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Embedding Bratteli–Vershik systems in cellular automata

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Abstract. Many dynamical systems can be naturally represented as Bratteli–Vershik (or adic) systems, which provide an appealing combinatorial description of their dynamics. If an adic system X is linearly recurrent, then we show how to represent X using a two-dimensional subshift of finite type Y; each 'row' in a Y-admissible configuration corresponds to an infinite path in the Bratteli diagram of X, and the vertical shift on Y corresponds to the 'successor' map of X. Any Y-admissible configuration can then be recoded as the space-time diagram of a one-dimensional cellular automaton Φ; in this way X is embedded in Φ (i.e. X is conjugate to a subsystem of Φ). With this technique, we can embed many odometers, Toeplitz systems, and constant-length substitution systems in one-dimensional cellular automata.

1. *Introduction*

The Bratteli–Vershik ('adic') transformations are a large and diverse class of symbolic dynamical systems which are of interest partly because many other dynamical systems can be naturally represented as adic transformations, thereby yielding an appealing combinatorial description of their dynamics. Cellular automata (CA) are another class of symbolic dynamical system with versatile representation capabilities. Recent work shows that certain simple adic systems arise naturally as subsystems of CA. For example, if $\Phi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is a non-injective, right-sided, left-permutative cellular automaton (CA), then many orbit closures of Φ are conjugate to odometers [**CPY07**]; a more precise characterization of some of these odometers appears in [**CY07**]. Similarly, in the *Game of Life* CA, the orbit closure of a certain configuration can be represented as a substitution system [**GM05**]. The main result of this paper is the following.

THEOREM 1. *Let* $(X_{\mathcal{B}}, V_{\mathcal{B}})$ *be a linearly recurrent adic system. Then there exists a one-dimensional cellular automaton which embeds* $(X_{\mathcal{B}}, V_{\mathcal{B}})$.

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For definitions of terms in Theorem 1 and notation, see §2. The proof of Theorem 1 is the content of §3. In §4 we give examples of dynamical systems whose adic representations satisfy the conditions of Theorem 1. These include: all aperiodic primitive substitution shifts; all stationary odometers; and many Toeplitz systems and Sturmian shifts. To illustrate Theorem 1, we give explicit constructions of the CA which embed the binary and ternary odometers. Finally, in §5, we show that the *Glider–Wall* CA embeds almost any odometer (not just the stationary ones covered by Theorem 1).

2. *Preliminaries*

2.1. *Symbolic dynamics.* Let $\mathcal{A}$ be a finite alphabet; elements in $\mathcal{A}$ will be denoted by $a, b, c, \ldots$. A *word* $\mathbf{a}$ from $\mathcal{A}$ is a finite concatenation of elements from $\mathcal{A}$. Let $\mathcal{A}^k$ and $\mathcal{A}^{\leq k}$ denote the set of all words of length k and length at most k, respectively. We include the empty word, and use boldface to denote words. If $\mathbf{a} = a_1 \cdots a_j$ and $\mathbf{a}^* = a_1^* \cdots a_k^*$, then $\mathbf{a}\mathbf{a}^* := a_1 \cdots a_j a_1^* \cdots a_k^*$. If $\mathbf{a} = a_1 \cdots a_j$, define $|\mathbf{a}| := j$. The space of all bi-infinite sequences from $\mathcal{A}$ is written as $\mathcal{A}^{\mathbb{Z}}$. Elements of the latter are written $\mathbf{a} = (\cdots a_{-1} \cdot a_0 a_1 \cdots)$. If $\mathbb{N} := \{0, 1, 2, \ldots\}$, and $\mathbb{N}_+ := \{1, 2, \ldots\}$, elements of $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{N}_+}$ are defined analogously. If $k \in \mathbb{Z}$, the notation $\{x_n\}_{n=\infty}^{k}$ is used to denote the left-infinite sequence $(\cdots x_{k+2}\, x_{k+1}\, x_k)$, called a *tail* of $\mathbf{x}$. (Note that we have *reversed* the usual ordering of an $\mathbb{N}$-indexed sequence here.) If $\mathbf{x} = \{x_n\}_{n=\infty}^{1}$ and $\mathbf{y} = \{y_n\}_{n=0}^{\infty}$, then by $\mathbf{x} \cdot \mathbf{y}$ we mean the bi-infinite sequence $(\cdots x_2\, x_1 \cdot y_0\, y_1 \cdots)$. If $\mathbb{L} = \mathbb{Z}$, $\mathbb{N}$, or $\mathbb{N}_+$, then $\mathcal{A}^{\mathbb{L}}$ is a *Cantor space*: a zero-dimensional compact perfect metric space, when $\mathcal{A}$ is endowed with the discrete topology and $\mathcal{A}^{\mathbb{L}}$ with the product topology. If $b \in \mathcal{A}$, let $[b]_j := \{\mathbf{a} \mid \ a_j = b\}$; these (clopen) sets form a countable basis for the topology on $\mathcal{A}^{\mathbb{L}}$. The (left) *shift map* $\sigma : \mathcal{A}^{\mathbb{L}} \to \mathcal{A}^{\mathbb{L}}$ is the map defined as $(\sigma(\mathbf{x}))_n = x_{n+1}$. Let $X \subset \mathcal{A}^{\mathbb{Z}}$; and let $T : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a homeomorphism. We say that (X, T) is a *subsystem of* $(\mathcal{A}^{\mathbb{L}}, T)$ if X is a closed and T-invariant of $\mathcal{A}^{\mathbb{L}}$; in the special case $T = \sigma$, (X, σ) is called a *subshift*. A (one-dimensional) *cellular automaton* is a continuous, σ-commuting map $\Phi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$. The Curtis–Hedlund–Lyndon theorem [**Hed69**] states that every CA is given by a *local rule* $\phi : \mathcal{A}^{l+r+1} \to \mathcal{A}$ for some $l \geq 0$ (the *left radius of* ϕ) and $r \geq 0$ (the *right radius of* ϕ), where for all $x \in \mathcal{A}^{\mathbb{Z}}$, and all $i \in \mathbb{Z}$,

