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First-Passage Percolation with Exponential Times on a Ladder

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We consider first-passage percolation on a ladder, *i.e.*, the graph $\mathbb{N} \times \{0,1\}$, where nodes at distance 1 are joined by an edge, and the times are exponentially i.i.d. with mean 1. We find an appropriate Markov chain to calculate an explicit expression for the *time constant* whose numerical value is ≈ 0.6827. This time constant is the long-term average inverse speed of the process. We also calculate the average residual time.

1. Introduction

Consider a graph G with vertex (node) set V and edge set $E \subset V \times V$. An (undirected) edge $e = \langle v, v' \rangle = \langle v', v \rangle$ joins vertex v and v'. A path $\pi(v, v')$ between v and v', if it exists, is an alternating sequence of vertexes and edges $(v_0, e_1, v_1, \ldots, e_n, v_n)$ such that $e_i = \langle v_{i-1}, v_i \rangle$ for $i = 1, \ldots, n$, $v_0 = v$ and $v_n = v'$.

Associate with each edge e a non-negative random variable ξ_e. Let the time of a path $\pi(v, v')$ be

$$T\pi(v, v') = \sum_{e \in \pi(v,v')} \xi_e \quad \text{and let} \quad T(v, v') = \inf_{\pi(v,v')} T\pi(v, v')$$

be the shortest time of any path between v and v'. This may be called the first-passage time of v' (with respect to v), and is the subject of investigation in first-passage percolation; see, *e.g.*, [9].

We will think of this as a model for a contagious disease and refer to the first-passage time of a node as the time of infection. Usually, we think of some node v (or more generally a set of nodes) as infected at time zero. The times $T(w) = T(v, w)$, $w \in V$, tell us when nodes $w \in V$ are infected.

One natural question is: How fast is the spread of the infection? The present paper considers the equivalent question of the inverse speed of percolation on a ladder, that is: How long does it take for the infection to spread one more step up the ladder, averaged

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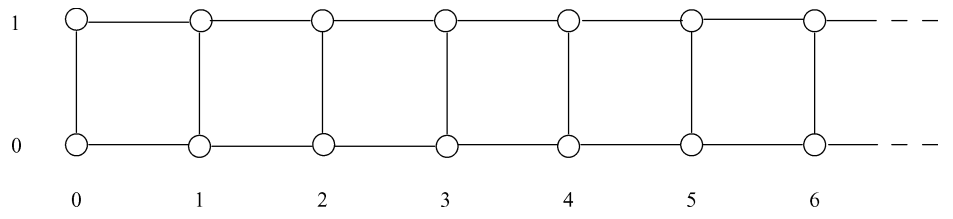


Figure 1. Part of the ladder L.

over time? We also calculate the average inverse speed in another sense, namely the average residual time.

2. First-passage percolation on the ladder

First-passage percolation on a ladder has previously been studied in [5], which gives a method of calculating the time constant when the times associated with edges have a discrete distribution, as well as a method for getting arbitrarily good bounds for the same quantity when the distribution is continuous (and well-behaved). Our paper gives an exact formula for the time constant only when the times are exponential, and the method we use relies heavily on the special properties of this distribution.

It has been brought to our attention that during the referee period of the present paper, a paper by Schlemm [7] has appeared, which also contains the percolation rate (inversion of the time constant), although it is achieved through other methods.

Let L be a ladder, by which we mean a graph with vertex set $V = \mathbb{N} \times \{0, 1\}$, and where the edge set E consists of $\langle v, v' \rangle$ such that v and v' are at distance one from each other: see Figure 1. By $\mathbb{N}$ we mean the set of non-negative integers $\{0, 1, 2, \ldots\}$.

We will consider the model where the nodes $(0, 0)$ and$(0, 1)$ are infected at time zero and each edge is associated with a random variable that has an exponential distribution with mean 1. Random variables associated with different edges are independent.

For a node $v = (x, y)$ we will say that v is at height $x \in \mathbb{N}$ and level $y \in \{0, 1\}$. Recall that $T[(x, y)]$ is the infection time of node (x, y). Let

$$N_t^{(i)} = \sup\{x : T[(x, i)] \leqslant t\}, \quad i = 0, 1,$$

that is, let $N_t^{(0)}$ and $N_t^{(1)}$ denote the height of the infection at level 0 and 1, respectively, at time t.

