A NOTE ON RELATIVISTIC SECOND QUANTIZATION

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In 1942 Dirac† gave in connexion with a new formulation of quantum electrodynamics a relativistic second quantization of bosons, i.e. particles satisfying Bose statistics. In this quantization, particles of negative energies are considered, but it seems quite natural that they will be present in a relativistic theory of second quantization. It is obvious that one can perform a similar quantization on particles satisfying Fermi-Dirac statistics, and such a quantization will be useful in a theory in which particles interact through the emission and absorption of Fermi-Dirac particles‡. In this note, such a quantization is given. It is very similar to the quantization of bosons, but is given here for comparison and for future reference. Dirac's notation of bra and ket vectors§ will be used here, and it is hoped that the following illustrates once again the only too obvious power of such a notation.

So far, the relativistic second quantization is restricted to free particles. In trying to extend to the case of interacting particles or particles capable of emissions and absorptions (not only capable of being emitted or absorbed, but also of emitting and absorbing other particles), one realizes that we do not yet have proper relativistic wave equations‖ and scalar products between states, when the number of particles involved is not fixed and the particles are capable of emissions and absorptions. In a proper relativistic theory, each of the particles must take a separate time coordinate, and since the number of particles is not fixed, the number of such independent time coordinates is also not fixed. This circumstance makes the construction of the wave equation a rather difficult task.

In the following we attempt to construct such wave equations. It is shown that in spite of the lack of expressions for relativistic scalar products between states (which in a relativistic theory can only be obtained after finding the wave equations), creation and annihilation operators can still be defined, and that by making use of them, consistent wave equations of the nature described above can be set up.

1. Quantization of free bosons

We start with the quantization of bosons, which is mainly a recapitulation of the results of Dirac. Let us first consider the non-relativistic case and denote the dynamical variable of a boson by q. Let us require, in addition to the symmetry of the wave functions

† Dirac, P. A. M., *Proc. Roy. Soc.* A, 180 (1942), 1.

‡ It is suggested that such particles are called 'fermions', which is simpler than 'Fermi-Dirac particles' or 'particles satisfying Fermi-Dirac statistics'.

§ Dirac, P. A. M., *Proc. Cambridge Phil. Soc.* 35 (1939), 416.

‖ It seems perhaps hardly necessary to point out the fact that any relativistic theory of second quantization on certain particles cannot proceed unless we know the wave equations which these particles satisfy, in full contrast with a non-relativistic second quantization.

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$\langle q'q''\ldots|\rangle$ and their conjugates $\langle|q'q''\ldots\rangle$, that all matrices $\langle q'q''\ldots q^u\,|\,\alpha\,|\,q^{u+1}\ldots q^{u+v}\rangle$ for the observable α are symmetrical between $q'q''\ldots q^u$ and between $q^{u+1}\ldots q^{u+v}$ separately. The unit matrix $\langle q'\ldots q^u\,|\,q^{u+1}\ldots q^{u+v}\rangle$ satisfying

$$\sum_v \sum_{q^{u+1}\ldots q^{u+v}} \langle q'\ldots q^u\,|\,q^{u+1}\ldots q^{u+v}\rangle\langle q^{u+1}\ldots q^{u+v}\,|\,=\langle q'\ldots q^u\,|, \tag{1·1}$$

$$\sum_v \sum_{q^{u+1}\ldots q^{u+v}} |\,q^{u+1}\ldots q^{u+v}\rangle\langle q^{u+1}\ldots q^{u+v}\,|\,q'\ldots q^u\rangle = |\,q'\ldots q^u\rangle, \tag{1·2}$$

is thus given by zero when $u\neq v$, and by

$$\langle q'\ldots q^u\,|\,q^{u+1}\ldots q^{2u}\rangle = \delta(q'\ldots q^u,\quad q^{u+1}\ldots q^{2u})\{P(q'\ldots q^u)\}^{-1} \tag{2}$$

when $u=v$, where $\delta(q'\ldots q^u, q^{u+1}\ldots q^{2u})$ is unity if $q^{u+1}\ldots q^{2u}$ is a permutation of $q'\ldots q^u$ and zero otherwise, and $P(q'\ldots q^u)$ is $u!(n_1!\,n_2!\ldots)^{-1}$, n_1 being the number of q's among $q'\ldots q^u$ taking a certain value of q, say q_1, etc. ($q_1, q_2, \ldots$ are the different eigenvalues of q). From (2) we get

$$\langle q'\ldots q^u\,|\,q^{u+1}\ldots q^{2u}\rangle = (u!)^{-1}\Sigma\,\delta_{q'q^a}\delta_{q''q^b}\ldots\delta_{q^uq^z}, \tag{3}$$

where $q^aq^b\ldots q^z$ is a permutation of $q^{u+1}\ldots q^{2u}$, and the summation is taken over all permutations.

If we define operators ξ_q and $\bar{\xi}_q$ by

$$\xi_q\,|\,q'q''\ldots q^u\rangle = (u+1)^{\frac{1}{2}}\,|\,qq'q''\ldots q^u\rangle, \tag{4·1}$$

$$\langle q'q''\ldots q^u\,|\,\bar{\xi}_q = (u+1)^{\frac{1}{2}}\langle qq'q''\ldots q^u\,|, \tag{4·2}$$

we find, with the help of (2) or (3), $\quad \xi_q\bar{\xi}_q = n_q, \tag{5·1}$

$$\bar{\xi}_{q_1}\xi_{q_2} - \xi_{q_2}\bar{\xi}_{q_1} = \delta_{q_1q_2}, \tag{5·2}$$

$$\xi_{q_1}\xi_{q_2} - \xi_{q_2}\xi_{q_1} = \bar{\xi}_{q_1}\bar{\xi}_{q_2} - \bar{\xi}_{q_2}\bar{\xi}_{q_1} = 0, \tag{5·3}$$

$$\bar{\xi}_q\,|\,q'q''\ldots q^u\rangle = u^{-\frac{1}{2}}\sum_r \delta_{qq^r}\,|\,q'\ldots q^{r-1}q^{r+1}\ldots q^u\rangle, \tag{6·1}$$

$$\langle q'q''\ldots q^u\,|\,\xi_q = u^{-\frac{1}{2}}\sum_r \delta_{q^rq}\langle q'\ldots q^{r-1}q^{r+1}\ldots q^u\,|. \tag{6·2}$$

It follows that:

(i) ξ_q and $\bar{\xi}_q$ are Hermitian conjugates.

(ii) If for any u and v, $\langle q'\ldots q^u\,|\,\alpha\,|\,q^{u+1}\ldots q^{u+v}\rangle$ is symmetrical in $q'\ldots q^u$ and in $q^{u+1}\ldots q^{u+v}$ separately, so is any observable formed from α, ξ and $\bar{\xi}$.

(iii) For such α, there exists for each u and v a function $\phi(\xi,\bar{\xi})$, so that $\langle q'\ldots q^u\,|\,\alpha\,|\,q^{u+1}\ldots q^{u+v}\rangle$ and $\langle q'\ldots q^u\,|\,\phi(\xi,\bar{\xi})\,|\,q^{u+1}\ldots q^{u+v}\rangle$ are equal. In fact $\phi(\xi,\bar{\xi})$ is

$$(u!\,v!)^{-\frac{1}{2}}\sum_q \langle q^{(1)}q^{(2)}\ldots q^{(u)}\,|\,\alpha\,|\,q^{(u+1)}\ldots q^{(u+v)}\rangle\,\xi_{q^{(1)}}\xi_{q^{(2)}}\ldots\xi_{q^{(u)}}\bar{\xi}_{q^{(u+1)}}\ldots\bar{\xi}_{q^{(u+v)}}, \tag{7}$$

the summation over each of the q's being taken over all eigenvalues of q. Since the equality of the two representatives holds for a certain u and v, α and $\phi(\xi,\bar{\xi})$ are not the same observable. However, a combination of such $\phi(\xi,\bar{\xi})$ can make its representative the same as that of α for any given set of u and v.