$$[\Phi(x)]_i = \phi(x_{i-l}, x_{i-l+1}, \ldots, x_{i+r}).$$

The dynamical systems (X, S) and (Y, T) are *topologically conjugate* if there exists a homeomorphism $f : X \to Y$ with $f \circ S = T \circ f$. We say that $(\mathcal{A}^{\mathbb{Z}}, \Phi)$ *embeds* (Y, T) if (Y, T) is conjugate to a subsystem $(\mathcal{A}^{\mathbb{Z}}, \Phi)$. If $T : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is a continuous map, then a *space-time diagram* for $(\mathcal{A}^{\mathbb{Z}}, T)$ is an element $\mathbf{x} = \{\mathbf{x}_m\}_{m \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z} \times \mathbb{N}}$ where $\mathbf{x}_m = T^m(\mathbf{x}_0)$ for each $m > 0$. For more information about subshifts and cellular automata, see [**Kur03**].

2.2. *Bratteli diagrams.* A *Bratteli diagram* $\mathcal{B} = (\mathcal{V}, \mathcal{E})$ is an infinite directed graph with *vertex set* $\mathcal{V} = \bigsqcup_{n=0}^{\infty} \mathcal{V}_n$ and *edge set* $\mathcal{E} = \bigsqcup_{n=1}^{\infty} \mathcal{E}_n$, where all $\mathcal{V}_n$ and $\mathcal{E}_n$ are finite, and $\mathcal{V}_0 = \{\boxtimes\}$. (Here, '$\boxtimes$' is a special symbol reserved for the bottom level of the Bratteli diagram. We use this 'hourglass' symbol because, in the construction in §3, $\boxtimes$ acts as a 'clock' which 'drives' the successor map of the adic system.) If x is an edge in $\mathcal{E}_n$, the

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source $s(x)$ of x lies in $\mathcal{V}_{n-1}$ and the *range* $r(x)$ of x lies in $\mathcal{V}_n$. We assume that $s^{-1}(v) \neq \emptyset$ for each $v \in \mathcal{V}$ and $r^{-1}(v) \neq \emptyset$ for all $v \in \mathcal{V}\backslash\{\boxtimes\}$. We will use $x, y \ldots$ when referring to edges, and $a, b, \ldots$ when referring to vertices, although edges in $\mathcal{E}_1$ will be given special labels $\boxtimes_i$, where i varies. A finite set of edges $\{x_{n+k}\}_{k=1}^{K}$, with $s(x_{n+k+1}) = r(x_{n+k})$ for $1 \leq k \leq K-1$, is called a *path* from $s(x_{n+1})$ to $r(x_{n+K})$. Similarly an *infinite path* in $\mathcal{B}$ is a sequence $\mathbf{x} = (\ldots, x_4, x_3, x_2, x_1)$, with $x_1 = \boxtimes_1 \in \mathcal{E}_1$, and $x_n \in \mathcal{E}_n$ for all $n \geq 2$, and with $s(x_{n+1}) = r(x_n)$ for all $n \geq 1$. We will sometimes indicate such a path as

$$\cdots a_{n+1} \overset{x_{n+1}}{\leftarrow} a_n \overset{x_n}{\leftarrow} \cdots \overset{x_2}{\leftarrow} a_1 \overset{\boxtimes_1}{\leftarrow} \boxtimes \tag{1}$$

