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UPPER BOUNDS FOR INFINITARY HARMONIC NUMBERS AND INFINITARY HARMONIOUS TUPLES

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Abstract

A positive integer N is said to be an infinitary harmonic number (IHN) if the harmonic mean of its infinitary divisors is integral. It is still an open problem whether or not there exist infinitely many IHNs. Cohen and Hagis ['Infinitary harmonic numbers', *Bull. Aust. Math. Soc.* **41** (1989), 151–158] showed that there exist at most finitely many IHNs with a fixed number of I-components. We give an explicit Nielsen-type upper bound for the number of IHNs. We also consider a class of general infinitary amicable numbers and obtain the Borho-type upper bound for them.

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1. Introduction

Let $d(n) = \sum_{d|n} 1$ and $\sigma(n) = \sum_{d|n} d$ denote respectively the number and the sum of the positive divisors of a positive integer n. Ore [7] studied the set of numbers S for which the ratio $H(n) = nd(n)/\sigma(n)$ is an integer. Such numbers are called *harmonic numbers*. Note that the set of perfect numbers is a subset of S.

Subsequently, this notion was generalised to the other types of divisor functions. Hagis and Lord [4] introduced the unitary analogue of the harmonic numbers. A positive integer d is called a unitary divisor of N if $d \mid N$ and $(d, N/d) = 1$. A positive integer N is said to be a *unitary harmonic number* (UHN) if the harmonic mean of the unitary divisors of N, given by

$$H^*(n) = \frac{Nd^*(N)}{\sigma^*(N)} = 2^{\omega(N)} \prod_{p^a \| N} \frac{p^a}{p^a + 1},$$

is an integer. Here, $\omega(N)$ is the number of distinct prime factors of N.

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Let

$$I = \{p^{2^\alpha} \mid p \text{ is prime and } \alpha \geq 0\}.$$

From the fact that the binary representation of a natural number is unique, it follows that every $N > 1$ can be written in exactly one way (up to the order of the factors) as the product of distinct elements from I. Each element of I in this product is called an I-component of N.

We denote the number of I-components of N by $J = J(N)$. Clearly, $J(N)$ is an additive function given by $J(1) = 0$ and $J(p^y) = \sum_{y_i=1} 1$ if $y = \sum_{i=0}^{\infty} y_i 2^i$.

A divisor d of N is said to be an I-divisor if every I-component of d is also an I-component of N. Let $d_\infty(N)$ and $\sigma_\infty(N)$ denote respectively the number and the sum of the I-divisors of N. It is easy to see that $d_\infty(N)$ and $\sigma_\infty(N)$ are multiplicative functions defined by

$$d_\infty(N) = \prod_{p^y \| N} 2^{J(p^y)} = 2^{J(N)} \quad \text{and} \quad \sigma_\infty(N) = \prod_{p^y \| N} \prod_{y_i=1} (1 + p^{2^i}) \quad \text{where } y = \sum_{i=0}^{\infty} y_i 2^i.$$

Hagis and Cohen [3] defined N to be an *infinitary harmonic number (IHN)* if the infinitary harmonic mean of N, defined by

$$H_\infty(n) = \frac{N d_\infty(N)}{\sigma_\infty(N)} = 2^{J(N)} \prod_{p^a \| N} \prod_{y_i=1} \frac{p^{2^i}}{p^{2^i} + 1},$$

is an integer. They showed that there exist at most finitely many IHNs with a specified number of I-components and only finitely many with a specified infinitary harmonic mean. In our first theorem, we establish an explicit upper bound for the number of IHNs which is similar to Nielsen's bound for odd perfect numbers [6]. (See also a recent paper by Stone [8].)

THEOREM 1.1. *Suppose that N is an IHN with $J = J(N)$ I-components and $H_\infty(N) = c$. Then:*

(a) $N \leq c^{c^2}$;
(b) $N \leq (2^{2^J - 1})^J$.

For an integer $k \geq 2$, we call a tuple of k positive integers $(M_i)_{i=1}^k$ an *infinitary amicable k-tuple* if the equation

$$\sigma_\infty(M_1) = \cdots = \sigma_\infty(M_k) = M_1 + \cdots + M_k$$

is satisfied. This is a generalisation of infinitary amicable pairs, defined by Cohen [2]. We also introduce a new class of tuples of integers.

