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Rigidity for group actions on homogeneous spaces by affine transformations

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Abstract. We give a criterion for the rigidity of the action of a group of affine transformations of a homogeneous space of a real Lie group. Let G be a real Lie group, Λ a lattice in G, and Γ a subgroup of the affine group $\mathrm{Aff}(G)$ stabilizing Λ. Then the action of Γ on G/Λ has the rigidity property in the sense of Popa [On a class of type II_1 factors with Betti numbers invariants. *Ann. of Math. (2)* **163**(3) (2006), 809–899] if and only if the induced action of Γ on $\mathbb{P}(\mathfrak{g})$ admits no Γ-invariant probability measure, where $\mathfrak{g}$ is the Lie algebra of G. This generalizes results of Burger [Kazhdan constants for SL(3, **Z**). *J. Reine Angew. Math.* **413** (1991), 36–67] and Ioana and Shalom [Rigidity for equivalence relations on homogeneous spaces. *Groups Geom. Dyn.* **7**(2) (2013), 403–417]. As an application, we establish rigidity for the action of a class of groups acting by automorphisms on nilmanifolds associated to two-step nilpotent Lie groups.

1. *Introduction*

Property (T) for locally compact groups was introduced by Kazhdan in 1967 [**Kaz67**], and he used it to prove that certain lattices are finitely generated. It was first defined in terms of unitary representations (see [**Kaz67**, **BdlHV08**]). Property (T) has since become a basic notion in various areas of mathematics such as group theory, ergodic theory, and operator algebras.

The first examples of groups with property (T) are the special linear groups $\mathrm{SL}_n(\mathbb{R})$, for $n \geq 3$. Lattices in $\mathrm{SL}_n(\mathbb{R})$ are examples of discrete groups with property (T), since this property is inherited by lattices.

As for the relative property (T), it was first coined by Margulis for pairs of groups (G, H), where G is a locally compact group and H a subgroup of G [**Mar82**]. Yet, the relative property (T) of $(\mathbb{R}^2 \rtimes \mathrm{SL}_2(\mathbb{R}), \mathbb{R}^2)$ was implicitly used in Kazhdan's original paper.

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Burger [**Bur91**] studied the relative property (T) for pairs of groups of the form $(A \rtimes \Gamma, A)$, where A is a discrete abelian group and Γ is a subgroup of $\mathrm{Aut}(A)$. He gave a sufficient condition on the dual action of Γ on $\hat{A}$ to ensure that $(A \rtimes \Gamma, A)$ has the relative property (T). In the case where $A = \mathbb{Z}^n$ and Γ is a subgroup of $\mathrm{SL}_n(\mathbb{Z})$, it gives rise to an action on the n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. His result can be stated as follows.

THEOREM 1.1. [**Bur91**] *The pair of groups* $(\mathbb{Z}^n \rtimes \Gamma, \mathbb{Z}^n)$ *has relative property (T) if and only if there is no* Γ*-invariant probability measure on the projective space* $\mathbb{P}(\mathbb{R}^n)$.

Actually, the fact that the condition above is necessary was noticed in [**dC06**] (Proposition 3.1.9, which is primarily based on the so-called Furstenberg lemma [**Fur76**]).

In the spirit of relative property (T) for pairs of groups, Popa [**Pop06**] introduced the notion of relative property (T) for inclusions $B \subset N$ of von Neumann algebras (also called rigidity of inclusion).

Given a free ergodic action of a countable group Γ on a non-atomic standard probability space (X, μ) by measure preserving transformations, there is an associated canonical von Neumann algebra: the group measure space construction $L^\infty(X) \rtimes \Gamma$ [**MVN36**]. Following [**Gab08**], we say that the action $\Gamma \curvearrowright (X, \mu)$ has *the property (T) relative to the space* if the inclusion $L^\infty(X) \subset L^\infty(X) \rtimes \Gamma$ has the relative property (T). This notion first appeared in [**Pop06**], where such actions are called rigid. For an ergodic theoretical equivalent formulation of this property, see Proposition 2.3 below.

In the case of group actions by automorphisms on abelian groups, this notion is linked to the Kazhdan–Margulis relative property (T) as follows.

PROPOSITION 1.1. [**Pop06**, Proposition 5.1] *Let A be a discrete abelian group and let Γ be a subgroup of Aut(A). Let μ be the Haar measure on $\hat{A}$, the dual space of A. Then the following assertions are equivalent:*

(1) *the action $\Gamma \curvearrowright (\hat{A}, \mu)$ has the property (T) relative to the space;*
(2) *the pair of groups $(A \rtimes \Gamma, A)$ has the relative property (T).*

The previous proposition, together with Theorem 1.1, shows that the action $\Gamma \curvearrowright (\mathbb{T}^n, \mu)$ of a subgroup of $\mathrm{SL}_n(\mathbb{Z})$ has the property (T) relative to the space if, and only if, there is no Γ-invariant probability measure on $\mathbb{P}(\mathbb{R}^n)$.

Popa [**Pop06**, Problem 5.10.2] raised the following natural question: characterize the countable discrete groups for which there exists free ergodic probability measure preserving actions on (X, μ) that have the property (T) relative to the space. This question is still open. Yet, some results were obtained. It has been shown in [**Gab08**, Theorem 1.3] that every non-amenable free product of groups admits such actions. Besides, it was proved in [**Ioa11**, Theorem 4.3] that every non-amenable group admits free ergodic measure preserving actions with a weak form of the property (T) relative to the space.

The notion of property (T) relative to the space of group actions played a fundamental role in Popa's work on fundamental groups of von Neumann II_1 factors [**Pop06**]. It was recently used as a key ingredient to produce actions of $\mathbb{F}_\infty$ (the free group on infinitely many generators) whose II_1 factors and orbit equivalence relations have prescribed fundamental group [**PV10**].

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Recall that probability measure preserving actions of two countable groups Γ and Λ on two standard probability spaces (X, μ) and (Y, ν) are orbit equivalent if they define isomorphic partitions of the spaces into orbits. It is a standard result that free ergodic actions of infinite amenable groups are all orbit equivalent [**OW80**]. As for non-amenable groups, it has been proven, using the property (T) relative to the space of actions of $\mathbb{F}_2$ on $\mathbb{T}^2$, that every countable non-amenable group has uncountably many non-orbit equivalent actions [**Hjo05**, **GP05**, **Ioa11**, **Eps08**]. Moreover, new examples of standard equivalence relations with trivial outer automorphism groups were given thanks to this notion of property (T) relative to the space (see [**PV10**, **Gab08**]).

Ioana and Shalom studied in [**IS13**] the property (T) relative to the space for actions given by a group of translations on homogeneous spaces of algebraic groups. They gave mild conditions under which this action has the property (T) relative to the space. Here is one of their results.

THEOREM 1.2. [**IS13**, Theorem D] *Let G be a real algebraic group and Λ a lattice in G. Let $\Gamma \subset G$ be a countable subgroup and denote by H its Zariski closure. Assume that H has no proper normal cocompact algebraic subgroup and no non-trivial homomorphism into $\mathbb{R}^*$. Let η be a probability measure on G/Λ which is invariant under translations by elements from Γ.*

If the centralizer of Γ (equivalently, of H) in G is finite, then the action of Γ by translations on $(G/\Lambda, \eta)$ has the property (T) relative to the space.

In the case $\eta = m_{G/\Lambda}$, the converse is true: if the action $\Gamma \curvearrowright (G/\Lambda, \eta)$ has the property (T) relative to the space, then the centralizer of Γ in G is finite.

