Math. Proc. Camb. Phil. Soc. (1996), **120**, 339
Printed in Great Britain

Derivations mapping into the socle

BY MATEJ BREŠAR
PF, University of Maribor, Koroška 160, 62000 *Maribor, Slovenia*
AND PETER ŠEMRL
TF, University of Maribor, Smetanova 17, 62000 *Maribor, Slovenia*

(*Received* 12 *January* 1995; *revised* 10 *April* 1995)

1. *Introduction*

Over the last few years a number of results giving conditions on a derivation of a Banach algebra implying that its range is contained in the radical have been obtained (see survey articles of Mathieu [7] and Murphy [8]). If an algebra is semi-simple, these conditions, of course, imply that a derivation is zero. In this paper we consider inner derivations that are non-zero in general, but their ranges are rather special and 'small' in some sense.

In Section 2 we characterize elements in the socle of a semi-simple Banach algebra $\mathscr{A}$ by properties of their cosets in quotient algebras of $\mathscr{A}$ by primitive ideals. This characterization is needed in Section 3 where the main result is proved. It shows, in particular, that for an inner derivation d on a semi-simple unital Banach algebra $\mathscr{A}$ the following three conditions are equivalent:

(a) every element in the range of d has a finite spectrum,

(b) every element in the range of d is algebraic,

(c) the range of d is contained in the socle of $\mathscr{A}$.

Let us point out that we consider inner derivations only. The problem whether these conditions are equivalent for general derivations seems to be challenging.

An interest for studying derivations having only algebraic elements in the range is coming from the ring theory (see, for example, [5] and references given there).

The main source of inspiration for the present work, however, has been Aupetit's book [2]. Inner derivations on Banach algebras having values with exactly one point in the spectrum were considered in [2, theorem 5.2.1]. The ranges of such derivations are contained in the radical. In [2, lemma 5.6.5] it is proved that if an element a of a Banach algebra $\mathscr{A}$ satisfies $a+F \subset F$, where F is the set of all elements in $\mathscr{A}$ with finite spectrum, then the elements in the range of the inner derivation $x \mapsto ax-xa$ have at most n points in the spectrum for some positive integer n (thus, the condition (a) holds). This result was used to prove that a is algebraic modulo the radical [2, theorem 5.6.6]. Our results give some further information about elements a satisfying $a+F \subset F$. We also consider an analogous condition $a+\mathrm{Alg}(\mathscr{A}) \subset \mathrm{Alg}(\mathscr{A})$, $\mathrm{Alg}(\mathscr{A})$ being the set of all algebraic elements in a semisimple Banach algebra $\mathscr{A}$. The result concerning this condition is proved in Section 4 which deals with applications of the main result.

All algebras in this paper will be unital. Let $\mathscr{A}$ be a semi-simple Banach algebra containing minimal one-sided ideals (right or left). Then the sum of all minimal left

https://doi.org/10.1017/S0305004100074892 Published online by Cambridge University Press

ideals coincides with the sum of all minimal right ideals of $\mathscr{A}$, and is called the socle of $\mathscr{A}$. It will be denoted by $\operatorname{soc}(\mathscr{A})$. If $\mathscr{A}$ does not have minimal one-sided ideals, we define $\operatorname{soc}(\mathscr{A}) = \{0\}$. The following characterization is very useful: $a \in \operatorname{soc}(\mathscr{A})$ if and only if $a\mathscr{A}a$ is a finite dimensional space [**1**, theorem 7.2]. We will denote the set of all primitive ideals in $\mathscr{A}$ by $\operatorname{Prim}(\mathscr{A})$. The quotient mapping belonging to $P \in \operatorname{Prim}(\mathscr{A})$ will be denoted by π_P. As usual, we will also use the symbol π_P to denote an irreducible representation of $\mathscr{A}$ on a Banach space X corresponding to P, that is, a representation satisfying $\ker(\pi_P) = P$. Several times we will use the following relation [**9**, theorem 2.2.9]:

$$\sigma(a) = \bigcup_{P \in \operatorname{Prim}(\mathscr{A})} \sigma(\pi_P(a)).$$

2. *Characterization of the socle*

In this section we give a characterization of the elements belonging to the socle. This result is an essential tool needed for our main result. We believe, however, that it is of some independent interest. We will need the following lemma to prove it.