For an integer $k \geq 2$, we say a tuple of positive integers $(M_i)_{i=1}^k$ is an *infinitary harmonious k-tuple* if the equation

$$\frac{M_1}{\sigma_\infty(M_1)} + \cdots + \frac{M_k}{\sigma_\infty(M_k)} = 1 \tag{1.1}$$

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holds. Note that if $(M_i)_{i=1}^k$ is an infinitary amicable tuple, then

$$1 = \frac{M_1}{M_1 + \cdots + M_k} + \cdots + \frac{M_k}{M_1 + \cdots + M_k} = \frac{M_1}{\sigma_\infty(M_1)} + \cdots + \frac{M_k}{\sigma_\infty(M_k)},$$

so every infinitary amicable tuple is infinitary harmonious.

Borho [1] proved an upper bound for amicable pairs (M, N) in terms of $\Omega(MN)$, the number of prime factors of MN counted with multiplicity. In our next theorem, we give an upper bound for the number of infinitary harmonious k-tuples which is analogous to Borho's bound.

THEOREM 1.2. *For any infinitary harmonious k-tuple* $(M_i)_{i=1}^k$,

$$M_1 \cdots M_k \le k^{-k}(2^{2^L} - 2^{2^{L-1}}) \quad \text{where } L = J(M_1) + \cdots + J(M_k).$$

2. Preliminaries

In this section, we give some lemmas that are useful for the proof of the main results. In the first lemma due to Hagis and Cohen [3], we collect some elementary results concerning $H_\infty(N)$ and IHNs.

LEMMA 2.1.

(a) *Let* $J = J(N)$. *If* $N > 1$, *then*

$$\frac{2^{J+1}}{J+2} \le H_\infty(N) < 2^J. \tag{2.1}$$

(b) *If N is an odd IHN, then* $H_\infty(N)$ *is odd.*

(c) *If N is an odd IHN,* $p^y \parallel N$ *and* $p = 4m + 3$, *then y is even.*

We also need an upper bound for the minimum and maximum I-components of an IHN.

LEMMA 2.2. *Let* p^a *and* r^c *respectively be the minimum and maximum I-components of an IHN. Then,*

$$p^a \le \frac{J \cdot H_\infty(N)}{2^J - H_\infty(N)}, \quad r^c \ge \frac{(J-1)\cdot 2^J + H_\infty(N)}{2^J - H_\infty(N)}.$$

PROOF. We have

$$H_\infty(N) \ge 2^J \frac{p^a}{p^a+1}\frac{p^a+1}{p^a+2}\cdots\frac{p^a+J-1}{p^a+J} = \frac{2^J p^a}{p^a+J}$$

and

$$H_\infty(N) \le 2^J \frac{r^c-J+1}{r^c-J+2}\frac{r^c-J+2}{r^c-J+3}\cdots\frac{r^c}{r^c+1} = \frac{2^J(r^c-J+1)}{r^c+1}.$$

□

The next lemma identifies IHNs with a few selected values of J and $H_\infty(N)$.

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LEMMA 2.3. *Let N be an IHN with $J = J(N)$ I-components and let $H_\infty(N) = c$.*

(a) *If $J \le 3$, then $N \in \{1, 6, 45, 60, 90, 15925\}$.*
(b) *If $c \le 5$, then $N \in \{1, 6, 45, 60, 90\}$.*

PROOF. (a) Let $T_{k,\ell}$ denote the set of IHNs which have k I-components and exactly ℓ prime factors. Clearly, $T_{k,1}$ is empty for every $k \ge 1$. If $N \in T_{2,2}$, then $N = p^a q^b$, where $q > p$ are primes and a, b are powers of 2. In this case, $H_\infty(N) = H^*(N)$, where $H^*(N)$ is the harmonic mean of the unitary divisors of N and, thus, N is a unitary harmonic number (UHN). Therefore, $T_{2,2} = \{6, 45\}$.

