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ARTICLE

A remark on unique continuation of $\bar{\partial}$

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Abstract

In this article, we extend our earlier unique continuation results of Pan and Zhang (2025, *Comm. Anal. Geom.* 33, 1975–2014) for the Schrödinger-type inequality $|\bar{\partial}u| \leq V|u|$ on a domain in $\mathbb{C}^n$, by removing the smoothness assumption on solutions $u = (u_1, \ldots, u_N)$. More specifically, we establish the unique continuation property for $W^{1,1}_{loc}$ solutions when the potential $V \in L^p_{loc}$, $p > 2n$, and for $W^{1,2n+\epsilon}_{loc}$ solutions when $V \in L^{2n}_{loc}$ with $N = 1$ or $n = 2$. Although the unique continuation property fails in general if $V \in L^p_{loc}$, $p < 2n$, we show that the property still holds for $W^{1,1}_{loc}$ solutions when V is a small constant multiple of $\frac{1}{|z|}$. The key idea that allows us to lower the regularity assumption is a careful analysis of how regularity and flatness are transferred to the sliced functions along complex radial directions.

1. Introduction

Let Ω be a domain in $\mathbb{C}^n$. Suppose $u : \Omega \to \mathbb{C}^N$ with $u \in W^{1,1}_{loc}(\Omega)$, and satisfies the following Schrödinger-type inequality:

$$|\bar{\partial}u| \leq V|u| \text{ a.e. on } \Omega \tag{1}$$

for some nonnegative locally Lebesgue integrable function V on Ω. The goal of the present article is to investigate the unique continuation property for (1). More precisely, we study whether a Sobolev function u satisfying (1) must vanish identically if u vanishes to infinite order in the L^1 sense at some $z_0 \in \Omega$. Here, a function $u \in L^q_{loc}(\Omega), q \geq 1$ is said to vanish to infinite order (or, be flat) in the L^q sense at a point $z_0 \in \Omega$, if for all $m \geq 1$,

$$\lim_{r \to 0} r^{-m} \int_{|z-z_0|<r} |u(z)|^q dv_z = 0.$$

While Example 1 illustrates the general failure of the unique continuation property for L^p_{loc} potentials with $p < 2n$, it was shown in Pan and Zhang [10], the property holds for smooth solutions to (1) with L^p_{loc} potentials, $p > 2n$, and L^{2n}_{loc} potentials if either $N = 1$ or $n = 2$. Although the smooth category remains of primary interest, our goal in this article is to extend the unique continuation

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property to general Sobolev solutions, which are of broader significance. See Examples 4 and 5 for cases of (1) that admit ample nonsmooth Sobolev solutions (vanishing to a finite order).

We first show that when the potential $V \in L^p_{loc}$ for some $p > 2n$, the unique continuation property holds for $W^{1,1}_{loc}$ solutions to (1). In particular, this extends a result of Bell and Lempert [2] for bounded potentials. We also refer to a more recent work of Shi [11], which establishes a related result under stronger assumptions on both the potentials and the regularity of the solutions.

Theorem 1.1. *Let Ω be a domain in $\mathbb{C}^n$. Suppose $u : \Omega \to \mathbb{C}^N$ with $u \in W^{1,1}_{loc}(\Omega)$, and satisfies $|\bar\partial u| \le V|u|$ a.e. on Ω for $V \in L^p_{loc}(\Omega)$, $p > 2n$. If u vanishes to infinite order in the L^1 sense at some $z_0 \in \Omega$, then u vanishes identically.*

The proof of the theorem makes use of the complex polar coordinate formula in Lemma 3.1, which reduces the unique continuation problem in higher-dimensional domains to a known one-dimensional result in Pan and Zhang [9] (see Theorem 2.1). While this approach is similar as in Pan and Zhang [10], when working with general Sobolev solutions, in contrast to the smooth case, the inequality (1) and the flatness property of Sobolev solutions do not immediately extend to their sliced counterparts. The novelty of the proof lies in showing that the minimal $W^{1,1}_{loc}$ Sobolev regularity of u is sufficient to guarantee that the sliced solutions satisfy all the assumptions in Theorem 2.1. In fact, the assumptions $u \in W^{1,1}_{loc}$ and $V \in L^p_{loc}$, $p > 2n$ here actually imply $u \in W^{1,p}_{loc}$, following a standard bootstrap argument. The key Lemma 3.3, together with this observation, then transforms a $W^{1,1}_{loc}$ solution of (1) to a family of $W^{1,2}_{loc}$ solutions to some Schrödinger-type inequalities along almost every complex radial direction. Moreover, we establish, in Lemma 3.4, an equivalence between the L^1 flatness of $W^{1,p}_{loc}$ functions and a much stronger geometric flatness given by (8), which further passes the flatness onto the sliced solutions. As a consequence, Theorem 2.1 readily applies to yield the desired unique continuation property.

Next, we consider the case when the potential V lies in the critical class L^{2n}_{loc}. The unique continuation results are proved in the following two cases: one for the case $N = 1$ (namely, u is a scalar function), and the other for the case $n = 2$. Since the bootstrap argument fails to improve the Sobolev regularity of solutions when $V \in L^{2n}_{loc}$ only, in the proofs of both theorems below, we rely on the assumption $u \in W^{1,2n+\epsilon}_{loc}$ in order to apply Lemmas 3.3 and 3.4. It would be desirable to know whether this regularity assumption on u can be weakened further.