We will let $N_t = \max\{N_t^{(0)}, N_t^{(1)}\}$ denote the height of the infection, that is, the largest height of any infected node, at time t.

We aim to calculate the time constant τ, defined as the limiting inverse percolation rate, $1/\tau = \lim_t N_t/t$. We will not do this directly, but calculate the percolation rate at 'time infinity', or rather when a process related to N_t has reached stationarity.

More explicitly, we will employ an observation by Alm and Janson [2] that arose in connection with their joint work on one-dimensional lattices [3]: namely, it might be fruitful to consider the *front* $F_t = |N_t^{(0)} - N_t^{(1)}|$ at time $t \geqslant 0$. Front thus means the absolute value of the difference in height between the highest infected nodes on each level: see Figure 2.

Now, the process $\{F_t, t \geqslant 0\}$ behaves like a continuous-time Markov chain on $\mathbb{N}$, with $F_0 = 0$. Suppose that this Markov chain tends to a stationary distribution $\Pi = (\pi_0,$

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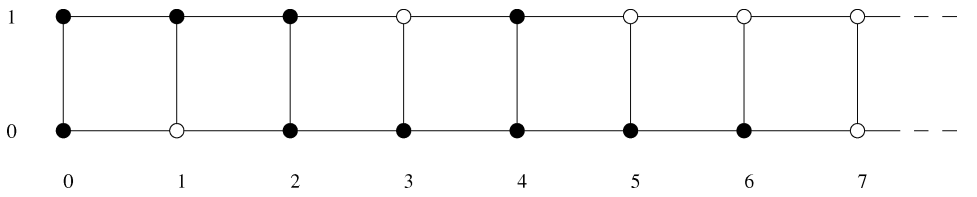


Figure 2. Infected nodes at time t marked as black. Here $N_t = 6$ and $F_t = 2$.

$\pi_1, \pi_2, \ldots)$ on $\mathbb{N}$. Then, at a late time t the process F will be in state 0 with probability π_0. From this state there are two possible nodes that may be the next infected ones that result in an increase of the N-process, that is, the intensity is 2 towards a state that increases the height of the infection, since these two nodes are associated with two distinct edges (and thus two distinct exponentially distributed mean 1 random variables). With probability $1 - \pi_0$ the F process is in some other state ($F \geqslant 1$), from which there is intensity 1 towards a state that increases the height of the infection. Thus, knowing the stationary distribution (or rather knowing π_0) gives us the following percolation rate at a late time:

$$2\pi_0 + 1(1 - \pi_0) = 1 + \pi_0. \tag{2.1}$$

Below we calculate this stationary distribution Π. To do so, it turns out that we may express any π_n in terms of π_0 in the form $\pi_n = a_n\pi_0 - b_n$, where both sequences of (positive) coefficients, a_n and b_n, satisfy a certain recursion (Claim 2) and can be expressed in terms of Bessel functions of the first and second kind. We stumbled upon the solution – which we might not otherwise have solved – when it was noted that the sequence b_n transformed into $B_n = (b_n - b_{n-1})/n$ is part of sequence A058797, as listed in [8], which is related to the Bessel functions. This relation to Bessel functions was a surprise, and we see no reason to expect it in this context. This method can be generalized to percolation on a more general 'ladder-like' graph; see [6]. The sequence b_n is also a subsequence of A056921.

Now, we start by looking at the intensity matrix Q of the process F_t. Consider the case in Figure 2, where $F_t = 2$. Now, there are two edges leading to the node $(5, 1)$, and if this is the next infected, the process will end up in state $F = 1$, *i.e.*, the intensity for this state is 2. There is one edge each to node $(6, 1)$ and $(7, 0)$ and if these are the next ones infected, the process will end up in state $F = 0$ or $F = 3$, respectively. This argument gives us the third row of the intensity matrix Q. We omit the rest of the details, but it is not hard to see that

$$Q = \begin{pmatrix} -2 & 2 & 0 & 0 & 0 & \\ 2 & -3 & 1 & 0 & 0 & \\ 1 & 2 & -4 & 1 & 0 & \cdots \\ 1 & 1 & 2 & -5 & 1 & \\ 1 & 1 & 1 & 2 & -6 & \\ 1 & 1 & 1 & 1 & 2 & \\ & & \vdots & & & \ddots \end{pmatrix}. \tag{2.2}$$