If $N \in T_{3,2}$, then from Lemma 2.1(a), $H_\infty(N) = 4, 5, 6$ or 7. We consider here the cases $H_\infty(N) = 6$ and 7; the other cases can be treated in a similar way. If $H_\infty(N) = 6$, then N is even, $3 \mid N$ and by Lemma 2.2, $p^a = 4, 5, 7$. If $p^a = 4, 5$ or 7, then N has three prime factors. If $H_\infty(N) = 7$, then $7 \mid N$. Assume first that $2 \mid N$. Then, $N = 2^{2^{a'}+2^a} 7^{2^b}$ or $N = 2^{2^a} 7^{2^{b'}+2^b}$. Hence, N must satisfy either

$$(2^{2^{a'}} + 1)(2^{2^a} + 1)(7^{2^b} + 1) = 2^{2^{a'}+2^a+3} 7^{2^b-1}$$

or

$$(2^{2^a} + 1)(7^{2^b} + 1)(7^{2^{b'}} + 1) = 2^{2^a+3} 7^{2^{b'}+2^b-1}.$$

If $m \ge 1$, then any odd prime divisor of the generalised Fermat number $7^{2^m} + 1$ is of the form $k \cdot 2^{m+1} + 1$ and must divide the right-hand side of the above equations, which is impossible. Now, assume that N is odd. Then, using Lemma 2.1(c) and Lemma 2.2, $7^2 \mid N$ and $p^a = 9, 13, 17$. If $p^a = 9$ or 17, then N has three prime factors. If $p^a = 13$, then $N = 7^{2^b} 13^{2^a+1}$ or $N = 7^{2^c+2^d} 13$, where $a \ge 1$, $b \ge 1$ and $d > c \ge 1$. In the former case, $H_\infty(N) > H_\infty(13^3 7^2) > 7$ and in the latter case, $H_\infty(N) > H_\infty(13 \cdot 7^6) > 7$. By considering the remaining cases of $H_\infty(N)$, one can show that no such IHN exists. Thus, $T_{3,2}$ is empty.

If $N \in T_{3,3}$, then $N = p^{2^a} q^{2^b} r^{2^c}$ with $p < q < r$ primes and $H_\infty(N) = H^*(N)$. Hence, N is a UHN and $T_{3,3} = \{60, 90, 15925\}$.

(b) Note that if $J \ge 4$, then from the inequality (2.1), we get $c > 5$. Since $H_\infty(15925) = 7$, the result follows from part (a). □

The next two lemmas due to Goto [5] and Suzuki [9] respectively play an important role in the proofs of the upper bounds for IHNs and infinitary harmonious tuples.

LEMMA 2.4. *Let r, a, b be positive integers, and let $x_1, \dots, x_r$ be integers such that $1 \le x_1 < x_2 < \cdots < x_r$ and*

$$1 \le \prod_{j=1}^{r-1} \left(1 + \frac{1}{x_j}\right) < \frac{a}{b} \le \prod_{j=1}^{r} \left(1 + \frac{1}{x_j}\right).$$

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Then,

$$\prod_{j=1}^{r} x_j \le (b+1)^{2^r-1} - (b+1)^{2^{r-1}-1}$$

with equality if and only if $x_j = m_j$, $1 \le j \le r$, *where*

$$m_i = \begin{cases} (b+1)^{2^{j-1}} & \text{for } j = 1, \dots, r-1, \\ (b+1)^{2^{j-1}} - 1 & \text{for } j = r. \end{cases}$$

LEMMA 2.5. *Let k and R be positive integers. Consider a sequence of integers*

$$\mathcal{M} = (m_j)_{j=1}^{R}, \quad 1 < m_1 \le m_2 \le \cdots \le m_R,$$

a decomposition of the index set

$$\mathcal{J} = \{1, \dots, R\}, \quad \mathcal{J} = \bigcup_{i=1}^{k} \mathcal{J}_i \quad \text{where } \mathcal{J}_1, \dots, \mathcal{J}_k \text{ are disjoint},$$

and a tuple of integers

$$a_1, \dots, a_k, b_1, \dots, b_k \ge 1$$

satisfying $a_i \ge b_i$ *for all i. If a pair of inequalities*

$$\sum_{i=1}^{k} \frac{b_i}{a_i} \prod_{\substack{j=1 \\ j \in \mathcal{J}_i}}^{R} \left(1 - \frac{1}{m_j}\right) \le 1 \quad \text{and} \quad \sum_{i=1}^{k} \frac{b_i}{a_i} \prod_{\substack{j=1 \\ j \in \mathcal{J}_i}}^{R-1} \left(1 - \frac{1}{m_j}\right) > 1$$

holds, then

$$a \prod_{j=1}^{R} m_j \le (a+1)^{2^R} - (a+1)^{2^{R-1}} \quad \text{where } a = a_1 \cdots a_k.$$

