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SOME ASYMMETRIC INEQUALITIES

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1. *Introduction.* Let $\alpha, \beta, \gamma, \delta$ be real numbers with $\Delta = |\alpha\delta - \beta\gamma| > 0$, and let ξ, η denote the linear forms

$$\xi = \alpha x + \beta y, \quad \eta = \gamma x + \delta y.$$

A well-known theorem of Minkowski states that for any real numbers θ, ϕ there exist integers x, y satisfying

$$|(\xi+\theta)(\eta+\phi)| \leqslant \tfrac{1}{4}\Delta. \tag{1}$$

More recently Davenport and Heilbronn (1) have shown there exist integers x, y satisfying

$$0 \leqslant (\xi+\theta)(\eta+\phi) < \Delta, \tag{2}$$

with, in fact,

$$\xi+\theta \geqslant 0, \quad \eta+\phi \geqslant 0.$$

It is known that in both (1) and (2) the constants on the right are best possible and that these results are not 'isolated', since there exists, in each case, an infinity of non-equivalent forms for which these results would cease to hold if the constants on the right were multiplied by any factor less than unity.

My object in this paper is to determine a (continuous) positive function $g(\mu)$, for $\mu \leqslant 1$, such that integers x, y always exist satisfying

$$-\mu g(\mu)\Delta < (\xi+\theta)(\eta+\phi) \leqslant g(\mu)\Delta; \tag{3}$$

and to determine for which values of μ this result is best possible in the sense that, for some product $(\xi+\theta)(\eta+\phi)$, (3) is only satisfied with equality on the right. Clearly $\mu > -1$ for (3) to be significant, and the restriction $\mu \leqslant 1$ imposes no real loss of generality, since we can replace $(\xi+\theta)(\eta+\phi)$ by $-(\xi+\theta)(\eta+\phi)$.

Inequalities of this type, with $\mu \geqslant 0$, have already been considered by Davenport ((2), Lemma 3), by Davenport and Heilbronn (1) and by Cassels (3). Cassels establishes the solubility of various inequalities of the form

$$-R\Delta \leqslant (\xi+\theta)(\eta+\phi) \leqslant S\Delta,$$

but the values of R and S are, in most cases, not explicit. They depend on a parameter which is, essentially, any positive value of the homogeneous product $\xi\eta$ arising from coprime integers x, y. Although one can deduce some (and possibly all) of my results

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from those of Cassels, the deduction is not an immediate one. Davenport ((2), Lemma 3) showed that the inequality

$$-\tfrac{1}{4}\mu^{\frac{1}{2}}\Delta < (\xi+\theta)(\eta+\phi) \leqslant \tfrac{1}{4}\mu^{-\frac{1}{2}}\Delta \tag{4}$$

is always soluble, which corresponds to $g(\mu) = 1/(4\sqrt{\mu})$ in (3). He pointed out that this is also best possible for $\mu = \frac{1}{2}$, as well as for $\mu = 1$, when it reduces to Minkowski's result (1). I repeat Davenport's proof with a slight modification, and show that (4) is best possible for an enumerable infinity of values of μ, all satisfying $\mu \geqslant \frac{1}{3}$. Moreover, these cases cluster at $\mu = 1$, thus showing, as we have already mentioned, that the result (1) is not isolated. More explicitly, I prove

THEOREM 1. *For any real θ, ϕ and any positive $\mu \leqslant 1$, integers x, y exist satisfying*

$$-\tfrac{1}{4}\mu^{\frac{1}{2}}\Delta < (\xi+\theta)(\eta+\phi) \leqslant \tfrac{1}{4}\mu^{-\frac{1}{2}}\Delta. \tag{4}$$

This is true with strict inequality unless $\mu = m/(m+2)$ $(m = 1, 2, 3, \ldots)$, and, with $C > 0$†,

$$(\xi+\theta)(\eta+\phi) \sim C[\{2x+m(2y+1)\}^2 - m(m+2)(2y+1)^2], \tag{5}$$

in which cases it is not.

For $0 \leqslant \mu < \frac{1}{3}$, we can improve on this result and, to provide, with Theorem 1, a transition from (1) to (2), I prove

THEOREM 2. *For any real μ with $0 \leqslant \mu \leqslant \frac{1}{3}$, integers x, y exist satisfying*

$$-\frac{\mu\Delta}{\{(1+\mu)(1+9\mu)\}^{\frac{1}{2}}} < (\xi+\theta)(\eta+\phi) \leqslant \frac{\Delta}{\{(1+\mu)(1+9\mu)\}^{\frac{1}{2}}}. \tag{6}$$

This is true with strict inequality unless either $\mu = 1/(4n-1)$ $(n = 1, 2, 3, \ldots)$ and

$$(\xi+\theta)(\eta+\phi) \sim C[n(2x+1)^2 - (n+2)(2y+1)^2],$$

or $\mu = 0$ and

$$(\xi+\theta)(\eta+\phi) \sim Cxy,$$

in which cases it is not.

We may observe that in this result there is again an enumerable infinity of best possible cases which now cluster at $\mu = 0$, thus showing, as we have already mentioned, that (2) also is not isolated.

For negative values of μ, writing $\mu = -\nu^2$, I prove

THEOREM 3. *For any real ν with $0 \leqslant \nu < 1$, integers x, y exist satisfying*

$$\frac{\nu^2\Delta}{\{(1-\nu)^3(1+3\nu)\}^{\frac{1}{2}}} < (\xi+\theta)(\eta+\phi) \leqslant \frac{\Delta}{\{(1-\nu)^3(1+3\nu)\}^{\frac{1}{2}}}. \tag{7}$$

This is true with strict inequality unless either

(i) $\nu = 0$ *and* $(\xi+\theta)(\eta+\phi) \sim Cxy$ *or* $C(x-y)(x+y+1)$,

(ii) $\nu = \frac{1}{3}$ *and* $(\xi+\theta)(\eta+\phi) \sim C[3(2x+1)^2 - (2y+1)^2]$,

or (iii) $\nu = \frac{1}{2}$ *and* $(\xi+\theta)(\eta+\phi) \sim C[x^2+xy-y^2]$,

in which cases it is not.

† The notion of equivalence expressed by $\sim$ in (5) is to be understood to mean that the non-homogeneous form on the left can be transformed into that on the right, for some positive value of C, by a non-homogeneous substitution

$$x = px' + qy' + h, \quad y = rx' + sy' + k,$$

with integral coefficients and determinant $ps - qr = \pm 1$.

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Finally, the first two critical cases of Theorem 2 are at $\mu = \frac{1}{3}$ and $\mu = \frac{1}{7}$. For part of the range between these values, I have obtained an improvement on the general result of that theorem, which may be expressed as

THEOREM 4. *For any real $q > 4{\cdot}573\ldots$, integers x, y exist satisfying*

$$-\frac{\Delta}{2q} < (\xi+\theta)(\eta+\phi) \leqslant \frac{(q^2-4)\Delta}{8q}. \tag{8}$$

This is true with strict inequality unless $q^2 = 16(m+1)/m$ ($m = 1, 2$ or 3) and

$$(\xi+\theta)(\eta+\phi) \sim C[4(m+1)x^2 - m(2y+1)^2],$$

in which cases it is not.

The number $4{\cdot}573\ldots = \sqrt{\{6+4\sqrt{3}+2\sqrt{(9+4\sqrt{3})}\}}$ (see Lemma 2 below).

The methods used to prove Theorems 1 and 3 of this note are variations of the well-known technique in the geometry of numbers of standardizing a lattice with respect to a point at which a form assumes a value at or near its minimum, but the proofs of Theorems 2 and 4 are arithmetical.