LEMMA 2·1. *Let $\mathscr{A}$ be a Banach algebra, and let $T: \mathscr{A} \to \mathscr{A}$ be a continuous linear mapping satisfying $\#\sigma(Tx) < \infty$ for all $x \in \mathscr{A}$. Here, $\#\sigma(u)$ denotes the cardinality of the spectrum of u. Then there exists a positive integer n such that $\#\sigma(Tx) \leqslant n$ for all $x \in \mathscr{A}$.*

Proof. Let $\mathscr{A}_k$ be the set of all $x \in \mathscr{A}$ satisfying $\#\sigma(Tx) \leqslant k, k = 1, 2, \ldots$. By [**2**, theorem 3.4.4], $\mathscr{A}_k$ is closed for every positive integer k. As we clearly have $\mathscr{A} = \bigcup_{k=1}^{\infty} \mathscr{A}_k$, we can apply Baire's theorem to find a positive integer n such that $\mathscr{A}_n$ contains an open ball $B(a, r)$. We fix $x \in \mathscr{A}$ and define an analytic function by

$$\phi(\lambda) = T(a + \lambda(x - a)).$$

For complex numbers λ satisfying $|\lambda| . \|x - a\| < r$ we have $\#\sigma(\phi(\lambda)) \leqslant n$. Using [**2**, theorem 3.4.25] we get that $\#\sigma(\phi(\lambda)) \leqslant n$ for all $\lambda \in \mathbb{C}$, and consequently, we have the desired relation $\mathscr{A} = \mathscr{A}_n$. ∎

PROPOSITION 2·2. *Let $\mathscr{A}$ be a semi-simple unital Banach algebra, and let $b \in \mathscr{A}$. Then the following two conditions are equivalent:*

(i) $b \in \operatorname{soc}(\mathscr{A})$,

(ii) $\pi_P(b) \in \operatorname{soc}(\mathscr{A}/P)$ *for every* $P \in \operatorname{Prim}(\mathscr{A})$ *and* $\pi_P(b) = 0$ *for all but finitely many* $P \in \operatorname{Prim}(\mathscr{A})$.

Proof. Assume first that $b \in \operatorname{soc}(\mathscr{A})$. Then $b\mathscr{A}b$ is finite dimensional, and consequently the same is true for $\pi_P(b)\pi_P(\mathscr{A})\pi_P(b)$ for every primitive ideal P. But $\pi_P(\mathscr{A})$ is semi-simple, and so $\pi_P(b)$ belongs to $\operatorname{soc}(\mathscr{A}/P)$ for every $P \in \operatorname{Prim}(\mathscr{A})$.

Every element of the form bx has a finite spectrum, for it belongs to $\operatorname{soc}(\mathscr{A})$. It follows that the linear mapping $x \mapsto bx$ satisfies the assumptions of Lemma 2·1, and so there exists a positive integer n such that $\#\sigma(bx) \leqslant n$ for all $x \in \mathscr{A}$.

Assume that there exist irreducible representations $\pi_1, \ldots, \pi_{n+1}$ of $\mathscr{A}$ on Banach spaces $X_1, \ldots, X_{n+1}$, respectively, such that the primitive ideals $P_k = \ker(\pi_k)$, $k = 1, \ldots, n+1$, are distinct, and $\pi_k(b) \neq 0$ for $k = 1, \ldots, n+1$. Then we can find vectors ξ_k, $\eta_k \in X_k$, $k = 1, \ldots, n+1$, such that

$$\pi_k(b)\,\xi_k = \eta_k \neq 0.$$

https://doi.org/10.1017/S0305004100074892 Published online by Cambridge University Press

As the equivalent representations have the same kernels, and since the commuting division algebras of the π_is are isomorphic to $\mathbb{C}$ [**3**, corollary 25.5], the so-called extended Jacobson density theorem [**4**, p. 283] tells us that there exists $x \in \mathscr{A}$ such that $\pi_k(x)\eta_k = k\xi_k$. But then we have $\pi_k(bx)\eta_k = k\eta_k$, which further implies that $\{1, \ldots, n+1\} \subset \sigma(bx)$. This contradiction completes the proof of the implication (i) $\Rightarrow$ (ii).