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Table 1. The beginning of the sequences $\{a_n\}$ and $\{b_n\}$.

n	1	2	3	4	5	6	7	8	9
a_n	3	11	56	340	2395	19231	173490	1737706	19136803
b_n	1	5	26	158	1113	8937	80624	807544	8893225

$\{F_t, t \geqslant 0\}$ is irreducible, so if we can find a distribution $\Pi = (\pi_0, \pi_1, \pi_2, \ldots)$ on $\mathbb{N}$ such that $\Pi Q = 0$, then Π is the unique stationary distribution for $\{F_t, t \geqslant 0\}$, by Markov chain theory. For calculations we henceforth assume that such a distribution Π exists.

Using the equations arising from columns one to three of Q, the condition $\Pi Q = 0$, and $\sum \pi_j = 1$, we get

$$\pi_1 = 3\pi_0 - 1, \quad \pi_2 = 11\pi_0 - 5, \quad \text{and} \quad \pi_3 = 56\pi_0 - 26. \tag{2.3}$$

Also, taking differences of equations arising from adjacent columns of Q, from column three and upwards, gives the equations

$$0 = \pi_{n-3} - (n+1)\pi_{n-2} + (n+3)\pi_{n-1} - \pi_n, \quad n \geqslant 4. \tag{2.4}$$

We set $\pi_n = a_n\pi_0 - b_n$, where $a_1, a_2, a_3, b_1, b_2, b_3$ are defined by (2.3) for $n = 1, 2, 3$, and by (2.4) for $n \geqslant 4$, with a_n or b_n in place of π_n. Table 1 gives the first nine numbers in the sequences $\{a_n\}$ and $\{b_n\}$.

Claim 1. *The sequences $\{a_n\}$ and $\{b_n\}$ both satisfy the recursion*

$$c_n = \frac{c_{n+1} - c_n}{n+1} - \frac{c_n - c_{n-1}}{n}, \quad n \geqslant 2. \tag{2.5}$$

Proof. We will use induction. First, check initial conditions, $a_2 = 11$ and $(56 - 11)/3 - (11 - 3)/2 = 11$, $b_2 = 5$ and $(26 - 5)/3 - (5 - 1)/2 = 5$, so equation (2.5) holds for $n = 2$. Rearranging (2.5) gives

$$c_{n+1} - c_n = (n+1)\left(c_n + \frac{1}{n}[c_n - c_{n-1}]\right), \quad n \geqslant 2, \tag{2.6}$$

and this relation is what we will prove. So assume (2.6) holds for some $n \geqslant 2$. We aim to show that it also holds for $n+1$. So, as $n+2 \geqslant 4$, we can use (2.4). Adding $\pi_n - \pi_{n-1}$ to both sides of (2.4), and substituting $n+2$ for n, gives

$$\begin{aligned} c_{n+2} - c_{n+1} &= c_{n-1} - (n+3)c_n + (n+5)c_{n+1} - c_{n+1} \\ &= (n+2)\left(c_{n+1} + \frac{1}{n+2}\left[2c_{n+1} - (n+2)c_n - \underbrace{(c_n - c_{n-1})}_{(*)}\right]\right). \end{aligned} \tag{2.7}$$

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From (2.6) and the induction hypothesis, we get $c_n - c_{n-1} = (nc_{n+1} - n(n+2)c_n)/(n+1)$, which we apply to (∗) above, and

$$(2.7) = (n+2)\left(c_{n+1} + \frac{1}{n+2}\left[\left(2 - \frac{n}{n+1}\right)c_{n+1} - \left(n+2 - \frac{n(n+2)}{n+1}\right)c_n\right]\right)$$
$$= (n+2)\left(c_{n+1} + \frac{1}{n+1}[c_{n+1} - c_n]\right). \qquad \square$$

Now set $C_n = (c_n - c_{n-1})/n$. Then, if c_n satisfies (2.5), *i.e.*, $c_n = C_{n+1} - C_n$, this implies both

$$C_{n+1} = c_n + C_n \tag{2.8}$$

and $C_n = C_{n+1} - c_n$. Substitute $n-1$ for n in the last equation to get

$$C_{n-1} = C_n - c_{n-1}. \tag{2.9}$$

Then, adding (2.8) and (2.9) and replacing $c_n - c_{n-1}$ with nC_n gives us the following.