(where $r(x_n) = a_n$). The set of all infinite paths in $\mathcal{B}$ will be denoted $X_{\mathcal{B}}$ (a subset of $\prod_{n\geq 1} \mathcal{E}_n$), and endowed with the subspace topology induced from the product topology on $\prod_{n\geq 1} \mathcal{E}_n$. Thus $X_{\mathcal{B}}$ is a compact zero-dimensional metric space.

Two Bratteli diagrams $\mathcal{B} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{B}' = (\mathcal{V}', \mathcal{E}')$ are *isomorphic* if there exists a pair of bijections $f_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}'$ and $f_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}'$ satisfying $f_{\mathcal{V}}(a) \in \mathcal{V}'_n$ if $a \in \mathcal{V}_n$, and $s(f_{\mathcal{E}}(x)) = f_{\mathcal{V}}(s(x))$, $r(f_{\mathcal{E}}(x)) = f_{\mathcal{V}}(r(x))$ whenever $x \in \mathcal{E}$. Let $\{n_k\}_{k=0}^{\infty}$ be a sequence of increasing integers with $n_0 = 0$. Then $\mathcal{B}' = (\mathcal{V}', \mathcal{E}')$ is a *telescoping* of $\mathcal{B} = (\mathcal{V}, \mathcal{E})$ if $\mathcal{V}'_k = V_{n_k}$ (with the vertex $v \in \mathcal{V}_{n_k}$ labelled as $v' \in \mathcal{V}'_k$), and the number of edges from $v'_k \in \mathcal{V}'_k$ to $v'_{k+1} \in \mathcal{V}'_{k+1}$ is the number of paths from $v_k \in \mathcal{V}_{n_k}$ to $v_{k+1} \in \mathcal{V}_{n_{k+1}}$. We consider two Bratteli diagrams $\mathcal{B}$ and $\mathcal{B}'$ *equivalent* if $\mathcal{B}'$ can be obtained from $\mathcal{B}$ by isomorphism and telescoping. We say that $\mathcal{B}$ is *simple* if there exists a telescoping $\mathcal{B}' = (\mathcal{V}', \mathcal{E}')$ of $\mathcal{B}$ so that, for any $a \in \mathcal{V}'_n$ and $b \in \mathcal{V}'_{n+1}$, there is at least one edge from a to b. If $\mathbf{x} = \{x_n\}_{n=\infty}^{1}$ and $\mathbf{x}' = \{x'_n\}_{n=\infty}^{1}$ are two elements in $X_{\mathcal{B}}$, we write $\mathbf{x} \sim \mathbf{x}'$ if the tails of $\mathbf{x}$ and $\mathbf{x}'$ are equal. It follows that $\sim$ is an equivalence relation. If $\mathcal{B}$ is simple, then each equivalence class for $\sim$ is dense in $X_{\mathcal{B}}$, and $X_{\mathcal{B}}$ has no isolated points, making it a Cantor space.

2.3. *Bratteli–Vershik adic systems.* Let $n \geq 1$. For each $a \in \mathcal{V}_n$, let $\mathcal{E}_n(a) = \{x \in \mathcal{E}_n \mid r(x) = a\}$. Describe $\mathcal{B}$ as *ordered* if there is a linear order $\geq$ on each $\mathcal{E}_n(a)$; elements of $\mathcal{E}_n(a)$ will then be labelled $1, 2, \ldots$ according to their order. If $a \in \mathcal{V}\backslash\{\boxtimes\}$, define $|a| := |\mathcal{E}_n(a)|$, so that $\mathcal{E}_n(a) = \{1, 2, \ldots |a|\}$.

