leqslant CnR^d$, for some $C = C(d,B)$.

The first claim is clear, since $|V_n|$ is a Poisson random variable with mean

$$\int_{[0,\sqrt[d]{n}]^d} g(x)dx \leqslant Bn.$$

For the second one, let $\ell = \lceil \sqrt[d]{n}/R \rceil$. We cover $[0,\sqrt[d]{n}]^d$ using translations by $R/2$ for each coordinate, accounting for $(2\ell-1)^d$ boxes of volume R^d. We observe that points at distance larger than R are not connected. Hence, $|E_n|$ is less than the sum of the number of edges in the covering small boxes.

These small boxes are partitioned into 2^d groups such that each group contains at most ℓ^d disjoint boxes with the same volume R^d.

The number of vertices in each box is stochastically dominated by Z, a Poisson random variable with mean BR^d, since the integral of g on a box is smaller than BR^d (as $g(x) \leqslant B$ for all x). Hence, the number of edges in a box is stochastically dominated by Z^2.

Moreover, in each group the numbers of edges are independent, as the boxes are disjoint. Therefore, using Chebyshev's inequality, the total number of edges in a group is w.h.p. smaller than

$$2\ell^d \mathbb{E}(Z^2) = 2\ell^d(BR^d)(BR^d+1).$$

Hence, $|E_n|$ is w.h.p. less than

$$2^{d+1}\ell^d(BR^d)(BR^d+1) \leqslant CnR^d,$$

for some $C = C(d,B)$ large enough. □

6. Some extensions

6.1. The one-dimensional case

When $d = 1$, RGGs are also called random interval graphs; see for instance [23]. We have the following result.

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Proposition 6.1. *Let $d = 1$. Consider the contact process on one-dimensional random geometric graphs $G(n,R,g)$ with g satisfying* (1.1). *Then there exist positive constants $\varkappa, K, c$ and C depending only on b and B, such that the following statements hold.*

(i) *If $R \leqslant \varkappa \log n$, then w.h.p. the number of vertices in the largest component is $o(n^{2/3})$. Thus $\log \tau_n = o(n)$ w.h.p.*

(ii) *If $R \geqslant K \log n$, then w.h.p. the graph is connected and*

$$cn \log(\lambda R) \leqslant \log(\tau_n) \leqslant Cn \log(\lambda R),$$

and

$$\frac{\tau_n}{\mathbb{E}(\tau_n)} \xrightarrow[n\to\infty]{(\mathcal{L})} \mathcal{E}(1).$$

Proof. For (i), it is sufficient to consider $R = \varkappa \log n$ with $\varkappa$ chosen later. We divide $[0,n]$ into $\lfloor n/R \rfloor$ intervals of length R, denoted by $I_1, \ldots, I_{\lfloor n/R \rfloor}$. Then the number of vertices in I_i is a Poisson random variable with mean $\int_{I_i} g(x)dx = \Theta(R)$. Therefore

$$\mathbb{P}(\#\{\text{vertices in } I_i\} \leqslant R^2 \text{ for all } i = 1, \ldots, \lfloor n/R \rfloor) = 1 - o(1), \tag{6.1}$$

as $R = \Theta(\log n)$. On the other hand, since $\int_{I_i} g(x)dx \leqslant BR$ for all i, the probability of an interval being empty is larger than e^{-BR}. Hence

$$\begin{aligned}\mathbb{P}(\text{there are at most } \lfloor \sqrt{n} \rfloor \text{ consecutive non-empty intervals}) &\geqslant 1 - \lfloor n/R \rfloor (1 - e^{-BR})^{\lfloor \sqrt{n} \rfloor} \\ &= 1 - o(1),\end{aligned} \tag{6.2}$$

with $R = \varkappa \log n$ and $\varkappa$ small enough.

We observe that if an interval is empty, then there is no edge between vertices in the left-hand side and the right-hand side of this interval. Thus, it follows from (6.1) and (6.2) that w.h.p. the number of vertices in any component is smaller than $R^2 \lfloor \sqrt{n} \rfloor = o(n^{2/3})$.