To prove the converse we assume that (ii) is satisfied. We have to show that $b\mathscr{A}b$ is finite dimensional. Let $\{P_1, \ldots, P_m\}$ be the set of all primitive ideals in $\mathscr{A}$ satisfying $\pi_{P_k}(b) \neq 0$. Denote by π_k the corresponding irreducible representation on Banach spaces X_k, $k = 1, \ldots, m$. Clearly, $\pi_k(b)$ is a finite rank operator for every $k = 1, \ldots, m$. Denoting its rank by r_k we have $\dim \pi_k(b\mathscr{A}b) \leqslant r_k^2$. We will show that $\dim b\mathscr{A}b \leqslant \sum_{k=1}^m r_k^2$. Assume on the contrary that $\dim b\mathscr{A}b \geqslant 1 + \sum_{k=1}^m r_k^2$. Let us denote the restriction of π_i to the subalgebra $b\mathscr{A}b$ by τ_i. Then we have $\dim \ker(\tau_i) \geqslant \dim b\mathscr{A}b - r_i^2$ which further yields

$$\dim\left(\bigcap_{i=1}^{m} \ker(\tau_i)\right) \geqslant 1.$$

So, there exists a non-zero element $x \in b\mathscr{A}b$ satisfying $\pi_{P_k}(x) = 0$ for $k = 1, \ldots, m$. Therefore, $\pi_P(x) = 0$ for every $P \in \operatorname{Prim}(\mathscr{A})$. Since $\mathscr{A}$ is semi-simple we have $x = 0$. This contradiction completes the proof.

3. *The main result*

In order to make the proof of the following result easier to read we recall some elementary facts about linear operators acting on an infinite-dimensional space X. Let $A : X \to X$ be a linear operator and λ a complex number such that $\ker(A-\lambda)$ is a finite-dimensional space. We choose a linear subspace Y of X such that $\ker((A-\lambda)^2) = \ker(A-\lambda) \oplus Y$. Clearly, the restriction of $A-\lambda$ to the subspace Y is injective operator mapping Y into $\ker(A-\lambda)$. It follows that $\ker((A-\lambda)^2)$ is finite-dimensional. Applying the inductive argument one can see that $\dim(\ker(A-\lambda)) < \infty$ implies that $\dim(\ker((A-\lambda)^r)) < \infty$ for every positive integer r. Similarly, if $\ker(A-\lambda)$ has a finite codimension in $\ker((A-\lambda)^2)$, then $\ker(A-\lambda)$ has a finite codimension in $\ker((A-\lambda)^r)$ for every positive integer r.

LEMMA 3·1. *Let n be a positive integer, X a complex vector space, and let $A : X \to X$ be a linear mapping such that the set $\{\xi_1, \ldots, \xi_n, A\xi_1, \ldots, A\xi_n\}$ is linearly dependent for any n-tuple of vectors $\xi_1, \ldots, \xi_n$ from X. Then A is of the form $A = \lambda I + A_1$, where $\lambda \in \mathbb{C}$ and A_1 is a finite-rank operator.*

Proof. Let ξ be any vector from X. Set $\xi_k = A^{2k-2}\xi$, $k = 1, \ldots, n$, and observe that according to our assumption the set $\{A^k\xi : k = 0, 1, \ldots, 2n-1\}$ must be linearly dependent. Kaplansky's theorem on locally algebraic operators [**2**, theorem 4.2.7] forces A to be algebraic. Let the polynomial p satisfying $p(A) = 0$ be of the form

$$p(\lambda) = (\lambda-\lambda_1)^{r_1} \ldots (\lambda-\lambda_k)^{r_k},$$

where $\lambda_1, \ldots, \lambda_k$ are distinct complex numbers. The polynomials $p_1, \ldots, p_k$ defined by

