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A Pólya–Vinogradov inequality for short character sums

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Abstract. In this paper, we obtain a variation of the Pólya–Vinogradov inequality with the sum restricted to a certain height. Assume χ to be a primitive character modulo q, $\varepsilon > 0$ and $N \leq q^{1-\gamma}$, with $0 \leq \gamma \leq 1/3$. We prove that

$$|\sum_{n=1}^{N} \chi(n)| \leq c(\tfrac{1}{3} - \gamma + \varepsilon)\sqrt{q}\log q$$

with $c = 2/\pi^2$ if χ is even and $c = 1/\pi$ if χ is odd. The result is based on the work of Hildebrand and Kerr.

1 Introduction

Studying the possible upper bounds of the quantity

$$S(N,\chi) := \left|\sum_{n=1}^{N} \chi(n)\right|,$$

with $N \in \mathbb{N}$ and χ a primitive Dirichlet character modulo q is of high interest, principally on primitive characters, as a result for these characters can easily be generalized to all non-principal characters. The famous Pólya–Vinogradov inequality tells us that

$$S(N,\chi) \ll \sqrt{q}\log q,$$

and aside from the implied constant, this is the best known result. The best known asymptotic constant is $\frac{69}{70\pi 3\sqrt{3}} + o(1)$, if χ is even [6] and $\frac{1}{3\pi} + o(1)$ if χ is odd [8]. For the best completely explicit constant, see [2, 3, 5]. It is interesting to note that it appears that $S(N,\chi)$ assumes its maximum for $N \approx q$; see the work by Bober et al. in [1] and by Hildebrand [8, Corollary 3], which proves that for even characters, we have that $N = o(q)$ implies $S(N,\chi) = o(\sqrt{q}\log q)$. Our work will continue in this direction using tools developed by Hildebrand in [7] and drawing inspiration from the work of Kerr in [4] that will lead us to prove the following result.

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Theorem 1.1 *Let χ be a primitive character modulo q, $\varepsilon > 0$ and $N \le q^{1-\gamma}$, with $0 \le \gamma \le 1/3$. We have*

$$S(N,\chi) \le c\Big(\frac{1}{3} - \gamma + \varepsilon\Big)\sqrt{q}\log q$$

with $c = 2/\pi^2$ if χ is even and $c = 1/\pi$ if χ is odd.

It is easy to see that Theorem 1.1, when $N \le q^{1-\gamma}$, improves on the previous bound in [6] for $\gamma \ge 0.0072$ if χ is even, although in this range both results are asymptotically superseded by [8, Corollary 3], and improves on the previous bound in [8] for any $\gamma > 0$ if χ is odd. Theorem 1.1 can be made completely explicit, though with smaller leading constants, following the approach used in [2]. We note that Theorem 1.1 is also interesting, as it could be viewed as a result between the standard Pólya–Vinogradov and the Burgess bound.

In Section 2, we will introduce a lemma, due to Hildebrand, that gives a good upper bound on a certain Gaussian sum and a second lemma related to the sum of trigonometric functions that is a generalization of a result due to Young and Pomerance. In Section 3, we will prove Theorem 1.1, drawing inspiration from Hildebrand's work [7]. Our main improvement will come from dividing a Gauss sum into three parts, instead of into two as done by Hildebrand, and control the first sum using certain bounds on sine and cosine in that range.

2 Two Important Lemmas

We will need the following result, that is, [7, Lemma 3].

Lemma 2.1 *Let $0 < \varepsilon < 1/2$ a fixed number. Then we have, for all primitive characters modulo q, $q^{1/3+\varepsilon} \le x \le q$ and real α,*

$$\Big|\sum_{n\le x}\chi(n)e(\alpha n)\Big| \ll_\varepsilon \frac{x}{\log q}.$$

We now prove a lemma that is a variation of two results from [11, 10].