In our main result below, we are able to generalize these results in three ways:

- G is an arbitrary real Lie group (and Λ is a lattice in G);
- Γ is an arbitrary countable group of affine transformations of G/Λ (see below for this notion);
- we give a simple necessary and sufficient condition for the property (T) relative to the space of the action $\Gamma \curvearrowright (G/\Lambda, \mu)$.

Moreover, our result shows that Burger's neat characterization of rigidity of group actions by automorphisms on tori extends to this general setting.

To state our result, we need to introduce some notation. Let G be a locally compact group and Λ a lattice in G. Let $\mathrm{Aut}(G)$ be the group of continuous automorphisms of G and $\mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G)$ the group of affine transformations of G. We denote by $\mathrm{Aut}_\Lambda(G)$ the subgroup of $\mathrm{Aut}(G)$ of all $\sigma \in \mathrm{Aut}(G)$ with $\sigma(\Lambda) = \Lambda$. The group of affine transformations of G/Λ,

$$\mathrm{Aff}_\Lambda(G) = G \rtimes \mathrm{Aut}_\Lambda(G),$$

acts in a natural way on G/Λ. Let μ be the G-invariant probability measure on G/Λ. Then μ is fixed by every $\gamma \in \mathrm{Aff}_\Lambda(G)$.

Assume now that G is a real Lie group with Lie algebra $\mathfrak{g}$. The group $\mathrm{Aff}(G)$ acts linearly on $\mathfrak{g}$. This gives rise to an induced action of $\mathrm{Aff}(G)$ on the projective space $\mathbb{P}(\mathfrak{g})$. Here is our main result.

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THEOREM 1.3. *Let G be a real Lie group with Lie algebra $\mathfrak{g}$ and let Λ be a lattice in G. Let G/Λ be the corresponding homogeneous space. Let μ be the G-invariant probability measure on G/Λ. Let Γ be a countable subgroup of $\mathrm{Aff}_\Lambda(G)$. Then the following properties are equivalent:*

(i) *$\Gamma \curvearrowright (G/\Lambda, \mu)$ has the property (T) relative to the space;*
(ii) *there is no Γ-invariant measure probability on $\mathbb{P}(\mathfrak{g})$;*
(iii) *the pair of groups $(\mathfrak{g}^* \rtimes \Gamma, \mathfrak{g}^*)$ has the relative property (T).*

We should mention that our approach is largely based on techniques developed in [**IS13**]. Up until recently, the examples of actions of groups which have the property (T) relative to the space were built using the relative property (T) of a pair $(A \rtimes \Gamma, A)$ (see [**Val05**, **Fer06**, **IS13**]). We give new examples of actions with the property (T) relative to the space which are not obtained from a pair of groups with relative property (T). For instance, let N be a simply connected and connected two-step free nilpotent Lie group. Let Λ be a lattice of N. Assume that the dimension of N is strictly bigger than 3. Then (see Example 4.5) the action of $\mathrm{Aut}_\Lambda(N)$ on N/Λ has the property (T) relative to the space.

The example given above is part of a family of nilmanifolds associated to finite graphs which were studied in [**DM05**]. We use Theorem 1.3 to give a fairly simple criterion for group actions by automorphisms on these nilmanifolds to have the property (T) relative to the space.

Let (S, E) be a finite non-oriented graph. Let N be the associated two-step nilpotent Lie group and $N/N_\mathbb{Z}$ the corresponding standard nilmanifold. Let V be a a complementary subspace to $[N, N]$ in N. The group $\mathrm{Aut}(N)$ has a decomposition as a semi-direct product $T \rtimes G$, where T (the group of the so-called shear automorphisms of N) is isomorphic to some $\mathbb{R}^n$ and G is a certain algebraic subgroup of GL(V) (see [**DM05**] for details). Let L be the semi-simple Levi component of G. Every element $n \in N$ can be written $n = (x, y)$, where $x \in V$ and $y \in [N, N]$. Then $(t, u) \in T \rtimes L$ acts on N as follows:

$$(t, u).(x, y) = (u(x), P(u)(y) + t(u(x))),$$

where $P(u) \in \mathrm{GL}([N, N])$ is a certain matrix whose coefficients are polynomials in those of u.

In this setting, we have the following result.

THEOREM 1.4. *Let Γ be a countable subgroup of $T \rtimes L \subset \mathrm{Aut}(N)$ stabilizing $N_\mathbb{Z}$. Assume that Γ is Zariski dense in $T \rtimes L$. The following assertions are equivalent:*

(i) *$\Gamma \curvearrowright (N/N_\mathbb{Z}, \mu)$ has the property (T) relative to the space, where μ is the N-invariant probability measure on $N/N_\mathbb{Z}$;*
(ii) *the linear action of Γ on $[\mathcal{N}, \mathcal{N}]$ has no non-zero fixed vector, where $\mathcal{N}$ denotes the Lie algebra of N.*

From this result, we deduce the following surprising fact. Let $H_\mathbb{R}$ be the Heisenberg group and $H_\mathbb{Z}$ the standard lattice. Despite the similarities with the action of the torus, the action $\mathrm{Aut}_{H_\mathbb{Z}}(H_\mathbb{R}) \curvearrowright H_\mathbb{R}/H_\mathbb{Z}$ does not have the property (T) relative to the space (see Corollary 4.1). In particular, any subequivalence relation of the equivalence relation induced by the action of $\mathrm{Aut}_{H_\mathbb{Z}}(H_\mathbb{R})$ on $H_\mathbb{R}/H_\mathbb{Z}$ does not have the property (T) relative to the space.

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The paper is organized as follows. Section 2 contains a few preliminary results. The proof of Theorem 1.3 is given in §3. In §4, we discuss the nilmanifolds associated to finite graphs and prove Theorem 1.4.

2. *Some preliminary lemmas*

The following lemma, which appears in [**IS13**], will be useful in what follows.

LEMMA 2.1. [**IS13**] *Let (X, μ) a standard probability space. Let $c > 0$ and η_n a sequence of probability measures on $X \times X$ with the following properties:*

- $p^i_*(\eta_n) \leq c\mu$, *where $p^i : X \times X \to X$ is the projection onto the i*th *coordinate $(i = 1, 2)$;*
- $\eta_n(\Delta_X) = 0$, *for all $n \in \mathbb{N}$;*
- $\eta_n(A \times (X \backslash A)) \underset{n\to\infty}{\longrightarrow} 0$, *for every Borel subset A of X.*

Then, for every Borel partition $(A_i)_{i=1}^{\infty}$ of X, we have

$$\eta_n\left(\bigcup_{i=1}^{\infty}(A_i \times A_i)\right) \underset{n\to\infty}{\longrightarrow} 1.$$

Our second lemma is an extension of Lemma F from [**IS13**]. The setting for this result and for the rest of this section is as follows. Let G be a real Lie group, $\mathfrak{g}$ be the Lie algebra of G and denote by $p : \mathfrak{g}\backslash\{0\} \to \mathbb{P}(\mathfrak{g})$ the canonical projection onto the projective space $\mathbb{P}(\mathfrak{g})$ of $\mathfrak{g}$.

Let $q : G \to \mathfrak{g}$ be any Borel map which equals the logarithm in some neighborhood U of $e \in G$. Define $r : (G \times G)\backslash\Delta \to G$, $(x, y) \mapsto xy^{-1}$.