The reason for distinguishing δ_{qrq} and δ_{qqr} in (6) is as follows. There are cases in which a scalar product between states a and b containing one particle only is given by

$$\sum_{q'q''}\langle a\,|\,q'\rangle W_{q'q''}\langle q''\,|\,b\rangle, \tag{8}$$

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where $W_{q'q'}$ plays the role of a weight function. The unit matrix is then determined by

$$\Sigma_{q'q''}\langle q' \mid q''\rangle W_{q''q'''}\langle q''' \mid \rangle = \langle q' \mid\rangle, \tag{9}$$

or

$$\langle q' \mid q''\rangle = (W^{-1})_{q'q''}. \tag{10}$$

If for u particles the weight $W_{q' \dots q^u, q^{u+1} \dots q^{2u}}$ is generalized to $W_{q'q^{u+1}}W_{q''q^{u+2}}\dots$ or some expression like the right-hand side of (3) with W in place of δ, formulae (3)–(6) remain but with δ replaced by W^{-1} and with the exception of the equation $\xi\bar{\xi} = n$. It is to be noted that when W is not self-Hermitian conjugate, ξ_q, $\bar{\xi}_q$ are not Hermitian conjugates and $\langle q' \dots \mid \xi_q, \bar{\xi}_q \mid q' \dots\rangle$ are not conjugate vectors. Such cases have not much interest. First $\langle a \mid a\rangle$ is not real. Secondly, there are two ways of defining the probability of q taking the value q' in a state a, i.e.

$$\sum_{q''}\langle a \mid q'\rangle W_{q'q''}\langle q'' \mid a\rangle \quad \text{and} \quad \sum_{q''}\langle a \mid q''\rangle W_{q''q'}\langle q' \mid a\rangle,$$

which are generally unequal. However, the self-Hermitian conjugate W cannot be entirely removed by a diagonalization, for after diagonalization the matrix elements may be $+1$ or -1.

In relativistic treatments, we consider wave functions $\langle q't'q''t'' \dots \mid\rangle$. A formal extension of the above by including t as one of the q's is useless and also faulty, since $\langle q't'q''t'' \dots \mid\rangle$ satisfy certain equations, while the non-relativistic $\langle q'q'' \dots \mid\rangle$ do not. Owing to this a different scalar product is needed. For definiteness, let us consider a group of free scalar particles with no mass. Let x denote the space-time four-vector, with $x_0 = t$ and $\mathbf{x} = (x_1x_2x_3)$. The scalar product of state a and b when there is only one particle present is then

$$\langle a \mid b\rangle = (4\pi i)^{-1}\int d\mathbf{x}\langle a \mid x\rangle(\overrightarrow{\Gamma} - \overleftarrow{\Gamma})\langle x \mid b\rangle, \tag{11·1}$$

where $\overrightarrow{\Gamma}$ is the operator $(\partial/\partial x_0)$ operating on expressions to its left and $\overleftarrow{\Gamma}$ the same operator operating on expressions to its right. When there may be more than one particle present, (11·1) is replaced by

$$\langle a \mid b\rangle = \sum_u (4\pi i)^{-u}\int\dots\int\langle a \mid x' \dots x^u\rangle(\overrightarrow{\Gamma}' - \overleftarrow{\Gamma}')(\overrightarrow{\Gamma}'' - \overleftarrow{\Gamma}'') \dots (\overrightarrow{\Gamma}^u - \overleftarrow{\Gamma}^u)\langle x' \dots x^u \mid b\rangle d\mathbf{x}' \dots d\mathbf{x}^u, \tag{11·2}$$

with obvious meanings for $\overrightarrow{\Gamma}'$, $\overleftarrow{\Gamma}'$, etc. Replacing a by some x we get an equation which is a relativistic extension of (1·1). From this we find $\langle x' \dots x^u \mid x^{u+1} \dots x^{u+v}\rangle$ is zero when $u \neq v$ and

$$\langle x' \dots x^u \mid x^{u+1} \dots x^{2u}\rangle = (-i)^u(u!)^{-1}\Sigma\Delta(x' - x^a)\Delta(x'' - x^b)\dots\dagger \tag{12}$$

when $u = v$, where $x^ax^b \dots$ is a permutation of $x^{u+1} \dots x^{2u}$, the summation is taken over all permutations and $\Delta(x)$ is the four-dimensional delta function. Obviously (12) satisfies (11·2) with b replaced by some x, and $\langle x' \dots \mid x^u \dots\rangle$ given by (12) is Hermitian.

† If we restrict $\langle x'|\rangle\langle x' \dots|\rangle$ to be of the type

$$\langle x'|\rangle = \frac{1}{2\pi i}\int e^{i(\mathbf{K}\mathbf{x} - k_0t)}\langle \mathbf{K}|\rangle\frac{d\mathbf{K}}{k_0}, \quad k_0 = +\{\mathbf{K}_1^2 + \mathbf{K}_2^2 + \mathbf{K}_3^2\}^{\frac{1}{2}}$$

(i.e. restricting ourselves to positive energy particles only), formula (12) remains but with $-i\Delta(x' - x'')$ replaced by

$$-\tfrac{1}{2}i\Delta(x' - x'') - \frac{1}{\pi}\frac{1}{(\mathbf{x}' - \mathbf{x}'')^2 - (t' - t'')^2}.$$

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With (12) as the unit matrix and (11·2) as the scalar product and with all matrices $\langle x' \dots | \alpha | x^{u+1} \dots \rangle$ satisfying the wave equations for $\langle x' \dots |\rangle$ and $\langle | x^{u+1} \dots \rangle$, to all non-relativistic formulae regarding the manipulation of $\langle q' \dots | \alpha | q^{u+1} \dots \rangle$, $\langle q' \dots |\rangle$, $\langle | q^{u+1} \dots \rangle$ there correspond relativistic formulae. In particular, if we define ξ_x and $\bar{\xi}_x$ by

$$\xi_x | x' \dots x^u \rangle = (u+1)^{\frac{1}{2}} | x x' \dots x^u \rangle, \tag{13·1}$$

$$\langle x' \dots x^u | \bar{\xi}_x = (u+1)^{\frac{1}{2}} \langle x x' \dots x^u |, \tag{13·2}$$

then owing to (12), which is exactly (3) except for a change of symbols, we get relations (5), (6) with q replaced by x and $\delta_{q_1 q_2}$ by $-i\Delta(x_1 - x_2)$ and with the exception of the formula (5·1). Owing to the property of $\Delta(x)$, ξ_x and $\bar{\xi}_x$ are Hermitian conjugates, and $\langle x' \dots | \xi_x, \bar{\xi}_x | x' \dots \rangle$ conjugate vectors. If we write part of the wave equations as

$$\langle | H_x | x x' \dots x^u \rangle = \langle x x' \dots x^u | H_x | \rangle = 0, \tag{14·1}$$

H_x being an operator operating on x, we get

$$\langle | H_x \xi_x = \bar{\xi}_x H_x | \rangle = 0. \tag{14·2}$$

If all states to be considered are those satisfying (14), we may write simply

$$H_x \xi_x = \bar{\xi}_x H_x = 0. \tag{15}$$

These results are mainly given by Dirac.