Theorem 1.2. *Let Ω be a domain in $\mathbb{C}^n$. Suppose $u : \Omega \to \mathbb{C}$ with $u \in W^{1,2n+\epsilon}_{loc}(\Omega)$ for some $\epsilon > 0$, and satisfies $|\bar\partial u| \le V|u|$ a.e. on Ω for some $V \in L^{2n}_{loc}(\Omega)$. If u vanishes to infinite order in the L^1 sense at some $z_0 \in \Omega$, then u vanishes identically.*

Theorem 1.3. *Let Ω be a domain in $\mathbb{C}^2$. Suppose $u : \Omega \to \mathbb{C}^N$ with $u \in W^{1,2n+\epsilon}_{loc}(\Omega)$ for some $\epsilon > 0$, and satisfies $|\bar\partial u| \le V|u|$ a.e. on Ω for some $V \in L^4_{loc}(\Omega)$. If u vanishes to infinite order in the L^1 sense at some $z_0 \in \Omega$, then u vanishes identically.*

In the special case $n = 1$, Theorem 1.2 may be compared with the earlier unique continuation result, Theorem 2.1, as follows. Whereas Theorem 2.1 assumes that $u \in W^{1,2}_{loc}$ and that u vanishes to infinite order in the L^2 sense at a point, Theorem 1.2 replaces this L^2 flatness assumption on the solution with L^1 flatness. This is achieved at the expense of the slightly stronger regularity condition $u \in W^{1,2+\epsilon}_{loc}$ for some $\epsilon > 0$.

Interestingly, for the Laplacian operator Δ, the unique continuation property for $W^{2,2}_{loc}(\Omega)$ solutions of the inequality

$$|\Delta u| \le V|\nabla u| \ \text{ on } \ \Omega \subset \mathbb{R}^d$$

with $V \in L^d_{loc}(\Omega)$ holds when $d = 2, 3, 4$, but fails in general when $d \ge 5$. See the works of Chanillo and Sawyer [4] and Wolff [12], [13]. In contrast, Theorem 1.2 shows that, in the critical

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$V \in L^{\dim_{\mathbb{R}} \Omega}_{loc}(\Omega)$ case, the unique continuation property for $\bar\partial$ is dimension-independent, highlighting a significant difference from the case of Δ.

Finally, we explore a special case where V is a constant multiple of $\frac{1}{|z|}$. Note that $\frac{1}{|z|} \notin L^{2n}_{loc}$ near 0. Although, as in the L^{2n}_{loc} potential case, the bootstrap argument does not in general improve the Sobolev regularity of solutions near 0, thanks to Lemma 4.4, the additional flatness of u at 0 allows us to eventually push u to fall in $W^{1,q}_{loc}$ for all $q < \infty$. In view of this, the $N = 1$ case in the following theorem is simply a direct consequence of [10, Theorem 5.1] for $W^{1,2}_{loc}$ solutions. For $N \geq 2$ case, one can use a similar approach as in Theorem 1.1 to weaken the smoothness assumption of u in [10, Theorem 1.5] to merely $W^{1,1}_{loc}$ regularity. It should also be mentioned that when $N \geq 2$, the unique continuation property fails in general if C is large, see Example 2.

Theorem 1.4. *Let Ω be a domain in $\mathbb{C}^n$ and $0 \in \Omega$. Let $u : \Omega \to \mathbb{C}^N$ with $u \in W^{1,1}_{loc}(\Omega)$, and satisfy $|\bar\partial u| \leq \frac{C}{|z|}|u|$ a.e. on Ω. Assume u vanishes to infinite order in the L^1 sense at $0 \in \Omega$.*

1) *If $N = 1$, then u vanishes identically.*
2) *If $N \geq 2$ and $C < \frac{1}{4}$, then u vanishes identically.*

2. Known results and examples

In this section, we summarize several known unique continuation properties for $W^{1,2}_{loc}$ functions satisfying (1) that will be used in our article, along with counter-examples corresponding to certain types of potentials.

First, let us clarify the meaning of the inequality (1). We say that a function $u = (u_1, \dots, u_N) \in W^{1,1}_{loc}(\Omega)$ satisfies (1) if

$$|\bar\partial u| := \left(\sum_{j=1}^{n} \sum_{k=1}^{N} |\bar\partial_j u_k|^2 \right)^{\frac{1}{2}} \leq V \left(\sum_{k=1}^{N} |u_k|^2 \right)^{\frac{1}{2}} := V|u| \text{ a.e. on } \Omega.$$

It is worth pointing out that every such $u \in W^{1,1}_{loc}(\Omega)$ satisfying $|\bar\partial u| \leq V|u|$ a.e. on Ω with $V \in L^p_{loc}(\Omega)$ is a weak solution to a Schrödinger-type equation of $\bar\partial$ below

$$\bar\partial u = u\mathcal{V} \text{ on } \Omega$$

for some $N \times N$ matrix $\mathcal{V} = (\mathcal{V}_{jk})$, with entries $\mathcal{V}_{jk}$ given by $(0,1)$ forms $\mathcal{V}_{jk} := \frac{\overline{u_j}\bar\partial u_k}{|u|^2} \in L^p_{loc}(\Omega)$. For this reason, we will occasionally refer to u as a solution to (1) whenever it satisfies (1).

The first theorem below discusses unique continuation properties to (1) with L^2_{loc} potentials.

Theorem 2.1 [9, Theorems 1.1 and 1.3]. *Let Ω be a domain in $\mathbb{C}^n$. Suppose $u : \Omega \to \mathbb{C}^N$ with $u \in W^{1,2}_{loc}(\Omega)$, and satisfies $|\bar\partial u| \leq V|u|$ a.e. on Ω for some $V \in L^2_{loc}(\Omega)$.*

1) *The weak unique continuation holds: if u vanishes in an open subset of Ω, then u vanishes identically.*
2) *If $n = 1$, then the (strong) unique continuation holds: if u vanishes to infinite order in the L^2 sense at some $z_0 \in \Omega$, then u vanishes identically.*

It is worthy pointing out that when $n \geq 1$, the (strong) unique continuation property fails in general if the potential does not belong to L^{2n}_{loc}.

Example 1. For each $1 \leq p < 2n$, let $\epsilon \in (0, \frac{2n}{p} - 1)$ and consider

$$|\bar\partial u| \leq V|u| := \frac{\epsilon}{2|z|^{\epsilon+1}}|u| \text{ on } B_1 \subset \mathbb{C}^n.$$

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Note that $V \in L^p(B_1)$, and $u_0 = e^{-\frac{1}{|z|^\epsilon}}$ is a nontrivial smooth solution to the above equation that vanishes to infinite order at 0.