Lemma 2.2 *Uniformly for $0 \le \gamma \le 1/3$ and real α, we have*

$$\sum_{q^\gamma \le n \le q^{1/3+\varepsilon}} \frac{1-\cos(\alpha n)}{n} \le \Big(\frac{1}{3} - \gamma + \varepsilon\Big)\log q + O(1) \tag{2.1}$$

and

$$\sum_{q^\gamma \le n \le q^{1/3+\varepsilon}} \frac{|\sin(\alpha n)|}{n} \le \frac{2}{\pi}\Big(\frac{1}{3} - \gamma + \varepsilon\Big)\log q + O(1). \tag{2.2}$$

Proof It is easy to show that we can assume $q^{1/3+\varepsilon} \ge 5q^\gamma + 6$. Define

$$\sigma(a,b)(\alpha) := \sum_{a\le n\le b}\frac{\cos(\alpha n)}{n}.$$

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We start by proving (2.1), noting that, assuming $C \geq 0$, the result would follow from

$$\sigma(q^\gamma, q^{1/3+\varepsilon})(\alpha) \geq -C. \tag{2.3}$$

Taking $\nu = p(2m+1)$, with p an odd integer and m any integer, [11, equation (10)] states that

$$\sigma(m+1,\nu)(\alpha) \geq \sigma(m+1,\nu)(\pi) - 4 - \frac{1}{2q},$$

and noting that

$$\sigma(m+1,\nu)(\pi) \geq -1,$$

we have

$$\sigma(m+1,\nu)(\alpha) \geq -5 - \frac{1}{2q}. \tag{2.4}$$

Taking $\overline{m} = \lceil q^\gamma \rceil$,

$$\overline{p} = \begin{cases} \left\lfloor \frac{q^{1/3+\varepsilon}}{2\overline{m}+1} \right\rfloor - 1 & \text{if } \left\lfloor \frac{q^{1/3+\varepsilon}}{2\overline{m}+1} \right\rfloor \text{ is even,} \\ \left\lfloor \frac{q^{1/3+\varepsilon}}{2\overline{m}+1} \right\rfloor & \text{if } \left\lfloor \frac{q^{1/3+\varepsilon}}{2\overline{m}+1} \right\rfloor \text{ is odd,} \end{cases}$$

and $\overline{\nu} = \overline{p}(2\overline{m}+1)$, we can rewrite the sum in (2.3), as we assumed $q^{1/3+\varepsilon} \geq 5q^\gamma + 6$, as follows:

$$\sigma(q^\gamma, q^{1/3+\varepsilon})(\alpha) = \sigma(q^\gamma, \overline{\nu})(\alpha) + \sigma(\overline{\nu}, q^{1/3+\varepsilon})(\alpha) - \frac{\cos(\alpha\overline{\nu})}{\overline{\nu}}. \tag{2.5}$$

Observing that $q^{1/3+\varepsilon} - \overline{\nu} \leq 4\overline{m} + 2$ and using $q^{1/3+\varepsilon} \geq 5q^\gamma + 6$, we obtain

$$\left| \sum_{\overline{\nu} \leq n \leq q^{1/3+\varepsilon}} \frac{\cos(\alpha n)}{n} \right| \leq \frac{q^{1/3+\varepsilon} - \overline{\nu}}{\overline{\nu}} \leq 2. \tag{2.6}$$

Thus, (2.3) follows applying (2.4) and (2.6) to (2.5).

We are thus left with proving (2.2). We need the Fourier expansion for $|\sin(\theta)|$ that can be found in [9, Problem 34, Part VI],

$$|\sin(\theta)| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\cos 2m\theta}{4m^2 - 1}.$$

Thus,

$$\sum_{q^\gamma \leq n \leq q^{1/3+\varepsilon}} \frac{|\sin(\alpha n)|}{n} = \frac{2}{\pi} \sum_{q^\gamma \leq n \leq q^{1/3+\varepsilon}} \frac{1}{n} - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{1}{4m^2-1} \sum_{q^\gamma \leq n \leq q^{1/3+\varepsilon}} \frac{\cos 2mn\alpha}{n},$$

and (2.2) now follows from (2.3). ∎

We note that, in the above result, the $O(1)$ could easily be made explicit, but for the following applications, this is not necessary.