The linear order on edges in each $\mathcal{E}_n(a)$ induces a partial ordering on paths from $\mathcal{V}_m$ to $\mathcal{V}_n$: the two paths

$$\mathbf{x} = a_n \overset{x_n}{\leftarrow} a_{n-1} \overset{x_{n-1}}{\leftarrow} \cdots \overset{x_{m+1}}{\leftarrow} a_m \quad \text{and} \quad \mathbf{x}' = a'_n \overset{x'_n}{\leftarrow} a'_{n-1} \overset{x'_{n-1}}{\leftarrow} \cdots \overset{x'_{m+1}}{\leftarrow} a'_m$$

from $\mathcal{E}_m$ to $\mathcal{E}_n$ are comparable with $\mathbf{x} < \mathbf{x}'$ if there is some $k \in [m+1, n]$ with $x_k < x'_k$, and $x_j = x'_j$ for $j \in [k+1, n]$.

Finally, two elements $\mathbf{x}, \mathbf{x}' \in X_{\mathcal{B}}$ are *comparable* with $\mathbf{x} < \mathbf{x}'$ if there is a k such that $x_n = x'_n$ for all $n > k$, and $x_k < x'_k$. Thus each equivalence class for $\sim$ is totally ordered. There is the obvious notion of ordered isomorphism of two ordered Bratteli diagrams $\mathcal{B}$, $\mathcal{B}'$: the isomorphism between $\mathcal{B}$ and $\mathcal{B}'$ also has to satisfy $f_{\mathcal{E}}(x) \leq f_{\mathcal{E}}(y)$ if $x \leq y$. If $\mathcal{B}'$ is a telescoping of the ordered Bratteli diagram $\mathcal{B}$, then the order induced on $\mathcal{B}'$ from the order on $\mathcal{B}$ makes $\mathcal{B}'$ an ordered Bratteli diagram. We say that the ordered Bratteli diagrams $\mathcal{B}$, $\mathcal{B}'$ are *equivalent* if $\mathcal{B}'$ is the image of $\mathcal{B}$ by telescoping and order isomorphism.

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An infinite path is *maximal* (*minimal*) if all the edges making up the path are maximal (minimal). If $\mathbf{x} = \{x_n\}_{n=\infty}^{1}$ is not maximal, let k be the smallest integer such that $x_k \in \mathcal{E}_k(a_k)$ is not a maximal edge, and let y_k be the successor of x_k in $\mathcal{E}_k(a_k)$. Then the *successor* $V_{\mathcal{B}}(\mathbf{x})$ is defined to be $V_{\mathcal{B}}(\mathbf{x}) = \cdots x_{k+2}\, x_{k+1}\, y_k\, 1 \cdots 1$. Similarly, every non-minimal path $\mathbf{x}$ has a unique *predecessor* $V_{\mathcal{B}}^{-1}(\mathbf{x})$. Let $X_{\min}$ ($X_{\max}$) $\subset X_{\mathcal{B}}$ be defined as the set of minimal (maximal) elements of $X_{\mathcal{B}}$. By compactness, these sets are non-empty. Simple ordered Bratteli diagrams which have a unique minimal and maximal element (called $\mathbf{x}_{\min}$ and $\mathbf{x}_{\max}$, respectively) are called *properly ordered*. If $\mathcal{B}$ is properly ordered, then $V_{\mathcal{B}}$ can be extended to a homeomorphism on $X_{\mathcal{B}}$ by setting $V_{\mathcal{B}}(\mathbf{x}_{\max}) = \mathbf{x}_{\min}$. We call $(X_{\mathcal{B}}, V_{\mathcal{B}})$ a *Bratteli–Vershik* or *adic* system. See [**DHS99**] for an introduction to adic systems. Note that $(X_{\mathcal{B}}, V_{\mathcal{B}})$ is a minimal system, since $V_{\mathcal{B}}$ orbits are equivalence classes for $\sim$. A *pointed* topological dynamical system is a triple $(X, T, \mathbf{x})$ where $T : X \to X$ is a homeomorphism of the compact metric space X and $\mathbf{x} \in X$. For example, $(X_{\mathcal{B}}, V_{\mathcal{B}}, \mathbf{x}_{\min})$ is pointed. We say that the two pointed systems $(X_i, T_i, \mathbf{x}_i)$, $i = 1, 2$ are *isomorphic* if there exists a topological conjugacy $f : X_1 \to X_2$ with $f(x_1) = x_2$. The pointed adic system associated with an equivalence class of properly ordered Bratteli diagrams is well defined up to isomorphism (see [**HPS92**, §4]).

