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Quadratic Langrangians and static gravitational fields

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Abstract. It is shown that the solutions of the field equations generated by the most general effective quadratic Lagrangian $aR^2 + bR_{kl}R^{kl}$ which are static, singularity-free, of class C^4 and asymptotically flat (of rank A^3) are (i) conformally flat when $3a + b = 0$, (ii) flat when $3a + b \neq 0$ ($b \neq 0$), and (iii) have zero scalar curvature when $b = 0$.

1. *Introduction.* The relativistic theory of gravitation operates within the arena of four-dimensional normal-hyperbolic Riemann spaces V_4. A V_4 is static (and called a V_4^*) if it admits a time-like hypersurface-orthogonal Killing vector, in which case coordinates may be introduced such that the metric $ds^2 = g_{kl}dx^k dx^l$ takes the particular form

$$ds^2 = g_{ab}dx^a dx^b + g_{44}(dx^4)^2, \tag{1}$$

where the g_{ab} and g_{44} do not depend upon x^4. (Indices a, b, c, d go from 1 to 3; those taken from the rest of the Roman alphabet go from 1 to 4.) If the V_4 is asymptotically flat

$$\lim_{r\to\infty} g_{kl} = \eta_{kl} = \text{diag}\,(-1, -1, -1, 1) \quad (r^2 = -\eta_{ab}x^a x^b), \tag{2}$$

granted that the cordinates x^k in (1) have been chosen appropriately. Of course, the sense in which the condition of asymptotic flatness is to be understood in the present context goes beyond (2). To make it more precise, let the g_{kl} in (1) be differentiable as often as required. Then the V_4^* will be said to be of rank A^f if for a suitable choice of coordinates there exist positive numbers ρ and M such that for $r > \rho$ and $1 \leqslant s \leqslant f$

$$|g_{kl} - \eta_{kl}| < Mr^{-1}, \quad |g_{kl,a_1a_2\ldots a_s}| < Mr^{-s-1}. \tag{3}$$

In particular, if the g_{kl} are analytic when $r > \rho$ the V_4^* is of rank A^∞. In the following work it is assumed throughout that the V_4^* is of class C^4 and rank A^3, and that $\det g_{kl} \neq 0$ everywhere. Under these conditions the V_4^* will be called a $V_4^\dagger$.

A formal expression of the usual vacuum field equations is this: the functional derivative of the scalar curvature R of V_4 vanishes, i.e.

$$\delta R/\delta g_{kl} = 0. \tag{4}$$

It is known that if a V_4 is a $V_4^\dagger$ then the equations (4) entail (5) that it is flat. Now, from time to time alternative field equations have been contemplated, namely those in which R is replaced in (4) by a more general invariant K of the Riemann tensor (for example (4)). The alternative considered most often is probably that where K is

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a homogeneous quadratic polynomial in the components of the Riemann tensor (a 'quadratic invariant' for short). In a V_4 the most general invariant of this kind is (3)

$$K = aR^2 + bR_{kl}R^{kl} + cR_{klmn}R^{klmn} + d(-g)^{-\frac{1}{2}}\epsilon^{klmn}R_{klpq}R_{mn}{}^{pq}, \tag{5}$$

($a, \ldots, d$ = constant). However, the functional derivatives of the last invariant on the right and of the invariant

$$K_0 = R^2 - 4R_{kl}R^{kl} + R_{klmn}R^{klmn}$$

vanish identically. (That this is so is shown most easily by invoking the calculus of 2-spinors (2).) One may therefore set c and d equal to zero without loss of generality, so that the most general quadratic invariant which needs to be contemplated in a V_4 is

$$K = aR^2 + bR_{kl}R^{kl}. \tag{6}$$

Writing P^{kl} for $\delta K/\delta g_{kl}$ one has (1)

$$\begin{aligned} P_{kl} = {} & b\,\Box R_{kl} - (2a+b)\,R_{;kl} - 2aRR_{kl} - 2bR^{kpql}R_{pq} \\ & + \tfrac{1}{2}g_{kl}[(4a+b)\,\Box R + aR^2 + bR_{pq}R^{pq}], \end{aligned} \tag{7}$$