As in [**IS13**], we consider the map

$$\rho : (G \times G)\backslash\Delta \to \mathbb{P}(\mathfrak{g}), \quad \rho(x, y) = p(q(r(x, y))).$$

We will denote by $\mathrm{Ad} : G \to GL(\mathfrak{g})$ the adjoint representation of G on $\mathfrak{g}$ as well as the associated action $G \to \mathrm{Aut}(\mathbb{P}(\mathfrak{g}))$ of G on $\mathbb{P}(\mathfrak{g})$.

Recall that the affine group $\mathrm{Aff}(G)$ is the semi-direct product $G \rtimes \mathrm{Aut}(G)$. The associated action of $\mathrm{Aff}(G)$ on $\mathbb{P}(\mathfrak{g})$ is given for all $\gamma = (g, \sigma) \in \mathrm{Aff}(G)$ and for all $Y \in \mathbb{P}(\mathfrak{g})$ by

$$\gamma.\bar{Y} = \mathrm{Ad}(g) \circ d\sigma_e(\bar{Y}),$$

where $d\sigma_e$ is the derivative of σ at the identity.

Now, we adapt Lemma F [**IS13**] to our setting.

LEMMA 2.2. *Let X be a Borel subset of G and let μ be a probability measure on X. Let $c > 0$ and let η_n be a sequence of probability measures on $X \times X$ with the following properties:*

- $p^1_*(\eta_n) \leq c\mu$;
- $\eta_n(\Delta_X) = 0$, *for all $n \geq 1$;*
- $\eta_n(A \times (X\backslash A)) \underset{n\to\infty}{\longrightarrow} 0$, *for every Borel subset A of X.*

Let Λ be a countable subgroup of G, $\varphi : X \to \Lambda$ a Borel map, and $\gamma = (g, \sigma) \in \mathrm{Aff}(G)$ with $g \in G$ and $\sigma \in \mathrm{Aut}(G)$.

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Set

$$D_\gamma = \{(x, y) \in X \times X \backslash \Delta_X \mid \rho(\gamma(x)\varphi(x), \gamma(y)\varphi(y)) = \gamma.(\rho(x, y))\}.$$

Then

$$\lim_{n\to\infty} \eta_n(D_\gamma) = 1.$$

Proof. By definition of q and of the adjoint representation Ad of G, we can find a neighborhood V of the group unit e in G such that

$$q(g\sigma(x)g^{-1}) = \mathrm{Ad}(g) \circ d\sigma_e(q(x)) \quad \text{for all } x \in V.$$

Set $A = \{(x, y) \in X \times X \mid xy^{-1} \in V\}$.

Choose a neighborhood W of e in G such that $WW^{-1} \subset V$. Since G is separable, there exists a sequence $(h_i)_{i\in\mathbb{N}}$ of elements of G such that

$$G = \bigcup_{i=1}^{\infty} Wh_i.$$

For every $i \geq 1$, let

$$A_i = \left(Wh_i \setminus \left(\bigcup_{j=1}^{i-1} Wh_j\right)\right) \cap X.$$

Then $(A_i)_{i=1}^{\infty}$ is a Borel partition of X and, since $\bigcup_{i=1}^{\infty}(A_i \times A_i) \subset A$, the previous lemma implies that $\eta_n(A) \underset{n\to\infty}{\longrightarrow} 1$.

Let $B = \{(x, y) \in X \times X \mid \varphi(x) = \varphi(y)\}$.

Since we can write X as a countable partition

$$X = \bigcup_{\lambda\in\Lambda} \{x \mid \varphi(x) = \lambda\},$$

we have again by the previous lemma that

$$\eta_n(B) \underset{n\to\infty}{\longrightarrow} 1.$$

We claim that $A \cap B \subset D_\gamma \cup \Delta_X$.

Indeed, let $(x, y) \in A \cap B$ with $x \neq y$. Since $(x, y) \in B$, we have by definition of ρ

$$\begin{aligned}\rho(\gamma(x)\varphi(x), \gamma(y)\varphi(y)) &= \rho(\gamma(x)\varphi(x), \gamma(y)\varphi(x)) \\ &= p(q(\gamma(x)\varphi(x)\varphi(x)^{-1}\gamma(y)^{-1})) \\ &= p(q(\gamma(xy^{-1}))) \\ &= p(q(g\sigma(xy^{-1})g^{-1})).\end{aligned}$$

Since $(x, y) \in A$, we have $xy^{-1} \in V$ and it follows that

$$\begin{aligned}\rho(\gamma(x)\varphi(x), \gamma(y)\varphi(y)) &= p(\mathrm{Ad}(g)\, d\sigma_e(q((xy^{-1})))) \\ &= \mathrm{Ad}(g) \circ d\sigma_e(\rho(x, y)) \\ &= \gamma.\rho(x, y).\end{aligned}$$

Since $\eta_n(A \cap B) \underset{n\to\infty}{\longrightarrow} 1$ and since $\eta_n(\Delta_X) = 0$, we obtain that $\eta_n(D_\gamma) \underset{n\to\infty}{\longrightarrow} 1$. This finishes the proof of the lemma. □

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We are now almost ready to prove the main result. We recall that Ioana (see [**Ioa10**, Theorem 4.4] and [**IS13**, Proposition 1]) gave the following purely ergodic theoretical characterization of the property (T) relative to the space. It appears to be, in our case, much more useful than the original characterization given in terms of a pair of von Neumann algebras.

PROPOSITION 2.3. [**IS13**] *A measure preserving action $\Gamma \curvearrowright (X, \mu)$ of a countable group Γ on a probability space (X, μ) has the property (T) relative to the space if and only if, for any sequence of Borel probability measures ν_n on $X \times X$ satisfying:*

(1) $p^i_* \nu_n = \mu$, *for all n and $i = 1, 2$, where $p^i : X \times X \to X$ denotes the projection onto the ith coordinate;*

(2) $\int_{X\times X} \varphi(x)\psi(y)\, d\nu_n(x, y) \underset{n\to\infty}{\longrightarrow} \int_X \varphi(x)\psi(x)\, d\mu(x)$, *for all bounded Borel functions φ, ψ on X;*

(3) $\|(\gamma \times \gamma)_* \nu_n - \nu_n\| \underset{n\to\infty}{\longrightarrow} 0$, *for every $\gamma \in \Gamma$,*

we have that $\nu_n(\Delta_X) \underset{n\to\infty}{\longrightarrow} 1$, where Δ_X denotes the diagonal in $X \times X$.

Note that in view of Proposition 1.1, a similar characterization was independently obtained in [**CT11**]. The authors actually studied the relative property (T) for pairs of groups of the form $(H \ltimes A, A)$, where A is a locally compact abelian group and H is a locally compact group acting on A.

3. *Proof of Theorem 1.3*

The equivalence between (ii) and (iii) follows from [**dC06**, Proposition 3.1.9.]. Hence, we are only interested in the equivalence between (i) and (ii).

We proceed by contrapositive. We show the following equivalence:

(i′) $\Gamma \curvearrowright (G/\Lambda, \mu)$ does not have the property (T) relative to the space;

(ii′) there exists a Γ-invariant measure probability on $\mathbb{P}(\mathfrak{g})$.

$(i') \Rightarrow (ii')$: Assume that $\Gamma \curvearrowright (G/\Lambda, \mu)$ does not have the property (T) relative to the space.

Choose a Borel fundamental domain $X \subset G$ for the action of Λ on G by right translations.