Despite Example 1, the unique continuation property can still be expected for some special forms of potentials not in L^{2n}_{loc}, for instance, when the potential is a multiple of $\frac{1}{|z|}$. In particular, Theorem 2.2 below establishes the property in the case $N = 1$, while Theorem 2.3 establishes the property in the case $n = 1$ for general $N \geq 1$, but with a small constant multiple.

Theorem 2.2 [10, Theorem 5.1]. *Let Ω be a domain in $\mathbb{C}^n$ and $0 \in \Omega$. Let $u : \Omega \to \mathbb{C}$ with $u \in W^{1,2}_{loc}(\Omega)$, and satisfies $|\bar\partial u| \leq \frac{C}{|z|}|u|$ a.e. on Ω for some constant $C > 0$. If u vanishes to infinite order in the L^2 sense at 0, then u vanishes identically.*

Theorem 2.3 [10, Theorem 6.1]. *Let Ω be a domain in $\mathbb{C}$ and $0 \in \Omega$. Let $u : \Omega \to \mathbb{C}^N$ with $u \in W^{1,2}_{loc}(\Omega)$, and satisfy $|\bar\partial u| \leq \frac{C}{|z|}|u|$ a.e. on Ω for some positive constant $C < \frac{1}{4}$. If u vanishes to infinite order in the L^2 sense at 0, then u vanishes identically.*

Note that when $N \geq 2$ and C is large, Theorem 2.3 no longer holds in general, as indicated by an example below of the first author and Wolff [8], or [1] by Alinhac and Baouendi.

Example 2. Let $v_0 : \mathbb{C} \to \mathbb{C}$ be the nontrivial smooth scalar function constructed in Pan and Wolff [8] that vanishes to infinite order at 0 and satisfies $|\triangle v_0| \leq \frac{C^\sharp}{|z|}|\nabla v_0|$ on $\mathbb{C}$ for some constant $C^\sharp > 0$. Letting $u_0 := (\partial \Re v_0, \partial \Im v_0)$, then $u_0 : \mathbb{C} \to \mathbb{C}^2$ is smooth, vanishes to infinite order at 0, and satisfies $|\bar\partial u_0| \leq \frac{C^\sharp}{2|z|}|u_0|$ on $\mathbb{C}$.

In the special case when the source dimension $n = 1$, the unique continuation property also holds when the potentials take on the following hybrid forms involving both powers of $\frac{1}{|z|}$ and Lebesgue functions:

$$|z|^{-\frac{\beta-1}{\beta}} V, \quad \text{where } V \in L^{2\beta}_{loc}(\Omega), 1 < \beta < \infty. \tag{2}$$

Note that none of these potentials belong to L^2_{loc}.

The following theorem establishes the unique continuation property for scalar solutions ($N = 1$) to equations with hybrid potentials.

Theorem 2.4 [10, Theorem 5.5]. *Let Ω be a domain in $\mathbb{C}$ containing 0 and $1 < \beta < \infty$. Suppose $u : \Omega \to \mathbb{C}$ with $u \in W^{1,2}_{loc}(\Omega)$, and satisfies*

$$|\bar\partial u| \leq |z|^{-\frac{\beta-1}{\beta}} V|u| \quad \text{a.e. on } \Omega$$

for some $V \in L^{2\beta}_{loc}(\Omega)$. If u vanishes to infinite order in the L^2 sense at 0, then u vanishes identically.

In the case when $\beta = 2$, one can even expect the above theorem to remain valid for vector solutions (i.e., $N \geq 1$).

Theorem 2.5 [10, Theorem 7.1]. *Let Ω be a domain in $\mathbb{C}$ and $0 \in \Omega$. Suppose $u : \Omega \to \mathbb{C}^N$ with $u \in W^{1,2}_{loc}(\Omega)$, and satisfies*

$$|\bar\partial u| \leq |z|^{-\frac{1}{2}} V|u| \quad \text{a.e. on } \Omega$$

for some $V \in L^4_{loc}(\Omega)$. If u vanishes to infinite order in the L^2 sense at 0, then u vanishes identically.

3. Properties of sliced functions

As will be seen in the next section, we convert the unique continuation property in the higher source-dimensional case to the complex one-dimensional case where Theorem 2.1 can be applied. For this

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purpose, we slice Sobolev solutions to (1) and the potential along almost all complex one-dimensional radial directions.

In order to justify this approach, the following complex polar coordinate formula (see [7, p. 260]) plays an essential role. The proof can be found, for instance, in [10, Lemma 4.2]. Denote by S^{2n-1} the unit sphere in $\mathbb{C}^n$. Let B_r be the open ball centered at 0 of radius r in $\mathbb{C}^n$, and D_r be the open disk centered at 0 of radius r in $\mathbb{C}$.

Lemma 3.1. *Let $u \in L^1(B_r)$. Then, for a.e. $\zeta \in S^{2n-1}$, $|w|^{2n-2}u(w\zeta)$ as a function of $w \in D_r$ is in $L^1(D_r)$, with*

$$\int_{|z|<r} u(z)dv_z = \frac{1}{2\pi}\int_{|\zeta|=1}\int_{|w|<r} |w|^{2n-2}u(w\zeta)dv_w dS_\zeta.$$

Corollary 3.2. *Let $u \in L^p(B_r)$ for some $p > n$ $(= \dim_{\mathbb{C}} B_r)$. Then, for a.e. fixed $\zeta \in S^{2n-1}$, $v(w) := u(w\zeta)$ as a function of $w \in D_r$ is in $L^q(D_r)$ for all $1 \le q < \frac{p}{n}$. In particular, if $u \in L^p(B_r)$ for some $p > 2n$, then $v \in L^2(D_r)$.*

Proof. By Lemma 3.1,

$$\int_{|z|<r} |u(z)|^p dv_z = \frac{1}{2\pi}\int_{|\zeta|=1}\int_{|w|<r} |w|^{2n-2}|u(w\zeta)|^p dv_w dS_\zeta.$$

