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Non-Degenerate Spheres in Three Dimensions

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Let P be a set of n points in $\mathbb{R}^3$, and let $k \leqslant n$ be an integer. A sphere σ is *k-rich* with respect to P if $|\sigma \cap P| \geqslant k$, and is *$\eta$-non-degenerate*, for a fixed fraction $0 < \eta < 1$, if no circle $\gamma \subset \sigma$ contains more than $\eta|\sigma \cap P|$ points of P.

We improve the previous bound given in [1] on the number of k-rich η-non-degenerate spheres in 3-space with respect to any set of n points in $\mathbb{R}^3$, from $O(n^4/k^5 + n^3/k^3)$, which holds for all $0 < \eta < 1/2$, to[1] $O^*(n^4/k^{11/2} + n^2/k^2)$, which holds for all $0 < \eta < 1$ (in both bounds, the constants of proportionality depend on η). The new bound implies the improved upper bound $O^*(n^{58/27}) \approx O(n^{2.1482})$ on the number of mutually similar triangles spanned by n points in $\mathbb{R}^3$; the previous bound was $O(n^{13/6}) \approx O(n^{2.1667})$ [1].

1. Introduction

1.1. Non-degenerate hyperplanes and spheres

The concept of *degeneracy* of a hyperplane was introduced by Elekes and Tóth [9]. Given a finite point set $P \subset \mathbb{R}^d$ and a constant $0 < \eta < 1$, a hyperplane π in $\mathbb{R}^d$ is said to be *η-degenerate* (with respect to P), if there exists some lower-dimensional affine subspace $\pi' \subset \pi$ such that

$$|\pi' \cap P| \geqslant \eta|\pi \cap P|.$$

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[1] The $O^*(\cdot)$ notation hides subpolynomial factors.

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If no such affine subspace π' exists, then π is said to be *η-non-degenerate*. A hyperplane π is called *k-rich* (with respect to P) if $|\pi \cap P| \geqslant k$. Elekes and Tóth [9] showed that for any dimension d there exists a constant $\eta_d < 1$ which depends on d, so that, for any $\eta < \eta_d$ and for any set of n points in $\mathbb{R}^d$, the number of η-non-degenerate k-rich hyperplanes is

$$O\left(\frac{n^d}{k^{d+1}} + \frac{n^{d-1}}{k^{d-1}}\right), \tag{1.1}$$

with the multiplicative constant depending on η and d, and that this bound is asymptotically best possible. We have $\eta_3 = 1$, $\eta_4 = 1/2$, and as d increases, η_d gets smaller and smaller. It is not known whether the bound applies to all $0 < \eta < 1$ in any dimension.

The motivation for studying non-degenerate hyperplanes comes from incidence problems between points and hyperplanes in three and higher dimensions. These problems face the major difficulty that without imposing any restrictions on the points and/or hyperplanes, the incidence graph can be the complete bipartite graph. This can already occur in three dimensions, if we place all the points on a common line and make all the planes pass through that line. We note, however, that in such constructions, the planes (or hyperplanes in higher dimensions) are highly degenerate. As Elekes and Tóth [9] showed, by restricting the hyperplanes to be non-degenerate, one can obtain better non-trivial bounds for point–hyperplane incidences.

In this paper we consider a related problem involving incidences between points and *spheres*. Given a finite point set $P \subset \mathbb{R}^d$ and a constant $0 < \eta < 1$, a $(d-1)$-sphere $\sigma \subset \mathbb{R}^d$ is called *η-degenerate* (with respect to P) if there exists some $(d-2)$-subsphere $\sigma' \subset \sigma$ such that

$$|\sigma' \cap P| \geqslant \eta |\sigma \cap P|.$$

Otherwise, σ is called *η-non-degenerate*. The notion of *k-richness* (with respect to P) is defined as in the case of hyperplanes.

Point–sphere incidences arise in several problems involving distances between points. For example, the study of Aronov, Pach, Sharir and Tardos [4] on the number of distinct distances determined by n points in $\mathbb{R}^3$ involves incidences between the points and a certain collection of spheres. In upper-bounding the number of these incidences, Aronov, Pach, Sharir and Tardos split the spheres into degenerate and non-degenerate ones and analyse the incidences of each class separately. Another study by the authors and others [1] considers the problem of bounding the number of similar triangles (or simplices) determined by n points in d dimensions. Here too the problem is reduced to incidences between points and spheres, and handling degenerate and non-degenerate spheres is a major step in the analysis.