Claim 2. *$A_n = (a_n - a_{n-1})/n$ and $B_n = (b_n - b_{n-1})/n$ both satisfy the recursion*

$$C_{n+1} + C_{n-1} = (n+2)C_n, \quad n \geqslant 3. \tag{2.10}$$

2.1. Bessel functions

The functions

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^{n+2k}$$

and

$$Y_n(x) = \frac{1}{\pi}\left[2\left(\gamma + \ln\frac{x}{2}\right)J_n(x) - \sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-n}\right.$$
$$\left. - \sum_{k=0}^{\infty}(\varsigma_k + \varsigma_{k+n})\frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^{2}\right],$$

where $n \in \mathbb{N}$, $\varsigma_m = \sum_{j=1}^{m}\frac{1}{j}$ and $\gamma = 0.577\cdots$ is Euler's constant, are known as Bessel functions of the first and second kind. They both satisfy the recursion

$$C_{n+1}(x) + C_{n-1}(x) = \frac{2n}{x}C_n(x). \tag{2.11}$$

Define the function

$$\Upsilon(n,m) = \pi[J_n(2)Y_m(2) - J_m(2)Y_n(2)]. \tag{2.12}$$

This function inherits recursion (2.11) in parameter n with $x = 2$, because for m fixed Υ is a linear combination of $J_n(2)$ and $Y_n(2)$, so that

$$\Upsilon(n+1,m) + \Upsilon(n-1,m) = n\Upsilon(n,m). \tag{2.13}$$

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Table 2. Connection between parameters a_n, b_n and $\Upsilon(n,m)$.

n	1	2	3	4	5	6	7
b_n	1	5	26	158	1113	8937	80624
$B_n = (b_n - b_{n-1})/n$		2	7	33	191	1304	10241
$\Upsilon(n,0)$	1	1	1	2	7	33	191
a_n	3	11	56	340	2395	19231	173490
$A_n = (a_n - a_{n-1})/n$		4	15	71	411	2806	22037
$2\Upsilon(n,3) + \Upsilon(n,0)$	−3	−1	1	4	15	71	411

Claim 3. *For all integers n and m the function* $\Upsilon(n,m)$ *is integer-valued.*

Proof. Clearly $\Upsilon(m,m) = 0$, so it suffices to show that $\Upsilon(m+1,m)$ is integer-valued, since recursion (2.13) takes care of all other values.

Two important relations for the Bessel functions are needed. The first is that

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x), \quad \forall n. \tag{2.14}$$

and the same holds with Y_n replacing J_n (see [10, 3.2(4), p. 45] and [10, 3.56(4), p. 66], respectively). The second is a fact relating to the Wronskian, namely that

$$\frac{2}{\pi x} = J_n(x)Y_n'(x) - J_n'(x)Y_n(x), \quad \forall n, x \neq 0 \tag{2.15}$$

(see [10, 3.63(1), p. 76]). By dividing both sides of equation (2.14) by $x = 2$ and using the resulting expression for $J_n'(2)$ and $Y_n'(2)$, respectively, in (2.15) results in

$$\begin{aligned}\frac{1}{\pi} &= J_n(2)\left[\frac{n}{2}Y_n(2) - Y_{n+1}(2)\right] - \left[\frac{n}{2}J_n(2) - J_{n+1}(2)\right]Y_n(2)\\ &= J_{n+1}(2)Y_n(2) - J_n(2)Y_{n+1}(2) = \frac{1}{\pi}\Upsilon(n+1,n),\end{aligned}$$

the last equality by the definition (2.12) of Υ. Hence, $\Upsilon(n+1,n) = 1$. □

Now we can evaluate the function $\Upsilon(n,m)$ for different values of n and m to find that $B_n = \Upsilon(n+2,0)$, $n = 2,3$, and $A_n = 2\Upsilon(n+2,3) + \Upsilon(n+2,0)$, $n = 2,3$: see Table 2. Since $\Upsilon(n+2,m)$ satisfies recursion (2.10) in n, this is enough to know that they agree for all $n \geqslant 2$.