Then, for a.e. $\zeta \in S^{2n-1}$,

$$\int_{|w|<r} |w|^{2n-2}|u(w\zeta)|^p dv_w < \infty. \tag{3}$$

Making use of Hölder's inequality,

$$\begin{aligned}\int_{|w|<r} |v(w)|^q dv_w &= \int_{|w|<r} |w|^{\frac{(2n-2)q}{p}} |v(w)|^q \cdot |w|^{-\frac{(2n-2)q}{p}} dv_w \\ &\le \left(\int_{|w|<r} |w|^{2n-2}|u(w\zeta)|^p dv_w\right)^{\frac{q}{p}} \left(\int_{|w|<r} |w|^{-\frac{(2n-2)q}{p}\frac{p}{p-q}} dv_w\right)^{\frac{p-q}{p}} \\ &= \left(\int_{|w|<r} |w|^{2n-2}|u(w\zeta)|^p dv_w\right)^{\frac{q}{p}} \left(\int_{|w|<r} |w|^{-\frac{(2n-2)q}{p-q}} dv_w\right)^{\frac{p-q}{p}}.\end{aligned}$$

Since $q < \frac{p}{n}$, we have $\frac{(2n-2)q}{p-q} < 2$ and thus $\int_{|w|<r} |w|^{-\frac{(2n-2)q}{p-q}} dv_w < \infty$. This, combined with (3), proves the corollary. □

In the rest of the article, we will repeatedly use the latter part of Corollary 3.2. The following example demonstrates that the property $v \in L^2$ fails in general when $u \in L^{2n}$ merely.

Example 3. Assume $n \ge 2$. For each $\epsilon \in \left(\frac{1}{2}, \frac{2n-1}{2n}\right)$, consider

$$V(z) := \frac{(-\ln|z|)^{\epsilon-1}}{|z|}, \quad z \in B_{\frac{1}{2}} \subset \mathbb{C}^n.$$

One checks directly that $V \notin L^p(B_{\frac{1}{2}})$ for any $p > 2n$, and $V \in L^{2n}(B_{\frac{1}{2}})$ since $\epsilon < \frac{2n-1}{2n}$.

Moreover, for each $\zeta \in S^{2n-1}$, the complex radial restriction of V is

$$\tilde{V}(w) := V(w\zeta) = \frac{(-\ln|w|)^{\epsilon-1}}{|w|}, \quad w \in D_{\frac{1}{2}}.$$

It is easy to see that $\tilde{V} \notin L^2_{loc}(D_{\frac{1}{2}})$ since $\epsilon > \frac{1}{2}$.

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Making use of Corollary 3.2, we establish the regularity of the resulting sliced functions for $W^{1,p}_{loc}$, $p > 2n$ functions below.

Lemma 3.3. *Suppose $u \in W^{1,p}(B_r)$ for some $p > 2n$ $(= \dim_{\mathbb{R}} B_r)$. Then, for a.e. $\zeta \in S^{2n-1}$, $v(w) := u(w\zeta)$ as a function of $w \in D_r$ belongs to $W^{1,2}(D_r)$. Moreover,*

$$\nabla v(w) = \zeta \cdot \nabla u(w\zeta), \quad w \in D_r \tag{4}$$

in the sense of distributions.

Proof. We only need to show (4). In fact, since $u \in W^{1,p}(B_r)$, $p > 2n$, once (4) holds, we can apply Corollary 3.2 to u and ∇u, respectively, and obtain $v \in W^{1,2}(D_r)$.

Let $u_j \in C^\infty(B_r) \cap W^{1,p}(B_r)$ be such that $u_j \to u$ in the $W^{1,p}(B_r)$ norm and $v_j(w) := u_j(w\zeta)$, $w \in D_r$. Then

$$\nabla v_j(w) = \zeta \cdot \nabla u_j(w\zeta), \quad w \in D_r. \tag{5}$$

Since u is continuous on B_r (and so is v on D_r), by the Sobolev embedding theorem, there exists some constant $C > 0$ such that

$$\|v_j - v\|_{C(D_r)} \le \|u_j - u\|_{C(B_r)} \le C\|u_j - u\|_{W^{1,p}(B_r)} \to 0$$

as $j \to 0$. In particular,

$$v_j \to v \ \text{ on } \ D_r \tag{6}$$

in the sense of distributions.

On the other hand, applying Lemma 3.1 to $|\nabla u - \nabla u_j|^p$, the function

$$g_j(\zeta) := \int_{|w|<r} |w|^{2n-2} |\nabla u(w\zeta) - \nabla u_j(w\zeta)|^p dv_w, \ \ \zeta \in S^{2n-1}$$

satisfies

$$\begin{aligned}\int_{|\zeta|=1} |g_j(\zeta)| dS_\zeta &= \int_{|\zeta|=1} \int_{|w|<r} |w|^{2n-2} |\nabla u(w\zeta) - \nabla u_j(w\zeta)|^p dv_w dS_\zeta \\ &= 2\pi \int_{|z|<r} |\nabla u(z) - \nabla u_j(z)|^p dv_z \to 0\end{aligned}$$

as $j \to \infty$. Hence, by passing to a subsequence if necessary (see, for instance, [3, Theorem 4.9]), one has, for a.e. $\zeta \in S^{2n-1}$,

$$g_j(\zeta) \to 0$$

as $j \to 0$. Making use of Hölder's inequality,

$$\begin{aligned}\int_{|w|<r} |\zeta \cdot \nabla u_j(w\zeta) - \zeta \cdot \nabla u(w\zeta)| dv_w &\le \int_{|w|<r} |w|^{-\frac{2n-2}{p}} \cdot |w|^{\frac{2n-2}{p}} |\nabla u_j(w\zeta) - \nabla u(w\zeta)| dv_w \\ &\le \left(\int_{|w|<r} |w|^{-\frac{2n-2}{p-1}} dv_w \right)^{\frac{p-1}{p}} \cdot (g_j(\zeta))^{\frac{1}{p}} \to 0\end{aligned}$$

as $j \to \infty$. Here, we used the fact that $p > 2n$, so $\frac{2n-2}{p-1} < 2$ particularly. This implies that

$$\zeta \cdot \nabla u_j(w\zeta) \to \zeta \cdot \nabla u(w\zeta) \ \text{ on } \ D_r \tag{7}$$

in the sense of distributions. Equation (4) is thus proved in view of (5)–(7). □

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We next examine how flatness extends to sliced functions along the radial directions. From the definition, it is immediate that if a function u is flat in the L^p sense at some z_0 and $p > q \geq 1$, then by Hölder's inequality, u is automatically flat in the L^q sense at z_0. For $W^{1,p}_{loc}$, $p > 2n$ functions, by establishing an equivalence between the L^1 flatness and a more refined notion of flatness (8), the following lemma shows that such flatness can be extended to their restrictions along radial directions.