Informally, similar to the case of incidences between points and hyperplanes, we may face a situation where many points lie on a common circle and many spheres pass through that circle, leading to a large number of incidences. This situation is easier to handle when the spheres are non-degenerate, because no circle can contain more than a constant fraction of the points on such a sphere; see below for details.

By lifting $\mathbb{R}^d$ to the standard paraboloid in $\mathbb{R}^{d+1}$ (see, *e.g.*, [8]), every sphere is transformed into a hyperplane, and the incidence relation, as well as degeneracy and

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non-degeneracy, are preserved. It thus follows that for any $\eta < \eta_{d+1}$, the number of k-rich η-non-degenerate $(d-1)$-spheres, with respect to a set of n points in $\mathbb{R}^d$, is

$$O\left(\frac{n^{d+1}}{k^{d+2}} + \frac{n^d}{k^d}\right). \tag{1.2}$$

See [1] for more details.

It has been conjectured (see [1]) that this bound is not tight. Supporting evidence comes from the fact that in the plane, where the spheres are circles, which are clearly non-degenerate (if they are k-rich and k is sufficiently large), we have an upper bound of $O^*(n^3/k^{11/2} + n^2/k^3 + n/k)$ on the number of k-rich circles [2, 5, 11], which is significantly better than the $O(n^3/k^4 + n^2/k^2)$ bound implied by lifting the circles into non-degenerate planes in $\mathbb{R}^3$. We recall that this improved bound holds for circles in any dimension; see [3].

At any rate, the bound for spheres in $\mathbb{R}^d$ should lie in between those for hyperplanes in $\mathbb{R}^d$ and for hyperplanes in $\mathbb{R}^{d+1}$. The second threshold follows from the lifting transform just discussed, and the first threshold follows from noting that an *inversion* of $\mathbb{R}^d$ takes a collection of hyperplanes in $\mathbb{R}^d$ into a collection of spheres (all passing through a common point), while preserving incidences, richness, degeneracy, and non-degeneracy. In particular, we obtain the lower bound (the best known, as far as we are aware)

$$\Omega\left(\frac{n^d}{k^{d+1}} + \frac{n^{d-1}}{k^{d-1}}\right)$$

on the number of k-rich η-non-degenerate spheres in $\mathbb{R}^d$, by taking the Elekes–Tóth lower bound construction for non-degenerate hyperplanes in $\mathbb{R}^d$, and by inverting space. For $k = d+1$ and $\eta > \frac{d}{d+1}$, there is also the trivial construction of n points in general position in $\mathbb{R}^d$. Such a set of points generates $\binom{n}{d+1}$ $(d+1)$-rich η-non-degenerate spheres. This construction is clearly best possible for $k = d+1$ and $\eta > \frac{d}{d+1}$, but for larger (albeit still small) values of k, we do not know whether it is possible to construct $\Omega(n^{d+1})$ k-rich non-degenerate spheres, and can only provide the general construction using inversion.

Narrowing this gap between the upper and lower bounds is an interesting problem in its own right. Moreover, any improvement of the upper bound will immediately improve the bounds on the problems studied in the papers [1, 4] mentioned above.

1.2. Our results

In this paper we improve the upper bound on the number of rich non-degenerate spheres in 3-space. The new bound is stated in the $O^*(\cdot)$ notation, which hides subpolynomial factors. That is, a bound of the form $O^*(f(n,k))$ means an upper bound of the form $C_\varepsilon n^\varepsilon k^\varepsilon f(n,k)$, which holds for any $\varepsilon > 0$ where C_ε depends on ε.

Our main result is as follows.

Theorem 1.1. *For any constant $\eta < 1$, for any set P of n points in $\mathbb{R}^3$, and for any $k \leqslant n$, the number of k-rich η-non-degenerate spheres is*

$$O^*\left(\frac{n^4}{k^{11/2}} + \frac{n^2}{k^2}\right),$$

where the implied constant of proportionality also depends on η.

