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SIZE AND LOGIC

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Abstract. We show how to develop a multitude of rules of nonmonotonic logic from very simple and natural notions of size, using them as building blocks.

§1. Introduction.

1.1. Context. The study of modal and temporal logic and the study of substructural logic went for many years along the following lines: on the one hand we had syntactic proof theoretic systems, mainly Gentzen or Hilbert systems and on the other hand we had semantical interpretations, mainly possible worlds or algebraic structures, and the community very thoroughly analyzed properties of one against matching properties of the other. The success of such depended on the correct identification of the relevant features in the semantics. In the case of nonmonotonic logic there is the syntactical consequence relation on the one hand and the preferential ordering on the other but the research is not yet in a similar comprehensive stage as in the other areas. In this paper we use the important semantical feature of size to display a detailed matching between syntactical and semantical conditions for nonmonotonic systems.

We show how one can develop a multitude of rules for nonmonotonic logics from a very small set of principles about reasoning with size. The notion of size gives an algebraic semantics to nonmonotonic logics, in the sense that α implies β iff the set of cases where $\alpha \wedge \neg\beta$ holds is a small subset of all α-cases. In a similar way, for example, Heyting algebras are an algebraic semantics for intuitionistic logic.

In our understanding, algebraic semantics describe the abstract properties corresponding model sets have. Structural semantics, on the other hand, give intuitive concepts like accessibility or preference, from which properties of model sets, and thus algebraic semantics, originate.

Varying properties of structural semantics (e.g., transitivity, etc.) result in varying properties of algebraic semantics, and thus of logical rules. We consider operations directly on the algebraic semantics and their logical consequences, and we see that simple manipulations of the size concept result in most rules of nonmonotonic logics. Even more, we show how to generate new rules from those manipulations. The result is one big table "Rules on size", which, in a much more modest scale, can be seen as a "periodic table" of the "elements" of nonmonotonic logic. Some simple underlying principles allow to generate them all—or most of them. On the other hand, our formal results on the (sometimes quite

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Size I

Table 1. Rules on size - Part I

	"Ideal"	"Filter"	$\mathcal{M}^+$	∇
	Optimal proportion			
(Opt)	$\emptyset \in \mathcal{I}(X)$	$X \in \mathcal{F}(X)$		$\forall x\alpha \rightarrow \nabla x\alpha$
	Monotony (Improving proportions). (iM): internal monotony, $(eM\mathcal{I})$: external monotony for ideals, $(eM\mathcal{F})$: external monotony for filters			
(iM)	$A \subseteq B \in \mathcal{I}(X)$ $\Rightarrow$ $A \in \mathcal{I}(X)$	$A \in \mathcal{F}(X)$, $A \subseteq B \subseteq X$ $\Rightarrow B \in \mathcal{F}(X)$		$\nabla x\alpha \wedge \forall x(\alpha \rightarrow \alpha') \rightarrow \nabla x\alpha'$
$(eM\mathcal{I})$	$X \subseteq Y \Rightarrow \mathcal{I}(X) \subseteq \mathcal{I}(Y)$			$\nabla x(\alpha : \beta) \wedge \forall x(\alpha' \rightarrow \beta) \rightarrow \nabla x(\alpha \vee \alpha' : \beta)$
$(eM\mathcal{F})$		$X \subseteq Y \Rightarrow \mathcal{F}(Y) \cap \mathcal{P}(X) \subseteq \mathcal{F}(X)$		$\nabla x(\alpha : \beta) \wedge \forall x(\beta \wedge \alpha \rightarrow \alpha') \rightarrow \nabla x(\alpha \wedge \alpha' : \beta)$
	Keeping proportions			
$(\approx)$	$(\mathcal{I} \cup disj)$ $A \in \mathcal{I}(X)$, $B \in \mathcal{I}(Y)$, $X \cap Y = \emptyset \Rightarrow$ $A \cup B \in \mathcal{I}(X \cup Y)$	$(\mathcal{F} \cup disj)$ $A \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$, $X \cap Y = \emptyset \Rightarrow$ $A \cup B \in \mathcal{F}(X \cup Y)$	$(\mathcal{M}^+ \cup disj)$ $A \in \mathcal{M}^+(X)$, $B \in \mathcal{M}^+(Y)$, $X \cap Y = \emptyset \Rightarrow$ $A \cup B \in \mathcal{M}^+(X \cup Y)$	$\nabla x(\alpha : \beta) \wedge \nabla x(\alpha' : \beta) \wedge \neg\exists x(\alpha \wedge \alpha') \rightarrow \nabla x(\alpha \vee \alpha' : \beta)$
	Robustness of proportions: $n * small \neq All$			
$(1 * s)$	$(\mathcal{I}_1)$ $X \notin \mathcal{I}(X)$	$(\mathcal{F}_1)$ $\emptyset \notin \mathcal{F}(X)$		(∇_1) $\nabla x\alpha \rightarrow \exists x\alpha$
$(2 * s)$	$(\mathcal{I}_2)$ $A, B \in \mathcal{I}(X) \Rightarrow A \cup B \neq X$	$(\mathcal{F}_2)$ $A, B \in \mathcal{F}(X) \Rightarrow A \cap B \neq \emptyset$		(∇_2) $\nabla x\alpha \wedge \nabla x\beta \rightarrow \exists x(\alpha \wedge \beta)$
$(n * s)$ $(n \geq 3)$	$(\mathcal{I}_n)$ $A_1, ., A_n \in \mathcal{I}(X)$ $\Rightarrow$ $A_1 \cup . \cup A_n \neq X$	$(\mathcal{F}_n)$ $A_1, ., A_n \in \mathcal{I}(X)$ $\Rightarrow$ $A_1 \cap . \cap A_n \neq \emptyset$	$(\mathcal{M}^+_n)$ $X_1 \in \mathcal{F}(X_2), ., X_{n-1} \in \mathcal{F}(X_n) \Rightarrow X_1 \in \mathcal{M}^+(X_n)$	(∇_n) $\nabla x\alpha_1 \wedge . \wedge \nabla x\alpha_n \rightarrow \exists x(\alpha_1 \wedge . \wedge \alpha_n)$
$(< \omega * s)$	$(\mathcal{I}_\omega)$ $A, B \in \mathcal{I}(X) \Rightarrow A \cup B \in \mathcal{I}(X)$	$(\mathcal{F}_\omega)$ $A, B \in \mathcal{F}(X) \Rightarrow A \cap B \in \mathcal{F}(X)$	$(\mathcal{M}^+_\omega)$ (1) $A \in \mathcal{F}(X)$, $X \in \mathcal{M}^+(Y) \Rightarrow A \in \mathcal{M}^+(Y)$ (2) $A \in \mathcal{M}^+(X)$, $X \in \mathcal{F}(Y) \Rightarrow A \in \mathcal{M}^+(Y)$ (3) $A \in \mathcal{F}(X)$, $X \in \mathcal{F}(Y) \Rightarrow A \in \mathcal{F}(Y)$ (4) $A, B \in \mathcal{I}(X) \Rightarrow A - B \in \mathcal{I}(X-B)$	(∇_ω) $\nabla x\alpha \wedge \nabla x\beta \rightarrow \nabla x(\alpha \wedge \beta)$
	Robustness of $\mathcal{M}^+$			
$(\mathcal{M}^{++})$			$(\mathcal{M}^{++})$ (1) $A \in \mathcal{I}(X)$, $B \notin \mathcal{F}(X) \Rightarrow A - B \in \mathcal{I}(X - B)$ (2) $A \in \mathcal{F}(X)$, $B \notin \mathcal{F}(X) \Rightarrow A - B \in \mathcal{F}(X - B)$ (3) $A \in \mathcal{M}^+(X)$, $X \in \mathcal{M}^+(Y) \Rightarrow A \in \mathcal{M}^+(Y)$	

subtle) relationships show—among other things—that the various variants of (AND) are very strong, stronger than many other rules like the various cautious monotony rules.

This is perhaps the right place to mention that, independently, and with different techniques, Hawthorne & Makinson (2007), have also examined connections between (AND), (CM), and other rules, in their article.

Our principles also provide a bridge between the qualitative and the quantitative approach to nonmonotonic logic. Whereas the first works with rules like (AND), the second uses, for example, some kind of probability, and then has to deal with problems like the

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lottery paradox. We introduce rules like (AND_n) which "cut" operations beyond a certain threshold, or, better, do not guarantee their success any more. As a matter of fact, our generalized quantifier approach in Schlechta (1990) was introduced precisely to be able to treat problems like exceptions and the lottery paradox in an abstract, nonnumerical way.

Our view as abstract size was inspired by the classical filter approach, as used, for example, in mathematical measure theory. There is a superficially different, but basically identical, approach to size by partial orders. The first discussion of this approach is due—to our knowledge—to Friedman & Halpern (1996). We will come back to this approach in Section 2.7.

Historical remarks: The first time that abstract size was related to nonmonotonic logics was, to our knowledge, in Schlechta (1990, 1995), and, independently, in Ben-David & Ben-Eliyahu (1994). More detailed remarks can be found in Gabbay & Schlechta (2009). But, again to our knowledge, connections are elaborated systematically and in fine detail here for the first time.

1.2. Overview. The main part of this paper is the big table "Rules on size" in Section 2.6. It shows connections and how to develop a multitude of logical rules known from nonmonotonic logics by combining a small number of principles about size. We use them as building blocks to construct the rules from. More precisely, "size" is to be read as "relative size", since it is essential to change the base sets.

These principles are some basic and very natural postulates, (Opt), (iM), $(eM\mathcal{I})$, $(eM\mathcal{F})$, and a continuum of power of the notion of "small", or, dually, "big", from $(1 * s)$ to $(< \omega * s)$. From these, we can develop the rest except, essentially, rational monotony, and thus an infinity of different rules.

The tables in Section 4 connect our rules to those known from preferential reasoning, where preferential structures define principal filters, generated by the set of minimal elements.

This is a conceptual paper, and it does not contain any more difficult formal results. The interest lies, in our opinion, in the simplicity, paucity, and naturalness of the basic building blocks. We hope that this schema brings more and deeper order into the rich fauna of nonmonotonic and related logics.

§2. Main table.

2.1. Notation.

(1) $\mathcal{P}(X)$ is the powerset of X, $\subseteq$ is the subset relation, $\subset$ the strict part of $\subseteq$, that is, $A \subset B$ iff $A \subseteq B$ and $A \neq B$. The operators $\wedge$, $\neg$, $\vee$, $\rightarrow$, and $\vdash$ have their usual, classical interpretation.

