# CONFORMAL TRANSFORMATIONS AND KINEMATICAL RELATIVITY

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1. *The Einstein relative-velocity formula.* Milne (3) has remarked that in kinematical relativity, whereas the Lorentz transformation is valid in $t$-time but fails in $\tau$-time, the Einstein relative-velocity formula is valid both in $t$-time and $\tau$-time, thus implying the validity of the latter in a more general system. It was thought that some interest might be attracted to a description of the most general system under which the Einstein formula holds.

The analysis which ensues suggests a Riemannian alternative to Einstein's general relativity theory. This in turn suggests a cosmology similar to that of Milne, but necessitating a modification at great distances to ensure convergence of the gravitational potential. The observable effects would, however, be at the very limits of possible observations.

In order that the Einstein relative-velocity formula may hold at any point there must be in the neighbourhood of the point a local metric. The general Riemannian geometry has a local metric, but this system is unnecessarily restricted because it implies that the metric at any one point may be compared with the metric at any other point. It is clear, then, that the most general system under which the Einstein relative-velocity formula is valid is the general Riemannian geometry (4-dimensional with metric signature $\pm 2$) subject to an arbitrary conformal transformation.

2. *Alternative Riemannian theory of gravitation.* The effects of a conformal change of metric on the geodesics and curvature of a space will be obtained. The geodesics will be considered first, and for simplicity the original space will be taken as flat.

Taking coordinates in the flat space-time a Lorentz frame may be chosen relative to which a small body considered is instantaneously at rest. Let it be supposed that the small body traverses a geodesic in the conformally modified space-time. Choose the $x$-axis in the direction of the spatial gradient of the scalar function of position which modifies the metric. To the first approximation the motion will be in the $xt$-plane.

Let $O$, $A$ be two events in space-time on the body, simultaneous at the initial time relative to the Lorentz frame chosen and with $A$ displaced from $O$ in the direction of the $x$-axis. Then if the body describes a geodesic in the modified metric and $O$, $A$ undergo metrically equal displacements to $O'$, $A'$, then the body will continue to move in a direction orthogonal to $O'A'$.

Put $OA = \delta x$. Let the original flat metric be

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2,$$

or in the $xt$-plane $\qquad ds^2 = dt^2 - dx^2.$

Let the modified metric be $\qquad ds' = \theta\, ds, \quad \theta = e^{\phi}.$

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Let $OO'$, $AA'$ represent a time interval $dt'$ in the modified metric. Then in flat space-time

$$OO' = e^{-\phi}\,dt', \quad AA' - OO' = -e^{-\phi}\frac{\partial\phi}{\partial x}\,\delta x\,dt'.$$

The inclination of $O'A'$ to the $x$-axis is then given by

$$-e^{-\phi}\frac{\partial\phi}{\partial x}\,\delta t'.$$

The orthogonal velocity will be equally inclined to the $t$-axis.

The change in velocity corresponding to the time $\delta t' = e^{\phi}\,\delta t$ is thus

$$-e^{-\phi}\frac{\partial\phi}{\partial x}\,\delta t' = -\frac{\partial\phi}{\partial x}\,\delta t.$$

Hence the acceleration induced, as measured in the original flat space-time, is

$$\frac{d^2x}{dt^2} = -\frac{\partial\phi}{\partial x}.$$

This conforms to the older potential theory provided that the index of the exponential in the metric modifying factor is identical with the gravitational potential at the point.

For a particle already in motion with velocity $v = \tanh u$ along the $x$-axis the correct acceleration is obtained by the Lorentz transformation

$$x' = x\cosh u + t\sinh u, \quad t' = x\sinh u + t\cosh u.$$

Using dots for derivatives and putting

$$\dot{x} = 0, \quad \dot{t} = 1, \quad \ddot{x} = -k, \quad \ddot{t} = 0,$$

this gives
$$\frac{d^2x'}{dt'^2} = \frac{\ddot{x}'\dot{t}' - \dot{x}'\ddot{t}'}{\dot{t}'^3} = \frac{-k\cosh^2 u + k\sinh^2 u}{\cosh^3 u} = -k\,\mathrm{sech}^3 u.$$

Also
$$k = \frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial\phi}{\partial t'}\frac{\partial t'}{\partial x} = \frac{\partial\phi}{\partial x'}\cosh u + \frac{\partial\phi}{\partial t'}\sinh u.$$