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Clearly, this is an improvement over the previous bound of $O(n^4/k^5 + n^3/k^3)$. Using this bound, we get an improved upper bound on the number of mutually similar triangles spanned by n points in $\mathbb{R}^3$.

Theorem 1.2. *Let P be a set of n points in $\mathbb{R}^3$, and let Δ be a triangle. Then the number of triples of points of P that span a triangle similar to Δ is $O^*(n^{58/27}) = O(n^{2.1482})$.*

The previous bound was $O(n^{13/6}) = O(n^{2.1667})$ [1].

Discussion. There are three conventional ways to present incidence bounds.

(a) An upper bound on the number of incidences between n points and m objects. For example, the above-mentioned Elekes–Tóth bound [9], when formulated in this way, asserts that the number of incidences between n points in $\mathbb{R}^d$ and m η_d-non-degenerate hyperplanes is $O(n^{d/(d+1)}m^{d/(d+1)} + nm^{(d-2)/(d-1)})$.

(b) An upper bound on the number of objects incident to at least k out of n points. This, for example, is the formulation of the Elekes–Tóth bound, which says that the number of k-rich η_d-non-degenerate hyperplanes spanned by a set of n points in $\mathbb{R}^d$ is $O(n^d/k^{d+1} + n^{d-1}/k^{d-1})$.

(c) An upper bound on the number of incidences between n points and objects incident to at least k of these points. For example, the Elekes–Tóth bound can be reformulated to say that the number of incidences between n points in $\mathbb{R}^d$ and any family of k-rich η_d-non-degenerate hyperplanes is $O(n^d/k^d + n^{d-1}/k^{d-2})$.

In practically all known instances, all three alternatives are equivalent and any one of them can be derived from any other. Theorem 1.1, when stated in form (a), reads as follows.

Theorem 1.3. *For any $\eta < 1$, the number of incidences I between n points in $\mathbb{R}^3$ and m η-non-degenerate spheres is*

$$I = O^*\left(n^{8/11}m^{9/11} + nm^{1/2}\right).$$

We will prove Theorem 1.3 rather than Theorem 1.1. Theorem 1.1 follows by noting that if each of the m spheres is also k-rich then $I \geqslant mk$. Solving the resulting inequality $mk = O^*(n^{8/11}m^{9/11} + nm^{1/2})$ for m yields the bound of Theorem 1.1.

We use in the proof the previous bound for the number of incidences between n point in $\mathbb{R}^3$ and m non-degenerate spheres, and strengthen it by *cutting*. The bound is

$$I = O\left(n^{4/5}m^{4/5} + nm^{2/3}\right), \tag{1.3}$$

which is form (a) of the Elekes–Tóth bound (1.2) in four dimensions (where the given spheres appear as hyperplanes).

2. Proof of Theorem 1.3

Let P be a set of n points in $\mathbb{R}^3$, and let $\mathcal{S}$ be a set of m η-non-degenerate spheres with respect to P in $\mathbb{R}^3$, for some fixed positive $\eta < 1$. Let $j > 0$ be some integer. We say that a

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sphere σ is *j-bad* with respect to P if σ contains some circle $\gamma \subset \sigma$, which contains at least j points of P, *i.e.*, $|\gamma \cap P| \geqslant j$. If no such circle exists, then σ is said to be *j-good*. Note that an η-non-degenerate sphere incident to exactly k points is, by definition, ηk-good.

Let $I(P, \mathcal{S})$ denote the number of incidences between P and $\mathcal{S}$. Denote by $I_\eta(n, m)$, or $I(n, m)$ for short, the maximum of $I(P, \mathcal{S})$ over all sets P of n points and $\mathcal{S}$ of m η-non-degenerate spheres in $\mathbb{R}^3$. Note that each sphere $\sigma \in \mathcal{S}$ contains some non-coplanar quadruple of points, and each such quadruple uniquely determines the sphere containing it, so we have $m = |\mathcal{S}| \leqslant \binom{n}{4}$. In fact, this bound decreases as η decreases, but it remains $O(n^4)$, with the constant of proportionality decreasing with η. Therefore, $I(n, m)$ is defined only for values of n and m satisfying (the appropriate variant of) this relationship.

The next lemma establishes a recurrence relation on $I(n, m)$.

