THIN CIRCULAR PLATES SUPPORTED AT SEVERAL POINTS ALONG THE BOUNDARY

By W. A. BASSALI

Communicated by R. G. Morris

Received 25 June 1956

ABSTRACT. The deflexion at any point of a thin circular plate subjected to certain loadings along the circumference or over the area of an eccentric circle is obtained when the plate is supported at several points along the boundary, the supports being symmetrically situated with respect to the common diameter of the plate and circle.

Introduction. A general method of handling problems of circular plates symmetrically loaded with respect to the centre and supported at several points along the boundary is given by Timoshenko (6), whose notation is adopted here. Several problems of this kind were discussed by Nadai (5). De Beer (4) used the same method to obtain the solution for a thin circular plate, symmetrically loaded and supported at a number of points regularly distributed along the boundary.

Chankvetadze (3) applied Muskhelishvili's complex variable method to the bending of a thin circular plate normally and uniformly loaded over the whole plate and supported at a number of points of its periphery. The complex variable method was applied by Dawoud and the author (2) to obtain solutions for the deflexion w of a circular plate supported along the entire boundary ($w = 0$) with uniform line loading along an eccentric circle under a general boundary condition including as special cases the rigidly clamped and simply supported boundaries. Using this solution the deflexion at any point of a circular plate having an eccentric circular patch symmetrically loaded with respect to its centre is obtained. The author also found (1) the solution for a linearly varying load over an eccentric circle under the same boundary condition. In this paper these solutions, in the case of a circular plate simply supported along the entire boundary, are used to solve the corresponding problems when the plate is subjected to the same loadings along the circumference or over the area of an eccentric circle and is supported only at several points along the boundary, the supports being symmetrically situated with respect to the common diameter of the plate and circle. The solutions obtained include as special cases those of Nadai (5) and De Beer (4). The application of Muskhelishvili's complex variable method to the problems of normally and partially loaded thin circular plates supported at several interior or boundary points will be considered later.

1. *Statement of the problem and its solution.* Consider an isotropic homogeneous thin circular plate of thickness $2h$ and flexural rigidity D, bounded by a circle Γ of centre O and radius c. In the Poisson–Kirchhoff theory of bending of thin plates (6) the stress couples and shears per unit length of arc are obtained in terms of the transverse deflexion w of the mid-surface of the plate, w being a solution of the biharmonic equation

$$D\nabla^4 w = p(r, \theta), \tag{1}$$

https://doi.org/10.1017/S0305004100032503 Published online by Cambridge University Press

where $p(r,\theta)$ is the normal load per unit area at the point $P=(r,\theta)$ of the plate and w is measured positively downwards.

Let AB be a diameter of Γ and let γ denote the boundary of a closed curve symmetrical with respect to AB (e.g. an eccentric circle with centre on AB). Let 1 and 2 refer to the regions inside γ and between γ and Γ respectively, and let $p_1(r,\theta)$, $p_2(r,\theta)$ be the transverse load intensities in these regions, supposed to be even functions of θ which is measured from BA. We suppose that the plate is supported ($w=0$) at the points $P_1, P_2, \ldots, P_k$ of the upper half of Γ and at their images $P_1', P_2', \ldots, P_k'$ in the diameter AB, and write γ_i for the angle defining the ith support. We denote the concentrated reactions at the points of support $P_1, P_2, \ldots, P_k$ by $N_1, N_2, \ldots, N_k$ respectively, a support at either A or B being considered as two coincident supports with equal reactions.

Let w_1 and w_2 be the deflexions in regions 1 and 2 respectively when the circular plate is simply supported along the entire boundary Γ and let w_1+w_0 and w_2+w_0 be the deflexions in the two regions when the plate is supported only at the points $P_1, P_2, \ldots, P_k$ and $P_1', P_2', \ldots, P_k'$. We assume that

$$\pi D w_2/c^2 = \sum_{n=0}^{\infty} A_n(\rho)\cos n\theta, \tag{2}$$

$$\pi D w_0/c^2 = \sum_{n=0}^{\infty} (C_n\rho^n + D_n\rho^{n+2})\cos n\theta, \tag{3}$$

where $\rho = r/c$, $A_n(\rho)$ is supposed to be a known real function of ρ that depends on the curve γ and the load intensities p_1 and p_2, and C_n, D_n are real constants to be determined. Since w_2 and $M_r = -(D/c^2)\{d^2+\nu\rho^{-2}(\rho d+d'^2)\}w_2$ (see (6), p. 259) both vanish on Γ, it follows that we must have

$$A_n(1) = [(d^2+\nu d)A_n(\rho)]_{\rho=1} = 0, \tag{4}$$

where $d=\partial/\partial\rho$, $d'=\partial/\partial\theta$ and ν is Poisson's ratio. The condition that

$$M_r^0 = -(D/c^2)\{d^2+\nu\rho^{-2}(\rho d+d'^2)\}w_0$$

vanishes on Γ gives

$$D_0 = D_1 = 0, \tag{5}$$

and

$$n(n-1)C_n + (n+1)(n+2/\beta)D_n = 0 \quad (n=2,3,\ldots), \tag{6}$$

where

$$\beta = (1-\nu)/(1+\nu). \tag{7}$$

The intensity V_r of the vertical reaction at the boundary Γ corresponding to the solution w_2+w_0 is given ((6), p. 259) by

$$V_r = -(D/c^3)[\{d^3+d^2-d+(2-\nu)dd'^2+(\nu-3)d'^2\}(w_2+w_0)]_{\rho=1}. \tag{8}$$