Thus
$$\frac{d^2x'}{dt'^2} = -\frac{\partial\phi}{\partial x'}\mathrm{sech}^2 u - \frac{\partial\phi}{\partial t'}\tanh u\,\mathrm{sech}^2 u = -\left(\frac{\partial\phi}{\partial x'} + v\frac{\partial\phi}{\partial t'}\right)(1-v^2).$$

The component of acceleration normal to the velocity is reduced by the factor $(1-v^2)$

$$\frac{d^2y'}{dt'^2} = -\frac{\partial\phi}{\partial y'}(1-v^2).$$

The above results were obtained from first principles, as they give an intuitive picture of the behaviour. An equivalent result could be obtained by the use of the 3-index symbols. Putting
$$g'_{ij} = e^{2\phi}\,g_{ij},$$

if $g_{ij}$ is constant
$$\frac{\partial g'_{ij}}{\partial x_k} = e^{2\phi}\,2\phi_k g_{ij}.$$

Putting
$$g_{11} = g_{22} = g_{33} = -1 = -g_{00},$$
$$[ii, k] = -e^{2\phi}\,\phi_k g_{ii} \quad (i \neq k),$$
$$\begin{Bmatrix} k \\ ii \end{Bmatrix} = -g^{kp}\phi_p g_{ii}.$$

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No contraction is intended with respect to the $i$'s. The result is equivalent to that obtained above.

3. *Values of the Riemann-Christofel tensor.* The change in values of the Riemann-Christofel tensor can be found as follows. A space of signature 4 is first considered with coordinates so that the metric is of the form

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2.$$

The change to any metric, or to the Lorentz metric in particular, is easily performed by expressing the result in tensor form. It is not necessary that the original space should be flat, only that the local metric is expressible in the above form. The local equation to a geodesic with metric $ds' = e^{\phi} ds$ is given by

$$\frac{d^2 x_i}{dx_j^2} = \frac{\partial \phi}{\partial x_i} \Big/ \left(\frac{dx_j}{ds}\right)^2.$$

It is convenient now to adopt a matrix technique. Let $V$ denote a vector function of position, regarded as a column vector operated on by matrices from the left. Let $X_{ij}$ denote a matrix operator corresponding to a rotation in the $x_i - x_j$-plane, so that, if $V_i$ denotes a unit vector in the $x_i$ direction,

$$X_{ij} V_i = V_j, \quad X_{ij} V_j = -V_i.$$

Such matrices $X_{ij}$ satisfy the equations

$$X_{ij} = -X_{ji}, \quad X_{ij} X_{ik} - X_{ik} X_{ij} = X_{jk}, \quad X_{ij} X_{kp} - X_{kp} X_{ij} = 0 \quad (i, j, k, p \text{ unequal}).$$

These operators can indicate the components of the Riemann-Christofel tensor $R_{ijkp}$ by showing the rotation of the frame of reference which is caused by traversing a small circuit. In terms of repeated differentiation this is expressed as

$$\frac{\partial^2 V}{\partial x_i \partial x_j} - \frac{\partial^2 V}{\partial x_j \partial x_i} = R_{ijkp} X_{kp} V.$$

For the derivative of a scalar function of position $\psi$, the change of metric gives the equation

$$\frac{\partial \psi}{\partial x_i'} = e^{-\phi} \frac{\partial \psi}{\partial x_i},$$

but for a vector function of position allowance must be made for the consequent rotation of the axes when differentiating with respect to $x_i'$. Then

$$\frac{\partial V}{\partial x_i'} = e^{-\phi} \left( \frac{\partial V}{\partial x_i} + \phi_j X_{ij} V \right),$$

where $\phi_j = \partial \phi / \partial x_j$. Such differentiation is repeated to give $\dfrac{\partial^2 V}{\partial x_1 \partial x_2} - \dfrac{\partial^2 V}{\partial x_2 \partial x_1}$. It must be observed in the second differentiation that $\phi_j$ is a vector and is modified by the operator $X_{ij}$. This operator is treated as a differential operator, so that, operating on a product, $UV$,

$$X_{ij}(UV) = (X_{ij} U) V + U(X_{ij} V).$$

In the second differentiation the first derivatives of $V$ are ignored, since it may be assumed that they are zero at the given point.