Lemma 2.1. *For any $0 < \eta < 1$, for any two positive integers n, m satisfying $m = O(n^4)$ as above, for any integer $1 < j < n$, and for any number $1 < r < \min\{m, n^{1/3}\}$, we have*

$$I(n, m) = O\left(r^3 \beta(r) \log^3 r \cdot I\left(\frac{n}{r^3}, \frac{m}{r}\right) + mjr^2 \beta(r) \log^3 r + \frac{n^4 \log n}{j^{11/2}} + \frac{n^3}{j^3} + \frac{n^2}{j}\right), \tag{2.1}$$

where the constant of proportionality depends on η, and where $\beta(r) = 2^{O(\alpha^2(r))}$ and $\alpha(r)$ is the extremely slowly growing inverse Ackermann function.

Proof. Let P be a set of n points in $\mathbb{R}^3$, and let $\mathcal{S}$ be a set of m η-non-degenerate spheres with respect to P. We partition $\mathcal{S}$ into two subsets consisting, respectively, of the j-good and the j-bad spheres, and bound the incidences of P with each of these subsets separately.

We bound the number of incidences with the j-bad spheres as follows. The number of j-rich circles with respect to P is $O((n^3 \log n)/j^{11/2} + n^2/j^3 + n/j)$ [2, 3, 5, 11].[2] Each circle γ can be incident to several spheres of $\mathcal{S}$. Since each such sphere σ is η-non-degenerate, we have $|\sigma \cap P| > \frac{1}{\eta}|\gamma \cap P|$, or $|\sigma \cap P| < \frac{1}{1-\eta}|(\sigma \setminus \gamma) \cap P|$. Any point of P not on γ can lie in at most one of these spheres, so

$$\sum_{\sigma \in \mathcal{S} | \sigma \supset \gamma} |\sigma \cap P| < \sum_{\sigma \in \mathcal{S} | \sigma \supset \gamma} \frac{1}{1-\eta} |(\sigma \setminus \gamma) \cap P| < \frac{n}{1-\eta}.$$

Hence, the total number of incidences between the spheres containing γ and P is $O(n)$. Multiplying by the number of circles, we get that the number of incidences between P and the j-bad spheres is

$$I_{\text{bad}} = O\left(\frac{n^4 \log n}{j^{11/2}} + \frac{n^3}{j^3} + \frac{n^2}{j}\right). \tag{2.2}$$

[2] This is form (b) of the bound $O\left(n^{6/11} m^{9/11} \log^{2/11}(n^3/m) + n^{2/3} m^{2/3} + n + m\right)$ on the number of incidences between n points and m circles in 3-space. This follows from the best-known bound, with the same asymptotic value, for the incidence problem in the plane, as derived by Marcus and Tardos [11]; the previous bounds [2, 3, 5] were slightly weaker.

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To bound the number of incidences with the j-good spheres, we construct a $\frac{1}{r}$-*cutting* of $\mathcal{S}$. We will use the simpler version of cutting, in which we sample spheres of $\mathcal{S}$ by choosing each sphere in the sample independently at random with probability $\frac{Cr}{n}\log r$, for an appropriate sufficiently large constant C, and then construct the vertical decomposition of the arrangement of the random sample (see, *e.g.*, [6], [7]). As follows from the analysis of [7], the result is a set of an expected number of $O(r^3\beta(r)\log^3 r)$ relatively open cells, each of constant description complexity, such that, with high probability, each cell is *crossed* by (*i.e.*, is intersected by but not contained in) at most m/r spheres of $\mathcal{S}$. We will assume that our cutting does satisfy these properties, *i.e.*, it consists of $O(r^3\beta(r)\log^3 r)$ cells, and each cell is crossed by at most m/r spheres of $\mathcal{S}$. We may also assume that each cell contains at most n/r^3 points of P. To enforce this, we take each cell that contains tn/r^3 points, for any $t > 1$, and partition it into $t+1$ subcells in some generic way, such that each subcell contains at most n/r^3 points. The number of cells in this refined cutting is still $O(r^3\beta(r)\log^3 r)$.