(2) $\mathcal{I}(X) \subseteq \mathcal{P}(X)$ and $\mathcal{F}(X) \subseteq \mathcal{P}(X)$ are dual abstract notions of size, $\mathcal{I}(X)$ is the set of "small" subsets of X, $\mathcal{F}(X)$ the set of "big" subsets of X. They are dual in the sense that $A \in \mathcal{I}(X) \Leftrightarrow X - A \in \mathcal{F}(X)$. "$\mathcal{I}$" evokes "ideal", "$\mathcal{F}$" evokes "filter" though the full strength of both is reached only in $(< \omega * s)$. "s" evokes "small", and "$(x * s)$" stands for "x small sets together are still not everything".

(3) If $A \subseteq X$ is neither in $\mathcal{I}(X)$, nor in $\mathcal{F}(X)$, we say it has medium size, and we define $\mathcal{M}(X) := \mathcal{P}(X) - (\mathcal{I}(X) \cup \mathcal{F}(X))$. $\mathcal{M}^+(X) := \mathcal{P}(X) - \mathcal{I}(X)$ is the set of subsets which are not small.

(4) $\nabla x \phi$ is a generalized first-order quantifier, it is read "almost all x have property ϕ". $\nabla x(\phi : \psi)$ is the relativized version, read: "almost all x with property ϕ have also

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Size II

Table 2. Rules on size - Part II

	various rules	AND	OR	Caut./Rat.Mon.
Optimal proportion				
(Opt)	(SC) $\alpha \vdash \beta \Rightarrow \alpha \mid\sim \beta$			
Monotony (Improving proportions)				
(iM)	(RW) $\alpha \mid\sim \beta, \beta \vdash \beta' \Rightarrow$ $\alpha \mid\sim \beta'$			
(cMI)	(PR') $\alpha \mid\sim \beta, \alpha \vdash \alpha',$ $\alpha' \wedge \neg\alpha \vdash \beta \Rightarrow$ $\alpha' \mid\sim \beta$ (μPR) $X \subseteq Y \Rightarrow$ $\mu(Y) \cap X \subseteq \mu(X)$		(wOR) $\alpha \mid\sim \beta, \alpha' \vdash \beta \Rightarrow$ $\alpha \vee \alpha' \mid\sim \beta$ (μwOR) $\mu(X \cup Y) \subseteq \mu(X) \cup Y$	
(eMF)				(wCM) $\alpha \mid\sim \beta, \alpha' \vdash \alpha,$ $\alpha \wedge \beta \vdash \alpha' \Rightarrow$ $\alpha' \mid\sim \beta$
Keeping proportions				
$(\approx)$	(NR) $\alpha \mid\sim \beta \Rightarrow$ $\alpha \wedge \gamma \mid\sim \beta$ or $\alpha \wedge \neg\gamma \mid\sim \beta$		$(disjOR)$ $\alpha \mid\sim \beta, \alpha' \mid\sim \beta'$ $\alpha \vdash \neg\alpha', \Rightarrow$ $\alpha \vee \alpha' \mid\sim \beta \vee \beta'$ $(\mu disjOR)$ $X \cap Y = \emptyset \Rightarrow$ $\mu(X \cup Y) \subseteq \mu(X) \cup \mu(Y)$	
Robustness of proportions: $n * small \neq All$				
$(1 * s)$	(CP) $\alpha \mid\sim \perp \Rightarrow \alpha \vdash \perp$	(AND_1) $\alpha \mid\sim \beta \Rightarrow \alpha \not\mid\sim \neg\beta$		
$(2 * s)$		(AND_2) $\alpha \mid\sim \beta, \alpha \mid\sim \beta' \Rightarrow$ $\alpha \not\mid\sim \neg\beta \vee \neg\beta'$	(OR_2) $\alpha \mid\sim \beta \Rightarrow \alpha \not\mid\sim \neg\beta$	(CM_2) $\alpha \mid\sim \beta \Rightarrow \alpha \not\mid\sim \neg\beta$
$(n * s)$ $(n \geq 3)$		(AND_n) $\alpha \mid\sim \beta_1, ., \alpha \mid\sim \beta_n$ $\Rightarrow$ $\alpha \not\mid\sim \neg\beta_1 \vee . \vee \neg\beta_n$	(OR_n) $\alpha_1 \mid\sim \beta, ., \alpha_{n-1} \mid\sim \beta$ $\Rightarrow$ $\alpha_1 \vee . \vee \alpha_{n-1} \not\mid\sim \neg\beta$	(CM_n) $\alpha \mid\sim \beta_1, ., \alpha \mid\sim \beta_{n-1}$ $\Rightarrow$ $\alpha \wedge \beta_1 \wedge . \wedge \beta_{n-2} \not\mid\sim$ $\neg\beta_{n-1}$
$(< \omega * s)$		(AND_ω) $\alpha \mid\sim \beta, \alpha \mid\sim \beta' \Rightarrow$ $\alpha \mid\sim \beta \wedge \beta'$	(OR_ω) $\alpha \mid\sim \beta, \alpha' \mid\sim \beta \Rightarrow$ $\alpha \vee \alpha' \mid\sim \beta$ (μOR) $\mu(X \cup Y) \subseteq \mu(X) \cup \mu(Y)$	(CM_ω) $\alpha \mid\sim \beta, \alpha \mid\sim \beta' \Rightarrow$ $\alpha \wedge \beta \mid\sim \beta'$ (μCM) $\mu(X) \subseteq Y \subseteq X \Rightarrow$ $\mu(Y) \subseteq \mu(X)$
Robustness of $\mathcal{M}^+$				
$(\mathcal{M}^{++})$				$(RatM)$ $\alpha \mid\sim \beta, \alpha \not\mid\sim \neg\beta' \Rightarrow$ $\alpha \wedge \beta' \mid\sim \beta$ $(\mu RatM)$ $X \subseteq Y,$ $X \cap \mu(Y) \neq \emptyset \Rightarrow$ $\mu(X) \subseteq \mu(Y) \cap X$

property ψ". To keep the table "Rules on size" simple, we write mostly only the nonrelativized versions. Formally, we have $\nabla x\phi :\Leftrightarrow \{x : \phi(x)\} \in \mathcal{F}(U)$ where U is the universe, and $\nabla x(\phi : \psi) :\Leftrightarrow \{x : (\phi \wedge \psi)(x)\} \in \mathcal{F}(\{x : \phi(x)\})$. Soundness and completeness results on ∇ can be found in Schlechta (1995).

(5) Analogously, for propositional logic, we define:
$\alpha \mid\sim \beta :\Leftrightarrow M(\alpha \wedge \beta) \in \mathcal{F}(M(\alpha))$,
where $M(\phi)$ is the set of models of ϕ.

(6) In preferential structures, $\mu(X) \subseteq X$ is the set of minimal elements of X. This generates a principal filter by $\mathcal{F}(X) := \{A \subseteq X : \mu(X) \subseteq A\}$. Corresponding properties about μ are not listed systematically.

(7) The usual rules (AND), and so forth, are named here (AND_ω), as they are in a natural ascending line of similar rules, based on strengthening of the filter/ideal properties.

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(8) For any set of formulas T, and any consequence relation $\mid\sim$, we will use $\overline{T} := \{\phi : T \vdash \phi\}$, the set of classical consequences of T, and $\overline{\overline{T}} := \{\phi : T \mid\sim \phi\}$, the set of consequences of T under the relation $\mid\sim$.

(9) We say that a set X of models is definable by a formula (or a theory) iff there is a formula ϕ (a theory T) such that $X = M(\phi)$, or $X = M(T)$, the set of models of ϕ or T, respectively.

(10) Most rules are explained in table "Logical rules" and "RW" stands for right weakening.

2.2. The groups of rules. The rules concern properties of $\mathcal{I}(X)$ or $\mathcal{F}(X)$, or dependencies between such properties for different X and Y. All X, Y, and so forth, will be subsets of some universe, say V. Intuitively, V is the set of all models of some fixed propositional language. It is not necessary to consider all subsets of V, the intention is to consider subsets of V, which are definable by a formula or a theory. So we assume all X, Y, and so forth, taken from some $\mathcal{Y} \subseteq \mathcal{P}(V)$, which we call the domain. In the former case, $\mathcal{Y}$ is closed under set difference, in the latter case not necessarily so. (We will mention it when we need some particular closure property.)

The rules are divided into five groups:

(1) (Opt), which says that "All" is optimal—that is, when there are no exceptions, then a soft rule $\mid\sim$ holds.

(2) Three monotony rules:

  (2.1) (iM) is inner monotony, a subset of a small set is small,

  (2.2) $(eM\mathcal{I})$ external monotony for ideals: enlarging the base set keeps small sets small,

  (2.3) $(eM\mathcal{F})$ external monotony for filters: a big subset stays big when the base set shrinks.

  These three rules are very natural if "size" is anything coherent over change of base sets. In particular, they can be seen as weakening.

(3) $(\approx)$ keeps proportions, it is here mainly to point the possibility out.

(4) A group of rules $x * s$, which say how many small sets will not yet add to the base set. The notation "$(< \omega * s)$" is an allusion to the full filter property, that filters are closed under $finite$ intersections.

(5) Rational monotony, which can best be understood as robustness of $\mathcal{M}^+$, see $(\mathcal{M}^{++})(3)$.

We will assume all base sets to be nonempty in order to avoid pathologies and in particular clashes between (Opt) and $(1 * s)$.

2.2.1. Regularities.

(1) The group of rules $(x * s)$ use ascending strength of $\mathcal{I}/\mathcal{F}$.

(2) The column $(\mathcal{M}^+)$ contains interesting algebraic properties. In particular, they show a strengthening from $(3 * s)$ up to rationality. They are not necessarily equivalent to the corresponding (I_x) rules, not even in the presence of the basic rules. The examples show that care has to be taken when considering the different variants.

(3) Adding the somewhat superflous (CM_2), we have increasing cautious monotony from (ωCM) to full (CM_ω).

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(4) We have increasing "or" from (ωOR) to full (OR_ω).
(5) The line $(2 * s)$ is only there because there seems to be no $(\mathcal{M}_2^+)$, otherwise we could begin $(n * s)$ at $n = 2$.

2.3. Direct correspondences. Several correspondences are trivial and are mentioned now. Somewhat less obvious (in)dependencies are given in Section 3. Finally, the connections with the $\mu-$rules are given in Section 4. In those rules, (I_ω) is implicit, as they are about principal filters. Still, the $\mu-$rules are written in the table "Rules on size" in their intuitively adequate place.