Lemma 3.4. *Suppose $u \in W^{1,p}(B_r)$ for some $p > 2n(= \dim_{\mathbb{R}} B_r)$. Then u vanishes to infinite order in the L^1 sense at 0 if and only if for every $m \geq 1$,*

$$|u(z)| = O(|z|^m) \ \text{ for all } \ |z| << 1. \tag{8}$$

In particular, if u vanishes to infinite order in the L^1 sense at $0 \in B_r$, then for a.e. $\zeta \in S^{2n-1}$, $v(w) := u(w\zeta)$ as a function of $w \in D_r$ vanishes to infinite order in the L^2 sense at $0 \in D_r$.

Proof. The backward direction is obvious by definition of the L^1 flatness. In fact, one can also easily check that, for any given $q \geq 1$, as long as $u \in L^q$ near 0 and satisfies (8), u vanishes to infinite order in the L^q sense at 0.

Assume u vanishes to infinite order in the L^1 sense at 0. Since $p > 2n$, by the Sobolev embedding theorem, u is continuous on B_r with

$$\sup_{|z|<r} |u| + \int_{|z|<r} |\nabla u|^p \leq C_0 \tag{9}$$

for some constant $C_0 > 0$. Fix some q with $2n < q < p$. For every $m \geq 1$, by the continuity of u and the L^1 flatness of u at 0, one has $u(0) = 0$ and

$$\int_{|z|<t} |u|^{\frac{qp}{p-q}} \leq C_0^{\frac{qp}{p-q}-1} \int_{|z|<t} |u| \leq O(t^{\frac{2pqm}{p-q}}), \ \text{ for all } \ t << 1. \tag{10}$$

Letting $f := u^2$, then $f \in W^{1,p}(B_r)$ with $\nabla f = 2u\nabla u$ on B_r. Moreover, making use of Hölder's inequality, (9), and (10),

$$\begin{aligned} \int_{|z|<t} |\nabla f|^q = 2^q \int_{|z|<t} |u|^q |\nabla u|^q &\leq 2^q \left(\int_{|z|<t} |u|^{\frac{qp}{p-q}} \right)^{\frac{p-q}{p}} \left(\int_{|z|<t} |\nabla u|^p \right)^{\frac{q}{p}} \\ &\leq 2^q C_0^{\frac{q}{p}} \left(\int_{|z|<t} |u|^{\frac{qp}{p-q}} \right)^{\frac{p-q}{p}} \leq O(t^{2qm}), \ \text{ for all } \ t << 1. \end{aligned} \tag{11}$$

On the other hand, since $f(0) = 0$ and $q > 2n$, there exists a constant $C_1 > 0$ such that

$$\sup_{|z| \leq \frac{t}{2}} |f| \leq C_1 t^{1-\frac{2n}{q}} \left(\int_{B_t} |\nabla f|^q \right)^{\frac{1}{q}}, \ \text{ for all } \ t < r.$$

See, for instance, [5, p. 283]. Together with (11), we get

$$|u(z)|^2 = |f(z)| \leq C_1 \left(\int_{B_{2|z|}} |\nabla f|^q \right)^{\frac{1}{q}} \leq O(|z|^{2m}), \ \text{ for all } \ |z| << 1.$$

This proves the forward direction.

Finally, if u vanishes to infinite order in the L^1 sense at 0, then u satisfies (8) by the equivalence of the two types of flatness. Hence, for a.e. $\zeta \in S^{2n-1}$, (8) holds true for f as well. In particular, since $f \in L^2$ near 0 due to Lemma 3.3, f vanishes to infinite order in the L^2 sense at 0. □

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We would like to note that, although both Lemmas 3.3 and 3.4 are stated for scalar functions for simplicity of notations, they can be seamlessly extended to the case of vector-valued functions.

4. Proof of the main theorems

In this section, we shall prove Theorems 1.1–1.4. Let us start by stating a local ellipticity lemma of $\bar\partial$, which will be frequently used in the bootstrap argument.

Lemma 4.1 [10, Lemma 3.1]. *Let Ω be a domain in $\mathbb{C}^n$ and $1 < p < \infty$. Let V be a $\bar\partial$-closed $(0,1)$ form on Ω with $L^p_{loc}(\Omega)$ coefficients. Then every solution to $\bar\partial f = V$ on Ω in the sense of distributions belongs to $W^{1,p}_{loc}(\Omega)$.*

Proof of Theorem 1.1. Without loss of generality, let $z_0 = 0$, and r be small such that $B_r \subset \Omega$. Since $u \in W^{1,1}_{loc}(\Omega)$, $u \in L^{\frac{2n}{2n-1}}_{loc}(B_r)$ by the Sobolev embedding theorem. As an application of Hölder's inequality, $Vu \in L^{\frac{2n}{(2n-1)+\frac{2n}{p}}}_{loc}(B_r)$. It follows from the inequality (1) and Lemma 4.1 that $u \in W^{1,\frac{2n}{(2n-1)+\frac{2n}{p}}}_{loc}(B_r) \subset L^{\frac{2n}{(2n-1)-(1-\frac{2n}{p})}}_{loc}(B_r)$. This indicates a gain in the integrability of u from $L^{\frac{2n}{2n-1}}_{loc}(B_r)$ to $L^{\frac{2n}{(2n-1)-\epsilon_0}}_{loc}(B_r)$ with $\epsilon_0 := 1 - \frac{2n}{p} > 0$, as a consequence of the condition $p > 2n$.