2. Free Fermi-Dirac particles (fermions)

Let us now consider the non-relativistic second quantization of fermions. Here we restrict ourselves to antisymmetrical $\langle q' \dots q^u |\rangle$ and $\langle | q' \dots q^u \rangle$, and to

$$\langle q' \dots q^u | \alpha | q^{u+1} \dots q^{u+v} \rangle,$$

which is antisymmetrical in $q' \dots q^u$ and in $q^{u+1} \dots q^{u+v}$ separately. Equation (1) is retained, and (2) replaced by

$$\langle q' \dots q^u | q^{u+1} \dots q^{2u} \rangle = (u!)^{-1} \delta(q' \dots q^u, q^{u+1} \dots q^{2u}) (-1)^{k(q' \dots q^u)} (-1)^{k(q^{u+1} \dots q^{2u})}, \tag{16}$$

where $k(q' \dots q^u)$ is the number of interchanges required to bring $q' \dots q^u$ to a standard order.

To avoid the appearance of a standard order, we replace (16) by

$$\langle q' \dots q^u | q^{u+1} \dots q^{2u} \rangle = (u!)^{-1} \Sigma \delta_{q' q^a} \delta_{q'' q^b} \dots (-1)^{k(q^a q^b \dots, q^{u+1} \dots q^{2u})}, \tag{17}$$

where $q^a q^b \dots$ is a permutation of $q^{u+1} \dots q^{2u}$, $k(q^a q^b \dots, q^{u+1} \dots q^{2u})$ is the number of interchanges to bring $q^a q^b \dots$ into the order $q^{u+1} \dots q^{2u}$, and the summation is taken over all permutations. Let us define ξ_q and $\bar{\xi}_q$ by

$$\xi_q | q' q'' \dots q^u \rangle = (u+1)^{\frac{1}{2}} | q q' q'' \dots q^u \rangle, \tag{18·1}$$

$$\langle q' q'' \dots q^u | \bar{\xi}_q = (u+1)^{\frac{1}{2}} \langle q q' q'' \dots q^u |. \tag{18·2}$$

Then we find by (17)

$$\xi_q \bar{\xi}_q = n_q, \tag{19·1}$$

$$\bar{\xi}_{q_1} \xi_{q_2} + \xi_{q_2} \bar{\xi}_{q_1} = \delta_{q_1 q_2}, \tag{19·2}$$

$$\xi_{q_1} \xi_{q_2} + \xi_{q_2} \xi_{q_1} = \bar{\xi}_{q_1} \bar{\xi}_{q_2} + \bar{\xi}_{q_2} \bar{\xi}_{q_1} = 0, \tag{19·3}$$

$$\bar{\xi}_q | q' q'' \dots q^u \rangle = u^{-\frac{1}{2}} \sum_r \delta_{q q^r} (-1)^{r-1} | q' \dots q^{r-1} q^{r+1} \dots q^u \rangle, \tag{20·1}$$

$$\langle q' q'' \dots q^u | \xi_q = u^{-\frac{1}{2}} \sum_r \delta_{q' q} (-1)^{r-1} \langle q' \dots q^{r-1} q^{r+1} \dots q^u |. \tag{20·2}$$

Further, we find that ξ_q and $\bar{\xi}_q$ satisfy (i), (ii), (iii) which were given for the case of bosons, but with 'antisymmetrical' in place of 'symmetrical'.

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In order to extend to relativistic second quantizations, let us consider for definiteness a group of free Dirac-electrons. The wave equation for a free Dirac-electron is

$$\{(\partial/\partial x_0) + \alpha_r(\partial/\partial x_r) + i\kappa\beta\}\psi = 0, \tag{21}$$

where α, β are Dirac matrices, $\hbar$, c are taken to be unity, and κ is the mass. Writing the solution for ψ for a state a as $\langle x_0\mathbf{x}\rho \mid a\rangle$ and its complex conjugate as $\langle a \mid x_0\mathbf{x}\rho\rangle$, where ρ is the spin suffix, we see that the scalar product of the state a and the state b with one electron present can be defined by

$$\langle a \mid b\rangle = \sum_{\rho} \int d\mathbf{x} \langle a \mid x_0\mathbf{x}\rho\rangle \langle x_0\mathbf{x}\rho \mid b\rangle, \tag{22·1}$$

the right-hand side being independent of x_0. Replacing a by $x_0'\mathbf{x}'\rho'$ and considering the above as an equation in $\langle x_0'\mathbf{x}'\rho' \mid x_0''\mathbf{x}''\rho''\rangle$, we find

$$\langle x_0'\mathbf{x}'\rho' \mid x_0''\mathbf{x}''\rho''\rangle = \left(\frac{\partial}{\partial x_0'} I - \alpha_r \frac{\partial}{\partial x_r'} - i\kappa\beta\right)_{\rho'\rho''} D(x' - x''), \tag{23}$$

where I is the unit matrix and $D(x)$ the function

$$(2\pi)^{-3} \int d\mathbf{k}\, e^{i(\mathbf{kx})} (\mathbf{k}^2 + \kappa^2)^{-\frac{1}{2}} \sin\{x_0(\mathbf{k}^2 + \kappa^2)^{\frac{1}{2}}\}.$$

Replacing b in (22·1) by some $x_0'\mathbf{x}'\rho'$, we get another equation for $\langle x_0'\mathbf{x}'\rho' \mid x_0''\mathbf{x}''\rho''\rangle$, which gives the same solution (23). If we extend to cases with more than one electron, the scalar product of state a and the state b is given by

$$\langle a \mid b\rangle = \sum_{u} \sum_{\rho'\rho''\ldots\rho^u} \int \ldots \int d\mathbf{x}' \ldots d\mathbf{x}^u \langle a \mid x'\rho' x''\rho'' \ldots x^u\rho^u\rangle \langle x'\rho' \ldots x^u\rho^u \mid b\rangle. \tag{22·2}$$

Let us introduce for simplicity in writing a notation y to denote the three symbols $x_0\mathbf{x}\rho$. Replacing a by some y, we find that $\langle y' \ldots y^u \mid y^{u+1} \ldots y^{u+v}\rangle$ is zero when $u \neq v$ and

$$\langle y' \ldots y^u \mid y^{u+1} \ldots y^{2u}\rangle = (u!)^{-1} \Sigma \langle y' \mid y^a\rangle \langle y'' \mid y^b\rangle \ldots (-1)^{k(y^a y^b \ldots, y^{u+1} \ldots y^{2u})} \tag{24}$$

when $u = v$. Obviously this value of $\langle y' \ldots y^u \mid y^{u+1} \ldots y^{u+v}\rangle$ will satisfy the equation (22·2) with b replaced by some y, and is Hermitian. With (24) as the unit matrix, (22·2) as the scalar product, and with all $\langle y' \ldots y^u \mid \alpha \mid y^{u+1} \ldots y^{u+v}\rangle$ satisfying the wave equations for $\langle y' \ldots y^u \mid\rangle$ and $\langle \mid y^{u+1} \ldots y^{u+v}\rangle$, all manipulations between the non-relativistic $\langle q' \ldots q^u \mid\rangle\langle\mid q^{u+1} \ldots q^{u+v}\rangle$, $\langle q' \ldots \mid \alpha \mid q^{u+1} \ldots\rangle$ can be properly extended to relativistic manipulations.