(1) The columns "Ideal" and "Filter" are mutually dual, when both entries are defined.
(2) The correspondence between the ideal/filter column and the ∇column is obvious, the latter is added only for completeness' sake, and to point out the trivial translation to (augmented) first-order logic. Note that the rules in the ∇column are object-level axioms.
(3) The ideal/filter and the AND columns correspond directly.
(4) We can construct logical rules from the $\mathcal{M}^+$column by direct correspondence, for example, for $(\mathcal{M}_\omega^+)$, (1):
Set $Y := M(\gamma)$, $X := M(\gamma \wedge \beta)$, $A := M(\gamma \wedge \beta \wedge \alpha)$.

- $X \in \mathcal{M}^+(Y)$ will become $\gamma \not\vdash \neg\beta$
- $A \in \mathcal{F}(X)$ will become $\gamma \wedge \beta \vdash \alpha$
- $A \in \mathcal{M}^+(Y)$ will become $\gamma \not\vdash \neg(\alpha \wedge \beta)$.

So we obtain $\gamma \not\vdash \neg\beta$, $\gamma \wedge \beta \vdash \alpha \Rightarrow \gamma \not\vdash \neg(\alpha \wedge \beta)$.
We did not want to make the table too complicated, so such rules are not listed in the table.

(5) Various direct correspondences:

- In the line (Opt), the filter/ideal entry corresponds to (SC),
- in the line (iM), the filter/ideal entry corresponds to (RW),
- in the line $(eM\mathcal{I})$, the ideal entry corresponds to (PR') and (ωOR),
- in the line $(eM\mathcal{F})$, the filter entry corresponds to (ωCM),
- in the line $(\approx)$, the filter/ideal entry corresponds to $(disjOR)$ and (NR),
- in the line $(1 * s)$, the filter/ideal entry corresponds to (CP),
- in the line $(2 * s)$, the filter/ideal entry corresponds to $(CM_2) = (OR_2)$.

(6) Note that one can, for example, write (AND_2) in two flavors:

- $\alpha \vdash \beta, \alpha \vdash \beta' \Rightarrow \alpha \not\vdash \neg\beta \vee \neg\beta'$, or
- $\alpha \vdash \beta \Rightarrow \alpha \not\vdash \neg\beta$

— this is $(CM_2) = (OR_2)$.

For reasons of simplicity, we mention only one.

2.4. Rational monotony. $(RatM)$ does not fit into adding small sets. We have exhausted the combination of small sets by $(< \omega * s)$, unless we go to languages with infinitary formulas.

The next idea would be to add medium size sets. But, by definition, $2 * medium$ can be all. Adding small and medium sets would not help either: Suppose we have a rule

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$medium + n * small \neq all$. Taking the complement of the first medium set, which is again medium, we have the rule $2 * n * small \neq all$. So we do not see any meaningful new internal rule. That is, without changing the base set.

(For a similar reason, rules like $A, B \in \mathcal{I}(X) \Rightarrow A \cup B \notin \mathcal{F}(X)$ will give us nothing new, as then $A \cup B \in \mathcal{I}(X)$ is the full filter/ideal property, and $A \cup B \in \mathcal{M}(X)$ means just that four subsets of X may add up to the full X.)

Probably, $(RatM)$ has more to do with independence: by default, all "normalities" are independent, and intersecting with another formula preserves normality as much as possible. One should not forget here either the double use of "small" sets in our context: Nonmonotonicity works with "small" exception sets, and many rules concern modifying the base set by a "small" subset. See also Remark 2.1. below.

Still, rational monotony has its natural place in an ascending chain of conditions, which can be seen by looking at the $\mathcal{M}^+$ conditions, especially at $(\mathcal{M}^+_\omega)$ (1)–(4), and $(\mathcal{M}^{++})$ (3). Further research might tell whether there are still deeper connections behind this formal series.

More remarks on rules beyond rationality can be found in Section 2.8.

2.5. Summary. We can obtain all rules except $(RatM)$ and $(\approx)$ from (Opt), the monotony rules—(iM), $(eM\mathcal{I})$, $(eM\mathcal{F})$—and $(x * s)$ with increasing x.

2.6. Main table. The following table is split in two, as it is too big for printing in one page.

REMARK 2.1. *There is, however, an important conceptual distinction to make here. Filters express "size" in an abstract way, in the context of nonmonotonic logics, $\alpha \mid\sim \beta$ iff the set of $\alpha \wedge \neg\beta$ is small in α. But here, we were interested in "small" changes in the reference set X (or α in our example). So we have two quite different uses of "size", one for nonmonotonic logics, abstractly expressed by a filter, the other for coherence conditions. It is possible, but not necessary, to consider both essentially the same notions. But we should not forget that we have two conceptually different uses of size here.*

2.7. The filter and the partial order view. Friedman and Halpern have presented an alternative abstract view on nonmonotonic logics, perhaps best accessible in Friedman & Halpern (1996). Their approach is via a partial order, essentially rewriting $A \in \mathcal{I}(X)$ as $A < X$. Thus, ideals and filters are replaced by a binary relation, expressing size. The (almost perfect) equivalence to the approach of Ben-David & Ben-Eliyahu (1994) was shown in Schlechta (1997).

A referee has suggested the following approach in the spirit of Friedman & Halpern (1996):

Define a two-place function f into a set with a linear order $\leq$, and with threshold elements s and l, $s \leq l$, and rewrite $A \in \mathcal{I}(X)$ as $f(A, X) \leq s$, and so forth. To speak about medium size elements, we need another element m between s and l.

In this way, we can translate

- (Opt) to: $f(\emptyset, X) = s$, $f(X, X) = l$
- (im) to: if $A \subseteq X \subseteq Y$, then $f(X, Y) = s \Rightarrow f(A, Y) = s$ and $f(A, Y) = l \Rightarrow f(X, Y) = l$
- (eMI) to: if $A \subseteq X \subseteq Y$, then $f(A, X) = s \Rightarrow f(A, Y) = s$
- (eMF) to: if $A \subseteq X \subseteq Y$, then $f(A, Y) = l \Rightarrow f(A, X) = l$.

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Rules I

Table 3. Logical rules, definitions and connections I

Logical rule		Corr.	Model set	Corr.	Size Rules
Basics					
(SC) Supraclassicality $\alpha \vdash \beta \Rightarrow \alpha \mid\!\sim \beta$ (REF) Reflexivity $T \cup \{\alpha\} \mid\!\sim \alpha$	(SC) $\overline{T} \subseteq \overline{\overline{T}}$	$\Rightarrow$ $\Leftarrow$	$(\mu \subseteq)$ $f(X) \subseteq X$	trivial	(Opt)
(LLE) Left Logical Equivalence $\vdash \alpha \leftrightarrow \alpha', \alpha \mid\!\sim \beta \Rightarrow$ $\alpha' \mid\!\sim \beta$	(LLE) $\overline{T} = \overline{T'} \Rightarrow \overline{\overline{T}} = \overline{\overline{T'}}$				
(RW) Right Weakening $\alpha \mid\!\sim \beta, \vdash \beta \rightarrow \beta' \Rightarrow$ $\alpha \mid\!\sim \beta'$	(RW) $T \mid\!\sim \beta, \vdash \beta \rightarrow \beta' \Rightarrow$ $T \mid\!\sim \beta'$			trivial	(iM)
(wOR) $\alpha \mid\!\sim \beta, \alpha' \vdash \beta \Rightarrow$ $\alpha \vee \alpha' \mid\!\sim \beta$	(wOR) $\overline{\overline{T} \cap \overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	$\Rightarrow$ $\Leftarrow$	(μwOR) $f(X \cup Y) \subseteq f(X) \cup Y$	$\Leftrightarrow$ (1)	$(eM\mathcal{I})$
$(disjOR)$ $\alpha \vdash \neg\alpha', \alpha \mid\!\sim \beta,$ $\alpha' \mid\!\sim \beta \Rightarrow \alpha \vee \alpha' \mid\!\sim \beta$	$(disjOR)$ $\neg Con(T \cup T') \Rightarrow$ $\overline{\overline{T} \cap \overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	$\Rightarrow$ $\Leftarrow$	$(\mu disjOR)$ $X \cap Y = \emptyset \Rightarrow$ $f(X \cup Y) \subseteq f(X) \cup f(Y)$	$\Leftrightarrow$ (4)	$(I \cup disj)$
(CP) Consistency Preservation $\alpha \mid\!\sim \bot \Rightarrow \alpha \vdash \bot$	(CP) $T \mid\!\sim \bot \Rightarrow T \vdash \bot$	$\Rightarrow$ $\Leftarrow$	$(\mu\emptyset)$ $f(X) = \emptyset \Rightarrow X = \emptyset$	trivial	(I_1)
			$(\mu\emptyset fin)$ $X \neq \emptyset \Rightarrow f(X) \neq \emptyset$ for finite X		(I_1)
	(AND_1) $\alpha \mid\!\sim \beta \Rightarrow \alpha \not\mid\!\sim \neg\beta$				(I_2)
	(AND_n) $\alpha \mid\!\sim \beta_1, \ldots, \alpha \mid\!\sim \beta_{n-1} \Rightarrow$ $\alpha \not\mid\!\sim (\neg\beta_1 \vee \ldots \vee \neg\beta_{n-1})$				(I_n)
(AND) $\alpha \mid\!\sim \beta, \alpha \mid\!\sim \beta' \Rightarrow$ $\alpha \mid\!\sim \beta \wedge \beta'$	(AND) $T \mid\!\sim \beta, T \mid\!\sim \beta' \Rightarrow$ $T \mid\!\sim \beta \wedge \beta'$			trivial	(I_ω)
(CCL) Classical Closure	(CCL) $\overline{\overline{T}}$ classically closed			trivial	$(iM) + (I_\omega)$
(OR) $\alpha \mid\!\sim \beta, \alpha' \mid\!\sim \beta \Rightarrow$ $\alpha \vee \alpha' \mid\!\sim \beta$	(OR) $\overline{\overline{T} \cap \overline{T'}} \subseteq \overline{T \vee T'}$	$\Rightarrow$ $\Leftarrow$	(μOR) $f(X \cup Y) \subseteq f(X) \cup f(Y)$	$\Leftrightarrow$ (2)	$(eM\mathcal{I}) + (I_\omega)$
$\overline{\overline{\alpha \wedge \alpha'}} \subseteq \overline{\overline{\alpha} \cup \{\alpha'\}}$	(PR) $\overline{\overline{T \cup T'}} \subseteq \overline{\overline{T} \cup T'}$	$\Rightarrow$ $\Leftarrow (\mu dp) + (\mu \subseteq)$ $\not\Leftarrow -(\mu dp)$ $\Leftarrow (\mu \subseteq)$ $T' = \phi$	(μPR) $X \subseteq Y \Rightarrow$ $f(Y) \cap X \subseteq f(X)$	$\Leftrightarrow$ (3)	$(eM\mathcal{I}) + (I_\omega)$
		$\Leftarrow$ $T' = \phi$	$(\mu PR')$ $f(X) \cap Y \subseteq f(X \cap Y)$		
(CUT) $T \mid\!\sim \alpha; T \cup \{\alpha\} \mid\!\sim \beta \Rightarrow$ $T \mid\!\sim \beta$	(CUT) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{T'} \subseteq \overline{\overline{T}}$	$\Rightarrow$ $\Leftarrow$	(μCUT) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(X) \subseteq f(Y)$	$\Leftarrow$ (8.1) $\not\Rightarrow$ (8.2)	$(eM\mathcal{I}) + (I_\omega)$

In this translation, (eMI) and (eMF) are special cases of the principle
$(Order2)$ if $A \subseteq X \subseteq Y$, then $f(A, Y) \leq f(A, X)$
and (im) can be written as
$(Order1)$ if $A \subseteq X \subseteq Y$, then $f(A, Y) \leq f(X, Y)$,
both can be put together to
$(Order)$ if $A \subseteq X \subseteq Y$, then $f(A, Y) \leq min\{f(X, Y), f(A, X)\}$,
and also (Opt) be added to obtain finally
$(Order^+)$ if $A \subseteq X \subseteq Y$, then $s = f(\emptyset, X) \leq f(A, Y) \leq min\{f(X, Y), f(A, X)\} \leq f(X, X) = l$.