As mentioned in the introduction, $\mathrm{Aff}_\Lambda(G)$ acts in a natural way on G/Λ. We transfer this to an action of $\mathrm{Aff}_\Lambda(G)$ on X by setting

$$\gamma.x = g\sigma(x)\omega(\gamma, x) \quad \text{for } x \in X,\ \gamma = (g, \sigma) \in \mathrm{Aff}_\Lambda(G),$$

where $\omega(\gamma, x)$ is the unique element in Λ such that $g\sigma(x)\omega(\gamma, x) \in X$. We denote by $\mathrm{B}(X)$ the algebra of bounded measurable complex valued functions on X.

Since $\Gamma \curvearrowright (X, \mu)$ does not have the property (T) relative to the space, there exists a sequence of probability measures ν_n on $X \times X$ with the following properties:

(1) $p^i_* \nu_n = \mu$, for all n and $i = 1, 2$;

(2) $\int_{X\times X} f(x)g(y)\, d\nu_n(x, y) \underset{n\to\infty}{\longrightarrow} \int_X f(x)g(x)\, d\mu$, for all $f, g \in \mathrm{B}(X)$;

(3) $\|(\gamma \times \gamma)_* \nu_n - \nu_n\| \underset{n\to\infty}{\longrightarrow} 0$, for all $\gamma \in \Gamma$;

(4) $\liminf_{n\to\infty} \nu_n(\Delta_X) < 1$.

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Upon passing to a subsequence, we can assume that $c = \inf_{n\in\mathbb{N}} c_n > 0$, where $c_n = 1 - \nu_n(\Delta_X)$.

Define a sequence of probability measures η_n on $X \times X$ by

$$\eta_n(A) = \frac{1}{c_n}\nu_n(A\backslash\Delta_X) \quad \text{for all Borel subsets } A \subset X \times X.$$

Property 1 above implies that

$$p_*^1\eta_n \leq \frac{1}{c}\mu \quad \text{for all } n.$$

As for the limit $\eta_n(A \times (X\backslash A)) \to 0$ for all Borel subsets $A \subset X$, it follows from property 2 applied to $f \equiv \mathbb{1}_A$ and $g \equiv \mathbb{1}_{X\backslash A}$.

Let $\gamma \in \Gamma$. Set $\varphi \equiv \omega(\gamma, .)$, where $\omega(\gamma, .) : X \to \Lambda,\ x \mapsto \omega(\gamma, x)$ is the Borel cocycle defined above. Lemma 2.2 shows that

$$\eta_n(\{(x, y) \in (X \times X)\backslash\Delta_X \mid \rho(\gamma.x, \gamma.y) = \gamma.(\rho(x, y))\}) \underset{n\to\infty}{\longrightarrow} 1. \qquad (*)$$

On the other hand, property 3 implies that

$$\|(\gamma \times \gamma)_*\eta_n - \eta_n\| \underset{n\to\infty}{\longrightarrow} 0.$$

Consider the sequence of probability measures on $\mathbb{P}(\mathfrak{g})$ defined by $\zeta_n = \rho_*\eta_n$. It follows from $(*)$ that we again have

$$\|\gamma_*\zeta_n - \zeta_n\| \underset{n\to\infty}{\longrightarrow} 0. \qquad (**)$$

Now, since $\mathbb{P}(\mathfrak{g})$ is a compact metric space, the space $\mathcal{M}(\mathbb{P}(\mathfrak{g}))$ of probability measures on $\mathbb{P}(\mathfrak{g})$, endowed with the weak-$*$ topology, is a compact metrizable space. So, passing to a subsequence, we can assume that ζ_n converges to a probability measure ζ on $\mathbb{P}(\mathfrak{g})$. By $(**)$, ζ is γ-invariant. Since this holds for every $\gamma \in \Gamma$, we have found a Γ-invariant probability measure on $\mathbb{P}(\mathfrak{g})$.

$(ii') \Rightarrow (i')$: Assume now that there exists a Γ-invariant probability measure ζ on $\mathbb{P}(\mathfrak{g})$.

We claim that the action of Γ on $(G/\Lambda, \mu)$ does not have the property (T) relative to the space.

Denote by H the Zariski closure of Γ in $\mathrm{GL}(\mathfrak{g})$. By [**Sha99**, Theorem 3.11], there exists a normal cocompact algebraic subgroup H_0 of the algebraic group H such that H_0 fixes every point in the support $\mathrm{supp}(\zeta)$ of ζ.

Since $\mathrm{supp}(\zeta) \neq \emptyset$, there exist $Y \in \mathfrak{g}\backslash\{0\}$ and a group homomorphism $\chi : H_0 \to \mathbb{R}^*$ such that

$$h.Y = \chi(h)Y \quad \text{for all } h \in H_0.$$

It follows that Y is fixed by the closure $\overline{[H_0, H_0]}$ of the commutator subgroup of H_0. Consider $x_n = \exp(Y/n)$, for all $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$, $x_n \in G$ is fixed by $\overline{[H_0, H_0]}$ for the action given by

$$\mathrm{A}(h_0)(x_n) = \exp(h_0.Y/n) = x_n \quad \text{for all } n \in \mathbb{N},\ \text{for all } h_0 \in [H_0, H_0].$$

In this setting, we need to introduce the following action of $\mathrm{Aff}(G)$ on G by

$$\mathrm{A} : \mathrm{Aff}(G) \to \mathrm{Aut}(G)$$

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with

$$\mathrm{A}(h)(x) = g\sigma(x)g^{-1} \quad \text{for } h = (g, \sigma) \in \mathrm{Aff}(G),\ x \in G.$$

So,

$$\mathrm{A}(h)(x_n) = \exp(h.Y/n) \quad \text{where } h \in \mathrm{Aff}(G), \text{ for all } n \in \mathbb{N}.$$

Since $x_n \to e$ and $x_n \neq e$ for n large enough, and since Λ is discrete, we can assume that $x_n \notin \Lambda$, for every n.

Consider the map

$$p_n : G/\Lambda \to G/\Lambda \times G/\Lambda$$
$$x \mapsto (x, x_n x),$$

for every n, and set $\nu_n = p_n^*\mu$, where we recall that μ is the G-invariant probability measure on G/Λ coming from a Haar measure on G. Then ν_n is a probability measure on $G/\Lambda \times G/\Lambda$ and, for all $\varphi \in B(G/\Lambda \times G/\Lambda)$, we have

$$\nu_n(\varphi) = \int_{x \in G/\Lambda} \varphi(x, x_n x)\, d\mu(x).$$

The measure ν_n being not necessarily invariant under Γ, we define a new sequence of measures obtained by averaging ν_n as follows.

First notice that $H/\overline{[H_0, H_0]}$ is amenable since $(H/\overline{[H_0, H_0]})/(H_0/\overline{[H_0, H_0]}) \cong H/H_0$ is amenable and $H_0/\overline{[H_0, H_0]}$ is an abelian normal subgroup of $H/\overline{[H_0, H_0]}$. So, by the Reiter condition (see [**BdlHV08**, Appendix G]), there exists $f_n \in \mathrm{L}^1_{+,1}(H/\overline{[H_0, H_0]}, \lambda)$ such that $\|h.f_n - f_n\| \to 0$, for all $h \in H$, where $h.f_n(.) = f_n(h^{-1}.)$ is the regular representation of H on $\mathrm{L}^2(H/\overline{[H_0, H_0]})$ and λ is a Haar measure on $H/\overline{[H_0, H_0]}$.