We now prove (ii). If $R = \Theta(n)$, then the graph contains a complete graph of size of order n. Thus, using Lemma 2.1(i), we get the lower bound on τ_n. Assume that $R = o(n)$. We divide $[0,n]$ into $\lfloor 2n/R \rfloor$ intervals of length $R/2$, denoted by $J_1, \ldots, J_{\lfloor 2n/R \rfloor}$. Then the numbers of vertices in these intervals form a sequence of independent Poisson random variables with mean larger than $bR/2$. For all $i \leqslant \lfloor 2n/R \rfloor$, we define

$$\{J_i \text{ is good}\} = \{\text{the number of vertices in } J_i \text{ is larger than } bR/4\}.$$

We have

$$\mathbb{P}(\mathrm{Poi}(bR/2) \geqslant bR/4) \geqslant 1 - \exp(cR),$$

for some constant $c = c(b) > 0$. Therefore

$$\mathbb{P}(J_i \text{ is good for all } i \leqslant \lfloor 2n/R \rfloor) \geqslant 1 - \lfloor 2n/R \rfloor e^{-cR} = 1 - o(1), \tag{6.3}$$

with $R \geqslant K \log n$ and K large enough. This implies that w.h.p. $G(n,R,g)$ contains as a subgraph a copy of $\mathcal{C}(\lfloor 2n/R \rfloor, \lfloor bR/4 \rfloor)$ (note that the vertices in the same interval or in adjacent ones are connected). Thus, similarly to Theorem 1.1, we get the lower bound on τ_n. The upper bound also follows from the same argument as in Theorem 1.1.

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For the connectivity, since all vertices in an interval J_i or in adjacent ones are connected, we observe that

$$\{J_i \text{ is good for all } i \leqslant \lfloor 2n/R \rfloor\} \subset \{J_i \text{ is non-empty for all } i \leqslant \lfloor 2n/R \rfloor\} \subset \{G_n \text{ is connected}\}.$$

Therefore, by (6.3), when $R \geqslant K \log n$ with K large enough, w.h.p. G_n is connected. In addition, its diameter is $d_n \leqslant \lceil 2n/R \rceil$.

For the convergence in law of $\tau_n/\mathbb{E}(\tau_n)$, we note that $\mathcal{C}(\ell, M)$ always contains a copy of $\mathcal{C}(\ell \lfloor M/M_1 \rfloor, M_1)$ if $M \geqslant M_1$. Moreover, w.h.p. G_n contains a copy of $\mathcal{C}(\lfloor 2n/R \rfloor, \lfloor bR/4 \rfloor)$. Therefore, w.h.p. G_n contains a copy of $\mathcal{C}(\lfloor cn \rfloor, M)$ for some $c > 0$ and M as in Lemma 4.2. Hence, G_n satisfies the hypothesis in Lemma 4.2 and the result follows. □

6.2. An equivalent model

We consider another version of random geometric graphs with density function f and connection radius r, denoted by $G'(n, r, f)$. It is defined as follows: place independently n points in $[0,1]^d$ according to f, then connect two points u and v by an edge if $\|u - v\| \leqslant r$. Suppose that

$$0 < b \leqslant f(x) \leqslant B < +\infty \quad \text{for all } x. \tag{6.4}$$

Similarly to the results for $G(n, R, g)$, we have the following.

Proposition 6.2. *The results of Theorem 1.1 and Proposition 6.1 hold for the graph $G'(n, r, f)$ with f satisfying* (6.4) *by replacing R^d by nr^d in the statements.*

Proof. First, we observe that the law of a Poisson point process with intensity g on a set A conditionally on its number vertices, say N, is the same as that of the process defined by placing independently N points in A with density $g/\int_A g$.

Therefore, the graph $G(n, R, g)$ conditionally on its size $|G|$ is isomorphic to $G'(|G|, r, f)$ with

$$r = R/\sqrt[d]{n} \quad \text{and} \quad f(x) = g(x\sqrt[d]{n}).$$

To prove the lower bound on τ_n, we consider $G_1 = G(n_1, R_1, g_1)$, where

$$n_1 = \lfloor n/(2B) \rfloor, \quad R_1 = r\sqrt[d]{n_1} \quad \text{and} \quad g_1(x) = f\left(\frac{x}{\sqrt[d]{n_1}}\right).$$

Since $|G_1|$ is a Poisson random variable with mean less than $n/2$, w.h.p. $|G_1|$ is less than n. Therefore w.h.p. G_1 can be coupled as a subgraph of $G'(n, r, f)$. This domination together with Theorem 1.1 and Proposition 6.1 imply the results for the lower bound on τ_n and the convergence in law of $\tau_n/\mathbb{E}(\tau_n)$ (note that the results of the connectivity and the diameter of the largest component or the size of the second largest component also hold in this model).