Defining ξ_y, $\bar{\xi}_y$ by

$$\xi_y \mid y'y'' \ldots y^u\rangle = (u+1)^{\frac{1}{2}} \mid yy'y'' \ldots y^u\rangle, \tag{25·1}$$

$$\langle y'y'' \ldots y^u \mid \bar{\xi}_y = (u+1)^{\frac{1}{2}} \langle yy'y'' \ldots y^u \mid, \tag{25·2}$$

we have (19), (20) with q replaced by y and $\delta_{q_1 q_2}$ by $\langle y_1 \mid y_2\rangle$, with the exception of the relation (19·1). Owing to the property of $\langle y_1 \mid y_2\rangle$ given by (23), ξ_y, $\bar{\xi}_y$ are Hermitian conjugates and $\langle y' \ldots \mid \xi_y, \bar{\xi}_y \mid y' \ldots\rangle$ conjugate vectors. Since part of the wave equations can be written as

$$\langle \mid H_y \mid yy' \ldots y^u\rangle = \langle yy' \ldots y^u \mid H_y \mid\rangle = 0, \tag{26·1}$$

where H_y is an operator operating on the variable y, we have

$$\langle \mid H_y\xi_y = \bar{\xi}_y H_y \mid\rangle = 0. \tag{26·2}$$

If all states satisfy (26), we may write

$$H_y\xi_y = \bar{\xi}_y H_y = 0. \tag{27}$$

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Thus one sees an entirely parallel situation between particles satisfying the two types of statistics†. It is hardly necessary to point out that the result in this section concerning the relativistic second quantization is entirely useless unless we consider other particles interacting with each other through the emission and absorption of such free Dirac-Fermi particles. All the equations in this section and in the previous one concerning non-relativistic second quantization were obtained long ago by Fock‡, but they appear here in a more elegant form. It should be noted that those observables β whose representative $\langle q' \dots q^u \,|\, \beta \,|\, q^{u+1} \dots q^{2u} \rangle$ is unchanged when $q' \dots q^u$ and $q^{u+1} \dots q^{2u}$ undergo the same permutation may be replaced by α described in (ii), (iii) (after formula (6)) so long as they operate on either symmetrical or antisymmetrical wave functions. In fact $\langle q' \dots q^u \,|\, \alpha \,|\, q^{u+1} \dots q^{2u} \rangle$ of this α is

$$(u!)^{-1} \Sigma \langle q' \dots q^u \,|\, \beta \,|\, q^a q^b \dots \rangle \tag{28·1}$$

in the symmetrical case, and

$$(u!)^{-1} \Sigma \langle q' \dots q^u \,|\, \beta \,|\, q^a q^b \dots \rangle (-1)^{k(q^a q^b \dots, q^{u+1} \dots q^{2u})} \tag{28·2}$$

in the antisymmetrical case, q^a, $q^b \dots$ and Σ having the same meaning as before. Combining this with (7), we express β as functions of ξ and $\bar{\xi}$. The result is, of course, well known.

Before concluding this section a word may be added on how the above can be extended to other free particles, whose dynamical variables are q and t, or simply y, say. The only difference is in the determination of $\langle y' \,|\, y'' \rangle$. This can be done by constructing from the wave functions $\langle a \,|\, y \rangle$, $\langle y \,|\, b \rangle$ for two states a and b, both containing one particle only, a four-current whose divergence vanishes and by letting $\langle a \,|\, b \rangle$ be the zero component integrated over all space. Replacing a by y we get an equation for $\langle y' \,|\, y'' \rangle$, and replacing b by y we get another. If they give the same solution and if the solution is Hermitian as in the above two cases, the rest are obviously repetitions of the above§.

3. PARTICLES WITH INTERACTIONS

Let us consider now the relativistic second quantization of particles interacting with each other or capable of emitting and absorbing each other. Here owing to the lack of an adequate wave equation (where each particle takes a separate time coordinate), we do not have expressions for $\langle a \,|\, b \rangle$ and cannot deduce therefrom $\langle y' \dots \,|\, y^{u+1} \dots \rangle$.

† We have not however, extended the $\langle \xi' | \rangle$ representations for bosons to fermions. For bosons, the definition of $\langle \xi' | \rangle$ is

$$\langle \xi' | \rangle = \sum_u (u!)^{-\frac{1}{2}} \sum_{q' \dots q^u} \xi_{q'} \xi_{q''} \dots \xi_{q^u} \langle q' q'' \dots q^u | \rangle,$$

$$\langle | \bar{\xi}' \rangle = \sum_u (u!)^{-\frac{1}{2}} \sum_{q' \dots q^u} \bar{\xi}_{q'} \bar{\xi}_{q''} \dots \bar{\xi}_{q^u} \langle | q' q'' \dots q^u \rangle.$$

To extend these to fermions, one must assume that the $\xi_{q'}$, $\xi_{q''} \dots$ in the above equations anti-commute, which spoils the advantages of introducing such representatives.

‡ Fock, V., *Z. Phys.* 75 (1932), 622.

§ This discussion can be applied to quantum electrodynamics with $\partial A_\mu / \partial x_\mu = 0$ (Chang, T. S., *Proc. Roy. Soc.* A, 185 (1946), 192). By using the above discussion, we may deduce the expressions for $\langle A_\mu x \,|\, A_\nu x' \rangle$, $\langle A_\mu x \,|\, Px' \rangle$, etc., given there.

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Suppose that the wave equations are constructed, that they are linear in each of the $(\partial/\partial t)$'s, and that as a consequence of them

$$\sum_{v}\sum_{q^{u+1}\dots}\langle a\,|\,y^{u+1}\dots y^{u+v}\rangle\,W\,\langle y^{u+1}\dots y^{u+v}\,|\,b\rangle$$

is independent of $t^{u+1}, \dots t^{u+v}$, where W is a certain matrix operating on $q^{u+1}\dots q^{u+v}$. On letting

$$\langle a\,|\,b\rangle = \sum_{v}\sum_{q^{u+1}\dots}\langle a\,|\,y^{u+1}\dots y^{u+v}\rangle\,W\,\langle y^{u+1}\dots y^{u+v}\,|\,b\rangle, \tag{29}$$

replacing a by some y, say $y'\dots y^u$, and putting $t^{u+1} = t^{u+2} = \dots = 0$, we obtain an equation for $\langle y'\dots y^u\,|\,q^{u+1}0, q^{u+2}0, \dots\rangle$. Let us determine this function and introduce $\langle y'\dots y^u\,|\,y^{u+1}\dots y^{u+v}\rangle$ as something satisfying the wave equations for $\langle y'\dots y^u\,|\rangle$ and $\langle|\,y^{u+1}\dots y^{u+v}\rangle$ and reducing at $t^{u+1} = t^{u+2} = \dots = 0$ to the above function. Then $\langle y'\dots y^u\,|\,y^{u+1}\dots y^{u+v}\rangle$ satisfies (29) with a replaced by $y'\dots y^u$. From the way it is introduced, it satisfies the wave equation for $\langle y'\dots y^u\,|\rangle$ and $\langle|\,y^{u+1}\dots y^{u+v}\rangle$ and reduces at $t' = t'' = \dots = t^{u+1} = \dots = 0$ to the $\langle q'\dots q^u\,|\,q^{u+1}\dots q^{u+v}\rangle$ satisfying equation (1·1) with a weight W. These properties of $\langle y'\dots y^u\,|\,y^{u+1}\dots y^{u+v}\rangle$ determine it completely.

In a similar way, we may construct a certain $\langle y'\dots y^u\,|\,y^{u+1}\dots y^{u+v}\rangle$ which satisfies (29) with b replaced by some y. From the properties of $\langle y'\dots y^u\,|\,y^{u+1}\dots y^{u+v}\rangle$, it follows that this $\langle y'\dots|\,y^{u+1}\dots\rangle$ and the $\langle y'\dots|\,y^{u+1}\dots\rangle$ in the above paragraph are the same. Comparing the complex conjugate of (29) with b replaced by y and the equation (29) with a replaced by y, we see further that $\langle y'\dots|\,y^{u+1}\dots\rangle$ is Hermitian if W is Hermitian.