Substituting from (2) and (3) in (8) and using (4) we get

$$V_r = -\frac{1}{\pi c}\sum_{n=0}^{\infty}[n^2(1-n)\sigma C_n + n(n+1)(4-n\sigma)D_n + \alpha_n]\cos n\theta, \tag{9}$$

where $\sigma = 1-\nu$ and

$$\alpha_n = [\{d^3+(\overline{\nu-2}n^2-1-\nu)d\}A_n(\rho)]_{\rho=1}. \tag{10}$$

https://doi.org/10.1017/S0305004100032503 Published online by Cambridge University Press

Let F be the resultant of the normal forces acting on the plate and M be the moment of these forces about the diameter perpendicular to AB. The conditions of equilibrium of the plate are

$$2\sum_{i=1}^{k} N_i = F, \quad 2c\sum_{i=1}^{k} N_i \cos\gamma_i = M. \tag{11}$$

The intensity of the reactive forces, measured positively downwards, at any point of Γ equals

$$-\frac{1}{\pi c}\sum_{i=1}^{k} N_i\left[1+\sum_{n=1}^{\infty}\{\cos n(\theta-\gamma_i)+\cos n(\theta+\gamma_i)\}\right] = -\frac{1}{\pi c}\left[\tfrac{1}{2}F+\frac{M}{c}\cos\theta+\sum_{n=2}^{\infty}\beta_n\cos n\theta\right], \tag{12}$$

where

$$\beta_n = 2\sum_{i=1}^{k} N_i\cos n\gamma_i. \tag{13}$$

Equating the coefficients of $\cos n\theta$ in (9) and (12) we have, using (5),

$$\alpha_0 = \tfrac{1}{2}F, \quad \alpha_1 = M/c \tag{14}$$

and

$$n^2(1-n)\,\sigma C_n + n(n+1)\,(4-n\sigma)\,D_n = \beta_n - \alpha_n \quad (n = 2, 3, \ldots). \tag{15}$$

Solving (6) and (15) we get

$$\left.\begin{aligned} C_n &= (n+2/\beta)\,(\alpha_n-\beta_n)/2(\nu+3)\,n^2\,(n-1),\\ D_n &= (\beta_n-\alpha_n)/2(\nu+3)\,n(n+1) \quad (n = 2, 3, \ldots),\end{aligned}\right\} \tag{16}$$

and hence we have

$$C_n + D_n = a_n(\alpha_n-\beta_n) = b_n - a_n\beta_n, \tag{17}$$

where

$$a_n = (n+n\beta+1)/\beta(\nu+3)\,n^2(n^2-1), \quad b_n = a_n\alpha_n. \tag{18}$$

We can write

$$a_n = (a_n' + \beta^{-1}a_n'')/(\nu+3), \tag{19}$$

where

$$a_n' = 1/n(n^2-1), \quad a_n'' = 1/n^2(n-1).$$

The conditions that $w_0 = 0$ at the points of support give

$$\sum_{n=0}^{\infty}(C_n+D_n)\cos n\gamma_i = 0 \quad (i = 1, 2, \ldots, k). \tag{20}$$

Substituting from (17) in (20) and using (5) we get

$$C_0 + C_1\cos\gamma_i - \sum_{n=2}^{\infty} a_n\beta_n\cos n\gamma_i + \sum_{n=2}^{\infty} b_n\cos n\gamma_i = 0 \quad (i = 1, 2, \ldots, k). \tag{21}$$

Substituting for β_n from (13) in (21) we obtain

$$C_0 + C_1\cos\gamma_i - \sum_{j=1}^{k} a_i^j N_j + d_i = 0 \quad (i = 1, 2, \ldots, k), \tag{22}$$

where

$$a_i^j = 2\sum_{n=2}^{\infty} a_n\cos n\gamma_i\cos n\gamma_j, \quad d_i = \sum_{n=2}^{\infty} b_n\cos n\gamma_i. \tag{23}$$

Now (11) and (22) are $k+2$ equations to determine the $k+2$ unknown quantities $C_0, C_1; N_1, N_2, \ldots, N_k$. If $k = 2$ then (11) determines N_1 and N_2, and substituting in (22) we get C_0 and C_1. The constants C_n and D_n $(n = 0, 1, 2, \ldots)$ are now obtained, and the

https://doi.org/10.1017/S0305004100032503 Published online by Cambridge University Press

solution w_0 to be superposed on the solutions w_1 and w_2 is now completely determined, if we know the values of α_n and β_n. These we now proceed to determine for special cases of distribution of supports and certain types of loading. In any case the deflexion w_c at the centre O of the plate is $C_0c^2/\pi D + (w_1)_O$ or $C_0c^2/\pi D + (w_2)_O$, according as O lies inside or outside γ.