Apply a bootstrap argument by repeating the preceding steps $m_0 := \left\lfloor \frac{2n-1}{\epsilon_0} \right\rfloor$ times. Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Next, we slightly adjust the above integrability-gaining steps to show that $u \in C(B_r)$ by splitting into two cases.

a) If $2n - 1 > m_0\epsilon_0$, then $u \in L^{\frac{2n}{(2n-1)-m_0\epsilon_0}}_{loc}(B_r)$, and $2n - 1 < (m_0+1)\epsilon_0$, which is equivalent to $(2n-1) - m_0\epsilon_0 + \frac{2n}{p} < 1$. Consequently, $Vu \in L^{\frac{2n}{(2n-1)-m_0\epsilon_0+\frac{2n}{p}}}_{loc}(B_r)$ and further $u \in W^{1,\frac{2n}{(2n-1)-m_0\epsilon_0+\frac{2n}{p}}}_{loc}(B_r)$. Since $\frac{2n}{(2n-1)-m_0\epsilon_0+\frac{2n}{p}} > 2n$, we conclude that $u \in C(B_r)$.

b) If $2n - 1 = m_0\epsilon_0$, then $u \in L^q_{loc}(B_r)$ for any $q < \infty$. Hence, $Vu \in L^q_{loc}(B_r)$ for any $q < p$. Since $p > 2n$, we may choose $q_0 > 2n$ with $Vu \in L^{q_0}_{loc}(B_r)$. It then follows that $u \in W^{1,q_0}_{loc}(B_r) \subset C(B_r)$.

Now that $u \in C(B_r)$ in both cases, one obtains $Vu \in L^p_{loc}(B_r)$ and finally $u \in W^{1,p}_{loc}(B_r)$.

For each fixed $\zeta \in S^{2n-1}$, let $v(w) := u(w\zeta)$ and $\tilde V(w) := V(w\zeta), w \in D_r$. Since $V \in L^p_{loc}(B_r)$, $p > 2n$, by Corollary 3.2, we have $\tilde V \in L^2_{loc}(D_r)$. On the other hand, it follows from Lemma 3.4 and the L^1 flatness assumption of u at 0 that v vanishes to infinite order in the L^2 sense at 0. Moreover, $v \in W^{1,2}_{loc}(D_r)$ by Lemma 3.3, and as a consequence of (4),

$$|\bar\partial v(w)| = |\zeta \cdot \bar\partial u(w\zeta)| \le V(w\zeta)|u(w\zeta)| = \tilde V(w)|v(w)|, \quad w \in D_r.$$

Hence, we can make use of Theorem 2.1 part 2) to v and obtain $v = 0$ on D_r for a.e. $\zeta \in S^{2n-1}$. Thus, $u = 0$ on B_r. Apply the weak unique continuation property in Theorem 2.1 part 1) to further get $u \equiv 0$ on Ω. □

One can make use of Theorem 1.1 to show the nonexistence of nontrivial flat $W^{1,1}_{loc}$ solutions to (1) when the potential $V \in L^p_{loc}, p > 2n$. Note that this example admits ample nonsmooth Sobolev functions that vanish to a finite order.

Example 4. For each $p > 2n$, let $\alpha \in (1 - \frac{2n}{p}, 1)$ and

$$V(z) := \frac{\alpha|z|^{\alpha-1}}{2(1+|z|^\alpha)} \quad z \in B_1 \subset \mathbb{C}^n.$$

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Then $V \in L^p(B_1)$. According to Theorem 1.1, there does not exist any nonconstant $W^{1,1}_{loc}(B_1)$ solution to

$$|\bar{\partial} u| = V|u| \text{ on } B_1$$

that vanishes to infinite order in the L^1 sense at 0. On the other hand, for every $k \in \mathbb{Z}^+$, the space of positive integers, one can directly verify that

$$u_k(z) = (1 + |z|^\alpha) z_1^k, \quad z \in B_1 \subset \mathbb{C}^n,$$

belongs to $W^{1,p}(B_1) \setminus C^\infty(B_1)$ and satisfies the equation.

We next prove Theorems 1.2 and 1.3 where the potential V is L^{2n}_{loc}.

Proof of Theorems 1.2 and 1.3. Let $z_0 = 0$ and $r > 0$ be small such that $B_r \subset \Omega$. For each fixed $\zeta \in S^{2n-1}$, let $\tilde{V}(w) := |w|^{\frac{n-1}{n}} V(w\zeta)$ and $v(w) := u(w\zeta), w \in D_r$. It follows from Lemma 3.1 that $\tilde{V} \in L^{2n}_{loc}(D_r)$ for a.e. $\zeta \in S^{2n-1}$. On the other hand, since $u \in W^{1,p}_{loc}(B_r)$ for some $p > 2n$, v vanishes to infinite order at 0 in the L^2 sense by Lemma 3.4. Moreover, as a consequence of Lemma 3.3, $v \in W^{1,2}_{loc}(D_r)$ and satisfies

$$|\bar{\partial} v(w)| \le |w|^{-\frac{n-1}{n}} \tilde{V}(w)|v(w)|, \quad w \in D_r.$$

For a.e. $\zeta \in S^{2n-1}$, employ Theorem 2.4 if $N = 1$; employ Theorem 2.5 if $n = 2$. In both cases, we get $v = 0$ on D_r and thus $u = 0$ on B_r. Apply the weak unique continuation property to get $u \equiv 0$. □

Similar to Example 4, one can construct ample nonsmooth Sobolev functions that satisfy (1) and vanish to a finite order (see below) in the case where the potential V is merely L^{2n}_{loc}. More interestingly, since the complex radial restriction of V in this example is no longer in L^2_{loc}, the slicing method in the proof of Theorem 1.1 does not apply.