We introduce the measures $\eta_{n,m}$ on $G/\Lambda \times G/\Lambda$ defined for $\varphi \in B(G/\Lambda \times G/\Lambda)$ by

$$\eta_{n,m}(\varphi) = \int_{x \in G/\Lambda} \int_{h \in H/\overline{[H_0, H_0]}} f_m(h)\varphi(x, \mathrm{A}(h)(x_n)x)\, d\lambda(h)\, d\mu(x).$$

We claim that there exists a sequence $(\eta_m)_m$ of $(\eta_{n,m})_{n,m}$ which satisfies properties 1–3 of Proposition 2.3 and such that $\liminf_{m\to\infty} \eta_m(\Delta_{G/\Lambda}) < 1$.

The projections of $\eta_{n,m}$ on each coordinate are equal to μ, for all n, m. Indeed, for every Borel subset $B \subset G/\Lambda$,

$$\begin{aligned}\eta_{n,m}(B \times G/\Lambda) &= \int_{h \in H/\overline{[H_0, H_0]}} \int_{x \in G/\Lambda} f_m(h)\mathbb{1}_{B \times G/\Lambda}(x, \mathrm{A}(h)(x_n)x)\, d\mu(x)\, d\lambda(h) \\ &= \int_{h \in H/\overline{[H_0, H_0]}} f_m(h)\mu(B)\, d\lambda(h) \\ &= \mu(B),\end{aligned}$$

so that property 1 is verified for every $n, m \in \mathbb{N}$.

To check property 3, let $\gamma \in \Gamma$. Write $\gamma = (\sigma, g)$, where $\sigma \in \mathrm{Aut}(G)$ and $g \in G$. For all $\varphi \in B(G/\Lambda \times G/\Lambda)$, we have

$$\begin{aligned}&(\gamma \times \gamma)_*\eta_{n,m}(\varphi) \\ &= \int_{x \in G/\Lambda} \int_{h \in H/\overline{[H_0, H_0]}} f_m(h)\varphi(\gamma(x), \gamma(\mathrm{A}(h)(x_n)x))d\lambda(h)\, d\mu(x) \\ &= \int_{x \in G/\Lambda} \int_{h \in H/\overline{[H_0, H_0]}} f_m(h)\varphi(\gamma(x), g\sigma(\mathrm{A}(h)(x_n))\sigma(x))\, d\lambda(h)\, d\mu(x),\end{aligned}$$

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since σ is an automorphism. Therefore,

$$\begin{aligned}
&(\gamma \times \gamma)_* \eta_{n,m}(\varphi) \\
&\quad = \int_{x \in G/\Lambda} \int_{h \in H/\overline{[H_0,H_0]}} f_m(h) \varphi(\gamma(x),\, g\sigma(\mathrm{A}(h)(x_n)) g^{-1} g\sigma(x))\, d\lambda(h)\, d\mu(x) \\
&\quad = \int_{x \in G/\Lambda} \int_{h \in H/\overline{[H_0,H_0]}} f_m(h) \varphi(\gamma(x),\, g\sigma(\exp(h.Y/n)) g^{-1} \gamma(x)) d\lambda(h)\, d\mu(x) \\
&\quad = \int_{x \in G/\Lambda} \int_{h \in H/\overline{[H_0,H_0]}} f_m(h) \varphi(x,\, \exp((\gamma h).Y/n)x)\, d\lambda(h)\, d\mu(x),
\end{aligned}$$

since μ is Γ-invariant.

As such,

$$\begin{aligned}
(\gamma \times \gamma)_* \eta_{n,m}(\varphi) &= \int_{x \in G/\Lambda} \int_{h \in H/\overline{[H_0,H_0]}} f_m(\gamma^{-1}.h') \varphi(x,\, \exp(h'.Y/n)x) d\lambda(h)\, d\mu(x), \\
&= \int_{x \in G/\Lambda} \int_{h' \in H/\overline{[H_0,H_0]}} f_m(\gamma^{-1}.h') \varphi(x,\, \mathrm{A}(h')(x_n)x)\, d\lambda(h')\, d\mu(x),
\end{aligned}$$

by the H-invariance of λ. Hence,

$$\begin{aligned}
&|(\gamma \times \gamma)_* \eta_{n,m}(\varphi) - \eta_{n,m}(\varphi)| \\
&\quad \le \left| \int_{x \in G/\Lambda} \int_{h \in H/\overline{[H_0,H_0]}} (f_m(\gamma^{-1}.h) - f_m(\gamma)) \varphi(x,\, \mathrm{A}(h)(x_n)x) d\lambda(h) d\mu(x) \right| \\
&\quad \le \int_{h \in H/\overline{[H_0,H_0]}} |f_m(\gamma^{-1}.h) - f_m(h)| \left(\int_{x \in G/\Lambda} |\varphi(x,\, \mathrm{A}(h)(x_n)x)| d\mu(x) \right) d\lambda(h) \\
&\quad \le \|\gamma.f_m - f_m\| \|\varphi\|.
\end{aligned}$$

In summary, we obtain

$$\|(\gamma \times \gamma)_* \eta_{n,m} - \eta_{n,m}\| \le \|\gamma.f_m - f_m\|.$$

Therefore, property 3 will be verified as soon as m tends to infinity.

We now verify property 2.

Let us first recall that since all standard probability spaces are Borel isomorphic, we can assume that G/Λ is a compact metric space.

Then, by approximating Borel bounded functions by continuous functions [**Con00**, Lemma 9.7], it is enough to check this property for functions of the form $\varphi \otimes \psi$, where φ and ψ are continuous functions on the standard Borel space G/Λ.

For $\varphi, \psi \in \mathrm{C}(G/\Lambda)$, we check using Lebesgue's dominated convergence theorem that, for any fixed $m \in \mathbb{N}$, we have

$$\int_{h \in H/\overline{[H_0,H_0]}} |f_m(h)| \int_{x \in G/\Lambda} |\psi(\mathrm{A}(h)(x_n)x) - \psi(x)| d\mu(x)\, d\lambda(h) \to 0,$$

since $x_n \to e$ when $n \to +\infty$. Therefore,

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$$\begin{aligned}\eta_{n,m}(\varphi\otimes\psi) &= \int_{(x,y)\in G/\Lambda\times G/\Lambda} \varphi(x)\psi(y)\, d\eta_{n,m}(x,\, y)\\ &= \int_{x\in G/\Lambda}\int_{h\in H/\overline{[H_0,H_0]}} f_m(h)\varphi(x)\psi(\mathrm{A}(h)(x_n)x)\, d\lambda(h)\, d\mu(x)\\ &\underset{n\to\infty}{\longrightarrow} \int_{x\in G/\Lambda}\varphi(x)\psi(x)\, d\mu(x) = p^*\mu(\varphi\otimes\psi),\end{aligned}$$

where $p : G/\Lambda \to G\Lambda \times G/\Lambda,\ x \mapsto (x, x)$.

We claim that there exists a sequence η_m that satisfies $\liminf_{m\to\infty} \eta_m(\Delta_{G/\Lambda}) < 1$. Let $X \subset G$ be a Borel fundamental domain for the action of Λ on G by right translations. We can assume that $\mu_G(X) = 1$, where μ_G is the Haar measure on G. Let U be a fixed neighborhood of e in G such that $\Lambda \cap U = \{e\}$. Then we have for any fixed $m \in \mathbb{N}$, using Lebesgue's dominated convergence theorem,

$$\int_{h\in H/\overline{[H_0,H_0]}} f_m(h)\mu_G(\{x\in X \mid x^{-1}\mathrm{A}(h)(x_n)x \notin U\})\, d\lambda(h) \to 0$$

as $n \to \infty$. Indeed, for every $h \in H/\overline{[H_0,\, H_0]}$,

$$\mu_G(\{x\in X \mid x^{-1}\mathrm{A}(h)(x_n)x \notin U\}) = 1 - \int_X \mathbb{1}_{\{y\in X\mid y^{-1}\mathrm{A}(h)(x_n)y\in U\}}(x) d\mu(x) \to 0,$$

by Lebesgue's dominated convergence theorem since, for all $x \in X$,

$$\mathbb{1}_{\{y\in X\mid A(h)(x_n)\in yUy^{-1}\}}(x) \to 1$$

as $n \to \infty$.