2. *Determination of β_n, C_0 and C_1 for certain distributions of supports.* We now consider the following special cases:

Case (*a*). Two supports at A and B. Then $\gamma_1 = 0$, $\gamma_2 = \pi$ and we obtain

$$N_1 = \tfrac{1}{4}(F + M/c), \quad N_2 = \tfrac{1}{4}(F - M/c).$$

Substituting in (13) we get

$$\left.\begin{aligned} \beta_n &= F \quad \text{if } n \text{ is even,} \\ \beta_n &= M/c \quad \text{if } n \text{ is odd.} \end{aligned}\right\} \tag{24}$$

The two conditions (21) now give

$$C_0 = \sum_{n=2m} (Fa_n - b_n), \quad C_1 = \sum_{n=2m+1} (Ma_n/c - b_n) \quad (m = 1, 2, \ldots). \tag{25}$$

Case (*b*). Three supports at points 120° apart, one support being at A. Here we have $\gamma_1 = 0°$, $\gamma_2 = 120°$; $N_1 = \frac{1}{6}(F + 2M/c)$, $N_2 = \frac{1}{3}(F - M/c)$ and

$$\left.\begin{aligned} \beta_n &= F \quad \text{if } n = 3m \\ \beta_n &= M/c \quad \text{if } n \neq 3m \end{aligned}\right\} \quad (m = 1, 2, \ldots). \tag{26}$$

The constants C_0 and C_1 are now given by

$$C_0 = \sum_{n=3m} (Fa_n - b_n), \quad C_1 = \sum_{n=3m\pm1} (Ma_n/c - b_n) \quad (m = 1, 2, \ldots). \tag{27}$$

Case (*c*). Four supports forming a square, $\gamma_1 = \frac{1}{4}\pi$, $\gamma_2 = \frac{3}{4}\pi$. In this case

$$N_1 = \tfrac{1}{4}(F + M\sqrt{2}/c), \quad N_2 = \tfrac{1}{4}(F - M\sqrt{2}/c).$$

Substitution in (13) now gives

$$\left.\begin{aligned} \beta_n &= (-1)^m F \quad \text{if } n = 4m \\ \beta_n &= (-1)^m M/c \quad \text{if } n = 4m \pm 1 \\ \beta_n &= 0 \quad \text{otherwise} \end{aligned}\right\} \quad (m = 1, 2, \ldots), \tag{28}$$

and hence

$$C_0 = \sum_{n=4m} (Fa_n - (-1)^m b_n), \quad C_1 = \sum_{n=4m\pm1} \left(\frac{M}{c} a_n - (-1)^m b_n\right). \tag{29}$$

Case (*d*). Four supports forming a square, $\gamma_1 = 0$, $\gamma_2 = \frac{1}{2}\pi$, $\gamma_3 = \pi$. In this case the reactions are statically indeterminate and the equilibrium conditions (11) give

$$N_1 = \tfrac{1}{4}(F + M/c - 2N_2), \quad N_3 = \tfrac{1}{4}(F - M/c - 2N_2),$$

and then

$$\left.\begin{aligned} \beta_n &= F \quad \text{if } n = 4m, \\ \beta_n &= F - 4N_2 \quad \text{if } n = 4m-2, \\ \beta_n &= M/c \quad \text{if } n = 2m-1. \end{aligned}\right\} \tag{30}$$

https://doi.org/10.1017/S0305004100032503 Published online by Cambridge University Press

The three conditions (22) give the following values for C_0, C_1 and N_2:

$$C_0 = \sum_{n=4m} (Fa_n - b_n), \quad C_1 = \sum_{n=2m+1} (Ma_n/c - b_n), \quad N_2 = \tfrac{1}{4}[F - \sum_{n=4m-2} b_n \Big/ \sum_{n=4m-2} a_n]. \tag{31}$$

We now give the following table for $\Sigma a'_n$, $\Sigma a''_n$ and Σa_n defined by (18) and (19) corresponding to values of n that occur in the four previous cases, taking $\nu = \frac{1}{3}$, $\beta = \frac{1}{2}$. Expressions for α_n and b_n will now be obtained for certain types of loading.

Table 1

Value of n in terms of m ($m = 1, 2, \ldots$)	$\Sigma a'_n$	$\Sigma a''_n$	Σa_n
$n = 2m$	$\log 2 - \frac{1}{2} = 0{\cdot}1931$	$\log 2 - \pi^2/24 = 0{\cdot}2819$	$\frac{1}{\nu+3}\left(\frac{2\log 2}{1-\nu} - \frac{\pi^2}{24\beta} - \frac{1}{2}\right) = 0{\cdot}2271$
$n = 2m+1$	$\frac{3}{4} - \log 2 = 0{\cdot}0569$	$2 - \log 2 - \pi^2/8 = 0{\cdot}0732$	$0{\cdot}0609$
$n = 3m$	$\frac{1}{2}(\log 3 - 1) = 0{\cdot}0493$	$\frac{1}{2}(\log 3 - \pi^2/27 - \sqrt{3}\,\pi/9) = 0{\cdot}0642$	$\frac{1}{\nu+3}\left(\frac{\log 3}{1-\nu} - \frac{\pi^2}{54\beta} - \frac{\sqrt{3}\,\pi}{18\beta} - \frac{1}{2}\right) = 0{\cdot}0533$
$n = 3m \pm 1$	$\frac{1}{4}(3 - 2\log 3) = 0{\cdot}2007$	$\frac{1}{4}(4 - \log 3 - 8\pi^2/27 + \sqrt{3}\pi/9) = 0{\cdot}2908$	$0{\cdot}2347$
$n = 4m$	$\frac{1}{4}(3\log 2 - 2) = 0{\cdot}0199$	$\frac{1}{4}(3\log 2 - \pi/2 - \pi^2/24) = 0{\cdot}0244$	$\frac{1}{\nu+3}\left(\frac{1{\cdot}5\log 2}{1-\nu} - \frac{\pi}{8\beta} - \frac{\pi^2}{96\beta} - \frac{1}{2}\right) = 0{\cdot}0206$
$n = 4m-2$	$\frac{1}{4}\log 2 = 0{\cdot}1733$	$\frac{1}{4}(\log 2 + \pi/2 - \pi^2/8) = 0{\cdot}2576$	$0{\cdot}2065$