It might be a question of taste which language one prefers. Above principle $(Order2)$ has the advantage to unite the intuitively close (eMI) and (eMF), whereas our separate notation underlines that our intuition might be false, as there are examples which separate them.

2.8. Discussion of other, related, rules. We discuss here briefly more rules found in the literature—and express our gratitude to the referee who pointed them out to us. One of them, (NR), fits into our "keeping proportions" line, and is integrated there, another

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Rules II

Table 4. Logical rules, definitions and connections II

Logical rule		Corr.	Model set	Corr.	Size-Rule
Cumulativity					
(wCM) $\alpha \mid\sim \beta, \alpha' \vdash \alpha, \alpha \wedge \beta \vdash \alpha' \Rightarrow$ $\alpha' \mid\sim \beta$				trivial	$(e\mathcal{MF})$
(CM_2) $\alpha \mid\sim \beta, \alpha \mid\sim \beta' \Rightarrow \alpha \wedge \beta \not\vdash \neg\beta'$					(I_2)
(CM_n) $\alpha \mid\sim \beta_1, \ldots, \alpha \mid\sim \beta_n \Rightarrow$ $\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{n-1} \not\vdash \neg\beta_n$					(I_n)
(CM) Cautious Monotony $\alpha \mid\sim \beta, \alpha \mid\sim \beta' \Rightarrow$ $\alpha \wedge \beta \mid\sim \beta'$	(CM) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T}} \subseteq \overline{\overline{T'}}$	$\Rightarrow$ $\Leftarrow$	(μCM) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) \subseteq f(X)$	$\Leftrightarrow$ (5)	$(\mathcal{M}^{+}_{\omega})(4)$
or $(ResM)$ Restricted Monotony $T \mid\sim \alpha, \beta \Rightarrow T \cup \{\alpha\} \mid\sim \beta$		$\Rightarrow$ $\Leftarrow$	$(\mu ResM)$ $f(X) \subseteq A \cap B \Rightarrow$ $f(X \cap A) \subseteq B$		
(CUM) Cumulativity $\alpha \mid\sim \beta \Rightarrow$ $(\alpha \mid\sim \beta' \Leftrightarrow \alpha \wedge \beta \mid\sim \beta')$	(CUM) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T}} = \overline{\overline{T'}}$	$\Rightarrow$ $\Leftarrow$	(μCUM) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) = f(X)$	$\Leftarrow$ (9.1) $\not\Rightarrow$ (9.2)	$(e\mathcal{MI}) + (I_\omega) + (\mathcal{M}^{+}_{\omega})(4)$
	$(\subseteq\supseteq)$ $T \subseteq \overline{\overline{T'}}, T' \subseteq \overline{\overline{T}} \Rightarrow$ $T' = \overline{\overline{T}}$	$\Rightarrow$ $\Leftarrow$	$(\mu \subseteq\supseteq)$ $f(X) \subseteq Y, f(Y) \subseteq X \Rightarrow$ $f(X) = f(Y)$	$\Leftarrow$ (10.1) $\not\Rightarrow$ (10.2)	$(e\mathcal{MI}) + (I_\omega) + (e\mathcal{MF})$
Rationality					
$(RatM)$ Rational Monotony $\alpha \mid\sim \beta, \alpha \not\mid\sim \neg\beta' \Rightarrow$ $\alpha \wedge \beta' \mid\sim \beta$	$(RatM)$ $Con(T \cup \overline{\overline{T'}}), T \vdash T' \Rightarrow$ $\overline{\overline{T}} \supseteq \overline{\overline{T'}} \cup T$	$\Rightarrow$ $\Leftarrow (\mu dp)$ $\not\Leftarrow -(\mu dp)$ $\Leftarrow T = \phi$	$(\mu RatM)$ $X \subseteq Y, X \cap f(Y) \neq \emptyset \Rightarrow$ $f(X) \subseteq f(Y) \cap X$	$\Leftrightarrow$ (6)	$(\mathcal{M}^{++})$
	$(RatM=)$ $Con(T \cup \overline{\overline{T'}}), T \vdash T' \Rightarrow$ $\overline{\overline{T}} = \overline{\overline{T'}} \cup T$	$\Rightarrow$ $\Leftarrow (\mu dp)$ $\not\Leftarrow -(\mu dp)$ $\Leftarrow T = \phi$	$(\mu =)$ $X \subseteq Y, X \cap f(Y) \neq \emptyset \Rightarrow$ $f(X) = f(Y) \cap X$		
	$(Log =')$ $Con(\overline{T'} \cup T) \Rightarrow$ $\overline{\overline{T \cup T'}} = \overline{\overline{T'}} \cup T$	$\Rightarrow$ $\Leftarrow (\mu dp)$ $\not\Leftarrow -(\mu dp)$ $\Leftarrow T = \phi$	$(\mu =')$ $f(Y) \cap X \neq \emptyset \Rightarrow$ $f(Y \cap X) = f(Y) \cap X$		
(DR) $\alpha \vee \beta \mid\sim \gamma \Rightarrow$ $\alpha \mid\sim \gamma$ or $\beta \mid\sim \gamma$	$(Log \parallel)$ $\overline{\overline{T \vee T'}}$ is one of $\overline{\overline{T}}$, or $\overline{\overline{T'}}$, or $\overline{\overline{T}} \cap \overline{\overline{T'}}$ (by (CCL))	$\Rightarrow$ $\Leftarrow$	$(\mu \parallel)$ $f(X \cup Y)$ is one of $f(X)$, $f(Y)$ or $f(X) \cup f(Y)$		
	$(Log\cup)$ $Con(\overline{\overline{T'}} \cup T), \neg Con(\overline{\overline{T'}} \cup \overline{\overline{T}}) \Rightarrow$ $\neg Con(\overline{T \vee T'} \cup T')$	$\Rightarrow (\mu \subseteq) + (\mu =)$ $\Leftarrow (\mu dp)$ $\not\Leftarrow -(\mu dp)$	$(\mu\cup)$ $f(Y) \cap (X - f(X)) \neq \emptyset \Rightarrow$ $f(X \cup Y) \cap Y = \emptyset$		
	$(Log\cup')$ $Con(\overline{\overline{T'}} \cup T), \neg Con(\overline{\overline{T'}} \cup \overline{\overline{T}}) \Rightarrow$ $\overline{\overline{T \vee T'}} = \overline{\overline{T}}$	$\Rightarrow (\mu \subseteq) + (\mu =)$ $\Leftarrow (\mu dp)$ $\not\Leftarrow -(\mu dp)$	$(\mu\cup')$ $f(Y) \cap (X - f(X)) \neq \emptyset \Rightarrow$ $f(X \cup Y) = f(X)$		
			$(\mu \in)$ $a \in X - f(X) \Rightarrow$ $\exists b \in X. a \notin f(\{a, b\})$		

one, (DR), is the unitary version of our $(Log\ \|)$, and put in the table "Logical rules" of Definition 4.

From Bezzazi *et al.* (1997), we cite:

- Disjunctive rationality
  $(DR)\ \alpha \vee \beta \ \vdash\!\sim \gamma \Rightarrow \alpha \ \vdash\!\sim \gamma$ or $\beta \ \vdash\!\sim \gamma$
  holds in ranked structures and corresponds to our $(Log\ \|)$.
- Negation rationality
  $(NR)\ \alpha \ \vdash\!\sim \beta \Rightarrow \alpha \wedge \gamma \ \vdash\!\sim \beta$ or $\alpha \wedge \neg\gamma \ \vdash\!\sim \beta$
  corresponds to the $(\mathcal{M}^+)$ variant of $(\approx)$, $(\mathcal{M}^+ \cup disj)$.

The following four rules, also taken from Bezzazi *et al.* (1997), all fall in a class we may call $(ULTRA)$. Looking at them from a size perspective, they say that some filters must be ultrafilters (i.e., there are no medium size sets, everything is either small or big). From the preferential perspective, it means that there is at most one minimal model.

- Determinacy preservation
  $(DP)\ \alpha \ \vdash\!\sim \beta \Rightarrow \alpha \wedge \gamma \ \vdash\!\sim \beta$ or $\alpha \wedge \gamma \ \vdash\!\sim \neg\beta$.
- Rational transitivity
  $(RT)\ \alpha \ \vdash\!\sim \beta, \beta \ \vdash\!\sim \gamma \Rightarrow \alpha \ \vdash\!\sim \gamma$ or $\alpha \ \vdash\!\sim \neg\gamma$.
- Rational contraposition
  $(RC)\ \alpha \ \vdash\!\sim \beta \Rightarrow \neg\beta \ \vdash\!\sim \alpha$ or $\neg\beta \ \vdash\!\sim \neg\alpha$.
- Weak determinacy
  $(WD)\ True \ \vdash\!\sim \neg\alpha \Rightarrow \alpha \ \vdash\!\sim \beta$ or $\alpha \ \vdash\!\sim \neg\beta$.

We think that these rules may well be adapted to certain situations, but will not be able to solve more general problems like the blonde Swedes problem. (Swedes are normally blonde and tall, but even not-blonde Swedes are tall. This is the problem of subideal situations, where still as much as possible of the ideal is preserved.)