$$p_j(\lambda) = \prod_{m \neq j} (\lambda-\lambda_m)^{r_m}, \quad j = 1, \ldots, k,$$

https://doi.org/10.1017/S0305004100074892 Published online by Cambridge University Press

have the greatest common divisor 1, so that there exist polynomials $q_1, \dots, q_k$ such that $\sum_{j=1}^{k} p_j q_j = 1$. Hence, for any vector $\xi \in X$ we have $\xi = \sum_{j=1}^{k} p_j(A) q_j(A) \xi$. From here it is easy to conclude that

$$X = \ker((A-\lambda_1)^{r_1}) \oplus \dots \oplus \ker((A-\lambda_k)^{r_k}).$$

Let us first show that there is at most one integer $j, 1 \leqslant j \leqslant k$, such that $\ker(A-\lambda_j)$ is infinite-dimensional. Assume on the contrary that there are at least two such integers. With no loss of generality we can assume that integers 1 and 2 are of the kind. It follows that we can find linearly independent vectors $\xi_1, \dots, \xi_n \in \ker(A-\lambda_1)$ and $\zeta_1, \dots, \zeta_n \in \ker(A-\lambda_2)$. Let $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ be complex numbers such that

$$\alpha_1(\xi_1+\zeta_1)+\dots+\alpha_n(\xi_n+\zeta_n)+\beta_1 A(\xi_1+\zeta_1)+\dots+\beta_n A(\xi_n+\zeta_n) = 0.$$

Then

$$(\alpha_1+\beta_1\lambda_1)\xi_1+\dots+(\alpha_n+\beta_n\lambda_1)\xi_n+(\alpha_1+\beta_1\lambda_2)\zeta_1+\dots+(\alpha_n+\beta_n\lambda_2)\zeta_n = 0.$$

It follows that $\alpha_1 = \dots = \alpha_n = \beta_1 = \dots = \beta_n = 0$, and consequently, the vectors

$$\xi_1+\zeta_1, \dots, \xi_n+\zeta_n, A(\xi_1+\zeta_1), \dots, A(\xi_n+\zeta_n)$$

are linearly independent which contradicts our assumption.

In the case that X is finite-dimensional there is nothing to prove. But if X is infinite-dimensional, then there exists exactly one integer $j, 1 \leqslant j \leqslant k$, such that $\ker(A-\lambda_j)$ is infinite-dimensional. Assume with no loss of generality that $j = 1$. In order to complete the proof we have to show that A is of the form $A = \lambda_1 + A_1$ with A_1 being a finite rank operator. In other words, we have to prove that $\ker(A-\lambda_1)$ has a finite co-dimension in $\ker(A-\lambda_1)^{r_1}$. We already know that it is enough to prove that $\ker(A-\lambda_1)$ has finite co-dimension in $\ker(A-\lambda_1)^2$. Moreover, we can assume without loss of generality that $\lambda_1 = 0$. Choose a linear subspace V in $\ker(A^2)$ such that $\ker(A^2) = \ker(A) \oplus V$. If V is infinite-dimensional then we can find linearly independent vectors $\xi_1, \dots, \xi_n \in V$. It is easy to see that $\xi_1, \dots, \xi_n, A\xi_1, \dots, A\xi_n$ are linearly independent as well. This contradiction completes the proof. ∎

Now we are ready to prove our main result.

THEOREM 3·2. *Let $\mathscr{A}$ be a unital Banach algebra, and let $a \in \mathscr{A}$. Then the following conditions are equivalent*:

(i) $\#\sigma([a,x]) < \infty$ *for all* $x \in \mathscr{A}$,

(ii) *there exists a positive integer n such that* $\#\sigma([a,x]) \leqslant n$ *for all* $x \in \mathscr{A}$,

(iii) $\pi_P(a) \in \mathbb{C}1 + \operatorname{soc}(\mathscr{A}/P)$ *for all* $P \in \operatorname{Prim}(\mathscr{A})$ *and* $\pi_P(a) \in \mathbb{C}1$ *for all but finitely many* $P \in \operatorname{Prim}(\mathscr{A})$.

