Proc. Camb. Phil. Soc. (1973), **73**, 7
PCPS 73–2
Printed in Great Britain

Local theorems and group extensions

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(*Received 7 January* 1972)

1. *Introduction.* The concept of a local system of a set W is defined in ((8), p. 166) and ((12), p. 126). Recall that a set $\mathscr{L}$ of subsets of W is a local system if $\cup \mathscr{L} = W$ and $\mathscr{L}$ is directed in the following sense: for every finite set $H_1, \ldots, H_n$ of elements of $\mathscr{L}$, there is an $M \in \mathscr{L}$ such that $H_i \subset M$ for $1 \leqslant i \leqslant n$. If Σ is a class of groups, $L(\Sigma)$ is the class of all groups G that possess a local system of Σ subgroups. Σ satisfies the local theorem, or is L-closed, if $L(\Sigma) \subset \Sigma$. Many classes of groups which satisfy the local theorem are discussed in ((12), pp. 126–144).

The following question is studied in this paper: if Σ and $\mathfrak{X}$ are both L-closed, is the class $\Sigma\mathfrak{X}$ L-closed? The same question is discussed for other, more restrictive types of local systems, which will be discussed below.

2. *Notation.* We use the following abbreviations throughout this paper.

$\mathfrak{A}$ = the class of Abelian groups.
$\mathfrak{S}$ = the class of simple groups.
$\mathfrak{F}$ = the class of finite groups.
$\mathfrak{F}_n$ = the class of finite groups of order $\leqslant n$.
$\mathscr{E}$ = the class consisting only of the trivial group.
$\mathfrak{D}$ = the class of all groups.

If G is a group, $\mathscr{C}(G)$ is the set of all countable subgroups of G. If Σ is a class of groups, $s(\Sigma)$, $ns(\Sigma)$, $q(\Sigma)$ denote respectively the class of groups which are subgroups, normal subgroups and quotients of Σ groups.

If Σ and $\mathfrak{X}$ are classes of groups, $\Sigma\mathfrak{X}$ is the class of all extensions of Σ groups by $\mathfrak{X}$ groups. While the class $\Sigma\mathfrak{X}$ is of primary interest here, some of our results are capable of handling the following more general classes.

DEFINITION 1. Let Σ, $\mathfrak{X}$, Γ, Λ be classes of groups. $(\Sigma\mathfrak{X}: \Gamma\Lambda)$ will denote the class of all groups G such that

(i) $G \in \Sigma\mathfrak{X}$, and

(ii) $Y = \cap \{N \lhd G \mid N \in \Sigma, G/N \in \mathfrak{X}\} \in L$ and $G/Y \in \Lambda$.

We note some special cases of $(\Sigma\mathfrak{X}; \Gamma\Lambda)$:

2·1 $(\Sigma\mathfrak{X}; \mathfrak{D}\mathfrak{D}) = \Sigma\mathfrak{X}$.

2·2 $(\mathfrak{D}\mathfrak{X}, \mathscr{E}\mathfrak{D}) = R\mathfrak{X}$, the class of residually $\mathfrak{X}$ groups.

2·3 $(\Sigma\mathfrak{X}, \mathscr{E}\mathfrak{D})$ = the class of all groups G such that for each $e \neq x \in G$ there is a normal Σ subgroup H of G such that $x \notin H$ and $G/H \in \mathfrak{X}$.

https://doi.org/10.1017/S030500410004740X Published online by Cambridge University Press

3. *Statement of results.*

3·1. *Local systems.* Our results in this section all make use of a technique of Mal'cev(9). Refinements of this method, due to P. Hall may be found in ((12), p. 131). It is this version of the Mal'cev method which we use here.

Mal'cev's Theorem. Let $\mathscr{L}$ be a local system of a set W and F a finite set. For each $H \in \mathscr{L}$, let f_H: $H \to F$. Then there is a function f: $W \to F$, such that for any finite subset K of W there is an $H \in \mathscr{L}$, such that $K \subset H$ and $f_H(k) = f(k)$ for all $k \in K$.

We use Mal'cev's theorem to prove

THEOREM 1. *If* $\Sigma, \mathfrak{X}, \Gamma, \Lambda$ *are* L *and* s *closed and* G *is a locally finite group in* $L(\Sigma\mathfrak{X}, \Gamma\Lambda)$, *then* $G \in (\Sigma\mathfrak{X}; \Gamma\Lambda)$.

The following two examples show that Theorem 1 cannot be significantly improved.

EXAMPLE 1. There are classes Σ and χ which are s, q and L closed such that
(i) $\Sigma \subset \mathfrak{A}$, $\mathfrak{X} \subset \mathfrak{A}$, and
(ii) $(L(\Sigma\mathfrak{X}) \cap \mathfrak{A}) \not\subseteq \Sigma\mathfrak{X}$.

EXAMPLE 1 shows that the local finiteness assumption in Theorem 1 cannot be weakened to 'locally Noetherian'. Nor can the assumption $s\mathfrak{X} = \mathfrak{X}$ be omitted. This is illustrated in

EXAMPLE 2. There is a locally finite group $G \in L(\mathfrak{F} \cap \mathfrak{A}\mathfrak{S})$ while $G \notin \mathfrak{A}\mathfrak{S}$.

Our other local theorem is

THEOREM 2. *If* Σ, Γ *and* Λ *are* s *and* L*-closed, then* $L(\Sigma\mathfrak{F}_n; \Gamma\Lambda) = (\Sigma\mathfrak{F}_n; \Gamma\Lambda)$.

The following corollary lists some special cases of Theorems 1 and 2.

COROLLARY 1. *Let* Σ *and* $\mathfrak{X}$ *be* s *and* L *closed.*
(i) *Every locally finite group in* $L(\Sigma\mathfrak{X})$ *is in* $\Sigma\mathfrak{X}$.
(ii) *Every locally finite* $L(R\mathfrak{X})$ *group is in* $R\mathfrak{X}$.
(iii) $L(\Sigma\mathfrak{F}_n) = \Sigma\mathfrak{F}_n$.
(iv) $L(R\mathfrak{F}_n) = R\mathfrak{F}_n$.

3·2. *Countably directed systems.* The set of all finitely generated subgroups of a group G is a local system of G. A larger local system is the set $\mathscr{C}(G)$ of countable subgroups of G. In (1), R. Baer considers questions of the type: if for each $H \in \mathscr{C}(G)$, $H \in \Sigma$, does it follow that $G \in \Sigma$? Similar types of questions are discussed in (11). We find it convenient here to isolate certain properties of the system $\mathscr{C}(G)$, and by doing so indicate some rather general methods of proving local theorems. The first property we consider is the 'countably directed' property.

DEFINITION 2. A local system $\mathscr{L}$ of a set W is a C_1-system of W if for every countable subset $\mathscr{X}$ of $\mathscr{L}_0$, there is an $M \in \mathscr{L}$ such that $H \subset M$ for all $H \in \mathscr{X}$.

C_1-systems will be called countably directed. The class $C_1(\Sigma)$ and the concept of C_1-closure are defined as for local systems. Note that $C_1(\Sigma) \subset L(\Sigma)$ and that $\mathscr{C}(G)$ is a C_1-system of G. The work of Neumann in (10) has been called to our attention by

https://doi.org/10.1017/S030500410004740X Published online by Cambridge University Press

both Professor J. Roseblade and Professor G. Sabbagh. What we have termed a C_1-system is called a σ-directed system in (10). We first indicate some negative results.