Let $(\varphi_n \otimes \psi_n)_n$ be a sequence of dense functions in $\{\varphi(x)\psi(y) \mid f,\, g \in \mathrm{C}(G/\Lambda)\}$. For every $m \in \mathbb{N}$, there exists $n_1(m) \in \mathbb{N}$ such that for all $n \geq n_1(m)$ we have

$$\left|\int_{G/\Lambda\times G/\Lambda} \varphi_1(x)\psi_1(y)\, d\eta_{n,m}(x,\, y) - \int_{G/\Lambda}\varphi_1(x)\psi_1(x)\, d\mu(x)\right| \leq \frac{1}{2^m}$$

and

$$\int_{h\in H/\overline{[H_0,H_0]}} f_m(h)\mu(\{x\in X \mid x^{-1}\mathrm{A}(h)(x_n)x \notin U\})\, d\lambda(h) \leq \frac{1}{2^m}.$$

In the same way, for every $m \in \mathbb{N}$, there exists $n_2(m) \geq n_1(m)$ such that for all $n \geq n_2(n_1(m))$ we have

$$\left|\int_{G/\Lambda\times G/\Lambda} \varphi_2(x)\psi_2(y) d\eta_{n,m}(x,\, y) - \int_{G/\Lambda}\varphi_2(x)\psi_2(x)\, d\mu(x)\right| \leq \frac{1}{2^m}.$$

We finally consider the sequence defined for all $m \in \mathbb{N}$ by $\eta_m = \eta_{n_m(m),m}$. Let $f,\, g \in \mathrm{C}(G/\Lambda)$. There exists a subsequence $(\varphi_{\theta(n)} \otimes \psi_{\theta(n)})_n$ of $(\varphi_n \otimes \psi_n)_n$ such that

$$\|f\otimes g - \varphi_{\theta(n)}\otimes\psi_{\theta(n)}\|_\infty \underset{n\to\infty}{\to} 0.$$

Let $\varepsilon > 0$. We have

$$\begin{aligned}|\eta_m(f\otimes g) - p^*\mu(f\otimes g)| &= |\eta_m(f\otimes g) - \eta_m(\varphi_{\theta(n)}\otimes\psi_{\theta(n)})\\ &\quad + \eta_m(\varphi_{\theta(n)}\otimes\psi_{\theta(n)}) - p^*\mu(\varphi_{\theta(n)}\otimes\psi_{\theta(n)})\\ &\quad + p^*\mu(\varphi_{\theta(n)}\otimes\psi_{\theta(n)}) - p^*\mu(f\otimes g)|\\ &\leq 2\|f\otimes g - \varphi_{\theta(n)}\otimes\psi_{\theta(n)}\|_\infty\\ &\quad + |\eta_m(\varphi_{\theta(n)}\otimes\psi_{\theta(n)}) - p^*\mu(\varphi_{\theta(n)}\otimes\psi_{\theta(n)})|\\ &\leq \varepsilon + |\eta_m(\varphi_{\theta(n)}\otimes\psi_{\theta(n)}) - p^*\mu(\varphi_{\theta(n)}\otimes\psi_{\theta(n)})|,\end{aligned}$$

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for n large enough. Since the second term tends to 0 as m tends to infinity, we have

$$\lim_{m\to\infty} \eta_m(f \otimes g) - p^*\mu(f \otimes g) \leq \varepsilon.$$

Therefore, the sequence η_m satisfies properties 1–3.

We claim that $\eta_m(\Delta_{G/\Lambda})$ does not converge to 1. Indeed, assume by contradiction that $\eta_m(\Delta_{G/\Lambda}) \to 1$, i.e.

$$\int_{h\in H/\overline{[H_0,H_0]}} f_m(h)\mu(\{x \in G/\Lambda \mid x = \mathrm{A}(h)(x_{n(m)})x\})\, d\lambda(h) \underset{m\to\infty}{\longrightarrow} 1.$$

In particular, for X the fundamental domain for the action of G on G/Λ fixed before, we have

$$\int_{h\in H/\overline{[H_0,H_0]}} f_m(h)\mu_G(\{x \in X \mid x^{-1}\mathrm{A}(h)(x_{n(m)})x \in \Lambda\})\, d\lambda(h) \underset{m\to\infty}{\longrightarrow} 1.$$

Recall that by construction of our subsequence we have

$$\int_{h\in H/\overline{[H_0,H_0]}} f_m(h)\mu_G(\{x \in X \mid x^{-1}\mathrm{A}(h)(x_{n(m)})x \notin U\})\, d\lambda(h) \underset{m\to\infty}{\longrightarrow} 0,$$

where U is a neighborhood of e such that $\Lambda \cap U = \{e\}$.

Thus,

$$\int_{h\in H/\overline{[H_0,H_0]}} f_m(h)\mu_G(\{x \in X \mid x^{-1}\mathrm{A}(h)(x_{n(m)})x \in \Lambda \cap U\})\, d\lambda(h) \underset{m\to\infty}{\longrightarrow} 1,$$

i.e.

$$\mu_G(\{x \in X \mid x_{n(m)} = e\}) \underset{m\to\infty}{\longrightarrow} 1.$$

This implies that $x_{n(m)} = e$ for m large enough. This contradicts the fact that $x_{n(m)} = \exp(Y/n(m)) \neq e$ for large m.

Hence, $\eta_m(\Delta_{G/\Lambda})$ does not converge to 1. This shows that the action of Γ on G/Λ does not have the property (T) relative to the space and the proof is complete. □

4. *Actions on nilmanifolds associated with graphs*

We give new examples of actions which have the property (T) relative to the space which do not arise from pairs of groups with relative property (T). Our examples are based on a class of nilpotent Lie groups associated with graphs which appear in [**DM05**].

4.1. *Action on the Heisenberg nilmanifold.*

First, let us consider the realization of the Heisenberg group $H_\mathbb{R}$ as $\mathbb{R}^3$ with the law

$$\begin{aligned}&\text{for all } (x, y, z), (x', y', z') \in \mathbb{R}^3\\ &\quad (x, y, z).(x', y', z') = (x + x', y + y', z + z' + (xy' - x'y)).\end{aligned}$$

The subset $H_\mathbb{Z} = \mathbb{Z}^3$ is a lattice. We consider the associated nilmanifold $H_\mathbb{R}/H_\mathbb{Z}$ with the Lebesgue measure.

The group of automorphisms of $H_\mathbb{R}$ stabilizing $H_\mathbb{Z}$ is, up to finite index (see [**BH11**]),

$$T \rtimes L \subset \mathrm{Aut}_{H_\mathbb{Z}}(H_\mathbb{R}),$$

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where

$$T \cong \mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z}), \quad L \cong \mathrm{SL}_2(\mathbb{Z}),$$

and the action of $(t, g) \in \mathrm{Aut}(H_{\mathbb{R}})$, with $t \in \mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z})$, $g \in \mathrm{SL}_2(\mathbb{Z})$, on $H_{\mathbb{R}}$ is given by

$$(t, g).(x, y, z) = (g(x, y), z + t(x, y)) \quad \text{for } (x, y, z) \in H_{\mathbb{R}}.$$

Observe that every $g \in \mathrm{Aut}_{H_{\mathbb{Z}}}(H_{\mathbb{R}})$ fixes pointwise the image of the center of $H_{\mathbb{R}}$ in $H_{\mathbb{R}}/H_{\mathbb{Z}}$. From Theorem 1.3, we deduce the following corollary.