For each cell τ separately, we bound the number of incidences between the points of $P \cap \tau$ and the spheres that cross τ, and then sum up the bounds over all cells. A cell τ has at most n/r^3 points and is crossed by at most m/r spheres. If a sphere contains more than j/η points of $P \cap \tau$, then, since it is a j-good sphere, it is locally η-non-degenerate in τ (*i.e.*, with respect to the points of $P \cap \tau$). The number of incidences between such spheres and the points of $P \cap \tau$ is at most $I(\frac{n}{r^3}, \frac{m}{r})$. The number of incidences with the other spheres in τ is $O(mj/r)$. The total, summed over all cells, is

$$\sum_{\tau} O\left(I\left(\frac{n}{r^3}, \frac{m}{r}\right) + \frac{mj}{r}\right) = O\left(r^3\beta(r)\log^3 r \cdot I\left(\frac{n}{r^3}, \frac{m}{r}\right) + mjr^2\beta(r)\log^3 r\right).$$

It remains to bound, for each cell τ, the number of incidences between the points of $P \cap \tau$ and the spheres of $\mathcal{S}$ that contain τ, which we do separately for cells of dimensions 2,1 and 0. (Cells of dimension 3 are 'exempt' from this analysis.) Any two-dimensional cell can be contained by at most one sphere, so these cells contribute a total of $O(n)$ incidences of this kind. For one-dimensional cells, we can restrict our attention to only these cells which are (portions of) edges in the arrangement of the sample spheres. (By choosing a generic coordinate frame, the other edges of the vertical decomposition will not contain points of P.) Each of the m spheres intersects the $O(r\log r)$ spheres of the sample along $O(r\log r)$ circles, and each such circle contains at most j points of P. Thus, the number of incidences contributed by the one-dimensional cells is $O(mjr\log r)$. As for zero-dimensional cells, which are simply the vertices of the arrangement of the sample spheres (as above, we may ignore the additional vertices created by the vertical decomposition), each of the m spheres σ contains $O(r^2\log^2 r)$ such vertices, which are intersection points of the $O(r\log r)$ circles of intersection of σ with the sample spheres. Hence, the total contribution of these cells is at most $O(mr^2\log^2 r)$ incidences. Thus, the total number of incidences, over all cells τ, between the points of $P \cap \tau$ and the spheres of $\mathcal{S}$ that contain τ is $O(n + mjr\log r + mr^2\log^2 r)$, which is asymptotically smaller than $O(n + mjr^2\beta(r)\log^3 r)$, and so, the total number of incidences with the good

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spheres is

$$I_{\text{good}} = O\left(r^3\beta(r)\log^3 r \cdot I\left(\frac{n}{r^3}, \frac{m}{r}\right) + mjr^2\beta(r)\log^3 r + n\right). \tag{2.3}$$

Summing both inequalities (2.2) and (2.3), we obtain the bound asserted in the lemma. □

Next, we simplify the recurrence by getting rid of j. We have, on one hand, $mjr^2\beta(r)\log^3 r$, which increases with j, and, on the other hand, the terms

$$\frac{n^4 \log n}{j^{11/2}} + \frac{n^3}{j^3} + \frac{n^2}{j},$$

which decrease with j, so we choose j so as to balance between them (ignoring the subpolynomial factors $\beta(r)\log^3 r$ and $\log n$). By comparing the increasing term with each of the decreasing terms, this leads to the choice

$$j = \max\left\{\frac{n^{8/13}}{m^{2/13}r^{4/13}}, \frac{n^{3/4}}{m^{1/4}r^{1/2}}, \frac{n}{m^{1/2}r}\right\},$$

which we simplify by putting

$$j = \frac{n^{8/13}}{m^{2/13}r^{4/13}} + \frac{n^{3/4}}{m^{1/4}r^{1/2}} + \frac{n}{m^{1/2}r}.$$

This implies

$$I(n,m) = O\left(r^3 l(r) I\left(\frac{n}{r^3}, \frac{m}{r}\right) + l(r)\left(n^{8/13}m^{11/13}r^{22/13}\log\left(\frac{n^4}{m}\right) + n^{3/4}m^{3/4}r^{3/2} + nm^{1/2}r\right)\right), \tag{2.4}$$

where $l(r) = \beta(r)\log^3 r$.

Our next and final step is to solve this recurrence, and derive the bound of Theorem 1.3.