EXAMPLE 4. There are classes Σ and $\mathfrak{X}$ such that $\{s, L\}\Sigma = \Sigma$ and $\{s, C_1\}\mathfrak{X} = \mathfrak{X}$, while $C_1(\Sigma\mathfrak{X}) \not\subseteq \Sigma\mathfrak{X}$.

In a somewhat different direction, we have

EXAMPLE 3. There is a locally finite group $G \in C_1(\mathfrak{A}\mathfrak{S})$ such that $G \notin \mathfrak{A}\mathfrak{S}$.

Observe that both of the classes in Example 3 are L-closed. We note that the groups that arise in both Examples 2 and 3 will be constructed by the same process.

A simple method for proving C_1-local theorems relies on

LEMMA 1. *If $\mathscr{L}$ is a C_1-system of a set W, and $\mathscr{L} = \bigcup_{n=1}^{\infty} \mathscr{L}_n$, then at least one of the $\mathscr{L}_n$ is a C_1-system of W.*

The proof of this lemma is to simply observe that at least one of the $\mathscr{L}_n$ must be co-final in $\mathscr{L}$ (recall $\mathscr{L}_n$ is co-final in $\mathscr{L}$ if $\cup \mathscr{L}_n = \cup \mathscr{L} = W$; this is equivalent to the following fact; for each $A \in \mathscr{L}$, there is a $B \in \mathscr{L}_n$ such that $A \subset B$). Clearly, any co-final subset of $\mathscr{L}$ is a C_1-system of W. Lemma 1, combined with Theorem 2, is used to prove

THEOREM 3. *Let Σ, Γ, Λ be classes such that*

(i) $\Sigma = \bigcup_{j=1}^{\infty} \Sigma_j$, *where for each j, $\{s, L\}\Sigma_j = \Sigma_j$, and*

(ii) Λ *and* Γ *are s and C_1-closed. Then* $C_1(\Sigma\mathfrak{F}; \Gamma\Lambda) = (\Sigma\mathfrak{F}; \Gamma\Lambda)$.

As a corollary of Theorem 3, we have

COROLLARY 2.

(i) *If* $\{s, L\}\Sigma = \Sigma$, *then* $C_1(\Sigma\mathfrak{F}) = \Sigma\mathfrak{F}$.

(ii) $C_1(R\mathfrak{F}) = R\mathfrak{F}$.

Part (ii) of this corollary has also been established by Neumann in ((10), p. 295).

As far as residual properties are concerned, we have been unable to go beyond residual finiteness. We leave as unresolved

3·2·1. For which C_1-closed classes Σ is $C_1(R\Sigma) \subset R\Sigma$?

A class related to the class $R\mathfrak{F}$ is the class $\mathscr{B}$ of all groups G which are subdirect products (direct product = restricted product) of $\mathfrak{F}$ groups. The class $\mathscr{B}$ is not C_1-closed. This fact is illustrated by an uncountable Abelian p-group without elements of infinite height, which is not a direct sum of cyclic groups. Such a group is not in $\mathscr{B}$, but each of its countable subgroups is a $\mathscr{B}$-group (for the existence of such a group, see ((2), p. 89)).

Using Corollary 2(ii), we are able to prove

COROLLARY 3. $C_1((\mathfrak{A} \vee \mathfrak{F})I) = (\mathfrak{A} \vee \mathfrak{F})I$.

The class $(\mathfrak{A} \vee \mathfrak{F})I$ is the class of groups possessing an invariant system with factors in $\mathfrak{A}$ or $\mathfrak{F}$ (for related local theorems, see ((12), pp. 136–140)). We have been unable to answer the related question: is $C_1(\mathfrak{F}I) = \mathfrak{F}I$?

https://doi.org/10.1017/S030500410004740X Published online by Cambridge University Press

We note, but will not prove here, that the class $\overline{\mathfrak{F}I}$ of all groups G whose chief factors are all finite is C_1-closed. This can be proved by using Lemma 1 with the techniques of ((12), p. 138).

Let Σ and $\mathfrak{X}$ be s and C_1-closed classes and let $\mathscr{L}$ be a C_1-system of $\Sigma\mathfrak{X}$ subgroups of G. Suppose that for each $H \in \mathscr{L}$ we can select a normal subgroup N_H of H in such a way that

(i) $N_H \in \Sigma$, $H/N_H \in \mathfrak{X}$,

(ii) $H, K \in \mathscr{L}$, $H \subset K$ implies $N_H \subset N_K$.

If such a selection exists, it is not difficult to show that $G \in \Sigma\chi$. Such a selection will always exist if, in addition to the above assumptions

3·2·2. $\mathfrak{X}$ is R closed.

3·2·3. $\mathfrak{X}$ is closed under finite direct products, and for each $H \in \mathscr{L}$, H has the minimum condition on normal Σ subgroups.

Our final theorem in this section relies on the fact that $\{x^H | H \in \mathscr{L}\}$ satisfies the property (ii) listed above.

THEOREM 4. If Σ and $\mathfrak{X}$ are classes with $\{ns, C_1\}\Sigma \subset \Sigma$ and $\{s, q, C_1\}\mathfrak{X} = \mathfrak{X}$ and $G \in C_1(\Sigma\mathfrak{X})$, then $G/R \in \mathfrak{X}$, where R is the product of all the normal Σ subgroups of G.

As a corollary, we have

COROLLARY 4. *If Σ is a radical class in the sense of Kurosh*(7) *with $C_1(\Sigma) = \Sigma$, and $\{s, q, C_1\}\mathfrak{X} = \mathfrak{X}$, then $C_1(\Sigma\mathfrak{X}) = \Sigma\mathfrak{X}$.*

We note that not all radical classes are C_1-closed. M. I. Kargapolov has shown in (6) that the class of SN^* groups is not C_1-closed.

3·3. *Countably complete systems.* We are able to obtain somewhat deeper results by imposing stronger conditions than that of 'countably directed' on a local system $\mathscr{L}$. The condition we have in mind is a 'completeness' condition.

DEFINITION 3. A local system $\mathscr{L}$ of a set W is a C_2-system of W if

(i) for every countable ascending chain $W_1 \subset W_2 \subset \ldots \subset W_n \subset \ldots$ of elements of $\mathscr{L}$, $\bigcup_{n=1}^{\infty} W_n \in \mathscr{L}$, and

(ii) every element of $\mathscr{L}$ is countable.

$C_2(\Sigma)$ and C_2 closure are defined in the obvious way. Observe that $C_2(\Sigma) \subset C_1(\Sigma)$ for all classes Σ. Further, $\mathscr{C}(G)$ is a C_2 system of G. If $s\Sigma = \Sigma$, then $C_2(\Sigma) = C_1(\Sigma)$, but in general the classes $C_2(\Sigma)$ and $C_1(\Sigma)$ are distinct. This fact is illustrated by comparing Example 3 with

THEOREM 5. $C_2(\mathfrak{A}\mathfrak{S}) \subset \mathfrak{A}\mathfrak{S}$.