Though we have the unit matrix $\langle y'\dots|\,y^{u+1}\dots\rangle$ and scalar products, we do not have formulae similar to (4), (5), (6) or (18), (19), (20) since $\langle y'\dots|\,y^{u+1}\dots\rangle$ is no longer expressible by equations similar to (12) or (24). Let us define ξ_y, $\bar{\xi}_y$ by (13) or (25) as before.

For comparison, let us see what corresponds to the above in a single time picture. We assume first that the wave equations are such that

$$\sum_{v}\sum_{q^{u+1}\dots}\langle a\,|\,q^{u+1}\dots q^{u+v}t\rangle\,W\,\langle q^{u+1}\dots q^{u+v}t\,|\,b\rangle$$

is independent of time t. After defining

$$\langle a\,|\,b\rangle = \sum_{v}\sum_{q^{u+1}\dots}\langle a\,|\,q^{u+1}\dots q^{u+v}t\rangle\,W\,\langle q^{u+1}\dots q^{u+v}t\,|\,b\rangle, \tag{30}$$

a similar piece of work shows that $\langle q'\dots q^u t'\,|\,q^{u+1}\dots q^{u+v}t''\rangle$ satisfying (30), with a replaced by some q and t', and the $\langle q'\dots t'\,|\,q^{u+1}\dots t''\rangle$ satisfying (30), with b replaced by some q and t'', are the same; they both satisfy the wave equations for $\langle q'\dots t'\,|\rangle$ and $\langle|\,q^{u+1}\dots t''\rangle$; they reduce at $t' = t'' = 0$ to the $\langle q'\dots|\,q^{u+1}\dots\rangle$ satisfying (1·1) and (1·2) respectively with a weight W. If we replace a in (30) by $q'\dots q^u t$, we see that $\langle q'\dots q^u t\,|\,q^{u+1}\dots q^{u+v}t\rangle$ is equal to the $\langle q'\dots|\,q^{u+1}\dots\rangle$ satisfying (1·1) with a weight W. Thus putting

$$|\,q^{u+1}\dots t\rangle = \bar{S}_t\,|\,q^{u+1}\dots t = 0\rangle,$$

$$\langle q'\dots t\,| = \langle q'\dots t = 0\,|\,S_t,$$

with $\bar{S}_t$ denoting the Hermitian conjugate of S_t, we have $S_t\bar{S}_t = 1$.

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The above enables us to compare the variables ξ, $\bar{\xi}$ in our many t picture with the corresponding observables in a picture with a single time variable. In the latter we let $\bar{\xi}'_{q0}$, ξ'_{q0} be defined by

$$\langle t=0, q'q''\dots \,|\, \bar{\xi}'_{q0} = (u+1)^{\frac{1}{2}} \langle t=0, qq'q''\dots \,|,$$

$$\xi'_{q0} \,|\, t=0, q'q''\dots\rangle = (u+1)^{\frac{1}{2}} \,|\, t=0, qq'q''\dots\rangle,$$

and $\bar{\xi}'_{qt}$, ξ'_{qt} by

$$\bar{\xi}'_{qt} = S_t^{-1}\bar{\xi}'_{q0}S_t, \quad \xi'_{qt} = \bar{S}_t\xi'_{q0}\bar{S}_t^{-1}, \tag{31}$$

so that

$$\langle tq'q''\dots q^u \,|\, \bar{\xi}'_{qt} = \langle tq'\dots q^u \,|\, S_t^{-1}\bar{\xi}'_{q0}S_t = \langle t=0, q'\dots q^u \,|\, \bar{\xi}'_{q0}S_t$$

$$= (u+1)^{\frac{1}{2}} \langle t=0, qq'\dots q^u \,|\, S_t = (u+1)^{\frac{1}{2}} \langle tqq'q''\dots q^u \,|, \tag{32·1}$$

and similarly

$$\xi'_{qt} \,|\, tq'q''\dots q^u\rangle = (u+1)^{\frac{1}{2}} \,|\, tqq'q''\dots q^u\rangle. \tag{32·2}$$

From (32), we recognize $\bar{\xi}'_{qt}$, ξ'_{qt} as the usual Schrödinger observables. If in the defining equation for $\bar{\xi}_y$

$$\langle y'y''\dots y^u \,|\, \bar{\xi}_y = (u+1)^{\frac{1}{2}} \langle yy'y''\dots y^u \,|$$

we put $t' = t'' = \dots = t$, we get (32). Thus $\bar{\xi}_y$ is a generalization of the Schrödinger observable $\bar{\xi}'_{qt}$ and reduces to the latter when we restrict the various t', $t''\dots$ in the labelling of its bra-representatives to be equal to t. Similarly ξ_y is a generalization of ξ'_{qt}.

In the following we attempt to construct wave equations with an indefinite number of particles and with each particle taking a separate time coordinate, which so far has not been tried. In this work we find it necessary to introduce $\bar{\xi}_y$ and ξ_y in a different way from (13) and (25). They are defined now by their operations on a ket vector (or a bra vector), and since $\langle y'\dots \,|\, y^{u+1}\dots\rangle$ is not yet found, we do not know the result of their operations on a bra vector (or a ket vector). We have, in fact, half of the relations given in the preceding sections.

Let us consider a group of bosons first, leaving the discussions of fermions to the end of the section. For reasons that will be obvious later, let us define ξ_y and $\bar{\xi}_y$ by

$$\langle y'y''\dots y^u \,|\, \bar{\xi}_y = (u+1)^{\frac{1}{2}} \langle yy'y''\dots y^u \,|, \tag{33·1}$$

$$\langle y'y''\dots y^u \,|\, \xi_y = u^{-\frac{1}{2}} \sum_r f(y^r, y) \langle y'\dots y^{r-1}y^{r+1}\dots y^u \,|, \tag{33·2}$$

where $f(y_1, y_2)$ is to be determined later. From (33),

$$\bar{\xi}_{y_1}\xi_{y_2} - \xi_{y_2}\bar{\xi}_{y_1} = f(y_1, y_2), \tag{34·1}$$

$$\xi_{y_1}\xi_{y_2} - \xi_{y_2}\xi_{y_1} = \bar{\xi}_{y_1}\bar{\xi}_{y_2} - \bar{\xi}_{y_2}\bar{\xi}_{y_1} = 0. \tag{34·2}$$

Let us see what becomes of the equation of the type

$$\{\sum_{y_i} H(y, y_i)\bar{\xi}_{y_i} + \sum_{y_i y_j y_k} H(y, y_i y_j y_k)\bar{\xi}_{y_i}\bar{\xi}_{y_j}\xi_{y_k}\}|\rangle = 0. \tag{35·1}$$

$H(y, y_i)$ and $H(y, y_i y_j y_k)$ may, of course, be looked upon as operators on y_i and on $y_i y_j y_k$, but without loss of generality they may be regarded as numbers. Adding $\langle y'y''\dots y^u \,|$ to the left of the left-hand side of (35·1) and using (33), we obtain what may be called wave equations. An equation of the type

$$\sum_{y_i} H(y, y_i)\bar{\xi}_{y_i} \,|\, \rangle = 0 \tag{35·2}$$

will give with proper choice of $H(y, y_i)$ an equation $\bar{\xi}_y H_y \,|\rangle = 0$, which holds for free particles. Assuming $H(y, y_i)$ to be so chosen, one sees that the inclusion of the second

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term in the bracket in (35·1) implies the inclusion of interactions between particles. One may of course introduce into (35·1) terms such as $\bar{\xi}_{y_i}\xi_{y_j}\xi_k$ or $\bar{\xi}_{y_i}\bar{\xi}_{y_j}\bar{\xi}_{y_k}$ which represent creation or annihilation of particles.