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Thence

$$\frac{\partial V}{\partial x_1'} = e^{-\phi}\left(\frac{\partial V}{\partial x_1} + \phi_2 X_{12} V + \phi_3 X_{13} V + \phi_4 X_{14} V\right),$$

$$\begin{aligned}\frac{\partial^2 V}{\partial x_1' \partial x_2'} = e^{-2\phi}\Big\{&\frac{\partial^2 V}{\partial x_1 \partial x_2} + (\phi_{22} - \phi_2^2) X_{12} V + (\phi_{23} - \phi_2\phi_3) X_{13} V + (\phi_{24} - \phi_2\phi_4) X_{14} V \\ &+ \phi_1^2 X_{12} V + \phi_3^2 X_{12} V + \phi_4^2 X_{12} V - \phi_2\phi_3 X_{13} V - \phi_2\phi_4 X_{14} V \\ &+ \phi_1\phi_2 X_{21} X_{12} V + \phi_1\phi_3 X_{21} X_{13} V + \phi_1\phi_4 X_{21} X_{14} V + \phi_3\phi_2 X_{23} X_{12} V \\ &+ \phi_3^2 X_{23} X_{13} V + \phi_3\phi_4 X_{23} X_{14} V + \phi_4\phi_2 X_{24} X_{12} V + \phi_4\phi_3 X_{24} X_{13} V + \phi_4^2 X_{24} X_{14} V\Big\}.\end{aligned}$$

By the use of the equations

$$X_{ij} X_{ik} - X_{ik} X_{ij} = X_{jk}, \quad X_{ij} X_{kp} - X_{kp} X_{ij} = 0,$$

it follows that

$$\begin{aligned}\left(\frac{\partial^2 V}{\partial x_1' \partial x_2'} - \frac{\partial^2 V}{\partial x_2' \partial x_1'}\right) e^{2\phi} = &\frac{\partial^2 V}{\partial x_1 \partial x_2} - \frac{\partial^2 V}{\partial x_2 \partial x_1} + (\phi_{11} + \phi_{22} + \phi_3^2 + \phi_4^2) X_{12} V \\ &+ (\phi_{23} - \phi_2\phi_3) X_{13} V + (\phi_{13} - \phi_1\phi_3) X_{32} V + (\phi_{24} - \phi_2\phi_4) X_{14} V + (\phi_{14} - \phi_1\phi_4) X_{42} V.\end{aligned}$$

Comparison with the equation

$$\frac{\partial^2 V}{\partial x_i \partial x_j} - \frac{\partial^2 V}{\partial x_j \partial x_i} = R_{ijkp} X_{kp} V$$

gives $e^{2\phi} R'_{1212} = R_{1212} - \phi_{11} - \phi_{22} - \phi_3^2 - \phi_4^2$, $e^{2\phi} R'_{1213} = R_{1213} - \phi_{23} + \phi_2\phi_3$,

with $$e^{2\phi} R'_{12ij} = R_{12ij}$$

for other combinations of $i, j$.

The tensor form of these equations is thus

$$R'_{ijkp} = e^{-2\phi}\left\{R_{ijkp} - \begin{vmatrix} g_{ik}, & \phi_{ip} - \phi_i\phi_p \\ g_{jk}, & \phi_{jp} - \phi_j\phi_p \end{vmatrix} - \begin{vmatrix} g_{jp}, & \phi_{jk} - \phi_j\phi_k \\ g_{ip}, & \phi_{ik} - \phi_i\phi_k \end{vmatrix} - \begin{vmatrix} g_{ik} & g_{ip} \\ g_{jk} & g_{jp} \end{vmatrix} g^{qr}\phi_q\phi_r\right\},$$

this being the general result for the metric

$$ds^2 = g_{ij} dx^i dx^j.$$

The first and second contractions give for the Ricci tensor $R'_{ik}$,

$$e^{2\phi} R'_{ik} = R_{ik} - 2(\phi_{ik} - \phi_i\phi_k) - g_{ik} g^{qr}(\phi_{qr} + 2\phi_q\phi_r),$$

and for the total curvature $R'$

$$e^{2\phi} R' = R - 6g^{ij}(\phi_{ij} + \phi_i\phi_j).$$

Further, apart from the factor $e^{-2\phi}$, the complete change in the Riemann-Christofel tensor is accounted for by the corresponding components of the Ricci tensor $R_{ik}$. It follows that the conformal curvature, i.e. that which remains of the curvature tensor after the removal of the components corresponding to the contracted Ricci tensor, is unaltered by the conformal transformation except for the introduction of the factor $e^{-2\phi}$. This result is of course well known.