As in section 3·1, our main result here depends on a finiteness condition, which is formulated in

DEFINITION 4. Let Σ and $\mathfrak{X}$ be classes of groups. A group G is a $\Sigma^f\mathfrak{X}$ group if there is a finite subset F of G such that $F^G \in \Sigma$ and $G/F^G \in \mathfrak{X}$.

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The relation $\Sigma^f\mathfrak{X} \subset \Sigma\mathfrak{X}$ always holds. If every Σ group is finitely generated, then $\Sigma^f\mathfrak{X} = \Sigma\mathfrak{X}$. Note also that $\mathfrak{S}^f\mathfrak{X} = \mathfrak{S}\mathfrak{X}$. Further, if $s\Sigma = \Sigma$, $G \in \Sigma\mathfrak{X}$ and G has the maximum condition for normal Σ subgroups, then $G \in \Sigma^f\mathfrak{X}$.

The main theorem here is

THEOREM 6. *If Σ and $\mathfrak{X}$ are C_1-closed, then $C_2(\Sigma^f\mathfrak{X}) \subset \Sigma^f\mathfrak{X}$.*

As consequences of Theorem 6 we have the following three corollaries. Corollaries 6 and 7 are easy consequences of the theorem, while Corollary 5 is proved in section 8.

COROLLARY 5. *If $\{ns, C_1\}\Sigma \subset \Sigma$, $C_1(\mathfrak{X}) \subset \mathfrak{X}$, $G \in C_2(\Sigma\mathfrak{X})$, and G has the maximum condition on normal Σ subgroups, then $G \in \Sigma\mathfrak{X}$.*

COROLLARY 6. *If every Σ group is finitely generated and $C_1(\mathfrak{X}) \subset \mathfrak{X}$, then $C_2(\Sigma\mathfrak{X}) \subset \Sigma\mathfrak{X}$.*

COROLLARY 7. *If χ is C_1-closed, then $C_2(\mathfrak{S}\mathfrak{X}) \subset \mathfrak{S}\mathfrak{X}$.*

We remark that Theorem 6 is proved in somewhat more generality than stated above. The techniques used to prove Theorems 5 and 6 are developed in Lemmas 3, 4 and 5. These lemmas can be used to prove C_2-local theorems other than those that occur here. All of the results in (1) and most of those in (11) can be proved using these lemmas. The essential property of C_2 systems used in proving these lemmas is the countable intersection property; i.e. the intersection of a countable number of C_2 systems is a C_2 system. This fact is the content of Lemma 2.

Finally, we have been unable to answer the following question:

3·3·1. If Σ and $\mathfrak{X}$ are s and L closed, is $\Sigma\mathfrak{X}$ C_2-closed?

Example 4 shows the answer to be negative, if the conditions on $\mathfrak{X}$ are weakened to $\{s, C_1\}\mathfrak{X} = \mathfrak{X}$, while Example 3 gives a negative answer if the conditions on $\mathfrak{X}$ are weakened to $L(\mathfrak{X}) \subset \mathfrak{X}$.

4. *Proofs of Theorems* 1 *and* 2. We develop some terminology that will be frequently used. First, we denote any function f that satisfies the conditions of Mal'cev's theorem by $f = \lim\{f_H | H \in \mathscr{L}\}$. Let $\mathscr{L}$ be a local system of G, and for each $H \in \mathscr{L}$, let $N_H \lhd H$. Define $g_H\colon H \to \{0, 1\}$ by $g_H(y) = 0$ of $y \in N_H$ and $g_H(y) = 1$ if $y \notin N_H$. Let

$$g = \lim\{g_H | H \in \mathscr{L}\}$$

and define $\lim\{N_H | H \in \mathscr{L}\} = \{x \in G | g(x) = 0\}$. The following facts are easily verified.

4·1. $\lim\{N_H | H \in \mathscr{L}\} \lhd G$.

4·2. If $\{s, L\}\Sigma = \Sigma$ and $N_H \in \Sigma$ for all $H \in \mathscr{L}$, then $\lim\{N_H | H \in \mathscr{L}\} \in \Sigma$.

4·3. If $x \in G$ and $x \notin N_H$ for all $H \in \mathscr{L}$, then $x \notin \lim\{N_H | H \in \mathscr{L}\}$.

We note that our function g is a rather special case of the more complicated functions discussed in ((12), pp. 126–144).

Proof of Theorem 1. Let $\mathscr{L}$ be a local system of G by $(\Sigma\mathfrak{X}, \Gamma\Lambda)$ subgroups. For each $H \in \mathscr{L}$, define $M_H = \cap\{N \lhd H | N \in \Sigma \text{ and } H/N \in \mathfrak{X}\}$. Then $M_H \in \Gamma$ and $H/M_H \in \Lambda$.

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Further, if $H, K \in \mathscr{L}$ with $H \subset K$, we have $M_H \subset M_K$. It follows that $\mathscr{M} = \{M_H | H \in \mathscr{L}\}$ is a local system of $M = \cup \{M_H | H \in \mathscr{L}\}$. Since $L(\Gamma) = \Gamma$, $M \in \Gamma$. If X is a finite subgroup of G, there is an $H \in \mathscr{L}$ such that $M \cap X = M_H \cap X$, and $X \subset H$. Thus

$$XM/M \simeq X/X \cap M = X/X \cup M_H \simeq XM_H/M_H \subset H/M_H \in \Lambda.$$

Since G is locally finite, it follows that $G/M \in L(\Lambda)$ and since, $L(\Lambda) = \Lambda$, $G/M \in \Lambda$.

If $N \lhd G$ with $N \in \Sigma$ and $G/N \in \mathfrak{X}$, then for all $H \in \mathscr{L}$, $M_H \subset N$. This forces $M \subset N$. Let $x \in G - M$. There is a co-final subset $\mathscr{N}$ of $\mathscr{L}$ such that for each $H \in \mathscr{N}$, $x \in H - M_H$. Evidently $\mathscr{N}$ is a local system of G, and for each $H \in \mathscr{N}$, there is a normal subgroup R_H of H such that $x \notin R_H$, $R_H \in \Sigma$ and $H/R_H \in \mathfrak{X}$. Let $R = \lim \{R_H | H \in \mathscr{N}\}$. Then by 4·1, 4·2 and 4·3 R is a normal Σ subgroup of G and $x \notin R$. We now show that $G/R \in \mathfrak{X}$. Let S be a finite subgroup of G. By Mal'cev's theorem there is an $H \in \mathscr{N}$ such that $S \subset H$ and $g(s) = g_H(s)$ for all $s \in S$. It follows that $S \cap R \subset R_H$ and $S - S \cap R \subset H - R_H$. These inclusions imply that $S \cap R = S \cap R_H$. Using the obvious isomorphisms, $SR/R \in \mathfrak{X}$. By the facts that $L(\chi) = \mathfrak{X}$ and that G is locally finite we have $G/R \in \mathfrak{X}$.

We have shown that $M = \cap \{Y \lhd G | Y \in \Sigma \text{ and } G/Y \in \mathfrak{X}\}$, and this completes the proof.