The properly ordered Bratteli diagram $\mathcal{B}$ is called *linearly recurrent* [**CDHM03**] if:

(LR1) *there exists a constant K such that $|\mathcal{E}_n| \leq K$ for each $n \geq 1$*; and

(LR2) *for $n \geq 1$, all minimal edges in $\mathcal{E}_n$ have the same source in $\mathcal{E}_{n-1}$.*

These two conditions imply that $|\mathcal{V}_n| \leq K$ for each $n \geq 1$ [**CDHM03**, Lemma 2]. An adic system $(X_{\mathcal{B}}, V_{\mathcal{B}})$ is called *linearly recurrent* if $\mathcal{B}$ is equivalent to a linearly recurrent Bratteli diagram. If $\mathcal{B}$ arises from a *substitution* (see §4), then (LR2) is equivalent to saying that the substitution is *left proper*. (A similar, but stronger, condition in substitution tiling systems is 'forcing the border'; see [**Kel95**].)

3. *Proof of Theorem 1*

In this section, $\mathcal{B}$ is any linearly recurrent Bratteli diagram. In this case, vertices in $\mathcal{V}$ can be labelled from a finite alphabet $\mathcal{A}$, so that for all $n \geq 1$, $\mathcal{V}_n \subset \mathcal{A}$. For each $n \geq 1$ there is a function $\boldsymbol{\tau}^n : \mathcal{A} \to \mathcal{A}^{\leq K}$ such that $\boldsymbol{\tau}^n$ completely describes $\mathcal{E}_n$ and its ordering. In other words, if $a \in \mathcal{V}_n$, we write $\boldsymbol{\tau}^n(a) = \tau_1^n(a)\tau_2^n(a) \cdots \tau_{l_a^n}^n(a)$, where $l_a^n = |\mathcal{E}_n(a)|$, and where $\boldsymbol{\tau}_k^n(a)$ is the source in $\mathcal{V}_{n-1}$ of the edge in $\mathcal{E}_n(a)$ which has the ordering label k. If $\boldsymbol{\tau}_a^n$ is the empty word, this is taken to mean that a does not appear as a vertex in $\mathcal{V}_n$. As $\mathcal{B}$ satisfies (LR2), we will use the label $\underline{a}$ for the source of all minimal edges in $\mathcal{E}_n$, for any $n \in \mathbb{N}$; thus the unique minimal element in $X_{\mathcal{B}}$ will be $(\cdots \overset{1}{\leftarrow} \underline{a} \overset{1}{\leftarrow} \underline{a} \overset{1}{\leftarrow} \underline{a} \overset{1}{\leftarrow} \boxtimes)$.

We shall use 'stars' such as $*$, $\star$, ... to indicate l_a^n, so that '$\tau_*^n(a)$' means the *last* letter of $\boldsymbol{\tau}^n(a)$, provided that there is no ambiguity.

Let $\mathcal{T} := \{\boldsymbol{\tau}^n\}_{n=1}^{\infty}$. Then (LR1) implies that $\mathcal{T}$ is finite. Let $\mathcal{A}_{\mathcal{B}}$ be the finite alphabet

$$\begin{aligned}\mathcal{A}_{\mathcal{B}} := \{\boxtimes\} \cup \{(a, \boldsymbol{\tau}^1, \boxtimes_x) \mid a \in \mathcal{A}, x \in [1, l_a^1]\} \\ \cup \{(a, \boldsymbol{\tau}^n, x) \mid a \in \mathcal{A}, n \in \mathbb{N}, x \in [1, l_a^n]\}.\end{aligned}$$

Each 3-tuple $(a, \boldsymbol{\tau}, x)$ represents a vertex–edge pair $a \overset{x}{\leftarrow}$ which occurs at a level of the Bratteli diagram of 'type' $\boldsymbol{\tau}$ (we do not have to keep track of the exact level, but only the level *type*, because $\mathcal{T}$ is finite by (LR1)). Thus, $(a, \boldsymbol{\tau}, x)$ represents a link in a path