If we add a $\langle y' \dots y^u|$ to the left of (35·1), and consider the resulting equation as determining $\langle y, y_i y_j y_k y' \dots y^u|$ as functions of $t, t_i, t_j, t_k, t', \dots t^u$, we see that we get integro-differential equations. To obtain purely differential equations we assume $H(y, y_i)$ to be a sum of terms which contain t, t_i through the factor $\delta(t-t_i)$ or its derivatives only, and $H(y, y_i y_j y_k)$ to be a sum of terms which contain t, t_i, t_j, t_k through the factor $\delta(t-t_i)\,\delta(t-t_j)\,\delta(t-t_k)$ or its derivatives to t_i, t_j, t_k only. $H(y, y_i)$ is determined by the wave equations for free particles and will satisfy this condition automatically. The condition on $H(y, y_i y_j y_k)$ will be assumed.

(35·1) gives us a set of equations for each value of y, and hence the consistency of such equations must be investigated. Put y equal to y_1 in (35·1), perform the integration to t_i, t_j, t_k, and assume for definiteness that the quantity between the bracket becomes

$$i(\partial\bar{\xi}_{y_1}/\partial t_1) + iA_{y_1}, \tag{36}$$

where A_{y_1} does not contain derivatives of ξ_y or $\bar{\xi}_y$ with respect to t. Since

$$-\frac{d\bar{\xi}_{y_1}}{dt_1}A_{y_2}|\rangle = \frac{d\bar{\xi}_{y_1}}{dt_1}\frac{d\bar{\xi}_{y_2}}{dt_2}\Big|\Big\rangle = \frac{d\bar{\xi}_{y_2}}{dt_2}\frac{d\bar{\xi}_{y_1}}{dt_1}\Big|\Big\rangle = -\frac{d\bar{\xi}_{y_2}}{dt_2}A_{y_1}|\rangle,$$

the condition of consistency is

$$\left(-\frac{d\bar{\xi}_{y_1}}{dt_1}A_{y_2} + \frac{d\bar{\xi}_{y_2}}{dt_2}A_{y_1}\right)\Big|\Big\rangle = 0, \tag{37}$$

or

$$\left\{-\frac{d\bar{\xi}_{y_1}}{dt_1}A_{y_2} + \frac{d\bar{\xi}_{y_2}}{dt_2}A_{y_1} + A_{y_2}\frac{d\bar{\xi}_{y_1}}{dt_1} - A_{y_1}\frac{d\bar{\xi}_{y_2}}{dt_2} + [A_{y_2}, A_{y_1}]\right\}\Big|\Big\rangle = 0,$$

or

$$-\left[\frac{d\bar{\xi}_{y_1}}{dt_1} + A_{y_1}, \frac{d\bar{\xi}_{y_2}}{dt_2} + A_{y_2}\right]\Big|\Big\rangle = 0, \tag{38}$$

where

$$[a, b] \equiv ab - ba.$$

Here (38) is a necessary and sufficient condition. If, in general, A_y contains also derivatives of ξ_y and $\bar{\xi}_y$ to t, (38) is a necessary condition, but not sufficient.

Let us work out the conditions on $H(y, y_i), f(y_1, y_2), H(y, y_i y_j y_k)$ for the operators (36) for different y to commute. The conditions are

$$\sum_i H(y_1, y_i) f(y_i, y_2) = 0, \tag{39}$$

$$\sum_{i_1, k_2} H(y_1, y_{i_1} y_{j_1} y_{k_1}) H(y_2, y_{i_2} y_{j_2} y_{k_2}) f(y_{i_1}, y_{k_2}) = 0, \tag{40·1}$$

$$\sum_{j_1, k_2} H(y_1, y_{i_1} y_{j_1} y_{k_1}) H(y_2, y_{i_2} y_{j_2} y_{k_2}) f(y_{j_1}, y_{k_2}) = 0. \tag{40·2}$$

We do not require these to be satisfied for all y, but only for those y coming into physical consideration, say for points lying outside each other's light cone in case the particles are also acted on by an electromagnetic field. If (39) and (40) are satisfied, (35·1) gives us a set of differential equations which may be consistent. They are believed to be sufficient to determine the wave functions with proper initial

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and boundary conditions, for they are sufficient when $H(y, y_i y_j y_k)$ is zero and we expect that this is unchanged after introducing $H(y, y_i y_j y_k)$.

If we assume (35·1) to be satisfied for all states, we have

$$\partial \bar{\xi}_y / \partial t + A_y = 0. \tag{41}$$

This enables us to compare with the unquantized theory for such particles if any, and to determine the coefficients $H(y, y_i y_j y_k)$.

For fermions, we replace (33) by

$$\langle y' y'' \dots y^u \,|\, \bar{\xi}_y = (u+1)^{\frac{1}{2}} \langle y y' y'' \dots y^u \,|, \tag{42·1}$$

$$\langle y' y'' \dots y^u \,|\, \xi_y = u^{-\frac{1}{2}} \Sigma f(y^r, y) (-1)^{r-1} \langle y' \dots y^{r-1} y^{r+1} \dots y^u \,|, \tag{42·2}$$

and then (34) is replaced by

$$\bar{\xi}_{y_1} \xi_{y_2} + \xi_{y_2} \bar{\xi}_{y_1} = f(y_1, y_2), \tag{43·1}$$

$$\xi_{y_1} \xi_{y_2} + \xi_{y_2} \xi_{y_1} = \bar{\xi}_{y_1} \bar{\xi}_{y_2} + \bar{\xi}_{y_2} \bar{\xi}_{y_1} = 0. \tag{43·2}$$

When we assume an equation of the type (35·1), where the operator can be reduced to the form (36), the consistency condition is that (36) for different y anticommute, and this is satisfied if (39) and (40) are satisfied.

The above scheme provides us equations to determine $\langle y' y'' \dots y^u \,|\rangle$ of a state, but it leaves open the determination of the scalar products of states and thus physical interpretations.

4. A SPECIAL CASE

Let us consider an idealistic case in which bosons are capable of creating new bosons of the same kind and annihilating some of themselves. Our purpose is to see if expressions for scalar products can be found.

In place of (35·1), we have

$$\{\sum_i H(y, y_i) \bar{\xi}_{y_i} + \sum_{ij} H^{(1)}(y, y_i y_j) \bar{\xi}_{y_i} \bar{\xi}_{y_j} + \sum_{ij} H^{(2)}(y, y_i y_j) \xi_{y_i} \bar{\xi}_{y_j}\} |\rangle = 0, \tag{44}$$

where without loss of generality, we take H, $H^{(1)}$, $H^{(2)}$ as numbers with $H^{(1)}(y, y_i y_j)$ symmetrical between y_i and y_j. If we assume that the operator in (44) is of the form (36), the condition of consistency is that for points y in physical consideration

$$\sum_i H(y_1, y_i) f(y_i, y_2) = 0, \tag{45}$$

$$\left.\begin{aligned} \sum_{i_1 i_2} H^{(1)}(y_1, y_{i_1} y_{j_1}) H^{(2)}(y_2, y_{i_2} y_{j_2}) f(y_{i_1}, y_{i_2}) = 0, \\ \sum_{j_1, i_2} H^{(2)}(y_1, y_{i_1} y_{j_1}) H^{(2)}(y_2, y_{i_2} y_{j_2}) f(y_{j_1}, y_{i_2}) = 0. \end{aligned}\right\} \tag{46}$$

To proceed further, we must assume a model, and shall consider the simplest, i.e. scalar particles without a mass.