Theorem 2.2 (*cf.* Theorem 1.3). *For any arbitrarily small $\varepsilon > 0$, there exists a constant A_ε, which depends on ε and η, such that we have*

$$I_\eta(n,m) \leqslant A_\varepsilon\left(n^{8/11+\varepsilon}m^{9/11+\varepsilon} + nm^{1/2}\right). \tag{2.5}$$

Proof. We use induction on m. Our end conditions are as follows.

(1) If m is smaller than some constant, say c, then $I \leqslant nm \leqslant cn$. Clearly, in this case (2.5) is also satisfied if we choose A_ε sufficiently large.

(2) If m is large enough relative to n, namely, if $m > Cn^4$, for some appropriate constant C depending on η, then $I = O(m) = O(n^4)$ (recall that only the case $m = O(n^4)$ can arise, so in the present situation $m = \Theta(n^4)$). This can be shown, *e.g.*, using the previous bound of $I = O(n^{4/5}m^{4/5} + nm^{2/3})$. In this case we also have $I = O(n^{8/11+\varepsilon}m^{9/11+\varepsilon} + nm^{1/2})$ as in (2.5).

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We now deal with the case where n and m are large, and $c < m < Cn^4$, in which case, we apply the induction hypothesis and rewrite (2.4) as

$$\begin{aligned} I(n,m) &\leqslant B\left(r^3 l(r) A_\varepsilon\left(\left(\frac{n}{r^3}\right)^{8/11+\varepsilon}\left(\frac{m}{r}\right)^{9/11+\varepsilon} + \left(\frac{n}{r^3}\right)\left(\frac{m}{r}\right)^{1/2}\right)\right. \\ &\quad \left. + l(r)\left(n^{8/13}m^{11/13}r^{22/13}\log\left(\frac{n^4}{m}\right) + n^{3/4}m^{3/4}r^{3/2} + nm^{1/2}r\right)\right) \\ &= B\cdot l(r) A_\varepsilon\left(\frac{n^{8/11+\varepsilon}m^{9/11+\varepsilon}}{r^{4\varepsilon}} + \frac{nm^{1/2}}{r^{1/2}}\right) \\ &\quad + B\cdot l(r)\left(n^{8/13}m^{11/13}r^{22/13}\log\left(\frac{n^4}{m}\right) + n^{3/4}m^{3/4}r^{3/2} + nm^{1/2}r\right), \end{aligned}$$

for an appropriate absolute constant B. We now choose r sufficiently large that $\frac{B\cdot l(r)}{r^{4\varepsilon}} < \frac{1}{2}$, and then choose A_ε sufficiently large that the last terms (in the second line of the bound) are at most $\frac{1}{2}A_\varepsilon(n^{8/11+\varepsilon}m^{9/11+\varepsilon} + nm^{1/2})$. This can be done because the terms $n^{8/13}m^{11/13}r^{22/13}\log\left(\frac{n^4}{m}\right)$ and $n^{3/4}m^{3/4}r^{3/2}$ are dominated by the terms $n^{8/11+\varepsilon}m^{9/11+\varepsilon} + nm^{1/2}$, for $m < Cn^4$. Thus,

$$I(n,m) \leqslant A_\varepsilon\left(n^{8/11+\varepsilon}m^{9/11+\varepsilon} + nm^{1/2}\right),$$

establishing the induction step and thereby completing the proof of the theorem, and consequently also the proof of Theorem 1.3. □

Remark. The reader may be left wondering where the exponents $8/11$ and $9/11$ 'popped up' from, since they do not appear explicitly in (2.4). The answer is that a solution of (2.4) with a leading term $n^\alpha m^\beta$ must satisfy the two inequalities $3\alpha + \beta > 3$, and $\alpha + 4\beta > 4$. The first inequality is needed to control the homogeneous part of the recurrence, *i.e.*, the term $r^3 l(r)\cdot I\left(\frac{n}{r^3}, \frac{m}{r}\right)$, and the second inequality is needed to ensure that the term $n^{8/13}m^{11/13}$ is dominated by $n^\alpha m^\beta$ when $m = O(n^4)$, as can be easily verified. The exponents $8/11 + \varepsilon$ and $9/11 + \varepsilon$, for arbitrarily small ε, are, in a sense, the best solution of these inequalities.