The Riemann-Christofel tensor separates into three parts which transform separately, the conformal curvature, the Ricci tensor, and the total curvature. For a Riemannian theory of gravitation one or more of these parts must be equated to zero in free space. There are only two reasonable possibilities. One is that adopted by Einstein of

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equating to zero in free space the Ricci tensor and the total curvature. The alternative is to equate the conformal curvature to zero, and to make in addition the total curvature zero in the absence of matter. Einstein's choice leads to a direct linking of the field equations to the law of conservation of matter. But the second alternative has certain other advantages, and will be examined here.

Assuming a modified metric corresponding to the factor $e^{\phi}$ from a flat space-time, the vanishing of the total curvature in the absence of matter gives

$$g^{ij}(\phi_{ij}+\phi_i\phi_j) = 0,$$

or

$$\Box^2(e^{\phi}) = 0,$$

where $\Box^2$ is the invariant second-order differential operator.

The older potential theory of gravitation requires

$$\Box^2\phi = 0.$$

Since in practical units $\phi$ involves the inverse square of the velocity of light, it is numerically very small, and the difference between the two equations is slight.

4. *Application to kinematical relativity.* In considering cosmological problems the distinction between $\phi$ and $\theta = e^{\phi}$ becomes significant. Since it is $\theta$ which satisfies the linear differential equation, the disposition of matter in the locality determines $\theta$ linearly. On the other hand, the gravitational deflexion is linearly dependent on the exponent $\phi$. For large variations of $\theta$ this distinction must be taken into account.

Assume as in kinematical relativity that matter is being propagated in all directions from a fixed event $O$ in a flat space-time. If $m_1$ and $m_2$ are two masses the distance apart is exactly proportional to the time interval $t$ from the fixed event. Hence the potential of $m_1$ due to $m_2$ is proportional to $t^{-1}$ and the potential at any event $P$ subsequent to $O$, due to all matter in the universe, is proportional to $t^{-1}$.

The modifying factor $\theta$ must consist of a term $A$, independent of all other matter, together with a term $Bt^{-1}$ depending on all other matter distributions. Put

$$d\tau = (A+Bt^{-1})\,dt.$$

Integrating

$$\tau = At + B\log t.$$

This leads to the transformation between the $t$ and $\tau$ scales of kinematical relativity if and only if $A = 0$, i.e. if and only if the modifying factor $\theta$ is exactly equal to the potential due to all matter in the universe.

A gravitational deflexion is proportional to $\delta\phi = \delta\theta/\theta$, and thus depends not only on the local matter distribution, but also is inversely proportional to the total potential due to all matter in the universe. This conforms with the principle of Mach that the inertia of a body is determined by the masses of the universe and their distribution: 'If therefore a body is removed sufficiently far from all other masses of the universe its inertia must be reduced to zero' (Einstein (2); see also Bondi (1), p. 29).

It is clear that the value of the gravitational constant $\gamma$ should thus give some estimate of the total matter distribution of the universe. If a scale is used so that $\delta\phi = \delta\theta$ in the present location, then the total potential due to all masses of the universe should be unity.

Let $T$ be the time interval from the event of creation, and use units such that the velocity of light is unity. A mass $m_1$ at distance $T\tanh u$ will then be receding with velocity $\tanh u$.

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The Lorentz transformation

$$x' = x\cosh u + t\sinh u, \quad t' = x\sinh u + t\cosh u,$$

shows that the potential at the event $x = 0, t = T$ of the mass $m_1$ is $\gamma m/T\sinh u$.

Let the mean density of matter at the present epoch be $\rho_0$. The density at epoch $t$ is $\rho_0 T^3/t^3$. Assume this to be constant relative to all frames and integrate for all matter corresponding to $t = T$ in a fixed frame. At distance $T\tanh u$ the apparent time interval from creation is $T/\cosh u$. The density as measured from a local frame should thus be $\rho_0\cosh^3 u$. Relative to the fixed frame the density would be $\rho_0\cosh^4 u$.