2. *The lemmas.* The problem for the homogeneous product $\xi\eta$, corresponding to that of determining $g(\mu)$ so that (3) is always soluble, has already been solved, and the result, which we shall need, may be stated as

LEMMA 1. *For any real c, integers x, y, with $(x, y) = 1$, exist satisfying*

$$\tfrac{1}{2}c\Delta < \xi\eta \leqslant \{(c^2+1)^{\frac{1}{2}} + c\}\Delta. \tag{9}$$

This is true with strict inequality unless $c = \pm 2/(p^2-4)^{\frac{1}{2}}$, where p (> 2) is an integer, and, with $C > 0$†,

$$\xi\eta \sim C[c(x^2+y^2+pxy)],$$

in which cases it is not.

For $c \leqslant 0$, Segre (4) gave a geometrical proof of this result and Robinson ((5), (6)) has since obtained the same solution, with some substantial improvements, by using continued fractions. More recently (7), I have obtained an arithmetical proof for all real c. We remark at once that we may allow equality on the left of (9) instead of on the right, without otherwise affecting the lemma.

The following weaker form of (9) will also be needed:

LEMMA 2. *For any real $q > 4{\cdot}573\ldots$, integers x, y, with $(x, y) = 1$, exist satisfying*

$$\tfrac{1}{2}\{(q^2+4)^{\frac{1}{2}} - 2\}q^{-1}\Delta < \xi\eta \leqslant \tfrac{1}{2}(q^2-4)q^{-1}\Delta. \tag{10}$$

Proof. Put $z^2 = q^2+4$, $z > 0$, and let

$$c = \{(q^2+4)^{\frac{1}{2}} - 2\}/q = (z-2)/q.$$

Then
$$\{(c^2+1)^{\frac{1}{2}} + c\} = [(z-2) + \{2z(z-2)\}^{\frac{1}{2}}]/q.$$

By Lemma 1, (10) is certainly true if

$$2(z-2) + 2\{2z(z-2)\}^{\frac{1}{2}} < q^2 - 4 = z^2 - 8,$$

that is (since $(z-1)^2 - 5 > 0$) if

$$(z-1)^4 - 18(z-1)^2 + 33 > 0.$$

† Here, of course, the notion of equivalence refers only to *homogeneous* integral unimodular substitutions. Further, it is essential that the sign of c be associated with these special forms.

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This is true if $z > 1+(9+4\sqrt{3})^{\frac{1}{2}}$, that is, if $q > 4{\cdot}573\ldots$, which is so. This completes the proof of the lemma.

We now establish two elementary inequalities.

LEMMA 3. *Let p, q be two positive numbers defined by $p^2 = 1+\mu$, $q^2 = 1+9\mu$, where $0 \leqslant \mu \leqslant \frac{1}{3}$. Then, for all real a with $-\frac{1}{8}pq \leqslant a \leqslant p/q$,*

$$-2\mu/(pq)+a+\{1-8a\mu/(pq)\}^{\frac{1}{2}} \leqslant 2/(pq), \tag{11}$$

and

$$2/(pq)+a-\{1+8a/(pq)\}^{\frac{1}{2}} \geqslant -2\mu/(pq). \tag{12}$$

Equality is needed in (11) *and* (12) *if and only if $a = p/q$.*

Proof. By the definition of p, (11) requires

$$2p/q-a \geqslant \{1-8a\mu/(pq)\}^{\frac{1}{2}}.$$

Now the left-hand side of this inequality is certainly positive, and the right-hand side is real. Hence, by squaring and arranging as a quadratic in a, we obtain the condition

$$a^2-\frac{4(1-\mu)}{pq}a+\frac{3-5\mu}{q^2} \geqslant 0;$$

that is, by factorizing this quadratic,

$$\left(\frac{p}{q}-a\right)\left(\frac{3-5\mu}{pq}-a\right) \geqslant 0. \tag{13}$$

But $p/q-a \geqslant 0$ anyway, and, since $\mu \leqslant \frac{1}{3}$,

$$\frac{3-5\mu}{pq}-a \geqslant \frac{2(1-3\mu)}{pq} \geqslant 0.$$

Hence each factor in (13) is positive or zero, whence (11), and with strict inequality unless $a = p/q$, in which case equality is attained.

By precisely the same argument, we find that the inequality (12) also follows from (13), and with strict inequality unless $a = p/q$ as before.

This completes the proof of the lemma.

The remaining seven lemmas are all concerned with the special cases in which equality is needed in one or other or more of the theorems.

LEMMA 4. *For all positive integral m, n the inequality*

$$-2m < n(2x+mY)^2-m(mn+2)\,Y^2 < 4n(m+1)-2m$$

has no solution in integers x, Y with Y odd.

Proof†. Put $y = x+mY$, so that $y-x$ is an odd multiple of m, and the form under consideration is $-2Q/m$, where

$$Q = x^2+y^2-2(mn+1)\,xy.$$

We thus have to show that the inequality

$$m^2-2mn(m+1) < Q < m^2 \tag{14}$$

has no solution in integers x, y with $x-y \equiv m \pmod{2m}$.

† The proof is based on a lemma of Davenport ((8), Lemma 3). For $m = 1$, $Y-nx = 2u+1$, $x = v$, $n = 2N$, where N is a positive integer, the result, but with zero on the left, is obtained otherwise by Davenport and Heilbronn ((1), Lemma 1). Again, for $n = 1$, $2m = k$, so that k is even, Cassels ((3), Theorem 3) proves the result geometrically.

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Suppose the contrary is true and choose that solution for which $|x|$ is least. Clearly $x \neq 0$, for then $y \equiv m \pmod{2m}$ and $Q = y^2 \geqslant m^2$. Hence, by changing the signs of x and y, if necessary, we may suppose $0 < x \leqslant |y|$.

Now
$$Q = \{y-(mn+1)x\}^2 - mn(mn+2)x^2, \tag{15}$$
and we suppose that Q, as given by this equation, satisfies (14).

Define the integer z by
$$|y-(mn+1)x| = (mn+1)x - z, \tag{16}$$
so that $z - x \equiv \pm(y-x) \equiv m \pmod{2m}$. Again, by (15) and (16),
$$Q = \{(mn+1)x-z\}^2 - mn(mn+2)x^2 = x^2+z^2-2(mn+1)zx,$$
and satisfies (14). Hence, by our assumption that $|x|$ is least, we must have $|z| \geqslant x (> 0)$. But $z - x \equiv m \pmod{2m}$, so that we must have either $z \geqslant x+m$ or $z \leqslant x-m$.

If $z \geqslant x+m$, by (16), $0 \leqslant (mn+1)x - z \leqslant m(nx-1)$, and
$$Q \leqslant m^2(nx-1)^2 - mn(mn+2)x^2 = -2mnx^2 - 2m^2nx + m^2 \leqslant m^2 - 2mn(m+1).$$

If $z \leqslant x-m$, then $(mn+1)x - z \geqslant m(nx+1)$, and
$$Q \geqslant m^2(nx+1)^2 - mn(mn+2)x^2 = 2mnx(m-x) + m^2.$$
This is greater than m^2 unless $x \geqslant m$. But if $x \geqslant m$ it follows, since $z \leqslant x-m$ and $|z| \geqslant x$, that $z \leqslant -x$, so that
$$Q \geqslant 2(mn+2)x^2 \geqslant 2(mn+2)m^2 > m^2.$$

Thus in each case we obtain a contradiction to (14), which proves the lemma.

LEMMA 5. *Let $\xi\eta = x^2 - m(m+2)y^2$ $(m = 1, 2, 3, \ldots)$. Then, for any real θ, ϕ, there exist integers x, y such that*
$$-\tfrac{1}{2}m < (\xi+\theta)(\eta+\phi) \leqslant 1 + \tfrac{1}{2}m. \tag{17}$$
This is true with strict inequality unless
$$(\xi+\theta)(\eta+\phi) \sim \{x + m(y+\tfrac{1}{2})\}^2 - m(m+2)(y+\tfrac{1}{2})^2,$$
in which cases it is not.