Proof of Theorem 2. The structure of this proof is the same as the preceding proof; accordingly we adopt the same terminology and indicate only the essential differences. Define M as above and observe that $M \in \Gamma$. Let X be a finitely generated subgroup of G. Then $X \in \Sigma\mathfrak{F}_n$. Now X has only a finite number of subgroups of index $\leqslant n$ ((14), p. 151). It follows that M_X has finite index in X. Since $M_X \subset M \cap X$, $M \cap X$ has finite index in X and consequently, $M \cap X$ is finitely generated. This implies the existence of $H \in \mathscr{L}$ such that $M \cap X = M_H \cap X$. As above, conclude that $G/M \in \Lambda$.

We let $x \in G - M$ and obtain the limit R as above. We now show that $G/R \in \mathfrak{F}_n$. If $G/R \notin \mathfrak{F}_n$, then there are $n+1$ elements $x_1, \ldots, x_{n+1}$ of G such that

$$A = \{x_i^{-1} x_j | 1 \leqslant i < j \leqslant n+1\}$$

intersects R in the empty set. Now, let $H \in \mathscr{N}$ be such that $A \subset N$ and $g(a) = g_H(a)$ for all $a \in A$. Then $A \cap R_H = \varnothing$ which implies $[H : R_H] \geqslant n+1$, a contradiction.

5. *Proofs of Theorem* 3 *and Corollary* 3.

Proof of Theorem 3. Let $\mathscr{L}$ be a C_1-system of $(\Sigma\mathfrak{F}, \Gamma\Lambda)$ subgroups of G. For each $H \in \mathscr{L}$, let $M_H = \cap \{Y \lhd H | Y \in \Sigma \text{ and } H/Y \in \mathfrak{F}\}$. Then $M_H \in \Gamma$, $H/M_H \in \Lambda$. Arguments similar to those in section 4 show that $M = \cup \{M_H | H \in \mathscr{L}\} \in \Gamma$ and $G/M \in \Lambda$. Also, if $N \lhd G$ with $N \in \Sigma$ and $G/N \in \mathfrak{F}$, then $M \subset N$.

Letting $x \in G - M$, we can find a co-final subset $\mathscr{N}$ of $\mathscr{L}$ such that for each $H \in \mathscr{N}$, $x \in H - M_H$. Thus, for each $H \in \mathscr{N}$, there is a normal subgroup R_H of H such that $R_H \in \Sigma$, $H/R_H \in \mathfrak{F}$ and $x \notin R_H$. By Lemma 1, there are positive integers i and n such that $\mathscr{K} = \{H \in \mathscr{N} | R_H \in \Sigma_i, H/R_H \in \mathfrak{F}_n\}$ is a C_1-system of G. If $R = \lim \{R_H | H \in \mathscr{K}\}$, we see that $x \notin R$, $R \in \Sigma_i$ and, as in Theorem 2, $G/R \in \mathfrak{F}_n$. We then conclude that $M = \cap \{Y \lhd G | Y \in \Sigma \text{ and } G/Y \in \mathfrak{F}\}$, which completes the proof.

Proof of Corollary 3. Let $\mathscr{L}$ be a C_1-system of G by $(\mathfrak{A} \vee \mathfrak{F})I$ subgroups. Let $H \in \mathscr{L}$ and define $\bar{H}$ to be the intersection of all the subgroups of finite index in H. Then $H/\bar{H}$

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is in $R\mathfrak{F}$. Let $\{H_\alpha|\alpha\in I\}$ be an invariant $\mathfrak{A}\vee\mathfrak{F}$ system of H. The groups $H_\alpha\cap\bar{H}$ are normal in H. If for some α, the factor $H_{\alpha+1}/H_\alpha$ is finite, then, $H_{\alpha+1}\cap\bar{H}/H_\alpha\cap\bar{H}$ is in the centre of $\bar{H}/H_\alpha\cap\bar{H}$, so that $H_{\alpha+1}\cap\bar{H}/H_\alpha\cap\bar{H}$ is Abelian. We deduce that $\bar{H}$ has an H-invariant system of subgroups with Abelian factors; i.e. $\bar{H}$ is an $\mathfrak{A}I(=SI)$ group. Now $\{\bar{H}|H\in\mathscr{L}\}$ is a local system for $Y=\cup\{\bar{H}|H\in\mathscr{L}\}$. A Mal'cev limit argument will show (for the details, see ((12), p. 138)) that Y has a G-invariant $\mathfrak{A}I$ system. The fact that $C_1(R\mathfrak{F})=R\mathfrak{F}$ is used to prove $G/Y\in R\mathfrak{F}$. Since $R\mathfrak{F}\subset\mathfrak{F}I$, the proof is complete.

6. *Proof of Theorem* 4. Let $\mathscr{L}$ be a C_1-system of $\Sigma\mathfrak{X}$ subgroups of G and let $x\in G$. We let $\mathscr{L}_x=\{H\in\mathscr{L}|x\in H\}$ and $\mathscr{N}_x=\{H\in\mathscr{L}_x|x^H\in\Sigma\}$. We note that if $\mathscr{N}_x$ is co-final in $\mathscr{L}_x$ (and hence in $\mathscr{L}$) then $x^G=\cup\{x^H|H\in\mathscr{N}_x\}\in C_1(\Sigma)\subset\Sigma$.

Let T be a countable subgroup of G and let $y\in T-R$. Then $y^G\notin\Sigma$ and this implies that $\mathscr{N}_y$ is not co-final in $\mathscr{L}$. Thus, there is a $K_y\in\mathscr{L}$ such that $y\in K_y$ and for all $H\in\mathscr{L}$ with $K_y\subset H$ we have $y^H\notin\Sigma$. Let K be a member of $\mathscr{L}$ that contains T and each of the K_y. Then for each y in $T-R$, $y^K\notin\Sigma$. The above conditions on K imply $T\cap N\subset T\cap R$. Thus $T/T\cap R$ is a quotient of $T/T\cap N\simeq TN/N\in\Lambda$. We now have $TR/R\in\Lambda$ and $G/R\in C_1(\Lambda)=\Lambda$.

7. *Proof of Theorem* 5. The essential property of C_2-systems is given in

LEMMA 2. *Let $\{\mathscr{L}_n|n=1,2,\ldots\}$ be a countable collection of C_2-systems of a set W. Then $\mathscr{L}=\bigcap_{n=1}^{\infty}\mathscr{L}_n$ is a C_2-system of W.*

Proof. Let T be a countable subset of W. For each n let $w_n(T)$ be a member of $\mathscr{L}_n$ such that $T\subset w_n(T)$. Observe that $w_n(T)$ is countable. Define the operators α_n by $\alpha_1=w_1$ and inductively $\alpha_n=w_1\ldots w_n\alpha_{n-1}$. Then for all n, we have $\alpha_n(T)\subset\alpha_{n+1}(T)$. Let $\bar{T}=\bigcup_{n=1}^{\infty}\alpha_n(T)$ and let r be a positive integer. Then for each $n\geqslant r$,

$$\alpha_n(T)\subset w_r\ldots w_n\alpha_n(T)\subset\alpha_{n+1}(T).$$

Thus for each $n\geqslant r$, there exists $M_n^r=\mathscr{L}_r$ such that $\alpha_n(T)\subset M_n^r\subset\alpha_{n+1}(T)$. We have then

$$\bar{T}=\bigcup_{n=1}^{\infty}\alpha_n(T)=\bigcup_{n=r}^{\infty}\alpha_n(T)\subset\bigcup_{n=r}^{\infty}M_n^r\subset\bigcup_{n=r+1}^{\infty}\alpha_{n+1}(T)=\bar{T}.$$

Thus $\bigcup_{n=r}^{\infty}M_n^r=\bar{T}$ and it follows that $\bar{T}\in\mathscr{L}_r$. Since r was arbitrary, $\bar{T}\in\bigcap_{n=1}^{\infty}\mathscr{L}_n$. It is clear that $T\subset\bar{T}$. The other properties of a C_2-system are easily seen to be satisfied by $\bigcap_{n=1}^{\infty}\mathscr{L}_n$.