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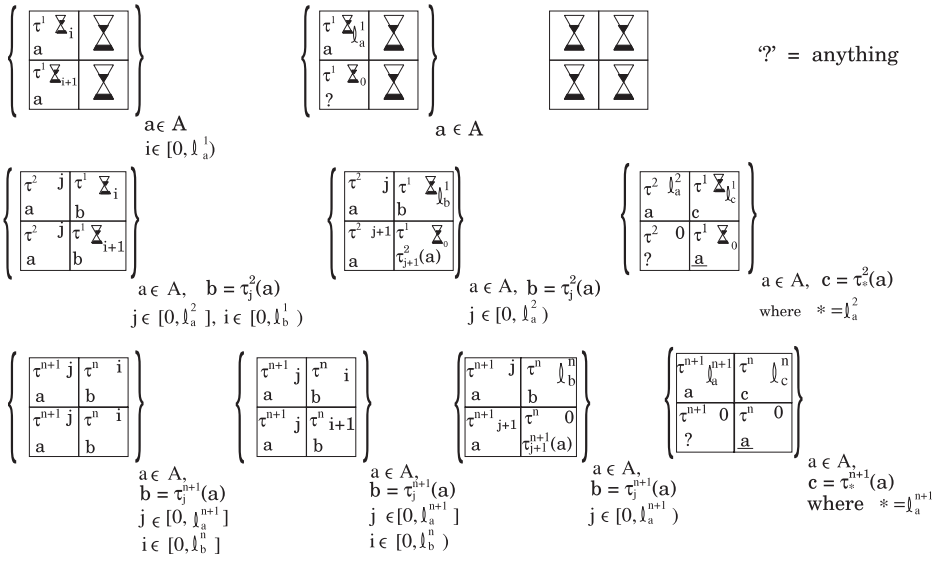


FIGURE 1. Admissible $\{0, 1\}^2$-blocks defining the subshift of finite type $\mathcal{S}$.

in $X_{\mathcal{B}}$. The 3-tuples $(a, \boldsymbol{\tau}^1, \boxtimes_x)$ represent links of the form $a \overset{\boxtimes_x}{\leftarrow} \boxtimes$. Thus, the path (1) can be represented with the sequence $(\ldots, (a_3, \boldsymbol{\tau}_3, x_3), (a_2, \boldsymbol{\tau}_2, x_2), (a_1, \boldsymbol{\tau}_1, \boxtimes_1), \boxtimes)$. Let $\tilde{\mathcal{Y}} \subset \mathcal{A}_{\mathcal{B}}^{\mathbb{Z}\times\mathbb{N}}$ be the set of all space-time diagrams for $(X_{\mathcal{B}}, V_{\mathcal{B}})$. Let $\boxtimes^\infty = (\boxtimes, \boxtimes, \ldots)$; we extend $\tilde{\mathcal{Y}}$ to $\mathcal{Y} \subset \mathcal{A}_{\mathcal{B}}{}^{\mathbb{Z}\times\mathbb{Z}}$ by letting

$$\mathcal{Y} := \{\mathbf{y} = \{(\mathbf{y}_m)\}_{m\in\mathbb{Z}} \mid \exists \tilde{\mathbf{y}} \in \tilde{\mathcal{Y}} \text{ such that } \forall m \in \mathbb{Z},\ \mathbf{y}_m = \tilde{\mathbf{y}}_m \cdot \boxtimes^\infty\}.$$

There is a natural homeomorphism from $\mathcal{Y}$ to $\tilde{\mathcal{Y}}$. Note that we use *positive* integers to denote column locations in $\mathbf{y}$, unless there is a possibility of confusion.

A (two-dimensional) *subshift* is a closed, shift-invariant subset $\mathcal{S} \subset \mathcal{A}^{\mathbb{Z}^2}$. If $\mathcal{S} \subset \mathcal{A}^{\mathbb{Z}^2}$ is a subshift, then for any finite $\mathbb{K} \subset \mathbb{Z}^2$, let $\mathcal{S}_{\mathbb{K}} := \{\mathbf{s}_{\mathbb{K}} \mid \mathbf{s} \in \mathcal{S}\}$ be the set of all $\mathcal{S}$-*admissible* $\mathbb{K}$-*blocks*. If $\mathsf{z} \in \mathbb{Z}^2$, then $\mathcal{S}_{\mathbb{K}} = \mathcal{S}_{\mathbb{K}+\mathsf{z}}$ (because $\mathcal{S}$ is shift-invariant). We say $\mathcal{S}$ is a *subshift of finite type* (SFT) if there is some finite *neighbourhood* $\mathbb{K} \subset \mathbb{Z}^2$ such that $\mathcal{S} = \{\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^2} \mid \mathbf{a}_{\mathbb{K}+\mathsf{z}} \in \mathcal{S}_{\mathbb{K}},\ \forall \mathsf{z} \in \mathbb{Z}^2\}$. For example, if $\Phi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is a one-dimensional CA, then the set of space-time diagrams for Φ is an SFT in $\mathcal{A}^{\mathbb{Z}^2}$.

Let $\mathbb{K} = \{0, 1\}^2$, and let $\mathcal{S} \subset \mathcal{A}_{\mathcal{B}}^{\mathbb{Z}^2}$ be the SFT defined using the $\mathbb{K}$-blocks in Figure 1. The following lemma is straightforward.