Proof. We can clearly find real numbers x_0, y_0 such that
$$\theta = \alpha x_0 + \beta y_0, \quad \phi = \gamma x_0 + \delta y_0.$$
We then have
$$(\xi+\theta)(\eta+\phi) = (x+x_0)^2 - m(m+2)(y+y_0)^2 = Q,$$
say. We can now find an integer y so that $|y+y_0| \leqslant \frac{1}{2}$. If, with this choice of y, we have $(y+y_0)^2 < 1/\{2(m+2)\}$, we choose x so that $|x+x_0| \leqslant \frac{1}{2}$. We then have
$$-\tfrac{1}{2}m < Q \leqslant \tfrac{1}{4} < 1 + \tfrac{1}{2}m,$$
whence (17) with strict inequality.

We may now suppose that $1/\{2(m+2)\} \leqslant (y+y_0)^2 \leqslant \frac{1}{4}$, and choose x to satisfy
$$\{m(m+2)(y+y_0)^2 - \tfrac{1}{2}m\}^{\frac{1}{2}} < x + x_0 \leqslant \{m(m+2)(y+y_0)^2 - \tfrac{1}{2}m\}^{\frac{1}{2}} + 1.$$
We then have
$$-\tfrac{1}{2}m < Q \leqslant 1 - \tfrac{1}{2}m + \{4m(m+2)(y+y_0)^2 - 2m\}^{\frac{1}{2}} \leqslant 1 - \tfrac{1}{2}m + m = 1 + \tfrac{1}{2}m,$$
which is (17) again. Moreover, we still have strict inequality except, perhaps, when $|y+y_0| = \frac{1}{2}$ and $x + x_0 = 1 + \frac{1}{2}m$. Since x, y are integers, these conditions require
$$y_0 \equiv \tfrac{1}{2} \pmod 1, \quad x_0 \equiv \tfrac{1}{2}m \pmod 1.$$
Hence
$$\begin{aligned} 4Q &\sim (2x+m)^2 - m(m+2)(2y+1)^2 \\ &\sim \{2x + m(2y+1)\}^2 - m(m+2)(2y+1)^2. \end{aligned}$$

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By Lemma 4 with $n = 1$, there is no solution in integers x, y of the inequality

$$-2m < 4Q < 2(m+2).$$

Hence, in these cases, and in these cases only, equality is required in (17).

This completes the proof of the lemma.

LEMMA 6. *Let* $Q(x,y) = p^2(x+fy)^2 - q^2y^2$, *where* $0 \leqslant f \leqslant \frac{1}{2}$, $p^2 = 1+\mu$, $q^2 = 1+9\mu$ *and* $0 \leqslant \mu \leqslant \frac{1}{3}$.

Then, for any real x_0, y_0, *there exist integers* x, y *for which*

$$-2\mu < Q(x+x_0, y+y_0) \leqslant 2. \tag{18}$$

This is true with strict inequality unless $\mu = 1/(4n-1)$ $(n = 1, 2, 3, \ldots)$ *and*

$$Q(x+x_0, y+y_0) \sim (4n-1)^{-1}[n(2x+1)^2 - (n+2)(2y+1)^2],$$

in which case it is not.

Proof. We have

$$Q = Q(x+x_0, y+y_0) = p^2(x+\alpha)^2 - q^2(y+y_0)^2,$$

where $\alpha = x_0 + f(y+y_0)$ is independent of x.

Choose y so that $|y+y_0| \leqslant \frac{1}{2}$. This fixes the value of α. If, with this choice of y, we have $(y+y_0)^2 < 2\mu/q^2$, we choose x so that $|x+\alpha| \leqslant \frac{1}{2}$. We then have

$$-2\mu < Q \leqslant \tfrac{1}{4}p^2 < 2,$$

which is (18) with strict inequality.

We can now suppose that $2\mu/q^2 \leqslant (y+y_0)^2 \leqslant \frac{1}{4}$, and choose x to satisfy

$$\{q^2(y+y_0)^2 - 2\mu\}^{\frac{1}{2}} < p(x+\alpha) \leqslant \{q^2(y+y_0)^2 - 2\mu\}^{\frac{1}{2}} + p.$$

We then have, by the definitions of p and q,

$$-2\mu < Q \leqslant -2\mu + p^2 + 2p\{q^2(y+y_0)^2 - 2\mu\}^{\frac{1}{2}} \leqslant 1 - \mu + p(q^2 - 8\mu)^{\frac{1}{2}} = 1 - \mu + p^2 = 2,$$

which is (18) again. Moreover, we still have strict inequality except, perhaps, when $|y+y_0| = \frac{1}{2}$ and $x+\alpha = \frac{3}{2}$. Then $y_0 \equiv \frac{1}{2} \pmod 1$, and two integer values of y exist for which $y+y_0 = \pm\frac{1}{2}$. For each of these values of y we must have, for equality in (18), $x+\alpha = \frac{3}{2}$. Hence the two congruences

$$\alpha = x_0 \pm \tfrac{1}{2}f \equiv \tfrac{1}{2} \pmod 1$$

must both obtain. By the definition of f, these can both be satisfied only if

$$f = 0, \quad x_0 \equiv \tfrac{1}{2} \pmod 1.$$

Hence $Q \sim p^2(x+\frac{1}{2})^2 - q^2(y+\frac{1}{2})^2$ and may require equality in (18), but for any other x_0, y_0, f strict inequality obtains.

In the case of the form $p^2(x+\frac{1}{2})^2 - q^2(y+\frac{1}{2})^2$, put $x = n+1, y = n$, where n is a positive integer, and substitute for p^2 and q^2 in terms of μ. We get

$$\begin{aligned} p^2(x+\tfrac{1}{2})^2 - q^2(y+\tfrac{1}{2})^2 &= \tfrac{1}{4}\{(1+\mu)(2n+3)^2 - (1+9\mu)(2n+1)^2\} \\ &= 2\{(n+1) - n(4n+3)\mu\}. \end{aligned}$$

If $1/(4n+3) < \mu < 1/(4n-1)$, this expression lies between -2μ and 2 so that we have obtained a solution of (18) with, in these cases, strict inequality. In the remaining cases, we have

$$\mu = 1/(4n-1)$$

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$(n = 1, 2, 3, \ldots)$, and

$$(4n-1)\,Q(x+x_0, y+y_0) \sim n(2x+1)^2 - (n+2)(2y+1)^2$$
$$\sim n(2x+2y+1)^2 - (n+2)(2y+1)^2.$$

By Lemma 4 with $m = 1$, there is no solution in integers x, y of the inequality

$$-2 < (4n-1)\,Q < 2(4n-1).$$

Hence, in these cases, and in these cases only, equality is needed in (18).

This completes the proof of the lemma.

LEMMA 7. *For any real x_0, y_0, integers x, y exist for which*

$$0 < (x+x_0)(y+y_0) \leqslant 1.$$

This is true with strict inequality unless $x_0 \equiv y_0 \equiv 0 \pmod 1$, in which case it is not.

This is obvious.

LEMMA 8. *For any real x_0, y_0, integers x, y exist for which*

$$\tfrac{1}{2} < 3(x+x_0)^2 - (y+y_0)^2 \leqslant \tfrac{9}{2}. \tag{19}$$

Equality is needed if and only if $x_0 \equiv y_0 \equiv \frac{1}{2} \pmod 1$.

Proof. Choose x and y so that $|y+y_0| \leqslant \frac{1}{2} \leqslant |x+x_0| \leqslant 1$. We then have

$$\tfrac{1}{2} \leqslant 3(x+x_0)^2 - (y+y_0)^2 \leqslant 3 < \tfrac{9}{2},$$

which is (19), and with strict inequality, unless $|x+x_0| = |y+y_0| = \frac{1}{2}$, so that $x_0 \equiv y_0 \equiv \frac{1}{2} \pmod 1$. Hence equality may be needed for a form equivalent to Q, where

$$Q = 3(x+\tfrac{1}{2})^2 - (y+\tfrac{1}{2})^2.$$

For $x = y = 0$, $Q = \frac{1}{2}$, and for $x = y = 1$, $Q = \frac{9}{2}$. Again, for all integers x, y, $4Q \equiv 2 \pmod 8$, but $4Q \not\equiv 0 \pmod 5$ unless, at the same time, $4Q \equiv 0 \pmod{25}$. Hence Q assumes no value between $\frac{1}{2}$ and $\frac{9}{2}$ and so does require equality in (19).