Lemma 3 will enable us to choose rather special types of C_2 systems of a group. The formulation of the lemma requires some special terminology.

Let W be a set. Denote by W^* the set of all functions σ that assign to each finite set $F\subset W$ a C_2 system $\sigma(F)$ of W. If $S\subset W$ and $\sigma\in W^*$, then S is σ-stable if for every finite $F\subset S$ we have $S\in\sigma(F)$.

LEMMA 3. *If A is a countable subset of W^*, then* stab $(A)=\{S\subset W|S$ *is σ-stable for all $\sigma\in A\}$ is a C_2-system of W.*

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Proof. Let $\sigma \in A$ and let $\mathscr{L} = \{S \subset W | S \text{ is } \sigma\text{-stable}\}$. $\mathscr{L}$ is clearly closed under countable ascending unions. Let T be a countable subset of W. Then $\mathscr{K} = \cap \{\sigma(F) | F \text{ is a finite subset of } T\}$ is a C_2-system of W by Lemma 2. Let $\pi(T)$ be a member of $\mathscr{K}$ that contains T. Then $\bigcup_{n=1}^{\infty} \pi^n(T)$ is σ-stable and $T \subset \bigcup_{n=1}^{\infty} \pi^n(T)$. Thus $\mathscr{L}$ is a C_2-system of W.

A second application of Lemma 2 completes the proof.

Some of the applications of Lemma 3 can be simplified by the following observations. Let $\mathscr{L}$ by C_2-system of W and suppose σ is a function that assigns to each element of W a countable subset of W. For each $x \in W$, define $\sigma^*(x) = \{H \in \mathscr{L} | \sigma(x) \subset H\}$. Then $\sigma^*(x)$ is a C_2-system of W. σ^* can be extended to finite subsets F of W by defining $\sigma^*(F) = \cap \{\sigma^*(x) | x \in F\}$. Accordingly, we define $\text{stab}(\sigma, \mathscr{L}) = \text{stab}(\sigma^*)$. It is now routine to verify

LEMMA 4. *If $\mathscr{L}$ is a C_2 system of a set W and σ is a function that assigns to each $x \in W$ some countable subset $\sigma(x)$ of W, then for each $S \in \text{stab}(\sigma, \mathscr{L})$ we have*

(i) $S \in \mathscr{L}$ *and*

(ii) *for each $x \in S$, $\sigma(x) \subset S$.*

As a first application of these techniques, we have

Proof of Theorem 5. Let $\mathscr{L}$ be a C_2 system of $\mathfrak{A}\mathfrak{S}$ subgroups of G. For $H \in \mathscr{L}$, let $A_H \lhd H$ such that $H/A_H \in \mathfrak{S}$, and $A_H \in \mathfrak{A}$.

Case 1. Assume $\mathscr{N} = \{H \in \mathscr{L} | H/A_H \in \mathfrak{A}\}$ is co-final in $\mathscr{L}$. Then $\mathscr{N}$ is a C_1 system of G. Consequently, there is, by Lemma 1, a prime p and a C_1 system $\mathscr{B}$ of G such that $\mathscr{B} \subset \mathscr{N}$ and $o(H/A_H) = p$ for all $H \in \mathscr{B}$. If $A = \lim \{A_H | H \in \mathscr{B}\}$, then A is a normal Abelian subgroup of G and by Corollary 1 (iii), $o(G/A) \leqslant p$. Further application of Mal'cev's theorem shows that $o(G/A) = p$.

Case 2. Assume $\mathscr{N}$ is not co-final in $\mathscr{L}$. Then there is a C_2-system $\mathscr{M} \subset \mathscr{N}$ such that for all $H \in \mathscr{M}$, H/A_H is non-Abelian. We denote the Hirsch–Plotkin radical of G by $\Psi(G)$; thus, $\Psi(G)$ is the unique maximal normal locally nilpotent subgroup of G. We show first that $\Psi(G)$ is Abelian, and secondly that $G/\Psi(G)$ is simple.

If $\Psi(G)$ is non-Abelian, there is an $x \in \Psi(G)$ and an $H \in \mathscr{M}$ such that $x \in H$ and x^H is non-Abelian. It follows that $x \notin A_H$ and that x^H has a non-Abelian simple quotient, an impossibility in a locally nilpotent group ((12), p. 119). Thus, $\Psi(G)$ is Abelian.

We observe that for each $H \in \mathscr{M}$, $A_H = \Psi(H)$. Now, if $x \notin \Psi(G)$, there is an

$$H = \sigma(x) \in \mathscr{M}$$

such that $x \notin \Psi(H)$. Define $\sigma(x) = \{e\}$ for all $x \in \Psi(G)$ and let $\mathscr{Y} = \text{stab}(\sigma, \mathscr{M})$. Let $S \in \mathscr{Y}$ and $x \in S - \Psi(G)$. Then $x^{\sigma(x)} \subset x^S$ by Lemma 4. Now $x^{\sigma(x)}$ has a non-Abelian simple quotient and therefore is not locally nilpotent. It follows that $x \notin \Psi(S)$. We have shown then that $\Psi(S) = S \cap \Psi(G)$.

If $S \in \mathscr{Y}$, $S\Psi(G)/\Psi(G) \simeq S/\Psi(G) \cap S = S/\Psi(S) \in \mathfrak{S}$. Thus, $\{S\Psi(G)/\Psi(G) | S \in \mathscr{Y}\}$ is a local system of simple subgroups of $G/\Psi(G)$. We now have that $G/\Psi(G)$ is simple and the proof is complete.

https://doi.org/10.1017/S030500410004740X Published online by Cambridge University Press

8. *Proof of Theorem* 6 *and Corollary* 5. Lemma 5 is a technical observation that will facilitate the proof of Lemma 6.

LEMMA 5. *If G is a group and $\mathscr{L}$ is a C_2-system of G, then $\mathscr{L}_1 = \{H \in \mathscr{L} | H$ is a subgroup of $G\}$ is also a C_2-system of G.*

Proof. The collection $\mathscr{C}(G)$ of countable subgroups of G is a C_2-system of G. Hence $\mathscr{L}_1 = \mathscr{L} \cap \mathscr{C}(G)$ is a C_2-system of G by Lemma 2.

Our next Lemma gives another application of Lemma 3.