LEMMA 2. *$\mathcal{Y}$ is the subset $\{\mathbf{y} \in \mathcal{S};\ y^0_0 = \boxtimes,\ y^0_{-1} = (a, \boxtimes_i)$ for some $a \in \mathcal{A}$ and $i \in [1, l^1_a]\}$.*

We will sometimes use the symbols $\boxed{\begin{matrix}\tau & j\\ a & \end{matrix}}$, $\boxed{\begin{matrix}\tau' & \boxtimes\\ a & \end{matrix}}$, ... or $\boxed{T_1}, \boxed{T_2}, \ldots$ to denote elements of $\mathcal{A}_{\mathcal{B}}$. For example, if $\mathcal{B}$ is the Bratteli diagram from Figure 2, then

$$\mathcal{A}_{\mathcal{B}} = \left\{\boxed{\begin{matrix}\tau & 1\\ a & \end{matrix}}, \boxed{\begin{matrix}\tau & 2\\ a & \end{matrix}}, \boxed{\begin{matrix}\tau & 3\\ a & \end{matrix}}, \boxed{\begin{matrix}\tau & 1\\ b & \end{matrix}}, \boxed{\begin{matrix}\tau & 2\\ b & \end{matrix}}, \boxed{\begin{matrix}\tau & \boxtimes\\ a & \end{matrix}}, \boxed{\begin{matrix}\tau & \boxtimes\\ b & \end{matrix}}, \boxed{\boxtimes}\right\},$$

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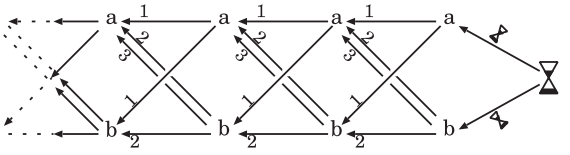


FIGURE 2. The Bratteli diagram corresponding to the substitution $\boldsymbol{\tau}(a) = abb$, $\boldsymbol{\tau}(b) = ab$.

where the dropped index in $\boxtimes$ indicates that there is a unique edge from each vertex in $\mathcal{V}_1$ to $\boxtimes$. A (possibly infinite) concatenation of letters from $\mathcal{A}_{\mathcal{B}}$ is a *fragment* of a space-time diagram if it appears in some $\mathbf{y} \in \mathcal{Y}$.

LEMMA 3. *If* $\begin{array}{|c|c|c|}\hline \mathbf{T}^m_{j+1} & \mathbf{T}^m_j & \mathbf{T}^m_{j-1} \\ \hline & \mathbf{T}^{m+1}_j & \mathbf{T}^{m+1}_{j-1} \\ \cline{2-3}\end{array}$ *is a fragment of* $\mathbf{y} \in \mathcal{Y}$, *then* $\begin{array}{|c|c|c|}\hline \mathbf{T}^m_{j+1} & \mathbf{T}^m_j & \mathbf{T}^m_{j-1} \\ \hline & & \mathbf{T}^{m+1}_{j-1} \\ \cline{3-3}\end{array}$ *determines* $\boxed{\mathbf{T}^{m+1}_j}$.

Proof. We prove the case where all tiles in $\begin{array}{|c|c|c|}\hline \mathbf{T}^m_{j+1} & \mathbf{T}^m_j & \mathbf{T}^m_{j-1} \\ \hline & \mathbf{T}^{m+1}_j & \mathbf{T}^{m+1}_{j-1} \\ \cline{2-3}\end{array}$ are from $\left\{ \boxed{\begin{matrix}\tau^n\ x \\ a\end{matrix}} \mid a \in \mathcal{A}, n \geq 1, x \in [1, l^n_a] \right\}$; other cases are similar. If $\mathbf{T}^m_{j-1} = \mathbf{T}^{m+1}_{j-1}$, then $\mathbf{T}^{m+1}_j = \mathbf{T}^m_j$, as is the case if $\mathbf{T}^m_{j-1} = \boxed{\begin{matrix}\tau^n\ x \\ a\end{matrix}}$ with $x < l^n_a$. If

$$\begin{array}{|c|c|}\hline \mathbf{T}^m_j & \mathbf{T}^m_{j-1} \\ \hline & \mathbf{T}^{m+1}_{j-1} \\ \cline{2-2}\end{array} \quad = \quad \begin{array}{|c|c|}\hline \tau^n\ x & \tau^{n-1}\ * \\ b & a \\ \hline & \tau^{n-1}\ 1 \\ & ? \\ \cline{2-2}\end{array}$$