3. Proofs of the theorems

PROOF OF THEOREM 1.1. We follow closely the argument of Goto [5]. Observe that

$$\frac{\sigma_\infty(N)}{N} = \prod_{p^y \| N} \prod_{y_i = 1} \left(1 + \frac{1}{p^{2^i}}\right) = \frac{d_\infty(N)}{H_\infty(N)} = \frac{2^J}{c}.$$

We arrange the J I-components of N in ascending order. Applying Lemma 2.4 with $x_\alpha = p^{2^\alpha}$, $a = 2^J$, $b = c$ and $r = J$,

$$N = \prod_{p^y \| N} \prod_{y_i = 1} p^{2^i} \le (c+1)^{2^J - 1} - (c+1)^{2^{J-1}-1} < (c+1)^{2^J - 1}. \tag{3.1}$$

From Lemma 2.3, the required inequalities hold if $J \le 3$ or $c \le 5$. Thus, we may assume that $J \ge 4$ and $c \ge 6$. From the inequality (2.1), $c + 1 \le 2^J$. Combining this with inequality (3.1) gives part (b).

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We now consider part (a). From $J \geq 4$, we easily get $J + 2 < 2^{(2J+3)/4}$. Thus, Lemma 2.1(a) implies that

$$c \geq \frac{2^{J+1}}{J+2} > 2^{(2J+1)/4}.$$

Therefore, $2^J < c^2/\sqrt{2}$. From $c \geq 6$, we easily deduce that $c + 1 < c^{\sqrt{2}}$. Hence, it follows that

$$N < (c+1)^{2^J} < (c+1)^{c^2/\sqrt{2}} < c^{c^2},$$

as desired. □

PROOF OF THEOREM 1.2. Let $(M_i)_{i=1}^k$ be an infinitary harmonious tuple. Using

$$\frac{M}{\sigma_\infty(M)} = \prod_{p^y \| M} \prod_{y_i=1} \frac{p^{2^i}}{1+p^{2^i}} = \prod_{p^y \| M} \prod_{y_i=1} \left(1 - \frac{1}{1+p^{2^i}}\right),$$

we can rewrite (1.1) as

$$\sum_{j=1}^{k} \prod_{p^y \| M_j} \prod_{y_i=1} \left(1 - \frac{1}{1+p^{2^i}}\right) = 1.$$

Arranging the I-components of $M_1, \ldots, M_k$ in ascending order and applying Lemma 2.5 with $m_\alpha = 1 + p^{2^\alpha}$, $a_i = b_i = 1$, we see that

$$\sigma_\infty(M_1) \cdots \sigma_\infty(M_k) = \prod_{j=1}^{k} \prod_{p^y \| M_j} \prod_{y_i=1} (1+p^{2^i}) \leq 2^{2^L} - 2^{2^{L-1}}, \tag{3.2}$$

where L is given by

$$L = \sum_{j=1}^{k} \sum_{p^y \| M_j} \sum_{y_i=1} 1 = J(M_1) + \cdots + J(M_k).$$

By using the AM-GM inequality in (1.1), we have

$$\frac{1}{k} \geq \sqrt[k]{\frac{M_1}{\sigma_\infty(M_1)} \cdots \frac{M_k}{\sigma_\infty(M_k)}}$$

or

$$\frac{\sigma_\infty(M_1) \cdots \sigma_\infty(M_k)}{M_1 \cdots M_k} \geq k^k.$$

Multiplying both sides by $M_1 \cdots M_k$, we obtain

$$\sigma_\infty(M_1) \cdots \sigma_\infty(M_k) \geq k^k M_1 \cdots M_k.$$

Combining this with (3.2), gives the required assertion. □

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