Similarly, for the upper bound on τ_n, we consider $G_2 = G(n_2, R_2, g_2)$ with

$$n_2 = \lfloor 2n/b \rfloor, \quad R_2 = r\sqrt[d]{n_2} \quad \text{and} \quad g_2(x) = f\left(\frac{x}{\sqrt[d]{n_2}}\right).$$

Then w.h.p. G_2 contains as a subgraph a copy of $G'(n, r, f)$. Thus, by applying Theorem 1.1 and Proposition 6.1, we get the desired results. □

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References

[1] Berger, N., Borgs, C., Chayes, J. T. and Saberi, A. (2005) On the spread of viruses on the internet. In *SODA '05: Proc. 16th Annual ACM–SIAM Symposium on Discrete Algorithms*, SIAM, pp. 301–310.
[2] Can, V. H. (2015) Metastability for the contact process on the preferential attachment graph. *Internet Math.*, to appear. doi:10.24166/im.08.2017
[3] Can, V. H. (2016) Processus de contact sur des graphes aléatoires. PhD thesis, University of Aix–Marseille.
[4] Can, V. H. and Schapira, B. (2015) Metastability for the contact process on the configuration model with infinite mean degree. *Electron. J. Probab.* **20** #26.
[5] Chatterjee, S. and Durrett, R. (2009) Contact processes on random graphs with degree power law distribution have critical value zero. *Ann. Probab.* **37** 2332–2356.
[6] Cranston, M., Mountford, T., Mourrat, J.-C. and Valesin, D. (2014) The contact process on finite homogeneous trees revisited. *ALEA: Lat. Am. J. Probab. Math. Statist.* **11** 385–408.
[7] Durrett, R. (1984) Oriented percolation in two dimensions. *Ann. Probab.* **12** 999–1040.
[8] Durrett, R. and Schonmann, R. H. (1988) Large deviation for the contact process and two dimensional percolation. *Probab. Theory Rel. Fields* **77** 583–603.
[9] Friedrich, T., Sauerwald, T. and Stauffer, A. (2013) Diameter and broadcast time of random geometric graphs in arbitrary dimensions. *Algorithmica* **67** 65–88.
[10] Ganesan, G. (2015) Infection spread in random geometric graphs. *Adv. Appl. Probab.* **47** 164–181.
[11] Grimmett, G. (1999) *Percolation*, Vol. 321 of Grundlehren der Mathematischen Wissenschaften, Springer.
[12] Grimmett, G. (1985) Long paths and cycles in a random lattice. *Ann. Discrete Math.* **33** 69–76.
[13] Gupta, P. and Kumar, P. R. (1999) Critical power for asymptotic connectivity in wireless networks. In *Stochastic Analysis, Control, Optimization and Applications* (W. M. McEneaney *et al.*, eds), Springer, pp. 547–566.
[14] Liggett, T. (1999) *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes.* Vol. 324 of Grundlehren der Mathematischen Wissenschaften, Springer.
[15] Liggett, T., Schonmann, R. and Stacey, A. (1997) Domination by product measure. *Ann. Probab.* **25** 71–95.
[16] Mountford, T. (1993) A metastable result for the finite multidimensional contact process. *Canad. Math. Bull.* **36** 216–226.
[17] Mountford, T., Mourrat, J.-C., Valesin, D. and Yao, Q. (2016) Exponential extinction time of the contact process on finite graphs. *Stochastic Process. Appl.* **126** 1974–2013.
[18] Mountford, T., Valesin, D. and Yao, Q. (2013) Metastable densities for contact processes on random graphs. *Electron. J. Probab.* **18** #103.
[19] Mourrat, J.-C. and Valesin, D. (2016) Phase transition of the contact process on random regular graphs. *Electron. J. Probab.* **21** #31.
[20] Ménard, L. and Singh, A. (2016) Percolation by cumulative merging and phase transition for the contact process on random graphs. *Ann. Sci. École Norm. Supér.* **49** 1189–1238.
[21] Penrose, M. (2003) *Random Geometric Graphs*, Oxford University Press.
[22] Preciado, V. M. and Jadbabaie, A. (2009) Spectral analysis of virus spreading in random geometric networks. In *Proc. IEEE Conference on Decision and Control*, IEEE, pp. 4802–4807.
[23] Scheinerman, E. R. (1990) An evolution of interval graphs. *Discrete Math.* **82** 287–302.

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