Moreover, if $\mathscr{A}$ is semi-simple then the above conditions are equivalent to the following two equivalent conditions:

(iv) $[a,x] \in \operatorname{soc}(\mathscr{A})$ *for all* $x \in \mathscr{A}$,

(v) $[a,x]$ *is algebraic for all* $x \in \mathscr{A}$.

Proof. The implication (i) ⇒ (ii) is a special case of Lemma 2·1. Next, let us prove the implication (ii) ⇒ (iii). Choose $P \in \operatorname{Prim}(\mathscr{A})$ and let π_P be an irreducible representation of $\mathscr{A}$ on a Banach space X with $\ker(\pi_P) = P$. Let $\{\xi_1, \dots, \xi_{n+1}\}$ be any subset of X. Assume that $\{\xi_1, \dots, \xi_{n+1}, \pi_P(a)\xi_1, \dots, \pi_P(a)\xi_{n+1}\}$ is a linearly independent

https://doi.org/10.1017/S0305004100074892 Published online by Cambridge University Press

set. Then the Jacobson density theorem implies the existence of an element x from $\mathscr{A}$ satisfying

$$\pi_P(x)\,\xi_j = 0 \quad \text{and} \quad \pi_P(x)\,\pi_P(a)\,\xi_j = -j\xi_j$$

for all $j \neq 1, \ldots, n+1$. It follows that

$$\pi_P([a,x])\,\xi_j = j\xi_j, \quad j = 1, \ldots, n+1.$$

Hence, $\{1, \ldots, n+1\} \subset \sigma(\pi_P([a,x])) \subset \sigma([a,x])$, which contradicts our assumption (ii). So, by Lemma 3·1, $\pi_P(a)$ can be written as $\pi_P(a) = \lambda I + A$, where $\lambda \in \mathbb{C}$ and A has finite rank. Consequently, $\pi_P(a) \in \mathbb{C}1 + \operatorname{soc}(\mathscr{A}/\mathscr{P})$.

Assume that there exist irreducible representations $\pi_1, \ldots, \pi_{n+1}$ of $\mathscr{A}$ on Banach spaces $X_1, \ldots, X_{n+1}$, respectively, such that the primitive ideals $P_k = \ker(\pi_k)$, $k = 1, \ldots, n+1$, are distinct, and $\pi_j(a) \notin \mathbb{C}I$. As $\pi_j(a)$ is not a scalar multiple of the identity operator we can find linearly independent vectors $\xi_j, \zeta_j \in X_j$ such that $\pi_j(a)\,\xi_j = \zeta_j, j = 1, \ldots, n+1$. Applying the extended Jacobson density theorem we get the existence of an element $x \in \mathscr{A}$ such that

$$\pi_j(x)\,\xi_j = 0 \quad \text{and} \quad \pi_j(x)\,\zeta_j = -j\xi_j, \quad j = 1, \ldots, n+1.$$

Consequently, $\pi_j([a,x])\,\xi_j = j\xi_j, j = 1, \ldots, n+1$, which further yields that $\{1, \ldots, n+1\} \subset \sigma([a,x])$. This contradiction completes the proof of (ii) $\Rightarrow$ (iii).

Assume now that (iii) is satisfied. Let $\{P_1, \ldots, P_m\}$ be the set of all primitive ideals in $\mathscr{A}$ satisfying $\pi_{P_j}(a) \notin \mathbb{C}1$. Then we have for an arbitrary $x \in \mathscr{A}$

$$\sigma([a,x]) = \bigcup_{P \in \operatorname{Prim}(\mathscr{A})} \sigma(\pi_P([a,x])) \subset \{0\} \cup \sigma([\pi_{P_1}(a), \pi_{P_1}(x)]) \cup \ldots \cup \sigma([\pi_{P_m}(a), \pi_{P_m}(x)]).$$

As $\pi_{P_j}(a) \in \mathbb{C}1 + \operatorname{soc}(\mathscr{A}/P_j), j = 1, \ldots, m$, we have $[\pi_{P_j}(a), \pi_{P_j}(x)] \in \operatorname{soc}(\mathscr{A}/P_j)$, and so, it follows that $\#\sigma([\pi_{P_j}(a), \pi_{P_j}(x)]) < \infty$. Thus, (iii) implies (i).