By relation (2.8), definition (2.12) and the above expression for A_n and B_n, we get

$$\begin{aligned}b_n &= \Upsilon(n+3,0) - \Upsilon(n+2,0) \qquad \text{and}\\ a_n &= 2[\Upsilon(n+3,3) - \Upsilon(n+2,3)] + \Upsilon(n+3,0) - \Upsilon(n+2,0).\end{aligned} \tag{2.16}$$

Henceforth we abbreviate $J_n = J_n(2)$ and $Y_n = Y_n(2)$, and hence, for example,

$$J_n = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}.$$

We need asymptotics for a_n and b_n.

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Claim 4.

$$\lim_{n\to\infty} \frac{b_n}{(n+2)!} = J_0 \quad \textit{and} \quad \lim_{n\to\infty} \frac{a_n}{(n+2)!} = 2J_3 + J_0. \tag{2.17}$$

Proof. Abramowitz and Stegun [1] list the following asymptotic relations, for fixed x and n tending to infinity:

$$J_n(x) \sim \frac{1}{\sqrt{2\pi n}} \left(\frac{ex}{2n}\right)^n \quad \text{and} \quad Y_n(x) \sim -\sqrt{\frac{2}{\pi n}} \left(\frac{2n}{ex}\right)^n.$$

Hence, by the well-known Stirling formula $n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n}$, we conclude that $J_n \sim 1/n!$ and $Y_n \sim -(n-1)!/\pi$. Thus

$$\Upsilon(n,m) \sim \pi \frac{1}{n!} Y_m + \pi J_m \frac{(n-1)!}{\pi} \sim J_m \cdot (n-1)!,$$

from which (2.17) follows. □

Now we are ready to find an expression for π_0. Since $\pi_n \to 0$ as $n \to \infty$, we get

$$\pi_0 = \lim_{n\to\infty} \frac{\pi_n + b_n}{a_n} = \frac{J_0}{2J_3 + J_0} = 0.4647184275\cdots. \tag{2.18}$$

This in turn gives us

$$\begin{aligned}
\pi_n &= a_n \pi_0 - b_n \\
&= 2[\Upsilon(n+3,3) - \Upsilon(n+2,3)]\pi_0 + [\Upsilon(n+3,0) - \Upsilon(n+2,0)][\pi_0 - 1] \\
&= \frac{2\pi}{2J_3 + J_0} \big([J_{n+3}Y_3 - J_3 Y_{n+3} - J_{n+2}Y_3 + J_3 Y_{n+2}] J_0 \\
&\quad - [J_{n+3}Y_0 - J_0 Y_{n+3} - J_{n+2}Y_0 + J_0 Y_{n+2}] J_3 \big) \\
&= \frac{2\pi(J_3 Y_0 - J_0 Y_3)}{2J_3 + J_0} (J_{n+2} - J_{n+3}) = \frac{2\Upsilon(3,0)}{2J_3 + J_0} (J_{n+2} - J_{n+3}) \\
&= \frac{2}{2J_3 + J_0} (J_{n+2} - J_{n+3}), \quad n \geqslant 1.
\end{aligned} \tag{2.19}$$

Then $\sum_0^N \pi_j = 1 - 2J_{N+3}/(2J_3 + J_0)$, and as $J_n \to 0$, as $n \to \infty$, we can verify that Π is indeed a distribution.

2.2. The time constant

Following the discussion before and after (2.1), we get the time constant

$$\tau = \frac{1}{1 + \pi_0} = \frac{2J_3 + J_0}{2J_3 + 2J_0} = 0.6827250759\cdots. \tag{2.20}$$

This can be compared with first-passage percolation on $\mathbb{N}$, *i.e.*, the infinite path graph (with exponential r.v.s with mean 1), where the time constant is necessarily 1. We may also compare it with $1/2$, which is the time constant if the rungs of the ladder were associated with random variables all identically 0, in other words, there is immediate infection between nodes on different levels at distance 1 from each other.

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Note that we started the process by infecting both $(0,0)$ and $(0,1)$. Consider the infection of the single node $(0,0)$ at time 0. Then, the front process $\hat{F}$ will not be a continuous-time Markov chain (as before), as the behaviour of the front is very different before and after the random time Z until there are infected nodes on both levels. Since we are interested in stationarity, we may simply disregard what happens before Z and view this as starting the front process F at time Z, $F_t = \hat{F}_{Z+t}$, $t \geqslant 0$. Then F is a continuous-time Markov chain with intensity matrix given by (2.2), as before, albeit started in a random state instead of $F_0 = 0$. Z is dominated by a mean 1 exponential random variable (the edge between $(0,0)$ and $(0,1)$). As F will converge to the stationary distribution Π, so will $\hat{F}$.