3. *Determination of α_n and b_n for a circular plate loaded along the circumference or over the area of an eccentric circle.* Let γ denote the boundary of an eccentric circle of centre $Q = (f, 0)$ and radius b and let $Q' = (f', 0)$ be the inverse of Q with respect to Γ, so that $ff' = c^2$. Let (r, θ), (R, Θ) and (R', Θ') be the polar coordinates of any point P of the plate referred to O, Q and Q' respectively. (See Fig. 1.) Since it is advantageous to work with dimensionless quantities, we introduce the ratios

$$u = f/c, \quad U = 1 - u^2, \quad v = b/c, \quad V = v^2, \quad W = U - V. \tag{32}$$

Let us now deal with the following types of loading.

Type I. Consider the case of a uniform line loading p per unit length distributed along the circumference of γ. In this case $p_1 = p_2 = 0$ and

$$F = 2\pi bp, \quad M = Ff = Fcu. \tag{33}$$

The solutions w_1 and w_2 corresponding to this loading were obtained by the author and Dawoud (2) and are given by

$$w_1 = \frac{F}{8\pi D}\left[(R^2 + b^2)\log\frac{b}{uR'} + c^2(1-\rho^2)\{(U + \beta W)\,S(\rho, \theta) - \tfrac{1}{2}\beta W\} + R^2 - b^2\right], \tag{34a}$$

$$w_2 = \frac{F}{8\pi D}\left[(R^2 + b^2)\log\frac{R}{uR'} + c^2(1-\rho^2)\{(U + \beta W)\,S(\rho, \theta) - \tfrac{1}{2}\beta W\}\right], \tag{34b}$$

https://doi.org/10.1017/S0305004100032503 Published online by Cambridge University Press

where $$S(\rho,\theta)=\sum_{n=0}^{\infty}\rho^n u^n\cos n\theta/(n+n\beta+1). \tag{35}$$

As $u\to 0$, $uR'\to c$ and the solutions (34 a) and (34 b) reduce respectively to the solutions (78) and (77) given by Timoshenko (6), p. 69.

Expanding the solution (34b) in the form (2) we get

$$\left.\begin{aligned} A_0(\rho)&=\tfrac{1}{8}F[(1-\rho^2)(1+\tfrac{1}{2}\beta W)+(1+\rho^2-W)\log\rho],\\ A_n(\rho)&=\frac{u^n}{8}F[(1+\rho^2-W)\phi_n-\rho(\phi_{n-1}+u^2\phi_{n+1})+(U+\beta W)\psi_n/(n+n\beta+1)]\\ &\qquad(\rho>u, n>0),\end{aligned}\right\} \tag{36}$$

where $$\phi_n=(\rho^n-\rho^{-n})/n,\quad \psi_n=\rho^n(1-\rho^2), \tag{37}$$

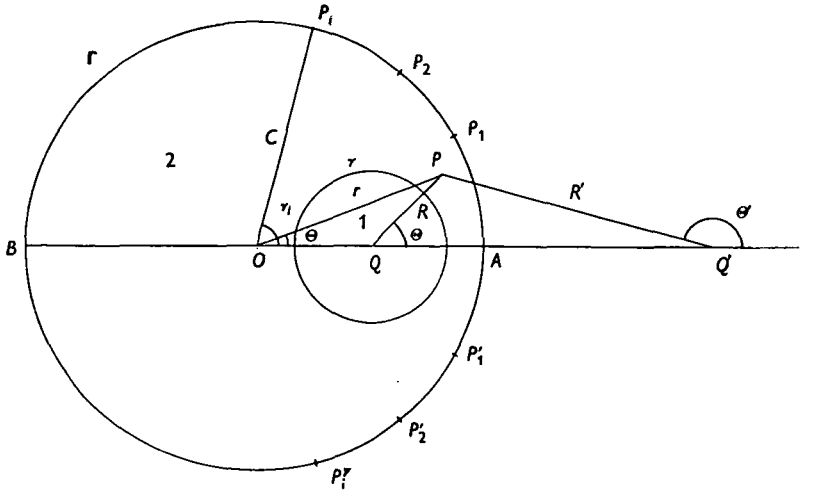


Fig. 1

and $\phi_0=\lim_{n\to 0}\phi_n=2\log\rho$. It can be easily verified that the functions $A_n(\rho)$ given by (36) satisfy the conditions (4). Substituting from (36) in (10) we get