In order to conclude the proof assume that $\mathscr{A}$ is semi-simple. If (iii) is fulfilled then for every $x \in \mathscr{A}$ the element $b = [a,x]$ satisfies the condition (ii) of Proposition 2·2, and consequently, a satisfies the condition (iv). Each element in $\operatorname{soc}(\mathscr{A})$ is algebraic. It follows that (iv) implies (v). Obviously, (v) yields (i). This completes the proof.

4. *Applications*

This section deals with applications of our main result. Elements a from $\mathscr{A}$ satisfying $a + F \subset F$, where F is the set of all elements in $\mathscr{A}$ having finite spectrum, have already been studied (see [**2**, lemma 5.6.5 and theorem 5.6.6]). Applying [**2**, lemma 5.6.5] one gets the following result containing some further information about such elements.

COROLLARY 4·1. *Let $\mathscr{A}$ be a unital Banach algebra and let $a \in \mathscr{A}$. If $a + F \subset F$ then a satisfies all the equivalent conditions of Theorem* 3·2.

Next we will consider an analogous condition $a + \operatorname{Alg}(\mathscr{A}) \subset \operatorname{Alg}(\mathscr{A})$, where $\operatorname{Alg}(\mathscr{A})$ denotes the set of all algebraic elements in $\mathscr{A}$. Every element a satisfying this condition must be algebraic for 0 is algebraic. To study such elements we need the following lemma.

LEMMA 4·2. *Let X be a Banach space and let $\mathscr{B}(X)$ be the algebra of all bounded linear*

https://doi.org/10.1017/S0305004100074892 Published online by Cambridge University Press

operators on X. Assume that $A \in \mathcal{B}(X)$ is of the form $A = \lambda I + A_0$ where $\lambda \in \mathbb{C}$ and A_0 is a finite rank operator. Then $A+B$ is algebraic for every algebraic operator $B \in \mathcal{B}(X)$.

Proof. Obviously, it is enough to prove this lemma in the special case when A is a rank one operator. Such an operator can be written as $A = u \otimes f, u \in X, f \in X^*$. Here, $u \otimes f$ is defined by $(u \otimes f)(x) = f(x)u$ for all $x \in X$. Let B be an algebraic operator of degree n. Choose any $x \in X$. It is easy to see that the linear span of $\{u, Bu, \dots, B^{n-1}u, x, Bx, \dots, B^{n-1}x\}$ is invariant for both operators A and B. Its dimension is at most $2n$, and so the vectors $x, (A+B)x, \dots, (A+B)^{2n}x$ are linearly dependent for every $x \in X$. Kaplansky's result [2, theorem 4.2.7] implies that $A+B$ is algebraic. This completes the proof. ∎

Note that this lemma has an equivalent reformulation. Let $\mathcal{A}$ be a primitive Banach algebra. If $a \in \operatorname{soc}(\mathcal{A}) + \mathbb{C}1$ then $a+b$ is algebraic for every algebraic element $b \in \mathcal{A}$.

COROLLARY 4·3. *Let $\mathcal{A}$ be a unital semi-simple Banach algebra. For $a \in \mathcal{A}$ the following conditions are equivalent:*

(i) *a is algebraic and $\#\sigma([a,x]) < \infty$ for all $x \in \mathcal{A}$,*
(ii) *a is algebraic and there exists a positive integer n such that $\#\sigma([a,x]) \leqslant n$ for all $x \in \mathcal{A}$,*
(iii) *$\pi_P(a) \in \mathbb{C}1 + \operatorname{soc}(\mathcal{A}/P)$ for all $P \in \operatorname{Prim}(\mathcal{A})$ and there exists a finite subset $K \subset \mathbb{C}$ such that $\pi_P(a) \in K1$ for all but finitely many $P \in \operatorname{Prim}(\mathcal{A})$,*
(iv) *a is algebraic and $[a,x] \in \operatorname{soc}(\mathcal{A})$ for all $x \in \mathcal{A}$,*
(v) *a is algebraic and $[a,x]$ is algebraic for all $x \in \mathcal{A}$,*
(vi) *$a + \operatorname{Alg}(\mathcal{A}) \subset \operatorname{Alg}(\mathcal{A})$.*