COROLLARY 4.1. *For every subgroup Γ of $Aff_{H_{\mathbb{Z}}}(H_{\mathbb{R}})$, the action $\Gamma \curvearrowright H_{\mathbb{R}}/H_{\mathbb{Z}}$ does not have the property (T) relative to the space.*

4.2. *Nilpotent Lie groups associated with graphs.* In this subsection, we introduce notation and recall results that will be used in the proof of Theorem 1.4 in §4.3.

Let (S, E) be a finite non-oriented graph, with S the set of vertices and E the set of edges. The edge between the vertices α and β will be denoted by $\alpha\beta$.

Let V be the $\mathbb{R}$-vector space with S as a basis. Let $(u_\alpha)_{\alpha \in S}$ be basis vectors.

Let W be the subspace of $\bigwedge^2 V$, the second exterior power of V generated by $\{u_\alpha \wedge u_\beta \mid \alpha, \beta \in S \text{ and } \alpha\beta \in E\}$. Let $(u_{\alpha\beta})_{\alpha\beta \in E}$ be basis vectors of W. Consider $\mathcal{N} = V \oplus W$. A Lie algebra structure is defined on $\mathcal{N}$ by

$$[u_\alpha, u_\beta] = \begin{cases} u_{\alpha\beta} \in W & \text{if } \alpha, \beta \in S \text{ and } \alpha\beta \in E, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\mathcal{N}$ is a two-step nilpotent Lie algebra.

The simply connected and connected Lie group N with Lie algebra $\mathcal{N}$ can be realized as $\mathcal{N}$ with the multiplication defined by

$$(v_1, w_1).(v_2, w_2) = (v_1 + v_2, w_1 + w_2 + \tfrac{1}{2}[v_1, v_2]),$$

for all $v_1, v_2 \in V$ and $w_1, w_2 \in W$.

Observe that the exponential map is simply the identity.

Example 4.2.

(i) The Heisenberg group is given by the graph with two vertices and one edge.

(ii) Two-step free nilpotent Lie groups are given by complete graphs.

Let $N_{\mathbb{Z}}$ be the subgroup of N generated by $\{u_\alpha \mid \alpha \in S\}$. Then $N_{\mathbb{Z}}$ is a cocompact lattice in N; we call $N/N_{\mathbb{Z}}$ the compact nilmanifold associated with (S, E).

Observe that $\mathrm{Aut}(N) = \mathrm{Aut}(\mathcal{N})$.

We choose a complementary subspace V of $[\mathcal{N}, \mathcal{N}]$ so that $\mathcal{N} = V \oplus [\mathcal{N}, \mathcal{N}]$ and denote by $\pi : \mathcal{N} \to V$ the associated projection. We define a group monomorphism

$$\begin{aligned} \mathrm{Hom}(V, [\mathcal{N}, \mathcal{N}]) &\to \mathrm{Aut}(\mathcal{N}) \\ \theta &\mapsto t_\theta \end{aligned}$$

by $t_\theta(x) = x + \theta(\pi(x))$, $x \in \mathcal{N}$. Set

$$T = \{t_\theta \mid \theta \in \mathrm{Hom}(V, [\mathcal{N}, \mathcal{N}])\}$$

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and

$$G = \{g \in \mathrm{Aut}(\mathcal{N}) \mid g(V) = V\}.$$

It is shown in [**DM05**] that

$$\mathrm{Aut}(\mathcal{N}) = T \rtimes G,$$

and that with respect to the decomposition $\mathcal{N} = V \oplus [\mathcal{N}, \mathcal{N}]$, we can write

$$T = \left\{ \begin{pmatrix} I & 0 \\ t & I \end{pmatrix} \mid t \in \mathrm{Hom}(V, [\mathcal{N}, \mathcal{N}]) \right\},$$
$$G = \left\{ \begin{pmatrix} g & 0 \\ 0 & P(g) \end{pmatrix} \mid g \in \mathrm{GL}(V) \right\},$$

where, for $g \in \mathrm{GL}(V)$, $P(g) \in \mathrm{GL}([\mathcal{N}, \mathcal{N}])$ is a certain matrix whose coefficients are polynomials in those of g.

As a result, to understand $\mathrm{Aut}(\mathcal{N})$, one has to study G and, for this, we recall the following definition from [**DM05**].

Definition 4.3. [**DM05**] A subset $S' \subset S$ is said to be *coherent* if for every pair $\alpha, \beta \in S'$ and for every $\gamma \in S'$ such that $\alpha\gamma \in E$, either $\gamma = \beta$ or $\beta\gamma \in E$.

We can decompose S into *maximal coherent components*, defining in this way an equivalence relation on S. We denote by $[S]$ the set of coherent components and, for any $[\alpha] \in [S]$, let $V_{[\alpha]}$ be the subspace of V generated by the elements of $[\alpha]$. We can then write $V = \bigoplus_{[\alpha]} V_{[\alpha]}$.

The following description of G was obtained in [**DM05**]: the connected component of the identity of G is

$$\left(\prod_{[\alpha] \in [S]} GL^+(V_{[\alpha]}) \right) \cdot M,$$

where M is the maximal closed nilpotent connected normal subgroup of G.

In what follows, we will only consider groups $\Gamma \subset \mathrm{Aut}_{N_{\mathbb{Z}}}(N)$ such that the projection on G is contained in

$$L \cong \prod_{[\alpha] \in [S]} SL(V_{[\alpha]}).$$

Let us look at a few examples.

Example 4.4.

(i) The Heisenberg group arises from the graph with two vertices and one edge; it has only one coherent component. So, in this case, $L = \mathrm{SL}_2(\mathbb{R})$ and

$$T \rtimes L \cong \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}) \rtimes \mathrm{SL}_2(\mathbb{R}).$$

(ii) The complete graph with $n \geq 3$ vertices has as automorphism group

$$T \rtimes L \cong \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^{n(n-1)/2}) \rtimes \mathrm{SL}_n(\mathbb{R}).$$

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4.3. *Proof of Theorem 1.4.* Recall that we have the following notation. Let $\mathcal{N}$ be the nilpotent Lie algebra associated to a finite graph (S, E). Let N be the corresponding simply connected nilpotent Lie group and $N/N_{\mathbb{Z}}$ the associated manifold.

Write $\mathcal{N} = V \oplus [\mathcal{N}, \mathcal{N}]$, for V the complementary subspace to $[\mathcal{N}, \mathcal{N}]$ in $\mathcal{N}$, and set $T \cong \mathrm{Hom}(V, [\mathcal{N}, \mathcal{N}])$ and $L = \prod_{[\alpha]\in[S]} SL(V_{[\alpha]})$ as in §4.2.

Observe that $L = \prod_{[\alpha]\in[S]} SL(V_{[\alpha]})$ has no non-trivial homomorphism into $\mathbb{R}^*$ and has no non-trivial cocompact normal subgroup.

Let us recall as well how $T \rtimes L$ acts on N identified with $\mathcal{N}$:

$$(t, g).(Y_1, Y_2) = (g(Y_1),\ P(g)(Y_2) + t(g(Y_1))), \qquad (*)$$

for $(t, g) \in T \rtimes L$ and $(Y_1, Y_2) \in \mathcal{N} = V \oplus [\mathcal{N}, \mathcal{N}]$.