$$\alpha_0=\tfrac{1}{2}F,\quad \alpha_n=Fu^n\left[1+\frac{\beta n(n-1)}{2(n+n\beta+1)}\{U-(n+1)V\}\right]\quad(n=1,2,\ldots), \tag{38}$$

which includes (14) on making use of (33). From (18) and (38) we easily find that

$$b_n=\frac{Fu^n}{2(\nu+3)}\left[\frac{\kappa}{n-1}-\frac{1+\kappa-W}{n}+\frac{u^2}{n+1}-\frac{2}{\beta n^2}\right]\quad(n=2,3,\ldots), \tag{39}$$

where $$\kappa=(\nu+3)/(1-\nu). \tag{40}$$

If the centre O of the plate lies inside γ then $0\leqslant u<v\leqslant 1-u$, $0\leqslant u<\frac{1}{2}$ and the deflexion w_c at the centre is given by

$$w_c=\frac{c^2}{\pi D}[C_0+\tfrac{1}{8}F\{1-v^2+\tfrac{1}{2}\beta W+(1-W)\log v\}]; \tag{41a}$$

while if O lies outside γ then $0\leqslant v<u\leqslant 1-v$, $0\leqslant v<\frac{1}{2}$ and

$$w_c=\frac{c^2}{\pi D}[C_0+\tfrac{1}{8}F\{1-u^2+\tfrac{1}{2}\beta W+(1-W)\log u\}]. \tag{41b}$$

https://doi.org/10.1017/S0305004100032503 Published online by Cambridge University Press

For a concentrated force F at Q we put $b = V = 0$, $W = U$ in (34b), (36), (38), (39) and (41b).

Type II. Let the load intensity p on the plate be given by

$$p_1 = p_0 R^{s-2} \ (s \geqslant 2) \text{ over region 1}, \quad p_2 = 0 \text{ over region 2}. \tag{42}$$

Hence
$$F = 2\pi p_0 b^s/s, \quad M = Fcu. \tag{43}$$

The solutions w_1 and w_2 in this case can be obtained from (34) by integration and we obtain (2)

$$w_1 = \frac{F}{8\pi D}\left[(R^2+b'^2)\log\frac{b}{uR'} + c^2(1-\rho^2)\{(U+\beta W')\,S(\rho,\theta) - \tfrac{1}{2}\beta W'\} + \left(1-\frac{1}{s}\right)(R^2-b^2) + 4\frac{R^{s'}-b^{s'}}{ss'^2 b^s}\right], \tag{44a}$$

$$w_2 = \frac{F}{8\pi D}\left[(R^2+b'^2)\log\frac{R}{uR'} + c^2(1-\rho^2)\{(U+\beta W')\,S(\rho,\theta) - \tfrac{1}{2}\beta W'\}\right], \tag{44b}$$

where $S(\rho,\theta)$ is given by (35) and

$$b'^2 = b^2\delta_s, \quad V' = V\delta_s, \quad W' = U - V', \quad s' = s+2 \quad \text{and} \quad \delta_s = s/s'. \tag{45}$$

For uniform loading over region 1 we put $s = 2, s' = 4$ and then $\delta_2 = \frac{1}{2}$.

The expressions for $A_n(\rho)$, α_n and b_n in this case are obtained from those in the previous case by putting W' and V' instead of W and V in (36), (38) and (39).

Putting $\rho = 0$ in (44) we find that

$$w_c = \frac{c^2}{\pi D}\left[C_0 + \tfrac{1}{8}F\left\{1 + \tfrac{1}{2}\beta W' + (1-W')\log v + \tfrac{1}{2}(\delta_s - 3)\,V' + \frac{u^2}{s}\left(\frac{4}{s'^2}\frac{u^s}{v^s} - 1\right)\right\}\right], \tag{46a}$$

or
$$w_c = \frac{c^2}{\pi D}[C_0 + \tfrac{1}{8}F\{1 + \tfrac{1}{2}\beta W' + (1-W')\log u - u^2\}], \tag{46b}$$

according as O lies inside or outside γ, and the value of C_0 in (46) is obtained from the corresponding value in (41) by replacing W and V by W' and V' respectively wherever they appear.

Type III. Let the load intensity on the plate be given by

$$p_1 = p_0 R^{s-3}\cos\Theta \ (s \geqslant 3) \text{ over region 1}, \quad p_2 = 0 \text{ over region 2}. \tag{47}$$

In this case we have
$$F = 0, \quad M = \pi p_0 b^s/s. \tag{48}$$

For the special case $s = 4$ we have a linearly varying load over the eccentric circle. The solutions w_1 and w_2 corresponding to the load intensities (47) were obtained by the author (1) and are given by

$$w_1 = \frac{Mc}{16\pi D}\left[(\rho\cos\theta - u)\left\{4\log\frac{uR'}{b} + \frac{4}{s} - \frac{s}{s-2}\frac{R^2}{b^2} + \frac{16R^s}{s(s^2-4)\,b^s}\right\} + \frac{b'^2\rho(\cos\theta - \rho u)}{u^2R'^2} - 2(1-\rho^2)\left\{(1+\beta)\,u + \rho\sum_{n=1}^{\infty}(1+\overline{1+2\beta}u^2 + \tfrac{1}{2}n\beta V')\frac{(\rho u)^{n-1}\cos n\theta}{n+n\beta+1}\right\}\right], \tag{49a}$$