This completes the proof of the lemma.

LEMMA 9. *For any real x_0, y_0, integers x, y exist such that*

$$4 < 4\{x+x_0+\tfrac{1}{2}(y+y_0)\}^2 - 5(y+y_0)^2 \leqslant 16. \tag{20}$$

Equality is needed if and only if $x_0 \equiv y_0 \equiv 0 \pmod 1$.

Proof. Choose y so that $\frac{1}{2} \leqslant |y+y_0| \leqslant 1$, and then choose x so that $\frac{3}{2} \leqslant |x+\alpha| \leqslant 2$, where $\alpha = x_0 + \frac{1}{2}(y+y_0)$. We then have

$$4 \leqslant 4(x+\alpha)^2 - 5(y+y_0)^2 \leqslant \tfrac{59}{4} < 16,$$

which is (20), and with strict inequality, unless $|y+y_0| = 1$ and $|x+\alpha| = \frac{3}{2}$. These conditions require $y_0 \equiv 0 \pmod 1$ and $\alpha = x_0 + \frac{1}{2} \equiv \frac{1}{2} \pmod 1$, so that $x_0 \equiv 0 \pmod 1$ also. Hence equality may be needed in (20) for a form equivalent to Q, where

$$Q = (2x+y)^2 - 5y^2 = 4(x^2+xy-y^2) \sim 4(x^2+y^2+3xy),$$

on putting $x = x'+y'$, $y = y'$.

By Lemma 1, with $p = 3$, the form $x^2+3xy+y^2$ assumes no value, for coprime integers x, y, between 1 and 5, and hence assumes no value, for any integers x, y, between 1 and 4.

This completes the proof of the lemma.

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LEMMA 10. *Let* $Q(x,y) = q^2(x+fy)^2 - 16y^2$, *where* $q^2 \geqslant 16$ *and* $0 \leqslant f \leqslant \frac{1}{2}$.

Then, for any real x_0, y_0, *integers* x, y *exist for which*

$$-4 < Q(x+x_0, y+y_0) \leqslant q^2 - 4. \tag{21}$$

This is true with strict inequality unless $q^2 = 16(m+1)/m$ $(m = 1, 2, 3, \ldots)$ *and*

$$Q(x+x_0, y+y_0) \sim (4/m)\,[4(m+1)\,x^2 - m(2y+1)^2],$$

in which cases it is not.

Proof. Choose y so that $|y+y_0| \leqslant \frac{1}{2}$, and then choose x so that $|x+\alpha| \leqslant \frac{1}{2}$ also, where $\alpha = x_0 + f(y+y_0)$. We then have, since $q^2 \geqslant 16$ anyway,

$$-4 \leqslant Q = Q(x+x_0, y+y_0) \leqslant \tfrac{1}{4}q^2 < q^2 - 4,$$

which is (21), and with strict inequality, unless $|y+y_0| = \frac{1}{2}$, so that $y_0 \equiv \frac{1}{2} \pmod 1$, and $|x+\alpha| = 0$. Since we can find y so that $y+y_0 = \pm\frac{1}{2}$, this last condition requires that, with either sign, $\alpha = x_0 \pm \frac{1}{2}f \equiv 0 \pmod 1$.

Hence, by the definition of f, $f = 0$, $x_0 \equiv 0 \pmod 1$,

and $$Q(x+x_0, y+y_0) \sim q^2x^2 - 4(2y+1)^2.$$

In this last expression, put $x = y = m$, where m is a positive integer, and it takes the value $(q^2-16)\,m^2 - 16m - 4$. This clearly lies between -4 and q^2-4 if we have

$$16(m+1)/m < q^2 < 16m/(m-1).$$

We have thus obtained a solution of (21) with, in these cases, strict inequality. In the remaining cases, we have $q^2 = 16(m+1)/m$ $(m = 1, 2, 3, \ldots)$, and

$$\begin{aligned} \tfrac{1}{2}mQ &\sim 8(m+1)\,x^2 - 2m(2y+1)^2 \\ &\sim 8(m+1)\,x^2 - 2m(2y-2x+1)^2 \\ &= 2\{2x + m(2y+1)\}^2 - m(2m+2)\,(2y+1)^2. \end{aligned}$$

By Lemma 4 with $n = 2$, there is no solution, in integers x, y, of the inequality

$$-2m < \tfrac{1}{2}mQ < 2(3m+4).$$

Hence, in these cases, and in these cases only, equality is needed in (21).

This completes the proof of the lemma, and we have now disposed of all the cases in which equality is needed in any of the theorems.

3. *Proof of Theorem* 1. We may suppose, without loss of generality, that $\Delta = 1$. Let M denote the lower bound of all values of the product $(\xi+\theta)\,(\eta+\phi)$ which exceed $-\frac{1}{4}\sqrt{\mu}$, where $0 < \mu \leqslant 1$. We can suppose that

$$M \geqslant 1/(4\sqrt{\mu}), \tag{22}$$

since otherwise (4) has a solution with strict inequality.

By the definition of M, for any $\epsilon_0 > 0$, integers x^*, y^* exist such that

$$(\xi^*+\theta)\,(\eta^*+\phi) = M/(1-\epsilon),$$

where $0 \leqslant \epsilon < \epsilon_0$, and we may suppose ϵ_0 is as small as we please. Take

$$X = (\xi - \xi^*)/(\xi^*+\theta), \quad Y = (\eta - \eta^*)/(\eta^*+\theta),$$

so that XY is the product of two homogeneous linear forms in the integral variables $x - x^*$, $y - y^*$ with determinant $D = (1-\epsilon)/M$.

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In the XY-plane, for all integral x, y, the points (X, Y) determine a lattice $\mathscr{L}$. Now

$$(X+1)(Y+1) = (\xi+\theta)(\eta+\phi)/(\xi^*+\theta)(\eta^*+\phi),$$

so that, by the definition of M and our supposition, for all integers x, y, we have

$$(X+1)(Y+1) \geqslant 1-\epsilon \quad \text{or} \quad \leqslant -\sqrt{\mu}(1-\epsilon)/(4M) = -\tau, \tag{23}$$

say, and similarly, since X, Y form a lattice, we have

$$(X-1)(Y-1) \geqslant 1-\epsilon \quad \text{or} \quad \leqslant -\tau. \tag{24}$$

Clearly $0 < \tau \leqslant \mu$ ($\leqslant 1$). Further, if (23) and (24) are both satisfied, it follows that the point (X, Y) does not lie inside a certain infinite region $\mathscr{K}$, say. We shall show that this consideration sets a lower limit on D, the determinant of the lattice $\mathscr{L}$.

We first prove† that there is no point of $\mathscr{L}$ other than 0 in the rectangle

$$|X+Y| < 2\{1+(1-\epsilon)^{\frac{1}{2}}\}, \quad |X-Y| < 2\sqrt{\tau}. \tag{25}$$

Suppose the contrary is true. Let such a point be (X, Y), where, by reflecting in the origin, if necessary, we may suppose

$$0 \leqslant X+Y < 2\{1+(1-\epsilon)^{\frac{1}{2}}\},$$

so that

$$-2 \leqslant X+Y-2 < 2(1-\epsilon)^{\frac{1}{2}}. \tag{26}$$

Now, by (25),

$$4(X-1)(Y-1) = (X+Y-2)^2-(X-Y)^2 \geqslant -(X-Y)^2 > -4\tau.$$

Hence, by (24), we must have $(X-1)(Y-1) \geqslant 1-\epsilon$, and so

$$(X+Y-2)^2 \geqslant 4(1-\epsilon).$$

Hence $|X+Y-2| \geqslant 2(1-\epsilon)^{\frac{1}{2}}$, so that, by (26),

$$-2 \leqslant X+Y-2 \leqslant -2(1-\epsilon)^{\frac{1}{2}}$$

and

$$0 \leqslant X+Y \leqslant 2\{1-(1-\epsilon)^{\frac{1}{2}}\} \leqslant 2\epsilon. \tag{27}$$

Again, by (26), since $(X-1)(Y-1) \geqslant 1-\epsilon$, we have

$$(X-Y)^2 = (X+Y-2)^2-4(X-1)(Y-1) \leqslant 4\epsilon.$$

Hence, by (27),

$$|X| < \delta, \quad |Y| < \delta, \tag{28}$$

where $\delta \to 0$ as $\epsilon \to 0$, that is, as $\epsilon_0 \to 0$. Hence, if X and Y are not both zero, for some integer m the point (mX, mY) lies inside (25) but not (28), which is a contradiction and so proves the assertion.