The total mass between distances $T\tanh u$ and $T\tanh(u+\delta u)$ is thus

$$4\pi T^2\tanh^2 u\ \rho_0\cosh^4 u\ \delta(T\tanh u) = 4\pi\rho_0 T^3\sinh^2 u\,\delta u.$$

The potential at $x = y = z = 0$, $t = T$ of the fixed frame is obtained by multiplying by $\gamma/T\sinh u$ to give

$$4\pi\rho_0\gamma T^2\sinh u\,\delta u.$$

Since the integral $\int\sinh u\,du$ is divergent for upper limit $u = \infty$, it must follow that the hypothesis that all frames of reference are equivalent and give a local apparent density $\rho_0$ after an apparent time $T$, applicable to all speeds up to the velocity of light, is not tenable, as it would lead to infinite potential and give infinite inertia to all masses.

The integral may be compared with observations to determine the possible size of the massive universe. Assume for simplicity that the density is homogeneous relative to all frames up to a velocity $\tanh u = \tanh\alpha$, at which point it ceases abruptly. The integral gives

$$1 = \int_0^\alpha 4\pi\rho_0\gamma T^2\sinh u\,du = 4\pi\rho_0\gamma T^2(\cosh\alpha - 1),$$

so that

$$\cosh\alpha = 1 + 1/4\pi\rho_0\gamma T^2.$$

Bondi ((1), p. 60) states that

$$\gamma\rho_0 T^2 \doteqdot 1,$$

with an unknown factor which may be as much as 10. Putting $\gamma\rho_0 T^2 = 1$, the maximum speed of recession would be about two-fifths of the velocity of light. The extreme value $\gamma\rho_0 T^2 = 0{\cdot}1$ gives a maximum speed of recession about seven-eighths of the velocity of light. The limit in either case would be beyond observational corroboration at present.

In fact, of course, the distribution of matter is not likely to end abruptly, the density is more likely to tail off gradually. It is of interest to note that there is a theoretically possible distribution with density $\rho_0\cosh^2 u$ at distance $R\tanh u$, which would indicate an infinity of mass in the universe, but a finite potential

$$\int 4\pi\rho_0\gamma T^2\sinh u\,du/\cosh^2 u.$$

This gives the correct total potential if

$$\rho_0\gamma T^2 = 1/4\pi.$$

Milne's theory would require some modification if there were an upper limit to the velocity of recession, or a tailing off of the density. First, there would be a centre of mass and a correspondingly privileged frame. Secondly, the gravitational effects of the universe on itself would have to be taken into account.

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Consider a particle with velocity $\tanh\beta$ relative to the privileged frame. The net gravitational attraction on the particle would be equal to that of a sphere of radius $T\tanh\beta$. Neglecting relativity corrections the attraction would be

$$\tfrac{4}{3}\pi\gamma\rho_0 T\tanh\beta.$$

The acceleration would thus be exactly proportional to the displacement from the centre of mass. The velocity of recession, although changing with time, would always remain proportional to the displacement.

The relativity corrections for large displacements would have effects in opposite directions. The increase of inertia with velocity would decrease the retardation of distant galaxies. On the other hand, the decrease in potential would decrease the inertia, and this effect might be more dominant. But to counteract this, observations of distant galaxies would see them at an earlier time when the time during which the retardation had been effective would be less. The actual values of these corrections would depend on the particular form of distribution assumed, but clearly the variation of velocity of recession from exact proportionality to distance could easily be beyond observational limits.

The result would follow, however, that the actual time since creation would be rather less than the $1{\cdot}8\times10^9$ years calculated from the velocities of recession. The effect would follow also that the velocities of recession would ultimately cease and the universe again would contract, but not uniformly to a point.

The decrease in density at great distances would seem to accord better with observations of densities of distant nebulae ((1), p. 41). If the solar system were off centre, say moving with velocity $\tanh\beta$, the densities of nebulae at distance $R\tanh u$ in the best and worst directions would be approximately

$$1-2\tanh u\sinh 2\beta,$$

if the distribution $\rho_0\cosh^2 u$ were assumed. This could be above or below the limits of observation.

REFERENCES

(1) BONDI, H. *Cosmology* (Cambridge, 1952).
(2) EINSTEIN, A. *S.B. preuss. Akad. Wiss.* (1917), p. 142.
(3) MILNE, E. A. *Proc. roy. Soc.* A, 200 (1950), 219.

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