We cite from Hawthorne & Makinson (2007), see also Hawthorne (1996, 2007) the following rules, satisfied by probabilistic consequences relation (defined as validity above a fixed value threshold):

- Very cautious monotony
  $(VCM)\ \alpha \ \vdash\!\sim \beta \wedge \gamma \Rightarrow \alpha \wedge \beta \ \vdash\!\sim \gamma$
  which we can deduce from (I_ω) the same way as (CM_ω), using other standard rules.
- Weak OR
  $(WOR)\ \alpha \wedge \beta \ \vdash\!\sim \gamma, \alpha \wedge \neg\beta \ \vdash\!\sim \gamma \Rightarrow \alpha \ \vdash\!\sim \gamma$
  corresponds to our $(disjOR)$.
- Weak AND
  $(WAND)\ \alpha \ \vdash\!\sim \gamma, \alpha \wedge \neg\beta \ \vdash\!\sim \beta \Rightarrow \alpha \ \vdash\!\sim \beta \wedge \gamma$
  will not be discussed here, as the prerequisite $\alpha \wedge \neg\beta \ \vdash\!\sim \beta$ entails that $\emptyset$ can be a big subset.

§3. Coherent systems.

3.1. Definition and basic facts. Note that whenever we work with model sets, the rule (LLE), left logical equivalence, $\vdash \alpha \leftrightarrow \alpha' \Rightarrow (\alpha \ \vdash\!\sim \beta \Leftrightarrow \alpha' \ \vdash\!\sim \beta)$
will hold. We will use this without further mentioning.

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DEFINITION 3.1. *A coherent system of sizes, $\mathcal{CS}$, consists of a universe U, $\emptyset \notin \mathcal{Y} \subseteq \mathcal{P}(U)$, and for all $X \in \mathcal{Y}$ a system $\mathcal{I}(X) \subseteq \mathcal{P}(X)$ (dually $\mathcal{F}(X)$, i.e., $A \in \mathcal{F}(X) \Leftrightarrow X - A \in \mathcal{I}(X)$). $\mathcal{Y}$ may satisfy certain closure properties like closure under $\cup$, $\cap$, complementation, and so forth. We will mention this when needed, and not obvious.*

We say that $\mathcal{CS}$ satisfies a certain property iff all $X, Y \in \mathcal{Y}$ satisfy this property. $\mathcal{CS}$ is called basic or Level 1 iff it satisfies (Opt), (iM), $(eM\mathcal{I})$, $(eM\mathcal{F})$, $(1 * s)$. $\mathcal{CS}$ is level x iff it satisfies (Opt), (iM), $(eM\mathcal{I})$, $(eM\mathcal{F})$, $(x * s)$.

FACT 3.4. *Note that, if for any Y $\mathcal{I}(Y)$ consists only of subsets of at most one element, then $(eM\mathcal{F})$ is trivially satisfied for Y and its subsets by (Opt).* □

FACT 3.5. *Let a $\mathcal{CS}$ be given s.t. $\mathcal{Y} = \mathcal{P}(U)$. If $X \in \mathcal{Y}$ satisfies $(\mathcal{M}^{++})$, but not $(< \omega * s)$, then there is $Y \in \mathcal{Y}$ which does not satisfy $(2 * s)$.*

Proof. We work with version (1) of $(\mathcal{M}^{++})$, we will see in Fact 3.12. that all three versions are equivalent.

As X does not satisfy $(< \omega * s)$, there are $A, B \in \mathcal{I}(X)$ s.t. $A \cup B \in \mathcal{M}^{+}(X)$. $A \in \mathcal{I}(X)$, $A \cup B \in \mathcal{M}^{+}(X) \Rightarrow X - (A \cup B) \notin \mathcal{F}(X)$, so by $(\mathcal{M}^{++})(1)$ $A = A - (X - (A \cup B)) \in \mathcal{I}(X - (X - (A \cup B))) = \mathcal{I}(A \cup B)$. Likewise $B \in \mathcal{I}(A \cup B)$, so $(2 * s)$ does not hold for $A \cup B$. □

FACT 3.6. *$(eM\mathcal{I})$ and $(eM\mathcal{F})$ are formally independent, though intuitively equivalent.*

Proof. Let $U := \{x, y, z\}$, $X := \{x, z\}$, $\mathcal{Y} := \mathcal{P}(U) - \{\emptyset\}$.

(1) Let $\mathcal{F}(U) := \{A \subseteq U : z \in A\}$, $\mathcal{F}(Y) = \{Y\}$ for all $Y \subset U$. (Opt), (iM) hold, $(eM\mathcal{I})$ holds trivially, so does $(< \omega * s)$, but $(eM\mathcal{F})$ fails for U and X.

(2) Let $\mathcal{F}(X) := \{\{z\}, X\}$, $\mathcal{F}(Y) := \{Y\}$ for all $Y \subseteq U$, $Y \neq X$. (Opt), (iM), $(< \omega * s)$ hold trivially, $(eM\mathcal{F})$ holds by Fact 3.4. $(eM\mathcal{I})$ fails, as $\{x\} \in \mathcal{I}(X)$, but $\{x\} \notin \mathcal{I}(U)$. □

FACT 3.7. *A level n system is strictly weaker than a level $n + 1$ system.*

Proof. Consider $U := \{1, \ldots, n+1\}$, $\mathcal{Y} := \mathcal{P}(U) - \{\emptyset\}$. Let $\mathcal{I}(U) := \{\emptyset\} \cup \{\{x\} : x \in U\}$, $\mathcal{I}(X) := \{\emptyset\}$ for $X \neq U$. (iM), $(eM\mathcal{I})$, $(eM\mathcal{F})$ hold trivially. $(n * s)$ holds trivially for $X \neq U$, but also for U. $((n + 1) * s)$ does not hold for U. □

REMARK 3.6. *Note that our schemata allow us to generate infinitely many new rules, here is an example:*

Start with A, add $s_{1,1}$, $s_{1,2}$ two sets small in $A \cup s_{1,1}$ ($A \cup s_{1,2}$ respectively). Consider now $A \cup s_{1,1} \cup s_{1,2}$ and s_2 s.t. s_2 is small in $A \cup s_{1,1} \cup s_{1,2} \cup s_2$. Continue with $s_{3,1}$, $s_{3,2}$ small in $A \cup s_{1,1} \cup s_{1,2} \cup s_2 \cup s_{3,1}$, and so forth.

Without additional properties, this system creates a new rule, which is not equivalent to any usual rules.

3.2. Implications between the finite versions.

FACT 3.8.

(1) $(I_n) + (eM\mathcal{I}) \Rightarrow (\mathcal{M}_n^{+})$,
(2) $(I_n) + (eM\mathcal{I}) \Rightarrow (CM_n)$,
(3) $(I_n) + (eM\mathcal{I}) \Rightarrow (OR_n)$.

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Proof.

(1) Let $X_1 \subseteq \ldots \subseteq X_n$, so $X_n = X_1 \cup (X_2 - X_1) \cup \ldots \cup (X_n - X_{n-1})$. Let $X_i \in \mathcal{F}(X_{i+1})$, so $X_{i+1} - X_i \in \mathcal{I}(X_{i+1}) \subseteq \mathcal{I}(X_n)$ by $(eM\mathcal{I})$ for $1 \leq i \leq n-1$, so by (I_n) $X_1 \in \mathcal{M}^+(X_n)$.

(2) Suppose $\alpha \mathrel{|\!\sim} \beta_1, \ldots, \alpha \mathrel{|\!\sim} \beta_{n-1}$, but $\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{n-2} \not\mathrel{|\!\sim} \neg\beta_{n-1}$. Then $M(\alpha \wedge \neg\beta_1), \ldots, M(\alpha \wedge \neg\beta_{n-1}) \in \mathcal{I}(M(\alpha))$, and $M(\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{n-2} \wedge \beta_{n-1}) \in \mathcal{I}(M(\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{n-2})) \subseteq \mathcal{I}(M(\alpha))$ by $(eM\mathcal{I})$. But $M(\alpha) = M(\alpha \wedge \neg\beta_1) \cup \ldots \cup M(\alpha \wedge \neg\beta_{n-1}) \cup M(\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{n-2} \wedge \beta_{n-1})$ is now the union of n small subsets, *contradiction*.

(3) Let $\alpha_1 \mathrel{|\!\sim} \beta, \ldots, \alpha_{n-1} \mathrel{|\!\sim} \beta$, so $M(\alpha_i \wedge \neg\beta) \in \mathcal{I}(M(\alpha_i))$ for $1 \leq i \leq n-1$, so $M(\alpha_i \wedge \neg\beta) \in \mathcal{I}(M(\alpha_1 \vee \ldots \vee \alpha_{n-1}))$ for $1 \leq i \leq n-1$ by $(eM\mathcal{I})$, so $M((\alpha_1 \vee \ldots \vee \alpha_{n-1}) \wedge \beta) = M(\alpha_1 \vee \ldots \vee \alpha_{n-1}) - \cup\{M(\alpha_i \wedge \neg\beta) : 1 \leq i \leq n-1\} \notin \mathcal{I}(M(\alpha_1 \vee \ldots \vee \alpha_{n-1}))$ by (I_n), so $\alpha_1 \vee \ldots \vee \alpha_{n-1} \not\mathrel{|\!\sim} \neg\beta$.

□

In the following example, (OR_n), $(\mathcal{M}_n^+)$, (CM_n) hold, but $(\mathcal{I}_n)$ fails, so by Fact 3.8. $(\mathcal{I}_n)$ is strictly stronger than (OR_n), $(\mathcal{M}_n^+)$, (CM_n).

EXAMPLE 3.8.

Let $n \geq 3$.

Consider $X := \{1, \ldots, n\}$, $\mathcal{Y} := \mathcal{P}(X) - \{\emptyset\}$, $\mathcal{I}(X) := \{\emptyset\} \cup \{\{i\} : 1 \leq i \leq n\}$, and for all $Y \subset X$ $\mathcal{I}(Y) := \{\emptyset\}$.

(Opt), (iM), $(eM\mathcal{I})$, $(eM\mathcal{F})$ (by Fact 3.4.), $(1 * s)$, $(2 * s)$ hold, (I_n) fails, of course.

(1) *(OR_n) holds:*

Suppose $\alpha_1 \mathrel{|\!\sim} \beta, \ldots, \alpha_{n-1} \mathrel{|\!\sim} \beta$, $\alpha_1 \vee \ldots \vee \alpha_{n-1} \not\mathrel{|\!\sim} \neg\beta$.