Since the area of the rectangle (25) is $8\sqrt{\tau}\{1+(1-\epsilon)^{\frac{1}{2}}\}$, it follows from Minkowski's fundamental theorem that the determinant of the lattice $\mathscr{L}$ satisfies

$$(1-\epsilon)/M \geqslant 2\sqrt{\tau}\{1+(1-\epsilon)^{\frac{1}{2}}\},$$

so that, by the definition of τ in (23), on squaring and simplifying, we have

$$M \leqslant (1-\epsilon)/[\sqrt{\mu}\{1+(1-\epsilon)^{\frac{1}{2}}\}^2].$$

This contradicts (22) unless $\epsilon = 0$, $\quad M = 1/(4\sqrt{\mu})$ (29)

and $\tau = \mu$.

† The proof is that given by Davenport ((2), Lemma 3), but he had K^2 in place of my τ, with $K \geqslant 1$.

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We have now shown that (4) has a solution, and with strict inequality except, perhaps, when (29) holds. We must now determine in which, if any, of these cases equality is actually needed in (4).

Now $X+Y=4$ touches $(X-1)(Y-1)=1$ at the point $A(2,2)$; and $Y-X=2\sqrt{\mu}$ touches $(X-1)(Y-1)=-\mu$ at the point

$$P(1-\sqrt{\mu}, 1+\sqrt{\mu}),$$

and touches $(X+1)(Y+1)=-\mu$ at the point $P-A$. Any other point on these two sides of the rectangle (25) violates either (23) or (24). Hence, if $\mathscr{L}$ has no point in $\mathscr{K}$, the points A and P form a basis of this lattice. We thus have

$$\begin{aligned} XY &= \{(1-\sqrt{\mu})x+2y\}\{(1+\sqrt{\mu})x+2y\} \\ &= (x+2y)^2-\mu x^2, \end{aligned}$$

so that, in fact, with $C>0$,

$$\xi\eta \sim C[(x+2y)^2-\mu x^2]. \tag{30}$$

Consider now the point mP of $\mathscr{L}$, where m is an integer. We have

$$\{m(1-\sqrt{\mu})-1\}\{m(1+\sqrt{\mu})-1\} = (m-1)^2-\mu m^2.$$

If $(m-2)/m<\mu<(m-1)/(m+1)$, this expression lies between $-\mu$ and 1 so that mP lies inside $\mathscr{K}$. This is equivalent to saying that, in these cases, (4) has a solution with strict inequality. In the remaining cases we have

$$\mu = m/(m+2),$$

where, for $0<\mu\leqslant 1$, $m=1,2,3,\ldots$. (The least value of μ, for $m=1$, is $\mu=\frac{1}{3}$. For $\mu<\frac{1}{3}$, the point $2P$ lies inside $\mathscr{K}$ and, as we shall see later, we can then improve on (4).) In the present cases, we now have, by (30),

$$\begin{aligned} \xi\eta &\sim C[(m+2)(x+2y)^2-mx^2] \\ &\sim C[\{x+(m+2)y\}^2-m(m+2)y^2] \\ &\sim C[x^2-m(m+2)y^2], \end{aligned}$$

where C is, at each step, a suitable positive number.

Now the form $x^2-m(m+2)y^2$ was considered in Lemma 5. Its discriminant $\Delta^2=4m(m+2)$ so that, since $\mu=m/(m+2)$, the inequalities (17) and (4), in the case of this form, are the same. It thus follows, by Lemma 5, that equality is needed in (4) if and only if $(\xi+\theta)(\eta+\theta)\sim C[\{2x+m(2y+1)\}^2-m(m+2)(2y+1)^2]$.

This completes the proof of the theorem.

4. *Proof of Theorem* 2. The main assertion of this theorem is that if $0\leqslant\mu\leqslant\frac{1}{3}$, the inequality (6) is always soluble. The method used to prove Theorem 1 is now inadequate, since the inequality provided by Minkowski's fundamental theorem is not sufficient to give the result. I have been unable to refine the geometrical argument sufficiently to prove Theorem 2 and give instead an elementary algebraic proof. Before doing so, however, it may be of interest to mention that the special cases in which equality is needed in (6) have the following geometrical significance. First, there is an infinity of cases in which the lattice in the XY-plane is generated by the points A and P which occurred in § 3, where now both P and $2P$ lie on $(X-1)(Y-1)=-\mu$. Further, the critical values of μ are found by considering the position of the lattice point $n(2P-A)$.

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(This has already been done, in fact, in Lemma 6.) Again, in the second case for which equality is needed in (6) when $\mu = 0$, we have $XY \sim xy$, and the corresponding lattice is simply that for which X, Y are integral.

For the proof of Theorem 2 we write

$$\xi\eta = (\alpha x + \beta y)(\gamma x + \delta y) = Q(x, y) = ax^2 + 2bxy + cy^2,$$

and we can then find real numbers x_0, y_0 such that

$$(\xi + \theta)(\eta + \phi) = Q(x + x_0, y + y_0),$$

Now, $Q(x, y)$ is an indefinite binary quadratic form with

$$b^2 - ac = \tfrac{1}{4}\Delta^2,$$

and we may, without loss of generality, suppose that $\Delta = 2$, so that

$$b^2 - ac = 1.$$

Now let $$p^2 = 1 + \mu, \quad q^2 = 1 + 9\mu,$$

where $p, q > 0$, and let $c = -4\mu/(pq)$. We then have

$$\{(c^2 + 1)^{\frac{1}{2}} + c\} = p/q.$$

By Lemma 1 with $\Delta = 2$ and $c = -4\mu/(pq)$, integers x, y, with $(x, y) = 1$, exist satisfying

$$-4\mu/(pq) \leqslant Q(x, y) < 2p/q, \tag{31}$$

allowing equality on the left instead of on the right, as we may. Further, after an integral unimodular substitution we may suppose in (31) that $x = 1$, $y = 0$.

Suppose first that $Q(1, 0) \neq 0$. We then have

$$Q(x, y) = a(x + fy)^2 - y^2/a, \tag{32}$$

where, by (31) and our supposition,

$$-4\mu/(pq) \leqslant a < 2p/q \quad (a \neq 0). \tag{33}$$

Further, by writing $$x = \pm x' + my', \quad y = y',$$

where m is a suitable integer, we may suppose in (32) that

$$0 \leqslant f \leqslant \tfrac{1}{2}. \tag{34}$$

If, on the other hand, $Q(1, 0) = 0$, we have in place of (32)

$$Q(x, y) = y(2x + \vartheta y). \tag{35}$$

Alternatively, by Lemma 1 with $c = 0$, we may suppose in (32) that

$$0 < a \leqslant 2. \tag{36}$$

We use (33) if $0 < \mu \leqslant \frac{1}{7}$, but if $\mu = 0$ or if $\frac{1}{7} < \mu \leqslant \frac{1}{3}$, we use (36). More explicitly we consider the following six cases:

(i)	$0 < a \leqslant p/q$,	$0 \leqslant \mu \leqslant \frac{1}{3}$.
(ii)	$p/q < a < 2p/q$,	$0 \leqslant \mu \leqslant \frac{1}{3}$.
(iii)	$-4\mu/(pq) \leqslant a < 0$,	$0 < \mu \leqslant \frac{1}{7}$.
(iv)	$Q(1, 0) = 0$,	$0 < \mu \leqslant \frac{1}{7}$.
(v)	$2p/q \leqslant a \leqslant 2$,	$\frac{1}{7} < \mu \leqslant \frac{1}{3}$.
(vi)	$a = 2$,	$\mu = 0$.