We prove the theorem by contrapositive. We show the following equivalence:

(i′) Γ has a non-zero fixed vector $Y \in [\mathcal{N}, \mathcal{N}]$;

(ii′) the action of Γ on $N/N_{\mathbb{Z}}$ does not have the property (T) relative to the space.

$(i') \Rightarrow (ii')$: Assume that Γ has a non-zero fixed vector $Y \in [\mathcal{N}, \mathcal{N}]$.

The Dirac measure corresponding to the image of Y in $\mathbb{P}(\mathcal{N})$ is then fixed by Γ. Hence, $\Gamma \curvearrowright N/N_{\mathbb{Z}}$ does not have the property (T) relative to the space, by Theorem 1.3.

$(ii') \Rightarrow (i')$: Assume that the action of Γ on $N/N_{\mathbb{Z}}$ does not have the property (T) relative to the space. By Theorem 1.3, Γ fixes a probability measure ξ on $\mathbb{P}(\mathcal{N})$. The stabilizer of ξ in $\overline{\Gamma}^{\mathrm{Zar}} = T \rtimes L$ is an algebraic group and contains a normal cocompact subgroup H which fixes every point in $\mathrm{supp}(\xi)$ [**Sha99**, Theorem 3.11].

Choose $Y \in \mathcal{N}$ such that its image in $\mathbb{P}(\mathcal{N})$ belongs to $\mathrm{supp}(\xi)$ and write $Y = (Y_1, Y_2)$ with $Y_1 \in V$, $Y_2 \in [\mathcal{N}, \mathcal{N}]$. Then

$$(t, g).Y = \lambda(t, g)Y \quad \text{for all } (t, g) \in H, \qquad (**)$$

for a homomorphism $\lambda : H \to \mathbb{R}^*$. We claim that $Y_1 = 0$.

Indeed, recall that, with respect to the decomposition $V = \bigoplus_{[\alpha]\in[S]} V_{[\alpha]}$, we have

$$L = \begin{pmatrix} SL(V_{[\alpha_1]}) & & & & & \\ & \ddots & & & & \\ & & SL(V_{[\alpha_k]}) & & & \\ & & & I_{[\alpha_{k+1}]} & & \\ & & & & \ddots & \\ & & & & & I_{[\alpha_r]} \end{pmatrix},$$

since some of the $V_{[\alpha]}$ may have dimension 1.

Write $Y_1 = (Y_{1,1}, \ldots, Y_{1,r})$ with $Y_{1,i}$ the component of Y_1 in $V_{[\alpha_i]}$. Assume first that $Y_{1,i} \neq 0$ for some $1 \leq i \leq k$. Since $\mathrm{SL}(V_{[\alpha_i]})$ is simple, and $H \cap \mathrm{SL}(V_{[\alpha_i]}) \neq \{1\}$, we have $\mathrm{SL}(V_{[\alpha_i]}) \subset H$; this is a contradiction as $\mathrm{SL}_n(\mathbb{R})$ stabilizes no line in $\mathbb{R}^n$. So,

$$Y_{1,1} = \cdots = Y_{1,k} = 0.$$

It remains to show that

$$Y_{1,k+1} = \cdots = Y_{1,r} = 0.$$

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Assume by contradiction that $Y_{1,i} \neq 0$ for some $k+1 \leq i \leq r$. Then, looking at the ith component in equation $(**)$, we see that $\lambda : H \to \mathbb{R}^*$ is trivial. It follows from $(*)$ and $(**)$ that

$$Y_2 = P(g)Y_2 + tY_1 \quad \text{for all } (t, g) \in T \rtimes L.$$

This is a polynomial equation in the matrix coefficients of t and g. Since it is verified for H, it holds for its Zariski closure denoted by Z:

$$Y_2 = P(g)Y_2 + tY_1 \quad \text{for all } (t, g) \in Z. \qquad (***)$$

Observe that Z is still a normal cocompact subgroup of $T \rtimes L$. We claim that $Z = T \rtimes L$. Once proved, it will follow from $(***)$ applied to $g = e$ that

$$Y_2 = Y_2 + tY_1 \quad \text{for all } t \in T$$

and hence $Y_1 = 0$.

To show that $Z = T \rtimes L$, observe that the projection of Z on L is L, since it has no non-trivial normal cocompact subgroup. So, it suffices to prove that $T \subset Z$.

For this, consider the group isomorphism $\varphi : T/T \cap Z \to TZ/Z$. Since $TZ = T \rtimes L$, and since $T/T \cap Z$ and $(T \rtimes L)/Z$ are σ-compact locally compact groups, the group isomorphism φ, which is continuous, is a homeomorphism.

The group $T/T \cap Z \cong TZ/Z$ is compact (as Z is cocompact in TZ); so, $T \cap Z$ is a normal cocompact Zariski closed subgroup of $T = \mathrm{Hom}(V, [\mathcal{N}, \mathcal{N}])$ which is isomorphic to some $\mathbb{R}^m$. Hence, $T \subset Z$. So, we have proved that $Y_1 = 0$.

We claim now that Y_2 is fixed by Γ. Since $Y_2 \in [\mathcal{N}, \mathcal{N}]$, this will finish the proof. To show this, let $p_2 : T \rtimes L \to L$ denote the canonical projection. Recall that L has no non-trivial homomorphism into $\mathbb{R}^*$ and has no non-trivial cocompact normal subgroup. It follows that $p_2(H) = L$. This shows that Y_2 is fixed by L and hence by Γ. □

We will denote by $(T \rtimes L)_{\mathbb{Z}}$ the subgroup of $T \rtimes L$ whose elements have integer matrix coefficients.

Example 4.5.

(i) Let N be the nilpotent Lie group associated with the complete graph with $n \geq 3$ vertices. Consider the following subgroup of automorphisms:

$$T \rtimes L \cong \begin{pmatrix} I_n & 0 \\ \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^{n(n-1)/2}) & I_{n(n-1)/2} \end{pmatrix} \rtimes \begin{pmatrix} \mathrm{SL}_n(\mathbb{R}) & 0 \\ 0 & P(\mathrm{SL}_n(\mathbb{R})) \end{pmatrix},$$

where P is an injective morphism. The action of $(T \rtimes L)_{\mathbb{Z}}$ on $\mathcal{N}$ has no fixed point. Hence, the action of $(T \rtimes L)_{\mathbb{Z}}$ on $N/N_{\mathbb{Z}}$ has the property (T) relative to the space, by Theorem 1.3.

(ii) Let N be the nilpotent Lie group associated with the 'star' graph; this is the graph given by a central vertex which is linked with n other vertices. Consider the following subgroup:

$$T \rtimes L \cong \begin{pmatrix} I_{n+1} & 0 \\ \mathrm{Hom}(\mathbb{R}^{n+1}, \mathbb{R}^n) & I_n \end{pmatrix} \rtimes \begin{pmatrix} \mathrm{SL}_n(\mathbb{R}) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{SL}_n(\mathbb{R}) \end{pmatrix}$$

of $\mathrm{Aut}(N)$, where $\mathrm{SL}_n(\mathbb{R})$ acts on $[\mathcal{N}, \mathcal{N}]$. There is obviously no fixed point in $[\mathcal{N}, \mathcal{N}]$. Hence, the action of $(T \rtimes L)_{\mathbb{Z}}$ on $N/N_{\mathbb{Z}}$ has the property (T) relative to the space, by Theorem 1.4.

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