Proof. To see that conditions (i), (ii), (iii), (iv), and (v) are equivalent we only have to show that an element $a \in \mathcal{A}$ satisfying the condition (iii) of Theorem 3·2 is algebraic if and only if there exists a finite subset $K \subset \mathbb{C}$ such that $\pi_P(a) \in K1$ for all but finitely many $P \in \operatorname{Prim}(\mathcal{A})$. Let us first assume that a is algebraic. Applying $\sigma(\pi_P(a)) \subset \sigma(a)$ and the assumption (iii) of Theorem 3·2 one can easily conclude that $\pi_P(a) \in \sigma(a)\,1$ for all but finitely many $P \in \operatorname{Prim}(\mathcal{A})$.

To prove the converse statement we assume the existence of a finite subset $K \subset \mathbb{C}$ such that $\pi_P(a) \in K1$ for all but finitely many $P \in \operatorname{Prim}(\mathcal{A})$. We denote by $P_1, \dots, P_m$ all primitive ideals satisfying $\pi_{P_j}(a) \notin K1$. For every j, $1 \leqslant j \leqslant m$, we have $\pi_{P_j}(a) \in \operatorname{soc}(\mathcal{A}/P_j) + \mathbb{C}1$, and consequently, $q_j(\pi_{P_j}(a)) = 0$ for some polynomial q_j. Introducing a new polynomial

$$q(\lambda) = \left(\prod_{j=1}^{m} q_j(\lambda)\right)\left(\prod_{\mu \in K} (\lambda - \mu)\right)$$

one can easily verify that

$$\pi_P(q(a)) = \left(\prod_{j=1}^{m} q_j(\pi_P(a))\right)\left(\prod_{\mu \in K} (\pi_P(a) - \mu 1)\right) = 0$$

for every $P \in \operatorname{Prim}(\mathcal{A})$. As $\mathcal{A}$ is semi-simple we have $q(a) \in \bigcap_{P \in \operatorname{Prim}(\mathcal{A})} P = \{0\}$.

The proof of the implication (vi) ⇒ (ii) goes through in exactly the same way as the proof of [2, lemma 5.6.5]. It remains to prove that (iii) implies (vi). For this purpose we choose an algebraic element $b \in \mathcal{A}$. Let p be a polynomial satisfying $p(b) = 0$. Let

https://doi.org/10.1017/S0305004100074892 Published online by Cambridge University Press

us denote the elements of K by $K = \{\lambda_1, \ldots, \lambda_k\}$. According to our assumption we have $\pi_P(a) \in K1$ for all $P \in \text{Prim}(\mathscr{A})$ but $P_1, \ldots, P_m$. Element $\pi_{P_j}(a)$ belongs to $\mathbb{C}1 + \text{soc}(\mathscr{A}/P_j)$ and $\pi_{P_j}(b)$ is algebraic, $j = 1, \ldots, m$. According to Lemma 4·2 we can find polynomials $q_j, j = 1, \ldots, m$, such that $q_j(\pi_{P_j}(a+b)) = 0$. Introducing a new polynomial q by

$$q(\lambda) = q_1(\lambda) \ldots q_m(\lambda)\, p(\lambda - \lambda_1) \ldots p(\lambda - \lambda_k)$$

one can easily verify that $\pi_P(q(a+b)) = 0$ for all $P \in \text{Prim}(\mathscr{A})$. It follows that $q(a+b) = 0$ which completes the proof. ∎

We will conclude by proving two results which characterize bounded finite rank operators on a Banach space X. The first one is a simple consequence of Lemma 3·1, Lemma 4·2, and Corollary 4·3.