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We note that when $\mu = 0$, $p = q = 1$, so that these are, in fact, the only cases we need to consider. We may also remark that the division at $\mu = \frac{1}{7}$ is somewhat arbitrary (see cases (iv) and (v) below).

By (32), we have

$$Q = Q(x+x_0, y+y_0) = a(x+\alpha)^2 - (y+y_0)^2/a, \tag{37}$$

where $\alpha = x_0 + f(y+y_0)$ is independent of x, and this is the appropriate non-homogeneous form to be considered in each case except (iv). In case (iv), however, we have, instead of (37), by (35),

$$Q = Q(x+x_0, y+y_0) = (y+y_0)\{2(x+x_0) + \vartheta(y+y_0)\}. \tag{38}$$

In order to prove the theorem we have to show, in each of the six cases, that integers x, y exist such that

$$-2\mu/(pq) < Q \leqslant 2/(pq), \tag{39}$$

where Q is defined by (37) or (38) as the case may be, and to discuss the cases when equality is actually needed.

Case (i). Suppose $0 < a \leqslant p/q$, $0 \leqslant \mu \leqslant \frac{1}{3}$.

In (37), choose y so that $|y+y_0| \leqslant \frac{1}{2}$. This fixes α. If, with this choice of y, we have $(y+y_0)^2 < 2a\mu/(pq)$, choose x so that $|x+\alpha| \leqslant \frac{1}{2}$. We then have

$$-2\mu/(pq) < Q \leqslant \tfrac{1}{4}a \leqslant p/(4q) < 2/(pq),$$

which is (39) with strict inequality.

We may thus suppose that $2a\mu/(pq) \leqslant (y+y_0)^2 \leqslant \frac{1}{4}$, and now choose x to satisfy

$$\{(y+y_0)^2 - 2a\mu/(pq)\}^{\frac{1}{2}} < a(x+\alpha) \leqslant \{(y+y_0)^2 - 2a\mu/(pq)\}^{\frac{1}{2}} + a.$$

We then have

$$\begin{aligned} -2\mu/(pq) < Q &\leqslant -2\mu/(pq) + a + 2\{(y+y_0)^2 - 2a\mu/(pq)\}^{\frac{1}{2}} \\ &\leqslant -2\mu/(pq) + a + \{1 - 8a\mu/(pq)\}^{\frac{1}{2}} \leqslant 2/(pq), \end{aligned}$$

by Lemma 3. Hence (39), and with strict inequality unless $a = p/q$, so that, with $C > 0$, $Q(x,y) \sim C[p^2(x+fy)^2 - q^2y^2]$.

By Lemma 6, equality is thus needed in (39), that is, in (6), if and only if, in this case, $\mu = 1/(4n-1)$ and

$$Q(x+x_0, y+y_0) = (\xi+\theta)(\eta+\phi) \sim C[n(2x+1)^2 - (n+2)(2y+1)^2],$$

for $n = 1, 2, 3, \ldots$.

Case (ii). Suppose $p/q < a < 2p/q$, $0 \leqslant \mu \leqslant \frac{1}{3}$.

As in case (i), with $|y+y_0| \leqslant \frac{1}{2}$, since $\frac{1}{4}a \leqslant p/(2q) < 2/(pq)$, we may suppose that $2a\mu/(pq) \leqslant (y+y_0)^2 \leqslant \frac{1}{4}$. We now choose x so that $\frac{1}{2} \leqslant |x+\alpha| \leqslant 1$, and have

$$Q \geqslant \tfrac{1}{4}a - 1/(4a) > p/(4q) - q/(4q) = -2\mu/(pq),$$

and

$$Q \leqslant a - 2\mu/(pq) < 2p/q - 2\mu/(pq) = 2/(pq),$$

whence (39) with strict inequality.

Case (iii). Suppose $-4\mu/(pq) \leqslant a < 0$, $0 < \mu \leqslant \frac{1}{7}$.

Choose y so that $|y+y_0| \leqslant \frac{1}{2}$. If, with this choice of y, we have $(y+y_0)^2 < -2a/(pq)$, choose x so that $|x+\alpha| \leqslant \frac{1}{2}$. We then have, since $\mu > 0 > a$,

$$2/(pq) > Q \geqslant \tfrac{1}{4}a \geqslant -\mu/(pq) > -2\mu/(pq).$$

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We may thus suppose that $-2a/(pq) \leqslant (y+y_0)^2 \leqslant \frac{1}{4}$ (so that $a \geqslant -\frac{1}{8}pq$), and choose x to satisfy $\{(y+y_0)^2+2a/(pq)\}^{\frac{1}{2}} < -a(x+\alpha) \leqslant \{(y+y_0)^2+2a/(pq)\}^{\frac{1}{2}} - a$.

We then have, since $a<0$,

$$2/(pq) > Q \geqslant 2/(pq)+a-2\{(y+y_0)^2+2a/(pq)\}^{\frac{1}{2}}$$
$$\geqslant 2/(pq)+a-\{1+8a/(pq)\}^{\frac{1}{2}} > -2\mu/(pq),$$

by Lemma 3. Whence (39) with strict inequality.

Case (iv). Suppose $Q(1,0)=0$, $0<\mu\leqslant\frac{1}{7}$.

In (38), choose y so that $|y+y_0| \leqslant \frac{1}{2}$, and choose x so that

$$|2(x+x_0)+\vartheta(y+y_0)| < 2 \quad \text{and} \quad Q \geqslant 0.$$

We then have, since $0<\mu\leqslant\frac{1}{7}$,

$$-2\mu/(pq) < 0 \leqslant Q \leqslant 1 < 2/(pq),$$

which is (39) with strict inequality.

We may observe that this argument fails for $\mu \geqslant 0{\cdot}245\ldots$, for we then have $pq \geqslant 2$.

Case (v). Suppose $2p/q \leqslant a \leqslant 2$, $\frac{1}{7} < \mu \leqslant \frac{1}{3}$. Then $2a\mu/(pq) \geqslant 4\mu/q^2 > \frac{1}{4}$. (This clearly fails if $\mu \leqslant \frac{1}{7}$.)

In (37) again, choose y so that $(y+y_0)^2 \leqslant \frac{1}{4} < 2a\mu/(pq)$, and choose x so that $|x+\alpha| \leqslant \frac{1}{2}$. We then have, since $\mu \leqslant \frac{1}{3}$,

$$-2\mu/(pq) < Q \leqslant \tfrac{1}{4}a \leqslant \tfrac{1}{2} < 2/(pq),$$

which is (39) with strict inequality.

Case (vi). Suppose $a=2$, $\mu=0$.

By cases (i) and (ii), we may suppose the inequality $0<Q(x,y)<2$ has no solution in integers x, y. By Lemma 1 with $c=0$, $Q(x,y) \sim Cxy$. Finally, by Lemma 7, equality is needed in (6), in this case, if and only if

$$Q(x+x_0, y+y_0) = (\xi+\theta)(\eta+\phi) \sim Cxy.$$

This completes the proof of the theorem.

5. *Proof of Theorem* 3. For the proof of this theorem we use an extension of the geometrical method employed to prove Theorem 1. We may again suppose that $\Delta = 1$. Let M now denote the lower bound of all values of the product $(\xi+\theta)(\eta+\phi)$ which exceed $\nu^2/\{(1-\nu)^3(1+3\nu)\}^{\frac{1}{2}}$, where $0<\nu<1$. (For $\nu=0$, we have the Davenport-Heilbronn theorem again, and this has already been included by the case $\mu=0$ of Theorem 3. Hence, for simplicity, we shall suppose $\nu>0$ in the proof, but we note that the case $\nu=0$ has its place in the answer.) We can suppose that

$$M \geqslant 1/\{(1-\nu)^3(1+3\nu)\}^{\frac{1}{2}}, \tag{40}$$

since otherwise (7) has a solution with strict inequality.