LEMMA 6. *Let G be a group and X be a countable subset of G.*

(i) *If $\mathscr{H}$ is a C_2-system of subgroups of X^G, then there is a C_2 system $\mathscr{L}$ of subgroups of G such that for all $S \in \mathscr{L}$, $X^S \in \mathscr{H}$.*

(ii) *If $N \lhd G$ and $\mathscr{H}$ is a C_2 system of subgroups of G/N, then there is a C_2 system $\mathscr{L}$ of subgroups of G such that for all $S \in \mathscr{L}$, $SN/N \in \mathscr{H}$.*

Proof. (i) According to Lemma 5, it suffices to demonstrate the existence of a C_2 system $\mathscr{L}$ of subsets of G satisfying (i).

Let $y \in X^G$. Then there is a finite subset F_y of G such that $y \in X^{F_y}$. Let

$$\sigma(y) = \{x^u | x \in X, u \in F_y\}.$$

Then $\sigma(y)$ is countable and, by Lemma 3, stab $(\sigma, \mathscr{H})$ is a C_2 system of X^G. Let $\mathscr{L} = \{T | T$ is a countable subset of G and $X^T \in$ stab $(\sigma, \mathscr{H})\}$. $\mathscr{L}$ is closed under ascending countable unions. If L is a countable subset of G, then $X^L \subset H$ for some $H \in$ stab $(\sigma, \mathscr{H})$. Since, by Lemma 4, $\sigma(x) \subset H$ for all $x \in H$, we have $H = X^F$, where $F = \cup \{F_y | y \in H\}$. It follows that $X^K = H$, where $K = F \cup L$. Accordingly, $K \in \mathscr{L}$ and $\mathscr{L}$ is a C_2 system of G.

(ii) Let $\mathscr{L} = \{S | S$ is countable and $SN/N \in \mathscr{H}\}$.

The proof of Theorem 6 requires the following notation: if Σ is a class of groups, we denote by $\sim \Sigma$ the class of all groups G such that $G = E$ or $G \notin \Sigma$. It is not hard to prove.

8·1. If $C_1(\Sigma) \subset \Sigma$, then $\sim \Sigma \subset C_2(\sim \Sigma)$.

Theorem 6 remains valid when the hypotheses, $C_1(\Sigma) \subset \Sigma$, $C_1(\chi) \subset \chi$, are replaced by the weaker assumptions $\sim \Sigma \subset C_2(\sim \Sigma)$, $\sim \mathfrak{X} \subset C_2(\sim \mathfrak{X})$, and we will prove the theorem in the latter context. Although we will not prove it here, we note that $\sim \Sigma \subset C_2(\sim \Sigma)$ does not always imply $C_1(\Sigma) \subset \Sigma$; the class Σ which demonstrates this fact is the class of completely reducible groups without centre.

Proof of Theorem 6. Suppose there is a group $G \in C_2(\Sigma' \mathfrak{X})$ while $G \notin \Sigma' \mathfrak{X}$. Let F be a finite subset of G and define $\sigma \in G^*$ as follows:

(i) If $F^G \notin \Sigma$, let $\mathscr{H}$ be a C_2 system of $\sim \Sigma$ subgroups of F^G. Let $\sigma(F)$ be the corresponding C_2 system of G given in Lemma 6 (i).

(ii) If $F^G \in \Sigma$, then $G/F^G \notin \mathfrak{X}$. Let $\mathscr{N}$ be a C_2 system of $\sim \mathfrak{X}$ subgroups of G/F^G. Let $\mathscr{L}$ be the corresponding C_2 system of G given in Lemma 6 (ii). Define a function ρ on G as follows:

if $g \notin F^G$, $\rho(g) = \{e\}$,

if $g \in F^G$, $\rho(g)$ is a finite subset of G such that $g \in F^{\rho(g)}$.

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Now let $\sigma(F) = \text{stab}\,(\rho, \mathscr{L})$

Evidently, $\sigma \in G^*$. Let $\mathscr{B} = \text{stab}\,(\sigma)$, and $V \in \mathscr{B}$. If F is a finite subset of V, then $V \in \sigma(F)$. If $\sigma(F)$ is defined by (i), then $F^V \in \mathscr{H}$ and consequently $F^V \notin \Sigma$. If $\sigma(F)$ is defined by (ii), then $V \in \text{stab}\,(\rho, \mathscr{L})$, and by 8·1, $\rho(x) \subset V$ for all $x \in V$. Thus $V \cap F^G = F^V$. Since $V \in \mathscr{L}$, $VF^G/F^G \notin \mathfrak{X}$. But $VF^G/F^G \simeq V/V \cap F^G = V/V \cap F^V \simeq VF^V/F^V$, which yields $VF^V/F^V \notin \mathfrak{X}$.

In either case, $V \notin \Sigma^f\mathfrak{X}$. If we let $\mathscr{Y}$ be a C_2 system of $\Sigma^f\mathfrak{X}$ subgroups of G, then $\mathscr{B} \cap \mathscr{Y}$ is also a C_2 system by Lemma 2. This is a contradiction, and the proof is complete.

Proof of Corollary 5. The proof follows the lines of Theorem 6; i.e. we suppose $G \notin \Sigma\mathfrak{X}$, let F be a finite subset of G and construct the function $\sigma \in G^*$ as in Theorem 6. Again, let $\mathscr{B} = \text{stab}\,(\sigma)$. Now $G \in C_2(\Sigma\mathfrak{X})$, so we have a C_2 system $\mathscr{M}$ of $\Sigma\mathfrak{X}$ subgroups of G. Put $\mathscr{K} = \mathscr{B} \cap \mathscr{M}$. It is shown in the proof of Theorem 6 that for each $V \in \mathscr{K}$ we have $V \notin \Sigma^f\mathfrak{X}$. We show here that each $V \in \mathscr{K}$ is in $\Sigma^f\mathfrak{X}$ and achieve a contradiction.

Let $V \in \mathscr{K}$ and N_V be a normal Σ subgroup of V such that $V/N_V \in \mathfrak{X}$. If $N_V \neq F^V$ for all finite $F \subset N_V$, we have a strictly increasing chain of Σ subgroups of V,

$$F_1^V \subset \cdots \subset F_n^V \subset \cdots,$$

where each F_i is a finite subset of N_V. Since $V \in \mathscr{B}$, we have $F_i^G \in \Sigma$ for each i, and it follows from the properties of $\mathscr{B}$ that, for each i, $V \cap F_i^G = F_i^V$. Since the sequence $F_1^G \subset \ldots \subset F_n^G \subset \ldots$ becomes stationary in a finite number of steps, we must have $F_1^V \subset \ldots \subset F_n^V \subset \ldots$ stationary in a finite number of steps, a contradiction. Thus $V \in \Sigma^f\mathfrak{X}$ and the proof is complete.

9. *Examples.*

Example 1. Let Σ be the class of locally cyclic groups and $\mathfrak{X}$ the class of periodic Abelian groups in which each p-primary component is trivial or has order p. Σ and $\mathfrak{X}$ are L, s, and q-closed.

Let B denote the Cartesian product of the groups $\{Z_p | p \text{ a prime}\}$ and T be the torsion subgroup of B. Thus $T \in \mathfrak{X}$. Now B/T is divisible, so there is a subgroup $M \subset B$ such that $T \subset M$ and $M/T \simeq Q$ (rationals under $+$). If K is a finitely generated subgroup of M, then $K = U \oplus F$ where $U \in \mathfrak{X}$ and $F \subsetneq Q$. Thus $F \in \Sigma$ and $K \in \Sigma\mathfrak{X}$. This shows $M \in L(\Sigma\mathfrak{X})$.