$$w_2 = \frac{Mc}{16\pi D}\left[(\rho\cos\theta - u)\left(4\log\frac{uR'}{R} - \frac{b'^2}{R^2}\right) + \frac{b'^2\rho(\cos\theta - \rho u)}{u^2R'^2} - 2(1-\rho^2)\left\{(1+\beta)\,u + \rho\sum_{n=1}^{\infty}(1+\overline{1+2\beta}u^2 + \tfrac{1}{2}n\beta V')\frac{(\rho u)^{n-1}\cos n\theta}{n+n\beta+1}\right\}\right], \tag{49b}$$

https://doi.org/10.1017/S0305004100032503 Published online by Cambridge University Press

where b', V', δ_s are defined by (45) and hence for a linearly varying load $\delta_4 = \frac{2}{3}$. As $u \to 0$, $uR' \to c$ and the solutions (49) reduce to

$$w_1 = \frac{Mc}{16\pi D}\rho\cos\theta\left[\frac{4}{s} - \frac{\rho^2 s}{(s-2)V} + \frac{16\rho^s}{s(s^2-4)v^s} + V' - 2\log V - \frac{2+\beta V'}{2+\beta}(1-\rho^2)\right], \quad (50a)$$

$$w_2 = \frac{Mc}{16\pi D}\rho\cos\theta\left[V'(1-\rho^{-2}) - 4\log u - \frac{2+\beta V'}{2+\beta}(1-\rho^2)\right]. \quad (50b)$$

Expanding the solution (49b) in the form (2), noticing that

$$c^2\rho(\cos\theta - \rho u)/uR'^2 = \sum_{n=1}^{\infty}(u\rho)^n\cos n\theta, \quad c^2u(\rho\cos\theta - u)/R^2 = \sum_{n=1}^{\infty}(u/\rho)^n\cos n\theta,$$

we find after some reduction that

$$\left.\begin{aligned} A_0(\rho) &= \frac{Mu}{8c}[(\rho^2-1)\beta + 2\log\rho], \\ A_n(\rho) &= \frac{Mu^{n-1}}{8c}\left[2u^2\phi_n - \rho(\phi_{n-1} + u^2\phi_{n+1}) - \frac{1+(1+2\beta)u^2}{n+n\beta+1}\psi_n + \tfrac{1}{2}nV'\left(\phi_n - \frac{\beta\psi_n}{n+n\beta+1}\right)\right], \end{aligned}\right\} \quad (51)$$

where ϕ_n and ψ_n are given by (37). It can be verified that (51) satisfies (4). Substituting from (51) in (10) we get

$$\alpha_0 = 0, \quad c\alpha_n = Mnu^{n-1}\left[1 + \frac{(n-1)\beta}{2(n+n\beta+1)}\{nU - 2u^2 - \tfrac{1}{2}n(n+1)V'\}\right] \quad (n = 1, 2, \ldots), \quad (52)$$

which includes (14) on making use of (48). From (18) and (52) we easily obtain

$$b_n = \frac{Mu^{n-1}}{2(\nu+3)c}\left[\frac{\kappa}{n-1} - \frac{1+\kappa-2U}{n} + \frac{u^2}{n+1} - \tfrac{1}{2}V'\right] \quad (n = 2, 3, \ldots). \quad (53)$$

The deflexion at the centre of the plate in this case is given by

$$w_c = \frac{c^2}{\pi D}\left[C_0 + \frac{Mu}{16c}\left\{4\log v + \frac{s}{s-2}\frac{u^2}{v^2} - \frac{16u^s}{s(s^2-4)v^s} - 2\left(1+\beta+\frac{2}{s}\right)\right\}\right], \quad (54a)$$

or

$$w_c = \frac{c^2}{\pi D}\left[C_0 + \frac{Mu}{16c}\{4\log u + (v'/u)^2 - 2(1+\beta)\}\right], \quad (54b)$$

according as O lies inside or outside γ.

Type IV. Circular plate loaded by a concentrated couple at any point. Putting $V' = 0$ in equations (49b), (50b), (51), (52), (53) and (54b) we get the corresponding results when the plate is acted upon by a concentrated couple M at Q, whose axis lies in the plane of the plate and is perpendicular to AB.

Equations (38) and (52) give α_n corresponding to the loadings I and III while (24), (26), (28) and (30) determine β_n for the special distributions of supports mentioned in § 2. Inserting these values in (16) we can get C_n and D_n when the plate is subjected to any of the types of loading discussed before and supported as in cases (a), (b), (c) or (d) of § 2. Substituting in (25), (27), (29) and (31) for Σa_n from Table 1 and for b_n from (39)

https://doi.org/10.1017/S0305004100032503 Published online by Cambridge University Press

or (53) we obtain the values of C_0 and C_1 in any case considered. For the loading (42) and two supports at A and B we easily get

$$C_0 = -\frac{F}{4(\nu+3)}\left[(1+\kappa)\left(u\log\frac{1+u}{1-u}+\log\frac{U}{4}\right)+2U-W'\log U+\frac{1}{\beta}\phi(u^2)\right], \tag{55}$$

$$C_1 = \frac{F}{4(\nu+3)}\left[(1+\kappa)\left(u\log\frac{U}{4}+\log\frac{1+u}{1-u}\right)-W'\log\frac{1+u}{1-u}+u(U-2V')+\frac{1}{\beta}\psi(u)\right], \tag{56}$$

where

$$\phi(x) = \frac{\pi^2}{6}-\sum_{1,2,\ldots}^{\infty}\frac{x^n}{n^2} = \int_1^x \frac{\log(1-x)}{x}dx,$$

$$\psi(x) = 4\sum_{1,3,5,\ldots}^{\infty}\frac{x^n}{n^2}-\tfrac{1}{2}\pi^2 x = 2\int_0^x \frac{1}{x}\log\frac{1+x}{1-x}dx-\tfrac{1}{2}\pi^2 x.$$