Hence, infecting just a single node initially makes no difference to the time constant. (In fact, starting with an initial infection of any finite number of nodes will not change the time constant.)

This time constant also gives an upper bound for the time constant for first-passage percolation with mean 1 exponential r.v.s on $\mathbb{Z}^2$, but not stronger than already existing bounds: see [4].

2.3. Residual times

The fact that we can write down the stationary distribution Π for the process F_t also gives us the opportunity to calculate another statistic for first-passage percolation on the ladder, namely the average residual time. As far as we know, this quantity has not previously been studied for percolation. Define the *residual time at t* as $R_t = \inf\{s : N_{t+s} = N_t + 1\}$, *i.e.*, the time it takes, after t, for the infection to spread one more step up the ladder. We are interested in $T = \mathbb{E}R_t$ for a late time t.

One might *a priori* think that T and τ are the same thing. If we were looking at percolation on $\mathbb{N}$, which is the same as a Poisson process, then they are. This is the waiting time paradox: on one hand, looking at a fixed time point increases the probability of choosing a long time interval, but on the other hand, typically half that interval has passed. For a Poisson process, these effects balance out. For percolation on a ladder, as shown in (2.20) and (2.23) below, they do not.

Now, to calculate T we will assume that F has its stationary distribution, so that we may disregard time and put $R = R_t$. Let $\gamma_n = \mathbb{E}[R|F = n]$. Hence, γ_n is the expected time it takes for the infection to spread another step given that the front is in state n.

If $F = 0$ then the residual time is the minimum of two mean 1 exponential random variables, so that $\gamma_0 = 1/2$. For $n \geqslant 1$, we write down a recursive formula for γ_n. If $F = n$ then there are $n+2$ possible edges, equally likely, along which the infection might spread. Only one of these results in an increase in the height of the infection, namely the one that gives $F = n+1$. Two edges result in $F = n-1$ and the remaining edges result in one of the states $\{0, 1, \ldots, n-2\}$, respectively. This gives the formula

$$\gamma_n = \frac{1}{n+2}\left(1 + 2\gamma_{n-1} + \sum_{j=0}^{n-2}\gamma_j\right). \tag{2.21}$$

By writing down the formula for γ_{n-1}, we see that

$$1 + \sum_{0}^{n-2}\gamma_j = (n+1)\gamma_{n-1} - \gamma_{n-2},$$

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which, inserted into (2.21), yields the incremental relation

$$(n+2)(\gamma_n - \gamma_{n-1}) = \gamma_{n-1} - \gamma_{n-2},$$

which together with the fact that $\gamma_1 = \frac{1}{3}(1 + 2\frac{1}{2}) = \frac{2}{3}$ in turn gives us the formulas

$$\gamma_n - \gamma_{n-1} = \frac{1}{(n+2)!}, \quad n \geqslant 1, \quad \text{and} \quad \gamma_n = \sum_{j=0}^{n+2} \frac{1}{j!}, \quad n \geqslant 0. \tag{2.22}$$

Then, as F_t is in state j with probability π_j where it remains on average γ_j amount of time before N_t increases, we get the average residual time:

$$\begin{aligned} T &= \sum_{n=0}^{\infty} \pi_n \gamma_n = \frac{1}{2J_3 + J_0} \left(J_0/2 + 2 \sum_{n=1}^{\infty} (J_{n+2} - J_{n+3}) \gamma_n \right) \\ &= \frac{1}{2J_3 + J_0} \left(\frac{1}{2} J_0 + 2J_3 \gamma_1 + 2 \sum_{n=1}^{\infty} (\gamma_{n+1} - \gamma_n) J_{n+3} \right) \\ &= \frac{1}{2J_3 + J_0} \left(\frac{1}{2} J_0 + \frac{4}{3} J_3 + 2 \sum_{n=1}^{\infty} \frac{J_{n+3}}{(n+3)!} \right) = 0.5953444665 \cdots . \end{aligned} \tag{2.23}$$

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