The treatment of such particles using one time coordinate is easy and is as follows. In order to obtain wave equations linear in $\partial/\partial t$, we let the wave functions consist of two components $\langle \mathbf{x}' \mathbf{x}'' \dots \sigma t \,|\rangle$, where σ is the index for the component and takes the values 1 and 2 only. Further we let G, $G^{(1)}$, $G^{(2)}$ be matrices

$$\frac{1}{4\pi}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \tag{47}$$

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operating on the σ coordinate. With

$$(\mathbf{x}'\mathbf{x}''\dots\sigma t\,|\,i(\partial/\partial t)\,|\rangle \quad \text{understood as} \quad i(\partial/\partial t)\,\langle \mathbf{x}'\mathbf{x}''\dots\sigma t\,|\rangle$$

and

$$\langle \mathbf{x}'\mathbf{x}''\dots\sigma\mathbf{x}\,|\,G^{(1)}\,|\rangle \quad \text{as} \quad \sum_{\sigma'}\langle \mathbf{x}'\mathbf{x}''\dots\sigma' t\,|\rangle\, G^{(1)}_{\sigma\sigma'}, \text{etc.},$$

we may write the wave equations without creations and annihilations as

$$\{i(\partial/\partial t)+G^{(1)}\int d\mathbf{x}\,\xi_{\mathbf{x}}\nabla^2\bar{\xi}_{\mathbf{x}}-G^{(2)}\}\,|\rangle = 0 \tag{48}$$

and the scalar product as

$$\sum_u\int\dots\int\langle|\,\mathbf{x}'\mathbf{x}''\dots\mathbf{x}^u\sigma t\rangle\, G\,\langle\mathbf{x}'\mathbf{x}''\dots\mathbf{x}^u\sigma t\,|\rangle\, d\mathbf{x}'\dots d\mathbf{x}^u, \tag{49}$$

where the matrix multiplications between $\langle|\,\mathbf{x}'\dots\sigma t\rangle$, G and $\langle\mathbf{x}'\dots\sigma t\,|\rangle$ are carried out. It is easy to verify from (48) that all $\langle\mathbf{x}'\mathbf{x}''\dots\sigma t\,|\rangle$ satisfy

$$\{(\nabla'^2+\nabla''^2+\dots)-(\partial^2/\partial t^2)\}\,\langle\mathbf{x}'\mathbf{x}''\dots\sigma t\,|\rangle = 0, \tag{50}$$

and that (49) does not change with t. Introducing creations and annihilations of particles, we replace (48) by

$$(i\partial/\partial t - H^0 - H')\,|\rangle = 0, \tag{51}$$

with

$$H^0 = -G^{(1)}\int d\mathbf{x}\,\xi_{\mathbf{x}}\nabla^2\bar{\xi}_{\mathbf{x}}+G^{(2)},$$

and

$$H' = \iiint\{H^{(1)}(\mathbf{x}_i\mathbf{x}_j\mathbf{x}_k)\,\xi_{\mathbf{x}_i}\xi_{\mathbf{x}_j}\bar{\xi}_{\mathbf{x}_k}+H^{(2)}(\mathbf{x}_i\mathbf{x}_j\mathbf{x}_k)\,\xi_{\mathbf{x}_i}\bar{\xi}_{\mathbf{x}_j}\bar{\xi}_{\mathbf{x}_k}\}\,d\mathbf{x}_i\,d\mathbf{x}_j\,d\mathbf{x}_k, \tag{52}$$

say. $H^{(1)}$ and $H^{(2)}$ are here two 2-2 matrixes operating on σ whose elements will be, as before, looked upon as pure numbers. If we retain (49) as the scalar product, i.e. we require (49) not to change with t, we find

$$\bar{H}'G = GH', \tag{53}$$

$\bar{H}'$ being the Hermitian conjugate of H'. This can be satisfied by letting

$$\bar{H}^{(2)}(\mathbf{x}_k\mathbf{x}_j\mathbf{x}_i)\,G = GH^{(1)}(\mathbf{x}_i\mathbf{x}_j\mathbf{x}_k), \tag{54}$$

$\bar{H}^{(2)}$ being the Hermitian conjugate of $H^{(2)}$.

When we let each of the particles take a time coordinate, we let each of them take a separate σ coordinate so that y now consists of $(\mathbf{x}, t, \sigma)$ or q consists of $(\mathbf{x}, \sigma)$. This spoils the apparent symmetry between $\mathbf{x}$ and t, but it seems unavoidable if we want equations linear in $(\partial/\partial t)$, different t coordinates and a symmetry between them. $H(y, y_i)$ in (44) considered as a matrix becomes

$$i\delta(\mathbf{x}-\mathbf{x}_i)\,(\partial/\partial t)\,\delta(t-t_i)+\delta(t-t_i)\,[G^{(1)}\nabla^2\delta(\mathbf{x}-\mathbf{x}_i)-G^{(2)}\delta(\mathbf{x}-\mathbf{x}_i)]. \tag{55}$$

Thus a solution of (45) for $f(\sigma_1 x_1, \sigma_2 x_2)$ is

$$\left.\begin{aligned} f(1x_1, 1x_2) &= -\tfrac{1}{2}i\left\{1-\frac{\partial^2}{\partial t_1^2}+2i\frac{\partial}{\partial t_1}\right\}\Delta(x_1-x_2),\\ f(2x_1, 2x_2) &= -\tfrac{1}{2}i\left\{1-\frac{\partial^2}{\partial t_1^2}-2i\frac{\partial}{\partial t_1}\right\}\Delta(x_1-x_2),\\ f(1x_1, 2x_2) &= f(2x_1, 1x_2) = -\tfrac{1}{2}i\left\{1+\frac{\partial^2}{\partial t_1^2}\right\}\Delta(x_1-x_2). \end{aligned}\right\} \tag{56}$$

The determination of $H^{(1)}$ and $H^{(2)}$ from (46) with this knowledge of $f(\sigma_1 x_1, \sigma_2 x_2)$ will not be pursued here.

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To search for the scalar product of two states a and b, we note first that we cannot assume (11·2) to hold, with $\langle a \mid x'x'' \ldots\rangle$ defined as the complex conjugate of $\langle x'x'' \ldots \mid a\rangle$, for the right-hand side of (11·2) will depend on the values of $x_0'x_0''\ldots$ in the integrand. Let us see if we may assume that the probability of having u particles at $x', x'', \ldots$ in a state a is something like the integrand of (11·2), say

$$\langle a \mid y'y'' \ldots y^u\rangle \prod_{r=1}^{u} G^r \langle y'y'' \ldots y^u \mid a\rangle, \tag{57}$$

where G^r is the G matrix operating on the σ coordinate of the rth particle and the matrix multiplications between $\langle a \mid y' \ldots\rangle$, G and $\langle y' \ldots \mid a\rangle$ are all carried out. This is all we want for a physical interpretation. In order that (57) may represent this probability, it is necessary that the summation $P(t)$ of this over all $\mathbf{x}$ and u with all different t replaced by a common t must be conserved. We see on general grounds that this cannot be so unless $H^{(1)}$ and $H^{(2)}$ in (44) are sums of terms which contain the factor $\delta(t-t_i)\,\delta(t-t_j)$ or its derivatives with respect to t_i, t_j, which we have assumed before. Let us therefore put

$$\left.\begin{aligned} H^{(1)}(y, y_i y_j) &= 4\pi(-1)^{\sigma-1} H^{\mathrm{I}}(q\, q_i q_j)\,\delta(t-t_i)\,\delta(t-t_j),\\ H^{(2)}(y, y_i y_j) &= 4\pi(-1)^{\sigma-1} H^{\mathrm{II}}(q\, q_i q_j)\,\delta(t-t_i)\,\delta(t-t_j),\end{aligned}\right\} \tag{58}$$

and calculate $dP(t)/dt$. The reason for introducing a factor $4\pi(-1)^{\sigma-1}$ will be seen later.