Case 1: *$\alpha_1 \vee \ldots \vee \alpha_{n-1} \vdash \neg\beta$, then for all i $\alpha_i \vdash \neg\beta$, so for no i $\alpha_i \mathrel{|\!\sim} \beta$ by $(1 * s)$ and thus (AND_1), contradiction.*

Case 2: *$\alpha_1 \vee \ldots \vee \alpha_{n-1} \not\mathrel{|\!\sim} \neg\beta$, then $M(\alpha_1 \vee \ldots \vee \alpha_{n-1}) = X$, and there is exactly 1 $k \in X$ s.t. $k \models \beta$. Fix this k. By prerequisite, $\alpha_i \mathrel{|\!\sim} \beta$. If $M(\alpha_i) = X$, $\alpha_i \vdash \beta$ cannot be, so there must be exactly 1 k' s.t. $k' \models \neg\beta$, but $card(X) \geq 3$, contradiction. So $M(\alpha_i) \subset X$, and $\alpha_i \vdash \beta$, so $M(\alpha_i) = \emptyset$ or $M(\alpha_i) = \{k\}$ for all i, so $M(\alpha_1 \vee \ldots \vee \alpha_{n-1}) \neq X$, contradiction.*

(2) *$(\mathcal{M}_n^+)$ holds:*

$(\mathcal{M}_n^+)$ is a consequence of $(\mathcal{M}_\omega^+)$, (3) so it suffices to show that the latter holds. Let $X_1 \in \mathcal{F}(X_2)$, $X_2 \in \mathcal{F}(X_3)$. Then $X_1 = X_2$ or $X_2 = X_3$, so the result is trivial.

(3) *(CM_n) holds:*

Suppose $\alpha \mathrel{|\!\sim} \beta_1, \ldots, \alpha \mathrel{|\!\sim} \beta_{n-1}$, $\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{n-2} \not\mathrel{|\!\sim} \neg\beta_{n-1}$.

Case 1: *For all i, $1 \leq i \leq n-2$, $\alpha \vdash \beta_i$, then $M(\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{n-2}) = M(\alpha)$, so $\alpha \mathrel{|\!\sim} \beta_{n-1}$ and $\alpha \not\mathrel{|\!\sim} \neg\beta_{n-1}$, contradiction.*

Case 2: *There is i, $1 \leq i \leq n-2$, $\alpha \not\vdash \beta_i$, then $M(\alpha) = X$, $M(\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{n-2}) \subset M(\alpha)$, so $\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{n-2} \vdash \neg\beta_{n-1}$. $Card(M(\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{n-2})) \geq n - (n-2) = 2$, so $card(M(\neg\beta_{n-1})) \geq 2$, so $\alpha \not\mathrel{|\!\sim} \beta_{n-1}$, contradiction.*

□

3.3. Implications between the ω versions.

FACT 3.9. *$(CM_\omega) \Leftrightarrow (\mathcal{M}_\omega^+)$ (4)*

Proof. " $\Rightarrow$ "

Suppose all sets are definable.

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Let $A, B \in \mathcal{I}(X)$, $X = M(\alpha)$, $A = M(\alpha \wedge \neg\beta)$, $B = M(\alpha \wedge \neg\beta')$, so $\alpha \mid\!\sim \beta$, $\alpha \mid\!\sim \beta'$, so by (CM_ω) $\alpha \wedge \beta' \mid\!\sim \beta$, so $A - B = M(\alpha \wedge \beta' \wedge \neg\beta) \in \mathcal{I}(M(\alpha \wedge \beta')) = \mathcal{I}(X - B)$.
"$\Leftarrow$"
Let $\alpha \mid\!\sim \beta$, $\alpha \mid\!\sim \beta'$, so $M(\alpha \wedge \neg\beta) \in \mathcal{I}(M(\alpha))$, $M(\alpha \wedge \neg\beta') \in \mathcal{I}(M(\alpha))$, so by prerequisite $M(\alpha \wedge \neg\beta') - M(\alpha \wedge \neg\beta) = M(\alpha \wedge \beta \wedge \neg\beta') \in \mathcal{I}(M(\alpha) - M(\alpha \wedge \neg\beta)) = \mathcal{I}(M(\alpha \wedge \beta))$, so $\alpha \wedge \beta \mid\!\sim \beta'$. □

FACT 3.10.

(1) $(I_\omega) + (eM\mathcal{I}) \Rightarrow (OR_\omega)$,
(2) $(I_\omega) + (eM\mathcal{I}) \Rightarrow (\mathcal{M}^+_\omega)$ *(1),*
(3) $(I_\omega) + (eM\mathcal{F}) \Rightarrow (\mathcal{M}^+_\omega)$ *(2),*
(4) $(I_\omega) + (eM\mathcal{I}) \Rightarrow (\mathcal{M}^+_\omega)$ *(3),*
(5) $(I_\omega) + (eM\mathcal{F}) \Rightarrow (\mathcal{M}^+_\omega)$ *(4) (and thus, by Fact 3.9.,* (CM_ω)*).*

Proof.

(1) Let $\alpha \mid\!\sim \beta$, $\alpha' \mid\!\sim \beta \Rightarrow M(\alpha \wedge \neg\beta) \in \mathcal{I}(M(\alpha))$, $M(\alpha' \wedge \neg\beta) \in \mathcal{I}(M(\alpha'))$, so by $(eM\mathcal{I})$ $M(\alpha \wedge \neg\beta) \in \mathcal{I}(M(\alpha \vee \alpha'))$, $M(\alpha' \wedge \neg\beta) \in \mathcal{I}(M(\alpha \vee \alpha'))$, so $M((\alpha \vee \alpha') \wedge \neg\beta) \in \mathcal{I}(M(\alpha \vee \alpha'))$ by (I_ω), so $\alpha \vee \alpha' \mid\!\sim \beta$.
(2) Let $A \subseteq X \subseteq Y$, $A \in \mathcal{I}(Y)$, $X - A \in \mathcal{I}(X) \subseteq_{(eM\mathcal{I})} \mathcal{I}(Y) \Rightarrow X = (X - A) \cup A \in \mathcal{I}(Y)$ by (I_ω).
(3) Let $A \subseteq X \subseteq Y$, let $A \in \mathcal{I}(Y)$, $Y - X \in \mathcal{I}(Y) \Rightarrow A \cup (Y - X) \in \mathcal{I}(Y)$ by $(I_\omega) \Rightarrow X - A = Y - (A \cup (Y - X)) \in \mathcal{F}(Y) \Rightarrow X - A \in \mathcal{F}(X)$ by $(eM\mathcal{F})$.
(4) Let $A \subseteq X \subseteq Y$, $A \in \mathcal{F}(X)$, $X \in \mathcal{F}(Y)$, so $Y - X \in \mathcal{I}(Y)$, $X - A \in \mathcal{I}(X) \subseteq_{(eM\mathcal{I})} \mathcal{I}(Y) \Rightarrow Y - A = (Y - X) \cup (X - A) \in \mathcal{I}(Y)$ by $(\mathcal{I}_\omega) \Rightarrow A \in \mathcal{F}(Y)$.
(5) Let $A, B \subseteq X$, $A, B \in \mathcal{I}(X) \Rightarrow_{(I_\omega)} A \cup B \in \mathcal{I}(X) \Rightarrow X - (A \cup B) \in \mathcal{F}(X)$, but $X - (A \cup B) \subseteq X - B$, so $X - (A \cup B) \in \mathcal{F}(X - B)$ by $(eM\mathcal{F})$, so $A - B = (X - B) - (X - (A \cup B)) \in \mathcal{I}(X - B)$.

□

We give three examples of independence of the various versions of $(\mathcal{M}^+_\omega)$.

EXAMPLE 3.11. *All numbers refer to the versions of* $(\mathcal{M}^+_\omega)$.
For easier reading, we rewrite for $A \subseteq X \subseteq Y$
$(\mathcal{M}^+_\omega)(1) : A \in \mathcal{F}(X)$, $A \in \mathcal{I}(Y) \Rightarrow X \in \mathcal{I}(Y)$,
$(\mathcal{M}^+_\omega)(2) : X \in \mathcal{F}(Y)$, $A \in \mathcal{I}(Y) \Rightarrow A \in \mathcal{I}(X)$.
We give three examples. Investigating all possibilities exhaustively seems quite tedious, and might best be done with the help of a computer. Fact 3.4. will be used repeatedly.

- *(1), (2), (4) fail, (3) holds:*
  Let $Y := \{a, b, c\}$, $\mathcal{Y} := \mathcal{P}(Y) - \{\emptyset\}$, $\mathcal{F}(Y) := \{\{a, c\}, \{b, c\}, Y\}$
  Let $X := \{a, b\}$, $\mathcal{F}(X) := \{\{a\}, X\}$, $A := \{a\}$, *and* $\mathcal{F}(Z) := \{Z\}$ *for all* $Z \neq X, Y$.
  (Opt), (iM), $(eM\mathcal{I})$, $(eM\mathcal{F})$ *hold,* (I_ω) *fails, of course.*
  (1) fails: $A \in \mathcal{F}(X)$, $A \in \mathcal{I}(Y)$, $X \notin \mathcal{I}(Y)$.
  (2) fails: $\{a, c\} \in \mathcal{F}(Y)$, $\{a\} \in \mathcal{I}(Y)$, *but* $\{a\} \notin \mathcal{I}(\{a, c\})$.
  (3) holds: If $X_1 \in \mathcal{F}(X_2)$, $X_2 \in \mathcal{F}(X_3)$, *then* $X_1 = X_2$ *or* $X_2 = X_3$, *so (3) holds trivially (note that* $X \notin \mathcal{F}(Y)$*).*
  (4) fails: $\{a\}, \{b\} \in \mathcal{I}(Y)$, $\{a\} \notin \mathcal{I}(Y - \{b\}) = \mathcal{I}(\{a, c\}) = \{\emptyset\}$.