COROLLARY 4·4. *Let X be a Banach space and let $\mathscr{B}(X)$ be the algebra of all linear bounded operators on X. For an operator $A \in \mathscr{B}(X)$ the following three conditions are equivalent*:

(i) *A is of the form $A = \lambda I + A_0$, where $\lambda \in \mathbb{C}$ and A_0 is a finite rank operator,*

(ii) *there exists a positive integer n such that $\{x_1, \ldots, x_n, Ax_1, \ldots, Ax_n\}$ is a linearly dependent set for all vectors $x_1, \ldots, x_n \in X$,*

(iii) *$A + B$ is algebraic for every algebraic operator $B \in \mathscr{B}(X)$.*

COROLLARY 4·5. *Let X be a Banach space and let $\mathscr{B}(X)$ be the algebra of all linear bounded operators on X. For an operator $A \in \mathscr{B}(X)$ the following two conditions are equivalent*:

(i) *A has finite rank,*

(ii) *$AB + C$ is algebraic for arbitrary algebraic operators $B, C \in \mathscr{B}(X)$.*

Proof. In case that X is finite dimensional there is nothing to prove. So, we will assume from now on that X is infinite-dimensional. Implication (i) ⇒ (ii) is a direct consequence of Corollary 4·4. So, assume that (ii) is satisfied. Replacing B by I and applying Corollary 4·4 once again we see that A must be of the form $A = \lambda I + A_0$ with A_0 being a finite rank operator. This further yields that $\lambda B + C + A_0 B$ is algebraic whenever B and C are of the kind. But $A_0 B$ is always a finite rank operator. Hence, by Corollary 4·4, $\lambda B + C$ is algebraic whenever B and C are algebraic. We have to show that $\lambda = 0$. Suppose this is not true. Then for every pair of algebraic operators B and C the operator $B + C$ is algebraic. Applying Corollary 4·4 once again we see that all algebraic operators belong to the set $\mathbb{C}I + \mathscr{F}(X)$, where $\mathscr{F}(X)$ denotes the ideal of all bounded finite rank operators. Showing that this is not true we will get a contradiction which completes the proof.

By [**6**, theorem 1.a.5] we can find a sequence of unit vectors $(x_n) \in X$ and a sequence of uniformly bounded functionals $(f_n) \subset X^*$ satisfying $f_n(x_m) = \delta_{n,m}$. Operator $A = \sum_{j=1}^{\infty} 2^{-j} x_{2j} \otimes f_{2j-1}$ is an example of an algebraic operator which is not a sum of a scalar multiple of the identity operator and a finite rank operator.

The authors were supported by grants from the Ministry of Science of Slovenia.

REFERENCES

[1] J. C. ALEXANDER. Compact Banach Algebras. *Proc. London Math. Soc.* **18** (1968), 1–18.

[2] B. AUPETIT. *A primer on spectral theory* (Springer-Verlag, 1991).

https://doi.org/10.1017/S0305004100074892 Published online by Cambridge University Press

[3] F. F. BONSALL and J. DUNCAN. *Complete normed algebras* (Springer, 1973).
[4] J. M. G. FELL and R. S. DORAN. *Representations of *-algebras, locally compact groups, and Banach *-algebraic bundles*, vol. 1 (Academic Press, 1988).
[5] C. LANSKI. Derivations with algebraic values on Lie ideals. *Comm. Algebra* **18** (1990), 1379–1399.
[6] J. LINDENSTRAUSS and L. TZAFRIRI. *Classical Banach spaces, Part I: sequence spaces* (Springer, 1977).
[7] M. MATHIEU. Where to find the image of a derivation, in *Functional analysis and operator theory*, Banach Center Publications, vol. 30 (Institute of Mathematics, Polish Academy of Sciences, Warsaw, 1994), 237–249.
[8] G. J. MURPHY. Aspects of the theory of derivations, in *Functional analysis and operator theory*, Banach Center Publications, vol. 30 (Institute of Mathematics, Polish Academy of Sciences, Warsaw, 1994), 267–275.
[9] C. E. RICKART. *General theory of Banach algebras* (van Nostrand, 1960).

https://doi.org/10.1017/S0305004100074892 Published online by Cambridge University Press