Example 5. Assume $n \ge 2$. For each $\epsilon \in \left(\frac{1}{2}, \frac{2n-1}{2n}\right)$, consider the equation

$$|\bar{\partial} u| = V|u| \text{ on } B_{\frac{1}{2}} \subset \mathbb{C}^n,$$

with

$$V(z) := \frac{\epsilon(-\ln|z|)^{\epsilon-1}}{2|z|}, \quad z \in B_{\frac{1}{2}}.$$

According to Example 3, $V \in L^{2n}(B_{\frac{1}{2}}) \setminus L^p(B_{\frac{1}{2}})$ for any $p > 2n$. Moreover, for each $\zeta \in S^{2n-1}$, its complex radial restriction

$$\tilde{V}(w) := V(w\zeta) = \frac{\epsilon(-\ln|w|)^{\epsilon-1}}{2|w|} \notin L^2_{loc}(D_{\frac{1}{2}}).$$

In particular, the slicing method in the proof of Theorem 1.1 fails to work. However, one can use Theorem 1.2 to conclude that there does not exist any nonconstant flat $W^{1,p}_{loc}(B_1), p > 2n$ solution to the equation.

On the other hand, for each $k \in \mathbb{Z}^+$,

$$u_k(z) := e^{-(-\ln|z|)^\epsilon} z_1^k, \quad z \in B_{\frac{1}{2}}$$

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is a nonsmooth $W^{1,p}(B_{\frac{1}{2}})$, $p > 2n$ solution to the equation that vanishes to a finite order at 0. Indeed, by a Harvey–Polking removable singularity theorem (see [6]), the continuous function

$$u_0(z) = e^{-(-\ln|z|)^\epsilon}, \quad z \in B_{\frac{1}{2}}$$

belongs to $W^{1,2n}(B_{\frac{1}{2}})$ and satisfies the equation. Moreover, for each $p > 2n$ and every $0 < r << 1$,

$$\int_{|z|<r} |\nabla u_0|^p dv_z \approx \int_{|z|<r} \frac{(-\ln|z|)^{p(\epsilon-1)}}{|z|^p} e^{-p(-\ln|z|)^\epsilon} dv_z \approx \int_{-\ln r}^{\infty} s^{p(\epsilon-1)} e^{-ps^\epsilon + (p-2n)s} ds = \infty.$$

Namely, $u_0 \notin W^{1,p}(B_{\frac{1}{2}})$ for every $p > 2n$. In particular, u_0 is not smooth near 0. Consequently, u_k is a nonsmooth solution. Noting that, for each $p > 2n$,

$$\begin{aligned}\int_{|z|<\frac{1}{2}} |\nabla u_k|^p dv_z &\approx \int_{|z|<\frac{1}{2}} \frac{(-\ln|z|)^{p(\epsilon-1)}}{|z|^p} e^{-p(-\ln|z|)^\epsilon} |z|^{pk} dv_z + \int_{|z|<\frac{1}{2}} e^{-p(-\ln|z|)^\epsilon} |z_1|^{p(k-1)} dv_z \\ &\approx \int_{\ln 2}^{\infty} s^{p(\epsilon-1)} e^{-ps^\epsilon - 2ns - p(k-1)s} ds + C < \infty,\end{aligned}$$

we have $u_k \in W^{1,p}(B_{\frac{1}{2}})$. In addition, since

$$u_k(z) \geq e^{-(-\ln|z|)} |z_1|^k = |z||z_1|^k$$

near 0, u_k vanishes to a finite order at 0 in the L^q sense for every $q \geq 1$.

Remark 4.2. In view of Theorems 1.1–1.3, the following two questions still remain open. With an approach similar as in the proof of Theorems 1.2 and 1.3, the resolution of Question 1 is reduced to that of Question 2.

1. Let Ω be a domain in $\mathbb{C}^n, n \geq 3$ and $N \geq 2$. Suppose $u : \Omega \to \mathbb{C}^N$ with $u \in W^{1,p}_{loc}(\Omega)$ for some $p > 2n$ and satisfies $|\bar\partial u| \leq V|u|$ a.e. on Ω for some $V \in L^{2n}_{loc}(\Omega)$. If u vanishes to infinite order in the L^1 sense at some $z_0 \in \Omega$, does u vanish identically?
2. Let Ω be a domain in $\mathbb{C}$ containing 0, and $n, N \in \mathbb{Z}^+$ with $n \geq 3, N \geq 2$. Suppose $u : \Omega \to \mathbb{C}^N$ with $u \in W^{1,2}_{loc}(\Omega)$ and satisfies $|\bar\partial u| \leq |z|^{-\frac{n-1}{n}} V|u|$ a.e. on Ω for some $V \in L^{2n}_{loc}(\Omega)$. If u vanishes to infinite order in the L^2 sense at $0 \in \Omega$, does u vanish identically?

Next, we prove Theorem 1.4 for $W^{1,1}_{loc}$ solutions to $|\bar\partial u| \leq \frac{C}{|z|}|u|$ a.e. on Ω, where C is a positive constant. Since $\frac{1}{|z|} \in L^\infty$ off 0, a bootstrap argument shows that $u \in W^{1,q}$ everywhere away from 0 for all $q < \infty$. However, this argument is not directly effective near 0 because $\frac{1}{|z|} \notin L^{2n}$ there. The following two lemmas demonstrate that, under the assumption of the L^1 flatness assumption of u at 0, the regularity of u can be improved to $W^{1,q}$ near 0 for all $q < \infty$.

Lemma 4.3. *Let $u \in L^1$ near $0 \in \mathbb{C}^n$, and vanishes to infinite order in the L^1 sense at 0. Then, for each $M > 0$, $\frac{u}{|z|^M} \in L^1$ near 0, and vanishes to infinite order in the L^1 sense at 0.*

Proof. For each $m \geq 1$ and $\epsilon > 0$, by the L^1 flatness of u at 0,

$$\int_{|z|<r} |u| dv_z \leq \epsilon r^{m+M} \quad \text{for all } r << 1.$$

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Then

$$\int_{|z|<r}\frac{|u|}{|z|^M}dv_z=\sum_{j=1}^{\infty}\int_{\frac{r}{2^j}<|z|<\frac{r}{2^{j-1}}}\frac{|u|}{|z|^M}dv_z\le\sum_{j=1}^{\infty}\frac{2^{Mj}}{r^M}\int_{\frac{r}{2^j}<|z|<\frac{r}{2^{j-1}}}|u|dv_z\le\sum_{j=1}^{\infty}\frac{2^{Mj}}{r^M}\int_{|z|<\frac{r}{2^{j-1}}}|u|dv_z$$
$$\le\epsilon\sum_{j=1}^{\infty}\frac{2^{Mj}}{r^M}\frac{r^{m+M}}{2^{(m+M)(j-1)}}=\epsilon 2^M r^m\sum_{j=1}^{\infty}2^{-m(j-1)}\le\epsilon 2^{M+1}r^m\ \text{ for all }\ r<<1.$$