3. Applications of the new bound

As already noted, non-degenerate spheres have been used in some previous works. In [4] they were used in the analysis of distinct distances in $\mathbb{R}^3$, and in [1] we used them to bound the number of mutually similar simplices in d dimensions. The improved bound of Theorem 1.1, once plugged into these analyses, gives improved results for the respective problems. In this section we apply Theorem 1.1 to the analysis of [1] of similar triangles in $\mathbb{R}^3$ and prove Theorem 1.2. Recall that the previous bound, established in [1], on the number of similar triangles spanned by n points in $\mathbb{R}^3$, is $O(n^{13/6}) = O(n^{2.1667})$. With the use of Theorem 1.1, the bound becomes $O^*(n^{58/27}) = O^*(n^{2.1482})$, as we prove next.

Proof of Theorem 1.2. Fix a parameter $k < n$, and denote by $M(n,k)$ the maximum possible number of k-rich non-degenerate spheres spanned by n points in $\mathbb{R}^3$ (for some fixed constant degeneracy factor η). As shown in [1], the number T of triangles spanned

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by n points in $\mathbb{R}^3$ and similar to a fixed triangle is upper-bounded by

$$T = O\bigl(n^2k^{1/3} + M(n,k)k^{4/3}\bigr),$$

for any choice of k. The optimal choice is of course the one that minimizes the expression on the right-hand side. In [1] we used the old bound $M(n,k) = O(n^4/k^5 + n^3/k^3)$, for which the optimal k is $k = n^{1/2}$, and the resulting bound is $T = O(n^{13/6})$. If we use the new bound, and assume $k < n^{4/7}$, then we get

$$T = O^*\left(n^2k^{1/3} + \frac{n^4}{k^{25/6}} + \frac{n^2}{k^{2/3}}\right) = O^*\left(n^2k^{1/3} + \frac{n^4}{k^{25/6}}\right).$$

We choose $k = n^{4/9}$, which is indeed smaller than $n^{4/7}$, and get

$$T = O^*(n^{58/27}),$$

as claimed. □

Theorem 1.1 can also be applied to yield a better result than the one achieved in [4] for the number, denoted by $g_3(n)$, of distinct distances determined by n points in $\mathbb{R}^3$, but this result has already been surpassed by Solymosi and Vu [12]. Specifically, the old bound in [4] was $g_3(n) = \Omega^*(n^{77/141}) = \Omega(n^{0.546})$. With the new bound we get $g_3(n) = \Omega^*(n^{5/9}) = \Omega(n^{0.555})$, but the Solymosi–Vu bound [12] is $g_3(n) = \Omega(n^{0.5644})$.[3] (The upper bound on the minimum number of distinct distances is $O(n^{2/3})$.)

4. Conclusion

We have shown an improved bound on the number of incidences between points and non-degenerate spheres in $\mathbb{R}^3$, and demonstrated its usefulness by improving the bound on the number of mutually similar triangles in $\mathbb{R}^3$. We believe that our main results (Theorems 1.1 and 1.3) can be improved still further. We also believe that our results can be extended to higher-dimensional non-degenerate spheres, and that they will find applications in various other problems of geometric incidences and repeated subconfigurations.

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[3] In their paper [12], they specify a slightly weaker bound of $g_3(n) = \Omega(n^{0.5643})$. Their analysis depends on the respective two-dimensional lower bound, *i.e.*, on the number of distinct distances between n points in the plane. This lower bound was later improved by Katz and Tardos [10], which consequently also improved the lower bound in three dimensions. In more detail, Solymosi and Vu [12] showed that $g_3(n) = \Omega^*(n^{\omega_3})$, where $\omega_3 \geqslant 3/(3 + 2/\omega_2)$, and ω_2 is any number such that $g_2(n) = \Omega^*(n^{\omega_2})$. As of today, the best estimate for ω_2, due to Katz and Tardos [10], is $\omega_2 \geqslant 0.8641$, which implies $\omega_3 \geqslant 0.5644$. Solymosi and Vu [12] used the weaker bound $\omega_2 \geqslant 0.8635$, which only implies $\omega_3 \geqslant 0.5643$.

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