As before, we obtain a lattice in the XY-plane of determinant $(1-\epsilon)/M$ such that every lattice point satisfies

$$(X+1)(Y+1) \geqslant 1-\epsilon \quad \text{or} \quad \leqslant \tau^2,$$
$$\text{and} \qquad (X-1)(Y-1) \geqslant 1-\epsilon \quad \text{or} \quad \leqslant \tau^2, \tag{41}$$

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where, by (40), $$0 < \tau^2 = (1-\epsilon)\nu^2/[M\{(1-\nu)^3(1+3\nu)\}^{\frac{1}{2}}] \leqslant \nu^2 < 1. \tag{42}$$

Hence we again have an infinite region $\mathscr{K}$ in which no lattice point other than 0 may lie. As before, we prove that any lattice point, except 0, in the unit square, $|X|<1, |Y|<1$, must satisfy

$$(X+1)(Y+1) \leqslant \tau^2,$$

and

$$(X-1)(Y-1) \leqslant \tau^2.$$

Let $\mathscr{K}_0$ denote that part of $\mathscr{K}$ in the unit square which satisfies

$$(X+1)(Y+1) > \tau^2, \quad (X-1)(Y-1) > \tau^2.$$

Then $\mathscr{K}_0$ contains no lattice point except 0. $\mathscr{K}_0$ is convex and symmetrical about 0 and we can investigate its critical determinant by the well-known process of Minkowski (9).

Let then $\mathscr{L}$, of determinant D, now denote a critical lattice of $\mathscr{K}_0$. Then D is the critical determinant of this region. Any lattice of smaller determinant must have points other than 0 inside $\mathscr{K}_0$ but $\mathscr{L}$ has not.

I first prove that $\mathscr{L}$ has at least three pairs of points, $\pm P$, $\pm Q$, $\pm R$, say, on the boundary $\mathscr{C}$ of $\mathscr{L}$. For suppose first that no point of $\mathscr{L}$ lies on $\mathscr{C}$, and let P, Q now denote an arbitrary basis of $\mathscr{L}$. Then, for all sufficiently small positive δ, the lattice generated by $(1-\delta)P$, $(1-\delta)Q$ has no point other than 0 inside $\mathscr{K}_0$. But this lattice has determinant $(1-\delta)^2 D < D$, which is a contradiction.

Similarly, if only the one pair of points $\pm P$ of $\mathscr{L}$ lies on $\mathscr{C}$, let Q now be any point which, together with P, generates $\mathscr{L}$. Then the lattice generated by P, $(1-\delta)Q$ leads to a contradiction again.

Suppose now that only the two pairs of points $\pm P$, $\pm Q$ of $\mathscr{L}$ lie on $\mathscr{C}$, and at Q draw the line QQ', say, parallel to the radius vector OP. By the convex property of the region, some points of QQ' lie outside $\mathscr{K}_0$. Let Q' be such a point sufficiently near Q. Then the lattice generated by P, Q' has determinant D and only the points $\pm P$ on $\mathscr{C}$. As before, this leads to a contradiction and establishes the result.

Furthermore, since the triangle PQR lies entirely inside or on the boundary of $\mathscr{K}_0$, each of the pairs of points Q, R; R, P; P, Q forms a basis of $\mathscr{L}$ and we can thus suppose, without loss of generality, that $R = P+Q$. Again, we may suppose that P, Q, R are all points of the curve $(X-1)(Y-1) = \tau^2$, any point of which, since $\tau > 0$, has coordinates

$$X = 1-\tau t, \quad Y = 1-\tau/t,$$

where $t > 0$.

If P, Q have parameters t, s respectively, where we may suppose $t > s$, we have, since $R = P+Q$ is on this curve,

$$\{1-\tau(t+s)\}\{1-\tau(t+s)/(ts)\} = \tau^2. \tag{43}$$

Hence $(1-\tau^2)ts = \tau(t+s)\{1+ts-\tau(t+s)\}$. Now put $x = t-s$, $y = t+s$, so that $x, y > 0$, and let $z = x/y$. We then have $4ts = y^2 - x^2$, and, by (43), on eliminating t, s and x,

$$\tau y(z^2-1) - (1-\tau^2)z^2 + (1+3\tau^2) - 4\tau/y = 0. \tag{44}$$

The area of the parallelogram $OPRQ$ is given by

$$\begin{aligned} A &= (1-\tau/t)(1-\tau s) - (1-\tau t)(1-\tau/s) \\ &= \tau(t-s)\{1+ts-\tau(t+s)\}/(ts) \\ &= (1-\tau^2)(t-s)/(t+s) = (1-\tau^2)z. \end{aligned}$$

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Our problem now is to minimize A, that is, z, subject to (44). The minimum value of A will be the required value D of the critical determinant of $\mathscr{K}_0$, and the corresponding points P, Q will generate the critical lattice $\mathscr{L}$.

Differentiating (44) we get, if $z' = dz/dy$,

$$\tau(z^2 - 1 + 4/y^2) + 2\{\tau y - (1-\tau^2)\} zz' = 0. \tag{45}$$

Hence, for $z' = 0$, by the definitions of x, y, z, we must have $ts = 1$. Thus R is a point on the line $Y = X$. Hence R is the point

$$(1-\tau, 1-\tau),$$

and $y = t + s = 1 + 1/\tau$.

Differentiating again and putting $z' = 0$, we get

$$-8\tau/y^3 + 2\{\tau y - (1-\tau^2)\} zz'' = 0.$$

But $y = 1 + 1/\tau$, $z > 0$, so that $z'' = 4/\{(1+\tau) y^3 z\} > 0$ and z is a minimum under these conditions.

By (45), since $z' = 0$ and $y = 1 + 1/\tau$, this minimum value of z is given by

$$z^2 = 1 - 4/y^2 = (1-\tau)(1+3\tau)/(1+\tau)^2.$$

Hence the critical determinant of $\mathscr{L}$ is given by

$$D = A = (1-\tau^2) z = \{(1-\tau)^3 (1+3\tau)\}^{\frac{1}{2}}.$$

Now, the lattice of determinant $(1-\epsilon)/M$ we supposed to have no point other than 0 inside $\mathscr{K}$ and thus, *a fortiori*, inside $\mathscr{K}_0$. Hence $(1-\epsilon)/M \geqslant D$, that is,

$$M \leqslant (1-\epsilon)/\{(1-\tau)^3 (1+3\tau)\}^{\frac{1}{2}}. \tag{46}$$

By (42) and (46), we now have

$$\tau^2\{(1-\tau)^3 (1+3\tau)\}^{-\frac{1}{2}} \geqslant \nu^2\{(1-\tau)^3 (1+3\tau)\}^{-\frac{1}{2}} = \chi(\nu),$$

say. But

$$\frac{\chi'(\nu)}{\chi(\nu)} = \frac{2}{\nu} + \frac{3}{2(1-\nu)} - \frac{3}{2(1+3\nu)} > 0,$$

for $0 < \nu < 1$. Hence $\tau \geqslant \nu$, which contradicts (42) unless $\tau = \nu$. But in this case (46) contradicts (40) unless $\epsilon = 0$ and

$$M = 1/\{(1-\nu)^3 (1+3\nu)\}^{\frac{1}{2}}. \tag{47}$$

We have now shown that (7) is always soluble, and with strict inequality except, perhaps, when $\tau = \nu$, $\epsilon = 0$ and (47) holds. We must now determine in which, if any, of these cases equality is actually needed in (7).