In particular, $\frac{u}{|z|^M}\in L^1$ near 0 and vanishes to infinite order in the L^1 sense at 0. □

Lemma 4.4. *Let Ω be a domain in $\mathbb{C}^n$ and $0\in\Omega$. Let $u:\Omega\to\mathbb{C}^N$ with $u\in W^{1,1}_{loc}(\Omega)$, and satisfy $|\bar\partial u|\le\frac{C}{|z|}|u|$ a.e. on Ω for some constant $C>0$. Assume u vanishes to infinite order in the L^1 sense at $0\in\Omega$. Then $u\in W^{1,q}_{loc}(\Omega)$ for every $q<\infty$.*

Proof. Near every point away from 0, since $\frac{1}{|z|}\in L^\infty$ there, one can apply the same argument as in the proof of Theorem 1.1 with p being any number less than ∞, to show that $u\in W^{1,q}$ near this point for every $q<\infty$.

To show that $u\in W^{1,q}$ near 0 for every $q<\infty$, we first claim if $u\in L^p$ near 0 for some $p>1$, then $\frac{u}{|z|}\in L^q$ for every $1<q<p$. Indeed, for each $q<p$, letting $\epsilon=\frac{p-q}{p-1}$, then $0<\epsilon<1$ and $\frac{q-\epsilon}{1-\epsilon}=p$. By Hölder's inequality and Lemma 4.3,

$$\int_{|z|<r}\left|\frac{u}{|z|}\right|^q dv_z=\int_{|z|<r}\frac{|u|^\epsilon}{|z|^q}\cdot|u|^{q-\epsilon}dv_z\le\left(\int_{|z|<r}\frac{|u|}{|z|^{\frac{q}{\epsilon}}}dv_z\right)^{\epsilon}\left(\int_{|z|<r}|u|^p dv_z\right)^{1-\epsilon}<\infty,\quad r<<1.$$

The claim is proved.

We are now ready to employ the bootstrap argument as in the proof of Theorem 1.1 near 0. Since $u\in W^{1,1}$ near 0, it follows from the Sobolev embedding theorem that $u\in L^{\frac{2n}{2n-1}}$ near 0. Consequently, the above proved claim gives $\frac{u}{|z|}\in L^q$ near 0 for any $q<\frac{2n}{2n-1}$. Lemma 4.1 further allows us to obtain $u\in W^{1,q}\subset L^{q'}$ near 0 for any $q'<\frac{2n}{2n-2}$. Repeating the process eventually leads to $u\in W^{1,q}$ near 0 for every $q<\infty$. □

Proof of Theorem 1.4. The $N=1$ case is an immediate consequence of Theorem 2.2 and Lemma 4.4. So we assume $N\ge 2$. Let r be small such that $B_r\subset\Omega$. For each fixed $\zeta\in S^{2n-1}$, let $v(w):=u(w\zeta)$, $w\in D_r$. Making use of Lemmas 3.3, 3.4, and 4.4, we have $v\in W^{1,2}_{loc}(D_r)$, which vanishes to infinite order in the L^2 sense at 0 and satisfies

$$|\bar\partial v(w)|=|\zeta\cdot\bar\partial u(w\zeta)|\le\frac{C}{|w|}|u(w\zeta)|=\frac{C}{|w|}|v(w)|,\quad w\in D_r.$$

Thus, for a.e. $\zeta\in S^{2n-1}$, we can apply Theorem 2.3 to get $v=0$ on D_r. Hence, $u=0$ on B_r. The weak unique continuation property further applies to give $u\equiv 0$. □

References

[1] S. Alinhac and S. Baouendi, *A counterexample to strong uniqueness for partial differential equations of Schrödinger's type*, Comm. Partial Differential Equations 19 (1994), nos. 9–10, 1727–1733.

[2] S. Bell and L. Lempert, *AC^∞ Schwarz reflection principle in one and several complex variables*, J. Differential Geom. 32 (1990), no. 3, 899–915.

[3] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Springer, New York, 2011, xiv+599 pp.

[4] S. Chanillo and E. Sawyer, *Unique continuation for $\triangle+v$ and the C. Fefferman-Phong class*, *Trans. Amer. Math. Soc.* 318 (1990), no. 1, 275–300.

[5] L. Evans, *Partial differential equations*, 2nd ed., Graduate Studies in Mathematics, 19, American Mathematical Society, Providence, RI, 2010, xxii+749 pp.

https://doi.org/10.1017/nmj.2026.10126 Published online by Cambridge University Press

[6] R. Harvey and J. Polking, *Removable singularities of solutions of linear partial differential equations*, Acta Math. 125 (1970), 39–56.
[7] L. Hörmander, *Notions of convexity*, Birkhäuser Boston, Inc., Boston, MA, 2007, viii+414 pp.
[8] Y. Pan and T. Wolff, *A remark on unique continuation*, J. Geom. Anal. 8 (1998), no. 4, 599–604.
[9] Y. Pan and Y. Zhang, *Unique continuation for $\bar{\partial}$ with square-integrable potentials*, New York J. Math 29 (2023), 402–416.
[10] Y. Pan and Y. Zhang, *Unique continuation of Schrödinger-type equations for $\bar{\partial}$*, Comm. Anal. Geom 33 (2025), no. 8, 1975–2014.
[11] Z. Shi, *A unique continuation property for $\left|\bar{\partial}u\right| \leq V\ |u|$*, Math. Scand 131 (2025), no. 1, 113–154.
[12] T. Wolff, *Unique continuation for $|\triangle u| \leq V\ |\nabla u|$ and related problems*, Revista Mat. Iberoam. 6 (1990), nos. 3–4, 155–200.
[13] T. Wolff, *A counterexample in a unique continuation problem*, Comm. Anal. Geom. 2 (1994), no. 1, 79–102.

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