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- *(2), (3), (4) fail, (1) holds:*
  Let $Y := \{a, b, c\}$, $\mathcal{Y} := \mathcal{P}(Y) - \{\emptyset\}$, $\mathcal{F}(Y) := \{\{a, b\}, \{a, c\}, Y\}$
  Let $X := \{a, b\}$, $\mathcal{F}(X) := \{\{a\}, X\}$, *and* $\mathcal{F}(Z) := \{Z\}$ *for all* $Z \neq X, Y$.
  (Opt), (iM), $(eM\mathcal{I})$, $(eM\mathcal{F})$ *hold,* (I_ω) *fails, of course.*
  (1) holds:
  Let $X_1 \in \mathcal{F}(X_2)$, $X_1 \in \mathcal{I}(X_3)$, *we have to show* $X_2 \in \mathcal{I}(X_3)$. *If* $X_1 = X_2$, *then this is trivial. Consider* $X_1 \in \mathcal{F}(X_2)$. *If* $X_1 \neq X_2$, *then* X_1 *has to be* $\{a\}$ *or* $\{a, b\}$ *or* $\{a, c\}$. *But none of these are in* $\mathcal{I}(X_3)$ *for any* X_3, *so the implication is trivially true.*
  (2) fails: $\{a, c\} \in \mathcal{F}(Y)$, $\{c\} \in \mathcal{I}(Y)$, $\{c\} \notin \mathcal{I}(\{a, c\})$.
  (3) fails: $\{a\} \in \mathcal{F}(X)$, $X \in \mathcal{F}(Y)$, $\{a\} \notin \mathcal{F}(Y)$.
  (4) fails: $\{b\}, \{c\} \in \mathcal{I}(Y)$, $\{c\} \notin \mathcal{I}(Y - \{b\}) = \mathcal{I}(\{a, c\}) = \{\emptyset\}$.
- *(1), (2), (4) hold, (3) fails:*
  Let $Y := \{a, b, c\}$, $\mathcal{Y} := \mathcal{P}(Y) - \{\emptyset\}$, $\mathcal{F}(Y) := \{\{a, b\}, \{a, c\}, Y\}$
  Let $\mathcal{F}(\{a, b\}) := \{\{a\}, \{a, b\}\}$, $\mathcal{F}(\{a, c\}) := \{\{a\}, \{a, c\}\}$, *and* $\mathcal{F}(Z) := \{Z\}$ *for all other* Z.
  (Opt), (iM), $(eM\mathcal{I})$, $(eM\mathcal{F})$ *hold,* (I_ω) *fails, of course.*
  (1) holds:
  Let $X_1 \in \mathcal{F}(X_2)$, $X_1 \in \mathcal{I}(X_3)$, *we have to show* $X_2 \in \mathcal{I}(X_3)$. *Consider* $X_1 \in \mathcal{I}(X_3)$. *If* $X_1 = X_2$, *this is trivial. If* $\emptyset \neq X_1 \in \mathcal{I}(X_3)$, *then* $X_1 = \{b\}$ *or* $X_1 = \{c\}$, *but then by* $X_1 \in \mathcal{F}(X_2)$ X_2 *has to be* $\{b\}$, *or* $\{c\}$, *so* $X_1 = X_2$.
  (2) holds: Let $X_1 \subseteq X_2 \subseteq X_3$, *let* $X_2 \in \mathcal{F}(X_3)$, $X_1 \in \mathcal{I}(X_3)$, *we have to show* $X_1 \in \mathcal{I}(X_2)$. *If* $X_1 = \emptyset$, *this is trivial, likewise if* $X_2 = X_3$. *Otherwise* $X_1 = \{b\}$ *or* $X_1 = \{c\}$, *and* $X_3 = Y$. *If* $X_1 = \{b\}$, *then* $X_2 = \{a, b\}$, *and the condition holds, likewise if* $X_1 = \{c\}$, *then* $X_2 = \{a, c\}$, *and it holds again.*
  (3) fails: $\{a\} \in \mathcal{F}(\{a, c\})$, $\{a, c\} \in \mathcal{F}(Y)$, $\{a\} \notin \mathcal{F}(Y)$.
  (4) holds:
  If $A, B \in \mathcal{I}(X)$, *and* $A \neq B$, $A, B \neq \emptyset$, *then* $X = Y$ *and, for example,* $A = \{c\}$, $B = \{b\}$, *and* $\{c\} \in \mathcal{I}(Y - \{b\}) = \mathcal{I}(\{a, c\})$.

□

3.4. Rational monotony.

FACT 3.12. *The three versions of* $(\mathcal{M}^{++})$ *are equivalent.*
(We assume closure of the domain under set difference. For the third version of $(\mathcal{M}^{++})$, *we use* (iM).)

Proof. For (1) and (2), we have $A, B \subseteq X$, for (3) we have $A \subseteq X \subseteq Y$. For $A, B \subseteq X$, $(X - B) - ((X - A) - B) = A - B$ holds.

(1) $\Rightarrow$ (2) : Let $A \in \mathcal{F}(X)$, $B \notin \mathcal{F}(X)$, so $X - A \in \mathcal{I}(X)$, so by prerequisite $(X - A) - B \in \mathcal{I}(X - B)$, so $A - B = (X - B) - ((X - A) - B) \in \mathcal{F}(X - B)$.

(2) $\Rightarrow$ (1) : Let $A \in \mathcal{I}(X)$, $B \notin \mathcal{F}(X)$, so $X - A \in \mathcal{F}(X)$, so by prerequisite $(X - A) - B \in \mathcal{F}(X - B)$, so $A - B = (X - B) - ((X - A) - B) \in \mathcal{I}(X - B)$.

(1) $\Rightarrow$ (3) :

Suppose $A \notin \mathcal{M}^+(Y)$, but $X \in \mathcal{M}^+(Y)$, we show $A \notin \mathcal{M}^+(X)$. So $A \in \mathcal{I}(Y)$, $Y - X \notin \mathcal{F}(Y)$, so by (1) $A = A - (Y - X) \in \mathcal{I}(Y - (Y - X)) = \mathcal{I}(X)$.

(3) $\Rightarrow$ (1) :

Suppose $A - B \notin \mathcal{I}(X - B)$, $B \notin \mathcal{F}(X)$, we show $A \notin \mathcal{I}(X)$. By prerequisite $A - B \in \mathcal{M}^+(X - B)$, $X - B \in \mathcal{M}^+(X)$, so by (3) $A - B \in \mathcal{M}^+(X)$, so by (iM) $A \in \mathcal{M}^+(X)$, so $A \notin \mathcal{I}(X)$. □

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Filter II

Table 5. Size and principal filter rules

(1)	$(eM\mathcal{I})$	$\Leftrightarrow$	(μwOR)
(2)	$(eM\mathcal{I}) + (I_\omega)$	$\Leftrightarrow$	(μOR)
(3)	$(eM\mathcal{I}) + (I_\omega)$	$\Leftrightarrow$	(μPR)
(4)	$(I \cup disj)$	$\Leftrightarrow$	$(\mu disjOR)$
(5)	$(\mathcal{M}^+_\omega)(4)$	$\Leftrightarrow$	(μCM)
(6)	$(\mathcal{M}^{++})$	$\Leftrightarrow$	$(\mu RatM)$
(7)	(I_ω)	$\Leftrightarrow$	(μAND)
(8.1)	$(eM\mathcal{I}) + (I_\omega)$	$\Rightarrow$	(μCUT)
(8.2)		$\not\Leftarrow$	
(9.1)	$(eM\mathcal{I}) + (I_\omega) + (\mathcal{M}^+_\omega)(4)$	$\Rightarrow$	(μCUM)
(9.2)		$\not\Leftarrow$	
(10.1)	$(eM\mathcal{I}) + (I_\omega) + (eM\mathcal{F})$	$\Rightarrow$	$(\mu \subseteq\supseteq)$
(10.2)		$\not\Leftarrow$	

FACT 3.13. *We assume that all sets are definable by a formula.*
$(RatM) \Leftrightarrow (\mathcal{M}^{++})$.

Proof. We show equivalence of $(RatM)$ with version (1) of $(\mathcal{M}^{++})$.
"$\Rightarrow$"

We have $A, B \subseteq X$, so we can write $X = M(\phi)$, $A = M(\phi \wedge \neg\psi)$, $B = M(\phi \wedge \neg\psi')$. $A \in \mathcal{I}(X)$, $B \notin \mathcal{F}(X)$, so $\phi \nvdash \psi$, $\phi \not\vdash \neg\psi'$, so by $(RatM)$ $\phi \wedge \psi' \nvdash \psi$, so $A - B = M(\phi \wedge \neg\psi) - M(\phi \wedge \neg\psi') = M(\phi \wedge \psi' \wedge \neg\psi) \in \mathcal{I}(M(\phi \wedge \psi')) = \mathcal{I}(X - B)$.

"$\Leftarrow$"

Let $\phi \nvdash \psi$, $\phi \not\vdash \neg\psi'$, so $M(\phi \wedge \neg\psi) \in \mathcal{I}(M(\phi))$, $M(\phi \wedge \neg\psi') \notin \mathcal{F}(M(\phi))$, so by $(\mathcal{M}^{++})$ (1) $M(\phi \wedge \psi' \wedge \neg\psi) = M(\phi \wedge \neg\psi) - M(\phi \wedge \neg\psi') \in \mathcal{I}(M(\phi \wedge \psi'))$, so $\phi \wedge \psi' \nvdash \psi$. □

§4. Size and principal filter logic. The connection of our size rules with logical rules defined by principal filters in preferential structures is shown in the following table "Logical rules" of Definition 4. Most of the table was already published in Gabbay & Schlechta (2009), it is repeated here for the reader's convenience.

Again, the following table is split in two, as it is too big for printing in one page.

The numbers in the first column "Corr.," meaning "Correspondence", refer to proposition 21 in Gabbay & Schlechta (2009), those in the second column "Corr." to Proposition 4.1.

Sometimes, supplementary conditions are noted in the "Correspondence" columns. $-(\mu dp)$ stands for "without (μdp)", $T = \phi$ (or $T' = \phi$) stands for "T (or T') is equivalent to a formula".

(1)–(7) of the following proposition (in different notation, as the more systematic connections were found only afterward) were already published in Gabbay & Schlechta (2009), we give it here in totality to complete the picture.

PROPOSITION 4.1. *If $f(X)$ is the smallest A s.t. $A \in \mathcal{F}(X)$, then, given the property on the left, the one on the right follows.*

Conversely, when we define $\mathcal{F}(X) := \{X' : f(X) \subseteq X' \subseteq X\}$, given the property on the right, the one on the left follows. For this direction, we assume that we can use the full powerset of some base set U—as is the case for the model sets of a finite language. This is perhaps not too restrictive, as we mainly want to stress here the intuitive connections, without putting any weight on definability questions.

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We assume (iM) *to hold.*

Note that there is no (μwCM), as the conditions $(\mu \ldots)$ imply that the filter is principal, and thus that (I_ω) holds—we cannot "see" (wCM) alone with principal filters, as (I_ω) will hold automatically in all filters, so a fortiori in all principal filters.