whence it follows by transvection with g^{kl} that

$$\tfrac{1}{2}P = (3a+b)\,\Box R. \tag{8}$$

It is to be noted that when $b = 0$ every term of P_{kl} contains R or its covariant derivative. One concludes that any investigation of the new field equations $P_{kl} = 0$ must pay heed to three distinct situations, i.e. those corresponding to (i) $3a+b = 0$, (ii) $3a+b \neq 0$ (with $b \neq 0$) and (iii) $b = 0$. The object of this note is to arrive at conclusions relating to the equations $P_{kl} = 0$ which form a counterpart to the result stated after equation (4). In fact I shall show that in case (i) the $V_4^\dagger$ must be conformally flat (the scale factor being appropriately behaved); in case (ii) the $V_4^\dagger$ must be flat; whilst in case (iii) one can in general conclude no more than that the scalar curvature must vanish.

2. *Case* (*i*): $3a+b = 0$. It suffices to set $a = -1$, so that

$$K = 3R_{kl}R^{kl} - R^2. \tag{9}$$

If $\mathfrak{R}$ denote the scalar density $K(-g)^{\frac{1}{2}}$ it may be confirmed by direct calculation that under a conformal transformation which takes g_{kl} into $\bar{g}_{kl} = e^{2q}g_{kl}$ $\mathfrak{R}$ goes into

$$\bar{\mathfrak{R}} = \mathfrak{R} + \mathfrak{D}^k{}_{,k} \tag{10}$$

where $\mathfrak{D}^k$ is a vector density. (In fact,

$$\mathfrak{D}^k = 12(-g)^{\frac{1}{2}}[(R^{kl} - \tfrac{1}{2}g^{kl}R) + q^{;kl} - g^{kl}(\Box q + q^{;m}q_{;m})]\,q_{;l}.) \tag{11}$$

That an equation of the form (10) obtains is implicit in the identity

$$K = \tfrac{3}{2}(C_{klmn}C^{klmn} - K_0),$$

where C_{klmn} is the conformal curvature tensor. From (10) it follows at once that the functional derivatives of $\mathfrak{R}$ and of $\bar{\mathfrak{R}}$ are identical. Therefore, given any solution g_{kl} of the field equations $P_{kl} = 0$, any metric conformal to g_{kl} is also a solution. A flat V_4

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is always a $V_4^\dagger$ and it trivially satisfies the field equations. Accordingly, if it be understood that the scale factor e^{2q} is suitably behaved, a sufficient condition for a $V_4^\dagger$ to satisfy the field equations is that it be conformally flat. It remains to show that this condition is also necessary.

Bearing in mind the conformal covariance of P_{kl} one can always make a conformal transformation with $e^{2q} = 1/g_{44}$ on a given $V_4^\dagger$, so that it suffices to consider metrics of the form

$$ds^2 = d\sigma^2 + (dx^4)^2, \quad d\sigma^2 = g_{ab}dx^a dx^b. \tag{12}$$

Let $V_3^\dagger$ be the three-dimensional subspace the metric of which is $d\sigma^2$, the dagger serving to emphasize that this subspace is of class C^4 and rank A^3 as a consequence of the properties of $V_4^\dagger$. Covariant derivatives in $V_3^\dagger$ are indicated by subscripts following a colon, and in particular, if X is any tensor in $V_3^\dagger$, $\Delta X = g^{ab}X_{:ab}$. Then (with $b = 3$ and $a = -1$) equation (7) may be written in the form

$$P_{ab} = 3\Delta R_{ab} - R_{:ab} - 7RR_{ab} + 12R_{ac}R^c_b - \tfrac{1}{2}g_{ab}(\Delta R + 9R_{cd}R^{cd} - 5R^2), \tag{13}$$