Table 2

Loading / Supports	Uniform line loading along γ	Uniform surface loading over the eccentric circle	Concentrated force	Concentrated couple	Linearly varying load over the eccentric circle
Along entire boundary	0·0824	0·0890	0·0955	−0·1804	−0·1711
Case (*a*)	0·2557	0·2610	0·2664	−0·4056	−0·3926
Case (*b*)	0·1295	0·1357	0·1418	−0·2217	−0·2108
Case (*c*)	0·1042	0·1109	0·1175	−0·1695	−0·1609
Case (*d*)	0·1018	0·1082	0·1146	−0·1917	−0·1817

For the loading (47) and two supports at A and B we obtain

$$C_0 = -\frac{M}{4(\nu+3)c}\left[(1+\kappa)\log\frac{1+u}{1-u}+2\left(u+\frac{1}{\beta u}\right)\log U-u\left(2+\frac{V'}{U}\right)\right], \tag{57}$$

$$C_1 = \frac{M}{4(\nu+3)c}\left[(1+\kappa)\log\frac{U}{4}+2\left(u+\frac{1}{\beta u}\right)\log\frac{1+u}{1-u}+3U-\frac{\pi^2-8}{2\beta}+\frac{u^2V'}{U}\right]. \tag{58}$$

The deflexion w at any point of the plate may be written in the forms $w = \lambda Fc^2/\pi D$ for the loadings I, II and $w = \lambda Mc/\pi D$ for the loadings III, IV, where λ is a dimensionless deflexion coefficient given by $\lambda = \lambda' + \lambda_0$, λ' being the deflexion coefficient had the plate been simply supported along the entire boundary and λ_0 is the part of the deflexion coefficient contributed by w_0. It is easily seen that at the centre of the plate $\lambda_0 = C_0/F$ or cC_0/M according as we are dealing with types I, II or III, IV. In Table 2 we give the values of λ' and λ at the centre of the plate corresponding to the four cases of distribution of supports and to the various types of loading mentioned before, taking $\nu = \frac{1}{3}, \beta = \frac{1}{2}, \kappa = 5; u = \frac{1}{2}, v = \frac{1}{3}$. Table 3 gives the magnitudes of the vertical reactions at the four supports in case *d* as calculated from (31) and the equilibrium conditions, with $\nu = \frac{1}{3}$, $u = \frac{1}{2}$, $v = \frac{1}{3}$.

4. *Circular plate concentrically loaded.* For all types of loading considered, putting $u = 0$ in (38), (39), (52) and (53) we get

$$\alpha_n = 0, \quad b_n = 0 \quad (n = 2, 3, \ldots). \tag{59}$$

https://doi.org/10.1017/S0305004100032503 Published online by Cambridge University Press

Hence if γ is concentric with the plate, equations (16) and (17) give

$$C_n = -\frac{(n+2/\beta)\beta_n}{2(\nu+3)n^2(n-1)}, \quad D_n = \frac{\beta_n}{2(\nu+3)n(n+1)}, \quad C_n+D_n = -\frac{(n+n\beta+1)\beta_n}{\beta(\nu+3)n^2(n^2-1)}. \tag{60}$$

In any of the four cases mentioned in § 2, putting $b_n = 0$ and remembering that $M = 0$ for loadings I, II and $F = 0$ for loadings III, IV, we easily find C_0 and C_1 in any required case on making use of Table 1.

Let us now calculate β_n for a circular plate symmetrically and concentrically loaded (types I, II) and supported at a number K of points regularly distributed along Γ. Let $\alpha = 2\pi/K$. If K is even, $2k$ say, then two of the supports may be taken at A and B. Hence

$$\gamma_i = (i-1)\alpha \quad (i = 1, 2, \ldots, k+1), \quad N_1 = N_{k+1} = F/2K, \quad N_i = F/K \quad (i = 2, 3, \ldots, k).$$

Table 3

Loading / Reactions	Type I	Type II ($s = 2$)	Concentrated force	Concentrated couple	Type III ($s = 4$)
$2N_1$ at A	$0{\cdot}5638F$	$0{\cdot}5651F$	$0{\cdot}5663F$	$0{\cdot}7589\,M/c$	$0{\cdot}7553\,M/c$
N_2 ($\gamma_2 = \frac{1}{2}\pi$)	$0{\cdot}1862F$	$0{\cdot}1849F$	$0{\cdot}1837F$	$-0{\cdot}2589\,M/c$	$-0{\cdot}2553\,M/c$
$2N_3$ at B	$0{\cdot}0638F$	$0{\cdot}0651F$	$0{\cdot}0663F$	$-0{\cdot}2411\,M/c$	$-0{\cdot}2447\,M/c$