In these possibly critical cases, we have $ts = 1$ and $t + s = 1 + 1/\nu$. Hence, writing $Z = \{(1-\nu)(1+3\nu)\}^{\frac{1}{2}}$, we have

$$t = (1+\nu+Z)/(2\nu), \quad s = (1+\nu-Z)/(2\nu).$$

It follows at once that P is the point

$$\{\tfrac{1}{2}(1-\nu-Z), \quad \tfrac{1}{2}(1-\nu+Z)\},$$

and we know that $R = P + Q$ is the point $(1-\nu, 1-\nu)$. Now the points P and Q form a basis of the lattice $\mathscr{L}$. We thus have

$$\begin{aligned} XY &= \tfrac{1}{2}\{(1-\nu-Z)x + (1-\nu+Z)y\}\{(1-\nu+Z)x + (1-\nu-Z)y\} \\ &= (1-\nu)\{(1+\nu)xy - \nu(x^2+y^2)\}, \end{aligned}$$

so that, in fact, with $C > 0$,

$$\xi\eta \sim C[(1+\nu)xy - \nu(x^2+y^2)]. \tag{48}$$

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Consider now the point mR of $\mathscr{L}$, where m is an integer. We have

$$\{m(1-\nu)-1\}\{m(1-\nu)-1\} = (m-1-m\nu)^2.$$

If $(m-2)/m < \nu < (m-1)/(m+1)$, this expression lies between ν^2 and 1 so that mR lies inside $\mathscr{K}$, by (41). Hence, if $\mathscr{L}$ is to be a critical lattice for $\mathscr{K}$ as well as for $\mathscr{K}_0$, which is a necessary condition for equality to be actually needed in (7), we must have

$$\nu = (m-1)/(m+1),$$

where, for $0 \leqslant \nu < 1$ ($m = 1, 2, 3, \ldots$). But this is not the only restriction on ν. For consider also the point $-nP$ of $\mathscr{L}$, where n is any integer. We have, by the definition of Z,

$$\begin{aligned}\{\tfrac{1}{2}n(\nu-1+Z)-1\}\{\tfrac{1}{2}n(\nu-1-Z)-1\} &= \tfrac{1}{4}\{(n+2-n\nu)^2 - n^2(1-\nu)(1+3\nu)\} \\ &= n^2\nu^2 - n(n+1)\nu + n + 1.\end{aligned}$$

If $1/n < \nu < 1/(n-1)$, this expression lies between ν^2 and 1, so that, by (41) again, $-nP$ lies inside $\mathscr{K}$. Hence we must also have

$$\nu = 1/n,$$

where $n = 1, 2, 3, \ldots$.

Clearly the only solutions in integers m, n of the equations

$$\nu = \frac{m-1}{m+1} = \frac{1}{n},$$

give us $\nu = \frac{1}{2}, \frac{1}{3}$ and, of course, $\nu = 0$. Hence there are only these three cases to consider. Putting $\nu = (m-1)/(m+1)$ ($m = 1, 2, 3$ only) in (48), we get

$$\xi\eta \sim C[2mxy - (m-1)(x^2+y^2)].$$

Case (i). If $m = 1$, $\nu = 0$ and, as we have already said, this corresponds to the case $\mu = 0$ of Theorem 2. We may note that from (48) we get $\xi\eta \sim Cxy$, which is only half the answer, but we obtain the other critical form $(\xi+\theta)(\eta+\phi) \sim C[(x-y)(x+y+1)]$ by letting $n \to \infty$, $nC \to C' \neq 0$, in Theorem 2. The reason why we have not obtained the critical lattice corresponding to $\xi\eta \sim C(x^2-y^2)$ in this case, is that our proof is incomplete if we allow $\nu = 0$. For a complete, and simpler, geometrical proof for the case $\nu = 0$, see Davenport and Heilbronn (1).

Case (ii). If $m = 2$, $\nu = \frac{1}{3}$ and

$$\begin{aligned}\xi\eta &\sim C[4xy - x^2 - y^2] \\ &= C[-(x-2y)^2 + 3y^2] \\ &\sim C[3x^2 - y^2].\end{aligned}$$

By Lemma 8, equality is needed in (7), in this case, if, and only if,

$$(\xi+\theta)(\eta+\phi) \sim C[3(2x+1)^2 - (2y+1)^2].$$

Case (iii). If $m = 3$, $\nu = \frac{1}{2}$ and

$$\begin{aligned}\xi\eta &\sim C[3xy - x^2 - y^2] \\ &\sim C[x^2 + xy - y^2] \\ &\sim C[(2x+y)^2 - 5y^2],\end{aligned}$$

putting $x = x'$, $y = x' + y'$ and completing the square. By Lemma 9, equality is needed in (7), in this case, if, and only if,

$$(\xi+\theta)(\eta+\phi) \sim C[x^2 + xy - y^2].$$

This completes the proof of the theorem.

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6. *Proof of Theorem* 4. We now revert to the algebraic methods used to prove Theorem 2. As before, we may suppose that $\Delta = 2$. By Lemma 2, after an integral unimodular substitution, we can write

$$\xi\eta = Q(x, y) = a(x+fy)^2 - y^2/a,$$

where, as in (34), $0 \leqslant f \leqslant \frac{1}{2}$, and, for any $q > 4{\cdot}573\ldots$,

$$\{(q^2+4)^{\frac{1}{2}} - 2\}/q < a < (q^2-4)/q. \tag{49}$$

To prove (8), we have to show, for all such q, that integers x, y exist for which

$$-1/q < Q \leqslant (q^2-4)/(4q), \tag{50}$$

where, as in (37), with $\alpha = x_0 + f(y+y_0)$ again,

$$Q = Q(x+x_0, y+y_0) = a(x+\alpha)^2 - (y+y_0)^2/a.$$

There are only two cases to consider.

Case (i). Suppose, in (49), that $\frac{1}{4}q < a < (q^2-4)/q$.

Choose y so that $|y+y_0| \leqslant \frac{1}{2}$, and then choose x so that $|x+\alpha| \leqslant \frac{1}{2}$ also. We then have

$$Q \geqslant -1/(4a) > -1/q,$$

and

$$Q \leqslant \tfrac{1}{4}a < (q^2-4)/(4q),$$

whence (50) with strict inequality.

Case (ii). Suppose now that $c < a \leqslant \frac{1}{4}q$, where (as in Lemma 2) $c = \{(q^2+4)^{\frac{1}{2}} - 2\}/q$. We note that

$$c - 1/c = -4/q. \tag{51}$$

Choose y so that $|y+y_0| \leqslant \frac{1}{2}$, as before. If, with this choice of y, we have $(y+y_0)^2 < a/q$, choose x so that $|x+\alpha| \leqslant \frac{1}{2}$ also. We then have, since $q^2 > 20{\cdot}91\ldots$ anyway,

$$-1/q < Q \leqslant \tfrac{1}{4}a \leqslant \tfrac{1}{16}q < (q^2-4)/(4q),$$

which is (50) with strict inequality.

We may thus suppose that $a/q \leqslant (y+y_0)^2 \leqslant \frac{1}{4}$, and choose x so that $\frac{1}{2} \leqslant |x+\alpha| \leqslant 1$. We then have, by (51),

$$Q \geqslant \tfrac{1}{4}a - 1/(4a) > \tfrac{1}{4}c - 1/(4c) = -1/q,$$

and

$$Q \leqslant a - 1/q \leqslant \tfrac{1}{4}q - 1/q = (q^2-4)/(4q),$$

whence (50) again. Moreover, we still have strict inequality except, perhaps, when $a = \frac{1}{4}q$, so that

$$Q(x, y) = \{q^2(x+fy)^2 - 16y^2\}/(4q).$$

By Lemma 10, equality is needed in (50), that is, in (8), if, and only if, $q^2 = 16(m+1)/m$, where, for $q > 4{\cdot}573\ldots$, $m = 1, 2$ or 3 only and

$$Q(x+x_0, y+y_0) = (\xi+\theta)(\eta+\phi) \sim C[4(m+1)x^2 - m(2y+1)^2].$$

This completes the proof of the theorem.

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