$$P_{44} = \tfrac{1}{2}(\Delta R - 3R_{cd}R^{cd} + R^2). \tag{14}$$

The Ricci tensor and scalar curvature on the right are now, of course, those of the $V_3^\dagger$. The scalar P vanishes identically in virtue of these equations, consistently with (8). Writing

$$S_{ab} = R_{ab} - \tfrac{1}{3}g_{ab}R, \tag{15}$$

the equation $P_{44} = 0$ takes the form

$$\Delta R = 3S_{ab}S^{ab}. \tag{16}$$

Writing $w = (-\det g_{ab})^{\frac{1}{2}}$ the left-hand member of the concomitant equation

$$\int w\Delta R d^3x = 3\int wS_{ab}S^{ab}d^3x, \tag{17}$$

in which the integrals are extended over the whole $V_3^\dagger$, may be converted into a surface integral. The latter vanishes since (3), with $s = 3$, entails that $r^2R_{,a}$ is zero at infinity. (The condition that the $V_4^\dagger$ be of rank A^3 is evidently unnecessarily stringent.) On the other hand, $S_{ab}S^{ab} \geqslant 0$ and therefore the integral on the right of (17) can vanish only if this invariant, and therefore S_{ab} itself, vanishes everywhere. In other words the $V_3^\dagger$ is an Einstein space and therefore, being three-dimensional, of constant Riemannian curvature. Being asymptotically flat it must therefore be flat everywhere. The necessity of the condition stated above has thus been established.

3. *Case* (*ii*): $3a + b \neq 0$ ($b \neq 0$). In view of equation (8) one must now have

$$\Box R = 0. \tag{18}$$

In the notation of Section 2, this may be written

$$\Delta R + (\tfrac{1}{2}lng_{44})^{:c}R_{:c} = 0, \tag{19}$$

where, however, R is still the scalar curvature of $V_4^\dagger$. The only acceptable solution of this equation is (3)

$$R = 0. \tag{20}$$

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Since $\delta R^2/\delta g_{kl}$ vanishes whenever $R = 0$ one is at liberty to replace K by

$$b(-\tfrac{1}{3}R^2 + R_{kl}R^{kl}).$$

One thus returns to the situation of Section 2 but with the additional restriction (20). In short, the acceptable solutions of $P_{kl} = 0$ must be conformally flat and have zero scalar curvature. With an appropriate choice of coordinates this means on the one hand that there exists a scalar f such that $g_{kl} = f^2 \operatorname{diag}(-1, -1, -1, 1)$ and, on the other hand, that f must obey the 3-dimensional flat space Laplace equation. The latter has no acceptable solutions other than $f =$ constant. Under the present circumstances any $V_4^\uparrow$ which satisfies the field equations must thus indeed be flat.

4. *Case (iii)*: $b = 0$. Equation (18) continues to hold and so therefore does equation (20). Then, however, P_{kl} vanishes, i.e. the vanishing of the scalar curvature of $V_4^\uparrow$ is the necessary and sufficient condition for the field equations to be satisfied in this case. That appears to be all one can say, unless further restrictions are placed upon the $V_4^\uparrow$. For instance, when spatial spherical symmetry obtains it must be flat.

REFERENCES

(1) BUCHDAHL, H. A. The Hamiltonian derivatives of a class of fundamental invariants. *Quart. J. Math. (Oxford)* **19** (1948), 150.
(2) BUCHDAHL, H. A. On the non-existence of a class of static Einstein spaces asymptotic at infinity to a space of constant curvature. *J. Math. and Phys.* **1** (1960), 537.
(3) COURANT, R. and HILBERT, D. *Methods of Mathematical Physics*, vol. II (Interscience, 1966), p. 326.
(4) EDDINGTON, A. S. *The mathematical theory of relativity* (Cambridge University Press, 1923), §62.
(5) FOCK, V. *The theory of space, time and gravitation* (Pergamon Press, 1959), p. 188.
(6) PETROW, A. S. *Einstein-Räume* (Akademie Verlag, 1964), §22.

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