Substituting in (13) we get

$$\beta_n = (F/K)[1 + \cos n\pi + 2\{\cos n\alpha + \cos 2n\alpha + \ldots + \cos(k-1)n\alpha\}],$$

which equals F if n is a multiple of K and zero otherwise. If the number of supports K is odd, $2k-1$ say, one support being at A, then

$$\gamma_i = (i-1)\alpha \quad (i = 1, 2, \ldots, k), \quad N_1 = F/2K, \quad N_i = F/K \quad (i = 2, 3, \ldots, k).$$

Substitution in (13) now gives

$$\beta_n = (F/K)[1 + 2\{\cos n\alpha + \cos 2n\alpha + \ldots + \cos(k-1)n\alpha\}],$$

which also equals F if n is a multiple of K and zero otherwise. For any number K of supports equations (60) therefore yield

$$\left.\begin{aligned} C_n &= -\frac{(n+2/\beta)F}{2(\nu+3)n^2(n-1)}, \quad D_n = \frac{F}{2(\nu+3)n(n+1)}, \\ C_n+D_n &= -\frac{(n+n\beta+1)F}{\beta(\nu+3)n^2(n^2-1)} \quad (n = K, 2K, \ldots); \\ C_n &= D_n = 0 \text{ for other values of } n. \end{aligned}\right\} \tag{61}$$

The conditions (20) together with (61) then give

$$C_0 = F\,\Sigma' a_n = \frac{F}{\beta(\nu+3)}\Sigma'\frac{n+n\beta+1}{n^2(n^2-1)}, \quad C_1 = 0, \tag{62}$$

https://doi.org/10.1017/S0305004100032503 Published online by Cambridge University Press

where the accent on Σ means that the summation is carried only over multiples of K. The solution w_0 to be superposed on w_1 and w_2 in this case can thus be represented by

$$\frac{\pi D}{c^2} w_0 = \frac{F}{\beta(\nu+3)} \Sigma' \frac{1}{n^2} \left[\frac{n+n\beta+1}{n^2-1} + \left(\frac{n\rho^2}{n+1} - \frac{n+2/\beta}{n-1} \right) \rho^n \cos n\theta \right], \tag{63}$$

which includes and is a generalization of equation (g) of Timoshenko ((6), p. 271).

The values given in finite terms in the last column of Table 1, corresponding to $n = 2m,\ 3m,\ 4m$ ($m = 1, 2, \ldots$) are the deflexion coefficients $\lambda_0 = C_0/F = \Sigma' a_n$ at the centre of the symmetrically loaded plate corresponding to $K = 2, 3, 4$ respectively. From equations (41 a) and (46 a) we easily find that when the plate is simply supported along the entire boundary, the deflexion coefficient λ' at the centre of the plate is either $(\beta+2)/16$ or $(2\beta+3)/64$ according as the load is applied at the centre or uniformly distributed over the whole plate. In particular for $K = 3$ and a uniformly distributed load we have

$$w_c = \frac{Fc^2}{\pi D} \left[\frac{1}{\nu+3} \left\{ \frac{\log 3}{1-\nu} - \frac{\pi}{54\beta} (\pi + 3\sqrt{3}) - \tfrac{1}{2} \right\} + \frac{2\beta+3}{64} \right], \tag{64}$$

which differs from the formula given by Chankvetadze ((3), equation (32), p. 78) because of some mistakes in his result, which should be written correctly as

$$w_{z=0} = \frac{qR^4}{D} \left[\frac{1}{6(3+\nu)} \left(\tfrac{3}{2} \log 3 + \frac{\pi}{2\sqrt{3}} - 3 \right) + \frac{1}{6(1-\nu)} \left(\tfrac{3}{2} \log 3 - \frac{\pi}{2\sqrt{3}} \right) - \frac{(1+\nu)\pi^2}{54(3+\nu)(1-\nu)} + \frac{5+\nu}{64(1+\nu)} \right]. \tag{65}$$

Noticing the difference in notation and remembering that $\beta = (1-\nu)/(1+\nu)$ it can be easily seen that (65) agrees with (64). Putting $\nu = \frac{1}{4}$ in (64) and remembering that $D = Eh^3/12(1-\nu^2)$ ((6), p. 3), we get

$$w_c = 0{\cdot}114 \frac{pc^4}{D} = 1{\cdot}280 \frac{pc^4}{Eh^3} = 0{\cdot}407 \frac{Fc^2}{Eh^3}, \tag{66}$$

which differs from the value given by Timoshenko (6) at the bottom of p. 271 and quoted by Chankvetadze (3) at the top of p. 78.

REFERENCES

(1) Bassali, W. A. *Proc. Camb. Phil. Soc.* 52 (1956), 734.
(2) Bassali, W. A. and Dawoud, R. H. *Proc. Camb. Phil. Soc.* 52 (1956), 584.
(3) Chankvetadze, G. G. *Engng Rev.* 14 (1953), 73.
(4) De Beer, C. *Ingénieur* (1946).
(5) Nadai, A. *Z. Phys.* 23 (1922), 366.
(6) Timoshenko, S. *Theory of plates and shells* (New York, 1940).

https://doi.org/10.1017/S0305004100032503 Published online by Cambridge University Press