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3 Proof of Theorem 1.1

We take χ primitive and start with

$$\chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) e\Big(\frac{an}{q}\Big) = \frac{1}{\tau(\overline{\chi})} \sum_{0<|a|<q/2} \overline{\chi}(a) e\Big(\frac{an}{q}\Big),$$

where $\tau(\overline{\chi})$ is the Gaussian sum. Summing over $1 \le n \le N$, we obtain

$$\sum_{n=1}^{N} \chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{0<|a|<q/2} \overline{\chi}(a) \sum_{n=1}^{N} e\Big(\frac{an}{q}\Big) = \frac{1}{\tau(\overline{\chi})} \sum_{0<|a|<q/2} \overline{\chi}(a) \frac{e\big(\frac{aN}{q}\big) - 1}{1 - e\big(\frac{-a}{q}\big)}.$$

Since $|\tau(\overline{\chi})| = \sqrt{q}$ for primitive characters and

$$\frac{1}{1 - e\big(\frac{-a}{q}\big)} = \frac{q}{2\pi i a} + O(1),$$

for $0 < |a| < q/2$, it follows that

$$S(N,\chi) \le \frac{\sqrt{q}}{2\pi} \Bigg| \sum_{0<|a|<q/2} \frac{\overline{\chi(a)}\big(e(\frac{aN}{q}) - 1\big)}{a} \Bigg| + O(\sqrt{q}).$$

Now we split the inner sum in three parts: Σ_1 with $0 < |a| < q^{\gamma}/(2\pi)$, Σ_2 with $q^{\gamma}/(2\pi) \le |a| \le q^{1/3+\varepsilon}$, and Σ_3 with $q^{1/3+\varepsilon} < |a| < q/2$. Therefore,

$$S(N,\chi) \le \frac{\sqrt{q}}{2\pi} \Big(\Big|\sum_1\Big| + \Big|\sum_2\Big| + \Big|\sum_3\Big| \Big) + O(\sqrt{q}). \tag{3.1}$$

We have

$$\Sigma_1 = \begin{cases} 2i \displaystyle\sum_{1\le a<q^{\gamma}/(2\pi)} \frac{\overline{\chi(a)} \sin(\frac{2\pi aN}{q})}{a} & \text{if } \chi \text{ is even,} \\ -2 \displaystyle\sum_{1\le a<q^{\gamma}/(2\pi)} \frac{\overline{\chi(a)}\big(1-\cos(\frac{2\pi aN}{q})\big)}{a} & \text{if } \chi \text{ is odd.} \end{cases}$$

Now observing that for $0 < |a| < q^{\gamma}/(2\pi)$, we have that $|\frac{2\pi aN}{q}| < 1$ and thus

$$\Big|\sin\Big(\frac{2\pi aN}{q}\Big)\Big| \ll \frac{2\pi aN}{q} \quad \text{and} \quad \Big|1 - \cos\Big(\frac{2\pi aN}{q}\Big)\Big| \ll \Big(\frac{2\pi aN}{q}\Big)^2,$$

where, remembering that $N \le q^{1-\gamma}$, easily gives $\Sigma_1 \ll 1$. We also have

$$\Sigma_2 = \begin{cases} 2i \displaystyle\sum_{q^{\gamma}/(2\pi)\le a\le q^{1/3+\varepsilon}} \frac{\overline{\chi(a)} \sin(\frac{2\pi aN}{q})}{a} & \text{if } \chi \text{ is even,} \\ -2 \displaystyle\sum_{q^{\gamma}/(2\pi)\le a\le q^{1/3+\varepsilon}} \frac{\overline{\chi(a)}\big(1-\cos(\frac{2\pi aN}{q})\big)}{a} & \text{if } \chi \text{ is odd,} \end{cases}$$

and applying Lemma 2.2, observing that the difference between having the lower bound of the sum starting from $q^{\gamma}/(2\pi)$ and not q^{γ} being bounded by a constant,

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we obtain

$$\Sigma_2 = \begin{cases} \frac{4}{\pi}\left(\frac{1}{3} - \gamma + \varepsilon\right)\log q + O(1) & \text{if } \chi \text{ is even,} \\ 2\left(\frac{1}{3} - \gamma + \varepsilon\right)\log q + O(1) & \text{if } \chi \text{ is odd.} \end{cases}$$

By partial summation and Lemma 2.1, we have

$$|\Sigma_3| \ll (\log q) \max_{q^{1/3+\varepsilon} \le x \le q} \left| \frac{1}{x} \sum_{a \le x} \overline{\chi}(a)\left(e\left(\frac{aN}{q}\right) - 1\right)\right| \ll_\varepsilon 1.$$

Thus, by the above upper bounds on $|\sum_i|$, for $i = 1, 2, 3$, and (3.1), we obtain the desired result.

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