To calculate $\{(\partial/\partial t^r)\,\langle y'y'' \ldots y^u \mid\rangle\}_{t'=t''=\ldots=t}$, we put $y = y^r$ in (44), add in the symbol $\langle y' \ldots y^{r-1} y^{r+1} \ldots y^u \mid$ and put all t equal. We may then replace ξ_y and $\bar{\xi}_y$ by the non-relativistic operators ξ_q and $\bar{\xi}_q$ described in § 1. On account of (33), (56), they satisfy

$$\left.\begin{aligned} \langle q'q'' \ldots q^u t \mid \bar{\xi}_q &= (u+1)^{\frac{1}{2}} \langle q\, q'q'' \ldots q^u t \mid,\\ \langle q'q'' \ldots q^u t \mid \xi_q &= u^{-\frac{1}{2}} \Sigma \langle q^r \mid q\rangle \langle q' \ldots q^{r-1} q^{r+1} \ldots q^u t \mid,\end{aligned}\right\} \tag{59}$$

where

$$\langle q_1 \mid q_2\rangle = (G^{-1})_{\sigma_1\sigma_2}\,\delta(\mathbf{x}_1 - \mathbf{x}_2). \tag{60}$$

If we denote by $\langle q' \ldots q^u t \mid q^{u+1} \ldots q^{u+v} t\rangle$ the right-hand side of (3) with $\delta_{q_1 q_2}$ replaced by this $\langle q_1 \mid q_2\rangle$, equations (1) and formulae (4)–(6) are valid (except (5·1)) with $\delta_{q_1 q_2}$ replaced by this $\langle q_1 \mid q_2\rangle$ and with a t appearing in all labels of the representatives $\langle q' \ldots \mid\rangle$, $\langle \mid q^{u+1} \ldots\rangle$, $\langle q' \ldots \mid \alpha \mid q^{u+1} \ldots\rangle$. Evidently we may drop out the t's in $\langle q' \ldots t \mid q^{u+1} \ldots t\rangle$ and $\langle q' \ldots t \mid \alpha \mid q^{u+1} \ldots t\rangle$. It is equally evident that $\langle q' \ldots \mid q^{u+1} \ldots\rangle$ will play the role of the unit matrix only when in taking the scalar product a weight $W_{q'q''} = G_{\sigma'\sigma''}$ or its generalization has been inserted. In the equation for

$$\{(\partial/\partial t^r)\,\langle y'y'' \ldots y^u \mid\rangle\}_{t'=t''=\ldots=t}$$

let us insert before the operators $\bar{\xi}_{q_i}\bar{\xi}_{q_j}$ and $\xi_{q_i}\bar{\xi}_{q_j}$ a factor $(-1)^{\sigma^r-1}\bar{\xi}_{q^r} n_{q^r}^{-1} \xi_{q^r} (4\pi)^{-1}$, since this factor is always unity. In this way we find

$$\begin{aligned} i\frac{\partial}{\partial t}\{\langle y' \ldots y^u \mid\rangle_{t'=t''=\ldots=t}\} = \left\{i\Sigma \frac{\partial}{\partial t^r}\langle y' \ldots y^u \mid\rangle\right\}_{t'=t''=\ldots=t} &= \sum_r [G^{(2)r} - G^{(1)r}(\nabla^2)^r]\,\langle q' \ldots q^u t \mid\rangle\\ - \sum_{ijk} \langle q' \ldots q^u t \mid H^{\mathrm{I}}(q_k q_i q_j)\,\xi_{q_k}\bar{\xi}_{q_i}\bar{\xi}_{q_j} \mid\rangle &- \sum_{ijk} \langle q' \ldots q^u t \mid H^{\mathrm{II}}(q_k q_i q_j)\,\xi_{q_k}\xi_{q_i}\bar{\xi}_{q_j} \mid\rangle. \end{aligned} \tag{61}$$

Thus $i\,dP(t)/dt$ is equal to

$$\begin{aligned} -\sum_{u,v}\int\ldots\int\left(\prod_{r=1}^{u} d\mathbf{x}^r\right)\langle \mid tq' \ldots q^u\rangle \prod_{r=1}^{u} G^r \langle q' \ldots q^u \mid \sum_{ijk}\{H^{\mathrm{I}}(q_k q_i q_j)\\ - H^{\mathrm{II}*}(q_j q_i q_k)\}\,\xi_{q_k}\bar{\xi}_{q_i}\bar{\xi}_{q_j} \mid q^{u+1} \ldots q^{u+v}\rangle \prod_{s=u+1}^{u+v} G^s \langle q^{u+1} \ldots q^{u+v} t \mid\rangle \prod_{s=u+1}^{u+v} d\mathbf{x}^s \end{aligned} \tag{62}$$

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plus a similar term, where c^* denotes the complex conjugate of a number c. Hence $P(t)$ is conserved if

$$H^{\mathrm{I}}(q_k q_i q_j) = H^{\mathrm{II}*}(q_j q_i q_k). \tag{63}$$

The above shows that under certain conditions (57) can still be used as a kind of probability and that the summation of (57) over all $\mathbf{x}$ and u with the second a replaced by b and all t by a common t can be considered as $\langle a \mid b \rangle$.

If $H^{(1)}$, $H^{(2)}$ are sums of terms which contain $\delta(t-t_i)\,\delta(t-t_j)$ or its derivatives as a factor, the symmetry between $\mathbf{x}$ and t narrows down our choice of $H^{(1)}$ and $H^{(2)}$ to very few alternatives. In that respect the theory is unsatisfactory. Though the theory has no apparent symmetry between $\mathbf{x}$ and t, it is entirely relativistic when $H^{(1)} = H^{(2)} = 0$, and we believe that it can remain so with a proper choice of $H^{(1)}$ and $H^{(2)}$.

A similar piece of work can be done for fermions, say electrons. There the symmetry between $\mathbf{x}$ and t can be more easily kept in sight. If we assume $H^{(1)}$, $H^{(2)}$ in (44) to be given by (58) without the factor $4\pi(-1)^{\sigma-1}$, it is easy to find the relation between H^{I}, H^{II} and the corresponding $H^{(1)}$, $H^{(2)}$ occurring in an equation of the type (51), i.e. the wave equation with a single time coordinate taking account of annihilations and creations. While the theory with a single time coordinate can never be relativistic, the present one can be made relativistic with a proper choice of $H^{(1)}$ and $H^{(2)}$. Here again, if $H^{(1)}$, $H^{(2)}$ are sums of terms which contain t, t_i, t_j through the factor $\delta(t-t_i)\,\delta(t-t_j)$ or its derivatives with respect to t_i, t_j, they must also be sums of terms which contain $\mathbf{x}$, $\mathbf{x}_i$, $\mathbf{x}_j$ through the factor $\delta(\mathbf{x}-\mathbf{x}_i)\,\delta(\mathbf{x}-\mathbf{x}_j)$ or its derivatives with respect to some component of $\mathbf{x}_i$ or $\mathbf{x}_j$. This is one condition on $H^{(1)}$, $H^{(2)}$ and other conditions remain to be investigated. This condition means physically that creations from and annihilations at a particle occur at the particle. It is obvious that particles may interact through such creations and annihilations.

Summary

The relativistic second quantization of free bosons is extended to fermions. To know how relativistic creation and annihilation operators operate on bra and ket vectors, it is necessary to have a relativistic scalar product which in the case of free particles can be constructed. For particles interacting with each other or capable of emitting and absorbing particles of the same kind, it is pointed out that adequate wave equations for an indefinite number of particles each taking a separate time coordinate have not yet been found and so no scalar product can be found. Thus creation and annihilation operators can be defined only by their operation on a bra vector or a ket vector but not by both. With the help of these, the relativistic wave equations referred to above are proposed and their consistency conditions studied. For the particular case of scalar particles, an illustration is given that such a theory may admit certain expressions as probabilities.

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