Proof. (1)
$(eM\mathcal{I}) \Rightarrow (\mu wOR)$:
$X - f(X)$ is small in X, so it is small in $X \cup Y$ by $(eM\mathcal{I})$, so $A := X \cup Y - (X - f(X)) \in \mathcal{F}(X \cup Y)$, but $A \subseteq f(X) \cup Y$, and $f(X \cup Y)$ is the smallest element of $\mathcal{F}(X \cup Y)$, so $f(X \cup Y) \subseteq A \subseteq f(X) \cup Y$.
$(\mu wOR) \Rightarrow (eM\mathcal{I})$:
Let $X \subseteq Y$, $X' := Y - X$. Let $A \in \mathcal{I}(X)$, so $X - A \in \mathcal{F}(X)$, so $f(X) \subseteq X - A$, so $f(X \cup X') \subseteq f(X) \cup X' \subseteq (X - A) \cup X'$ by prerequisite, so $(X \cup X') - ((X - A) \cup X') = A \in \mathcal{I}(X \cup X')$.
(2)
$(eM\mathcal{I}) + (I_\omega) \Rightarrow (\mu OR)$:
$X - f(X)$ is small in X, $Y - f(Y)$ is small in Y, so both are small in $X \cup Y$ by $(eM\mathcal{I})$, so $A := (X - f(X)) \cup (Y - f(Y))$ is small in $X \cup Y$ by (I_ω), but $X \cup Y - (f(X) \cup f(Y)) \subseteq A$, so $f(X) \cup f(Y) \in \mathcal{F}(X \cup Y)$, so, as $f(X \cup Y)$ is the smallest element of $\mathcal{F}(X \cup Y)$, $f(X \cup Y) \subseteq f(X) \cup f(Y)$.
$(\mu OR) \Rightarrow (eM\mathcal{I}) + (I_\omega)$:
Let again $X \subseteq Y$, $X' := Y - X$. Let $A \in \mathcal{I}(X)$, so $X - A \in \mathcal{F}(X)$, so $f(X) \subseteq X - A$. $f(X') \subseteq X'$, so $f(X \cup X') \subseteq f(X) \cup f(X') \subseteq (X - A) \cup X'$ by prerequisite, so $(X \cup X') - ((X - A) \cup X') = A \in \mathcal{I}(X \cup X')$.
(I_ω) holds by definition.
(3)
$(eM\mathcal{I}) + (I_\omega) \Rightarrow (\mu PR)$:
Let $X \subseteq Y$. $Y - f(Y)$ is the largest element of $\mathcal{I}(Y)$, $X - f(X) \in \mathcal{I}(X) \subseteq \mathcal{I}(Y)$ by $(eM\mathcal{I})$, so $(X - f(X)) \cup (Y - f(Y)) \in \mathcal{I}(Y)$ by (I_ω), so by "largest" $X - f(X) \subseteq Y - f(Y)$, so $f(Y) \cap X \subseteq f(X)$.
$(\mu PR) \Rightarrow (eM\mathcal{I}) + (I_\omega)$:
Let again $X \subseteq Y$, $X' := Y - X$. Let $A \in \mathcal{I}(X)$, so $X - A \in \mathcal{F}(X)$, so $f(X) \subseteq X - A$, so by prerequisite $f(Y) \cap X \subseteq X - A$, so $f(Y) \subseteq X' \cup (X - A)$, so $(X \cup X') - (X' \cup (X - A)) = A \in \mathcal{I}(Y)$.
Again, (I_ω) holds by definition.
(4)
$(I \cup disj) \Rightarrow (\mu disjOR)$:
If $X \cap Y = \emptyset$, then (1) $A \in \mathcal{I}(X), B \in \mathcal{I}(Y) \Rightarrow A \cup B \in \mathcal{I}(X \cup Y)$ and (2) $A \in \mathcal{F}(X), B \in \mathcal{F}(Y) \Rightarrow A \cup B \in \mathcal{F}(X \cup Y)$ are equivalent. (By $X \cap Y = \emptyset$, $(X - A) \cup (Y - B) = (X \cup Y) - (A \cup B)$.) So $f(X) \in \mathcal{F}(X)$, $f(Y) \in \mathcal{F}(Y) \Rightarrow$ (by prerequisite) $f(X) \cup f(Y) \in \mathcal{F}(X \cup Y)$. $f(X \cup Y)$ is the smallest element of $\mathcal{F}(X \cup Y)$, so $f(X \cup Y) \subseteq f(X) \cup f(Y)$.
$(\mu disjOR) \Rightarrow (I \cup disj)$:
Let $X \subseteq Y$, $X' := Y - X$. Let $A \in \mathcal{I}(X)$, $A' \in \mathcal{I}(X')$, so $X - A \in \mathcal{F}(X)$, $X' - A' \in \mathcal{F}(X')$, so $f(X) \subseteq X - A$, $f(X') \subseteq X' - A'$, so $f(X \cup X') \subseteq f(X) \cup f(X') \subseteq (X - A) \cup (X' - A')$ by prerequisite, so $(X \cup X') - ((X - A) \cup (X' - A')) = A \cup A' \in \mathcal{I}(X \cup X')$.
(5)
$(\mathcal{M}^+_\omega) \Rightarrow (\mu CM)$:

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$f(X) \subseteq Y \subseteq X \Rightarrow X - Y \in \mathcal{I}(X)$, $X - f(X) \in \mathcal{I}(X) \Rightarrow$ (by $(\mathcal{M}_\omega^+)$, (4)) $A := (X - f(X)) - (X - Y) \in \mathcal{I}(Y) \Rightarrow Y - A = f(X) - (X - Y) \in \mathcal{F}(Y) \Rightarrow f(Y) \subseteq f(X) - (X - Y) \subseteq f(X)$.

$(\mu CM) \Rightarrow (\mathcal{M}_\omega^+)$:

Let $X - A \in \mathcal{I}(X)$, so $A \in \mathcal{F}(X)$, let $B \in \mathcal{I}(X)$, so $f(X) \subseteq X - B \subseteq X$, so by prerequisite $f(X - B) \subseteq f(X)$. As $A \in \mathcal{F}(X)$, $f(X) \subseteq A$, so $f(X - B) \subseteq f(X) \subseteq A \cap (X - B) = A-\mathrm{B}$, and $A - B \in \mathcal{F}(X-\mathrm{B})$, so $(X - A) - B = X - (A \cup B) = (X - B) - (A - B) \in \mathcal{I}(X-\mathrm{B})$, so $(\mathcal{M}_\omega^+)$, (4) holds.

(6)

$(\mathcal{M}^{++}) \Rightarrow (\mu RatM)$:

Let $X \subseteq Y$, $X \cap f(Y) \neq \emptyset$. If $Y - X \in \mathcal{F}(Y)$, then $A := (Y - X) \cap f(Y) \in \mathcal{F}(Y)$, but by $X \cap f(Y) \neq \emptyset$ $A \subset f(Y)$, contradicting "smallest" of $f(Y)$. So $Y - X \notin \mathcal{F}(Y)$, and by $(\mathcal{M}^{++})$ $X - f(Y) = (Y - f(Y)) - (Y - X) \in \mathcal{I}(X)$, so $X \cap f(Y) \in \mathcal{F}(X)$, so $f(X) \subseteq f(Y) \cap X$.

$(\mu RatM) \Rightarrow (\mathcal{M}^{++})$:

Let $A \in \mathcal{F}(Y)$, $B \notin \mathcal{F}(Y)$. $B \notin \mathcal{F}(Y) \Rightarrow Y - B \notin \mathcal{I}(Y) \Rightarrow (Y - B) \cap f(Y) \neq \emptyset$. Set $X := Y-\mathrm{B}$, so $X \cap f(Y) \neq \emptyset$, $X \subseteq Y$, so $f(X) \subseteq f(Y) \cap X$ by prerequisite. $f(Y) \subseteq A \Rightarrow f(X) \subseteq f(Y) \cap X = f(Y) - B \subseteq A-\mathrm{B}$.

(7) Trivial in both directions.

(8.1) Let $f(X) \subseteq Y \subseteq X$. $Y - f(Y) \in \mathcal{I}(Y) \subseteq \mathcal{I}(X)$ by $(eM\mathcal{I})$. $f(X) \subseteq Y \Rightarrow X - Y \subseteq X - f(X) \in \mathcal{I}(X)$, so by (iM) $X - Y \in \mathcal{I}(X)$. Thus by (I_ω) $X - f(Y) = (X - Y) \cup (Y - f(Y)) \in \mathcal{I}(X)$, so $f(Y) \in \mathcal{F}(X)$, so $f(X) \subseteq f(Y)$ by definition.

(8.2) (μCUT) is too special to allow to deduce $(eM\mathcal{I})$. Consider $U := \{a, b, c\}$, $X := \{a, b\}$, $\mathcal{F}(X) = \{X, \{a\}\}$, $\mathcal{F}(Z) = \{Z\}$ for all other $X \neq Z \subseteq U$. Then $(eM\mathcal{I})$ fails, as $\{b\} \in \mathcal{I}(X)$, but $\{b\} \notin \mathcal{I}(U)$. (iM) and $(eM\mathcal{F})$ hold. We have to check $f(A) \subseteq B \subseteq A \Rightarrow f(A) \subseteq f(B)$. The only case where it might fail is $A = X$, $B = \{a\}$, but it holds there, too.

(9.1) Obviously $(\mu CM)+(\mu CUT) \Leftrightarrow (\mu CUM)$, so the result follows from (5.1) and (8.1).

(9.2) Consider the same example as in (8.2). $f(A) \subseteq B \subseteq A \Rightarrow f(A) = f(B)$ holds there, too, by the same argument as above.

(10.1) Let $f(X) \subseteq Y$, $f(Y) \subseteq X$. So $f(X), f(Y) \subseteq X \cap Y$, and $X - (X \cap Y) \in \mathcal{I}(X)$, $Y - (X \cap Y) \in \mathcal{I}(Y)$ by (iM). Thus $f(X), f(Y) \in \mathcal{F}(X \cap Y)$ by $(eM\mathcal{F})$ and $f(X) \cap f(Y) \in \mathcal{F}(X \cap Y)$ by (I_ω). So $X \cap Y - (f(X) \cap f(Y)) \in \mathcal{I}(X \cap Y)$, so $X \cap Y - (f(X) \cap f(Y)) \in \mathcal{I}(X), \mathcal{I}(Y)$ by $(eM\mathcal{I})$, so $(X - (X \cap Y)) \cup (X \cap Y - f(X) \cap f(Y)) = X - f(X) \cap f(Y) \in \mathcal{I}(X)$ by (I_ω), so $f(X) \cap f(Y) \in \mathcal{F}(X)$, likewise $f(X) \cap f(Y) \in \mathcal{F}(Y)$, so $f(X) \subseteq f(X) \cap f(Y)$, $f(Y) \subseteq f(X) \cap f(Y)$, and $f(X) = f(Y)$.

(10.2) Consider again the same example as in (8.2), we have to show that $f(A) \subseteq B$, $f(B) \subseteq A \Rightarrow f(A) = f(B)$. The only interesting case is when one of A, B is X, but not both. Let, for example, $A = X$. We then have $f(X) = \{a\}$, $f(B) = B \subseteq X$, and $f(X) = \{a\} \subseteq B$, so $B = \{a\}$, and the condition holds. □

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