Suppose $M \in \Sigma\mathfrak{X}$; then there is an $N \subset M$, $N \in \Sigma$, and $M/N \in \mathfrak{X}$. N is either torsion free or else periodic since $N \in \Sigma$. In the latter case, M/N is not periodic, implying $M/N \notin \mathfrak{X}$. Hence $N \cap T = \{0\}$, and $T \simeq T \oplus N/N \subsetneq M/N \in \chi$. This forces

$$T \oplus N/N \simeq M/N$$

since T is the maximum possible χ-group. Thus $M = T \oplus N$ and $N \simeq Q$, an impossibility since B is a reduced group. This shows $M \notin \Sigma\mathfrak{X}$.

Examples 2 *and* 3.

Any group as described in Example 3 will suffice for Example 2, if we take care to ensure that the G of Example 2 possesses a local system of finite $\mathfrak{U}\mathfrak{S}$-subgroups.

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We will construct locally finite groups N_α and subgroups $M_\alpha \lhd N_\alpha$ inductively for each ordinal $\alpha \leqslant \omega_1$ (the first uncountable ordinal). The groups constructed will have the following properties: let p be a fixed prime.

(*a*) $N_\alpha/M_\alpha \simeq Z_p$, the cyclic group of order p.

(*b*) M_α is a non-Abelian simple group.

(*c*) If α is a successor ordinal or o, then M_α is a direct factor of N_α.

(*d*) $N_\alpha \subset N_{\alpha+1}$.

(*e*) If β is a limit ordinal, then $N_\beta = \cup \{N_\alpha | \alpha < \beta\}$.

(*f*) If $\alpha < \beta$, then $N_\alpha \cap M_\beta = M_\alpha$.

(*g*) If $\alpha < \beta$ and $g \in N_\alpha - M_\alpha$, then $g^{N_\beta} = N_\beta$.

(*h*) If α is finite, then N_α is finite; if $\alpha < \omega_1$, then N_α is countable.

Suppose μ is a limit ordinal. Properties (*a*), (*b*), (*c*) and (*e*) imply $N_\mu \in L(\mathfrak{A}\mathfrak{S})$, and if $\mu = \omega_1$, then $N_\mu \in C_1(\mathfrak{A}\mathfrak{S})$. Properties (*e*), (*f*) and (*g*) imply that M_μ is the unique minimum normal subgroup of N_μ, and hence $N_\mu \notin \mathfrak{A}\mathfrak{S}$. Thus $G = N_\omega$ will suffice for Example 2 and $G = N_{\omega_1}$ will suffice for Example 3.

Construction of the N_α.

Let M_0 be some finite non-Abelian simple group and let $N_0 = M_0 \times Z_p$. Assume that the groups $M_\alpha \lhd N_\alpha$ have been constructed for all $\alpha < \mu \leqslant \omega_1$ and that properties (*a*)–(*h*) are satisfied.

Suppose μ is a limit ordinal. Define $N_\mu = \cup \{N_\beta | \beta < \mu\}$ in accordance with property (*e*). By property (*f*), the groups $\{M_\beta | \beta < \mu\}$ form a chain, and we define $M_\mu = \cup \{M_\beta | \beta < \mu\}$. (*f*) also guarantees that $M_\mu \neq N_\mu$ and, in view of (*a*), $N_\mu/M_\mu \simeq Z_p$. Since each $M_\beta \in \mathfrak{S}$, it follows that $M_\mu \in L(\mathfrak{S}) = \mathfrak{S}$. If $\alpha < \mu$, then $N_\alpha \cap M_\mu \supset M_\alpha$ and if $N_\alpha \cap M_\mu \neq M_\alpha$, it would follow that $N_\alpha \cap M_\mu = N_\alpha$. This implies $N_\alpha \cap M_\beta = N_\alpha$ for some $\alpha < \beta < \mu$, in contradiction with (*f*). Hence $N_\alpha \cap M_\mu = M_\alpha$. If $\alpha < \mu$ and $g \in N_\alpha - M_\alpha$, then

$$g^{N_\mu} = \cup \{g^{N_\beta} | \alpha < \beta < \mu\} = \cup \{N_\beta | \alpha < \beta < \mu\} = N_\mu$$

by (*g*). Thus (*a*)–(*h*) are satisfied by the groups $M_\alpha \lhd N_\alpha$, $\alpha \leqslant \mu$.

Suppose μ is a successor ordinal, $\mu = \rho + 1$. Let i: $N_\rho \to S$ be an embedding of N_ρ into a countable locally finite simple group S. If N_ρ is finite we choose S to be a finite alternating group. If N_ρ is infinite, we may choose S to be the universal countable locally finite group of Hall(4). Define an embedding u: $N_\rho \to S \times N_\rho/M_\rho$ by $gu = (gi, gM_\rho)$ for each $g \in N_\rho$. We will view N_ρ as a subgroup of $S \times N_\rho/M_\rho$ (via u) and define

$$N_\mu = S \times N_\rho/M_\rho$$

and let $M_\mu = S \times \{0\}$, the first coordinate subgroup of N_μ.

It remains only to verify that (*f*) and (*g*) are valid for all $\alpha < \beta \leqslant \mu$, the other properties being evident. The embedding u yields immediately that $N_\rho \cap M_\mu = M_\rho$ and (*f*) follows easily from this. If $g \in N_\rho - M_\rho$, then gu projects non-trivially on both (simple) factors of $S \times N_\rho/M_\rho$. Hence $g^{N_\mu} = N_\mu$. Suppose $\alpha < \mu$ and $g \in N_\alpha - M_\alpha$. Since

$$N_\alpha \cap M_\rho = M_\alpha,$$

it follows that $g \notin M_\rho$ and, as just noted $g^{N_\mu} = N_\mu$. Thus (*g*) is seen to hold, and the construction is complete.

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Example 4. Let $\mathscr{U}$ be the set of non-empty finite sequences whose terms belong to $\{0, 1\}$. Since $\mathscr{U}$ is countable we may associate a distinct prime number $p(u)$ with each $u \in \mathscr{U}$. If $X \subset \mathscr{U}$ let C_X be the direct product of the groups $\{Z_{p(u)} | u \in X\}$. If $X = \phi$, C_X will be the trivial group.

Let $\tilde{\mathscr{U}}$ be the set of all ω-term sequences whose terms belong to $\{0, 1\}$. From $\tilde{\mathscr{U}}$ choose distinct members indexed by the countable ordinals, $\{\tilde{u}_\alpha | \alpha < \omega_1\}$. If $\tilde{u} \in \mathscr{U}$, $\tilde{u} = (\epsilon_i)_{i=0}^{\infty}$, define $V(\tilde{u}) = \{(\epsilon_i)_{i=0}^{n} | n = 0, 1, 2, \ldots\} \subset \mathscr{U}$. There will be no loss in assuming that $\mathscr{U} = \cup \{V(\tilde{u}_\alpha) | \alpha < \omega_1\}$. Notice that $\tilde{u} \neq \tilde{v} \in \tilde{\mathscr{U}}$ implies $V(\tilde{u}) \cap V(\tilde{v})$ is finite.

Several definitions are now necessary:

9·1. If $\alpha < \omega_1$, let $C_\alpha = C_X$, where $X = V(\tilde{u}_\alpha)$.

9·2. Let $G = \overset{*}{\prod}\{C_\alpha | \alpha < \omega_1\}$ (free product).

9·3. If $\alpha < \omega_1$, H_α will be the subgroup of G generated by the free factors $\{C_\beta | \beta < \alpha\}$. Evidently, $\{H_\alpha | \alpha < \omega_1\}$ is a C_1-system of G.

9·4. If $\beta < \alpha < \omega_1$, A_β^α will denote the (unique) subgroup of C_β isomorphic with C_X, where $X = V(\tilde{u}_\beta) \cap V(\tilde{u}_\alpha)$.

9·5. If $\alpha < \omega_1$, let S_α be the subgroup of H_α that is generated by $\{A_\beta^\alpha | \beta < \alpha\}$.

9·6. If $\alpha < \omega_1$, put $N_\alpha = S_\alpha^{H_\alpha}$ and $K_\alpha = H_\alpha / N_\alpha$.

We are now in a position to define the classes Σ and $\mathfrak{X}$.

9·7. Σ will be the smallest s and L closed class containing each of the groups N_α, $\alpha < \omega_1$. Accordingly, $M \in \Sigma$ if and only if every finitely generated subgroup T of M is isomorphic to a subgroup of some N_α.

9·8. $\mathfrak{X}$ will be the smallest s and C_1 closed class containing each of the groups K_α, $\alpha < \omega_1$. In other words, $M \in \mathfrak{X}$ if and only if every countable subgroup T of M is isomorphic to a subgroup of one of the K_α.

Since each $H_\alpha \in \Sigma\mathfrak{X}$, we see that $G \in C_1(\Sigma\mathfrak{X})$. We now need to verify that $G \notin \Sigma\mathfrak{X}$.

Let N be an arbitrary Σ group. Denote by $\pi(N)$ the set of all primes p such that N has an element of order p. Let $\mathfrak{p} = \{p(u) | u \in \mathscr{U}\}$. We introduce a partial ordering $\prec$ on $\mathfrak{p}$ by $p(u_1) \prec p(u_2)$ if and only if the sequence u_1 is an initial segment of the sequence u_2. Suppose $p_1, p_2 \in \pi(N)$. For $i = 1$ and 2, let $b_i \in N$ have order p_i. Then $\langle b_1, b_2 \rangle$ is isomorphic to a subgroup of some N_α, $\alpha < \omega_1$ (from the definition of Σ). Suppose $x \in N_\alpha$ has prime order p. Applying the subgroup theorem for free products ((7), p. 17), since $x \in H_\alpha$, there is some $h \in H_\alpha$ such that $x^h \in C_\beta$ for some $\beta < \alpha$. Since $N_\alpha \lhd H_\alpha$, we have $x^h \in N_\alpha$. Now obviously $N_\alpha \cap C_\beta = A_\beta^\alpha$ and this forces $p = p(u)$ for some $u \in V(\tilde{u}_\beta) \cap V(\tilde{u}_\alpha)$. We conclude that $p_1, p_2 \in \{p(y) | y \in V(\tilde{u}_\alpha)\}$. It follows that p_1 and p_2 are comparable in $(\mathfrak{p}, \prec)$, and since p_1 and p_2 were arbitrary members of $\pi(N)$, $\pi(N)$ must be a chain in $(\mathfrak{p}, \prec)$. Hence there exists a $\tilde{v} \in \mathscr{U}$ such that $\pi(N) \subset \{p(u) | u \in V(\tilde{v})\}$.

Now let us assume that $G \in \Sigma\mathfrak{X}$, $N \lhd G$, $N \in \Sigma$, and $G/N \in \mathfrak{X}$. Let $\tilde{v}$ be as determined above. Evidently G/N contains elements of all orders $p(u)$ where $u \in \mathscr{U} - V(\tilde{v})$. Choose

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some $\alpha < \omega_1$ subject to the condition $\alpha > \tau$ where $\tilde{v} = \tilde{u}_\tau$ (if such a τ exists). From the definition of $\mathfrak{X}$, there is some group $K_\mu = H_\mu/N_\mu$ such that

(1) K_μ has elements of all orders $p(u)$ where $u \in \mathscr{U} - V(\tilde{v})$, and

(2) K_μ contains a subgroup isomorphic to $Q = C_\alpha/N \cap C_\alpha$.

K_μ is isomorphic to the free product of the groups $\{C_\beta/A^\alpha_\beta | \beta < \mu\}$. An easy consequence of the subgroup theorem for free products is that Q must be isomorphically contained in some C_γ/A^μ_γ, $\gamma < \mu$. $N \cap C_\alpha$ is finite because $V(\tilde{u}_\alpha) \cap V(\tilde{v})$ is a finite set. Thus Q is infinite. Hence the isomorphic containment $Q \subsetneq C_\gamma/A^\mu_\gamma$ is possible only if $\alpha = \gamma$ since, otherwise, $V(\tilde{u}_\alpha) \cap V(\tilde{u}_\gamma)$ would be finite. It follows that $\mu > \alpha$ and hence $\tilde{u}_\mu \neq \tilde{v}$.

Let $u \in \mathscr{U} - V(\tilde{v})$. By (1), K_μ has an element of order $p(u)$ and so some free factor C_β/A^μ_β, $\beta < \mu$, of K_μ has an element of order $p(u)$. This implies $u \notin V(\tilde{u}_\mu)$ since $A^\mu_\beta = C_X$ where $X = V(\tilde{u}_\beta) \cap V(\tilde{u}_\mu)$. Thus $\mathscr{U} - V(\tilde{v}) \subset \mathscr{U} - V(\tilde{u}_\mu)$ which forces $V(\tilde{u}_\mu) \subset V(\tilde{v})$. This contradiction shows that $G \notin \Sigma\mathfrak{X}$.

10. *Added May* 14, 1972. Professor G. Sabbagh has offered some relevant comments concerning the question 3·2·1. In particular Sabbagh has proved the following theorem in ((13), p. 184).

THEOREM. *If Σ is s-closed and is closed under the taking of ultrapowers, then $R\Sigma$ is C_1-closed. It is an immediate corollary that $R\mathfrak{F}$ is C_1-closed.*

Sabbagh has also noted the existence of a class Σ satisfying s and C_1-closure, but such that $C_1(R\Sigma) \not\subset R\Sigma$. Such a class is $\Sigma =$ the class of infinite cyclic groups. Any non-trivial group in $R\Sigma$ has an infinite cyclic direct summand. Yet there exists an indecomposable group, G, every countable subgroup of which is free Abelian so that $G \in C_1(R\Sigma) - R\Sigma$. Such a group is mentioned in ((3), p. 24).

K. Hickin has recently improved the techniques of section 7 and applied them to more general algebraic properties. These results are in the forthcoming paper (5). In connexion with residual properties, the following is proved: Suppose $\{s, C_1\}\Sigma = \Sigma$. Let Λ be the class of groups G such that every homomorphic image of G belongs to $R\Sigma$. Then $C_1(\Lambda) \subset \Lambda$.

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