with $x < l^n_b$, then $\mathbf{T}^{m+1}_j = \boxed{\begin{matrix}\tau^n\ x+1 \\ a\end{matrix}}$. If

$$\begin{array}{|c|c|c|}\hline \mathbf{T}^m_{j+1} & \mathbf{T}^m_j & \mathbf{T}^m_{j-1} \\ \hline & & \mathbf{T}^{m+1}_{j-1} \\ \cline{3-3}\end{array} \quad = \quad \begin{array}{|c|c|c|}\hline \tau^{n+1}\ y & \tau^n\ \star & \tau^{n-1}\ * \\ c & b & a \\ \hline & & \tau^{n-1}\ 1 \\ & & ? \\ \cline{3-3}\end{array}$$

with $y < l^{n+1}_c$, then $\mathbf{T}^{m+1}_j = \boxed{\begin{matrix}\tau^n\ 1 \\ \tau^{n+1}_{y+1}(c)\end{matrix}}$. Finally, if

$$\begin{array}{|c|c|c|}\hline \mathbf{T}^m_{j+1} & \mathbf{T}^m_j & \mathbf{T}^m_{j-1} \\ \hline & & \mathbf{T}^{m+1}_{j-1} \\ \cline{3-3}\end{array} \quad = \quad \begin{array}{|c|c|c|}\hline \tau^{n+1}\ \ast & \tau^n\ \star & \tau^{n-1}\ * \\ c & b & a \\ \hline & & \tau^{n-1}\ 1 \\ & & ? \\ \cline{3-3}\end{array},$$

then since all minimal edges in $\mathcal{E}_n$ have source $\underline{a}$, then we must have $\mathbf{T}^{m+1}_j = \boxed{\begin{matrix}\tau^n\ 1 \\ \underline{a}\end{matrix}}$. □

Let W_4 be the set of all fragments $\begin{array}{|c|c|c|}\hline \mathbf{T}^m_{j+1} & \mathbf{T}^m_j & \mathbf{T}^m_{j-1} \\ \hline & & \mathbf{T}^{m+1}_{j-1} \\ \cline{3-3}\end{array}$ appearing in $\mathcal{Y}$, where $m, j, \in \mathbb{Z}$. Let $f : W_4 \to : \mathcal{A}_{\mathcal{B}}$ be the function such that if $\begin{array}{|c|c|c|}\hline \mathbf{T}^m_{j+1} & \mathbf{T}^m_j & \mathbf{T}^m_{j-1} \\ \hline & \mathbf{T}^{m+1}_j & \mathbf{T}^{m+1}_{j-1} \\ \cline{2-3}\end{array}$ is the unique extension of

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$\boxed{T^m_{j+1}}\boxed{T^m_j}\boxed{T^m_{j-1}}$ $\boxed{T^{m+1}_{j-1}}$, then

$$f\left(\boxed{T^m_{j+1}}\boxed{T^m_j}\boxed{T^m_{j-1}}\ \boxed{T^{m+1}_{j-1}}\right) = \boxed{T^{m+1}_j}.$$

This also means that any (northwest diagonal) ray

$$\mathbf{R} := \left\{ \left(\boxed{T^{-k}_{2k+3}}\boxed{T^{-k}_{2k+2}}\boxed{T^{-k}_{2k+1}} \right) \;\middle|\; k \in \mathbb{N} \right\}$$

which is a fragment in $\mathcal{Y}$ determines the horizontal $\mathbf{T}_0 := \{T^0_k \mid k \in \mathbb{N}^+\}$. Let $\mathcal{R}$ be the collection of such rays $\mathbf{R}$, and let $F : \mathcal{R} \to \mathcal{A}_{\mathcal{B}}^{\mathbb{N}}$ be the function which maps these rays in $\mathcal{R}$ to their horizontal image $\mathbf{T}_0$. Call any 3-tuple $\boxed{T^{-k}_{2k+3}}\boxed{T^{-k}_{2k+2}}\boxed{T^{-k}_{2k+1}}$ in a ray a *step*. Note that any ray in $\mathcal{R}$ can be extended in a unique way to a 'two-sided' ray fragment from $\mathcal{Y}$: simply attach infinitely many steps of the form $\boxed{\ }\boxed{\ }\boxed{\ }$.

Let $\mathcal{A}_\Phi := \mathcal{A}_{\mathcal{B}}^3$ and $W_3 \subset \mathcal{A}_\Phi$ be the set of all horizontal 3-tuple fragments from $\mathcal{Y}$. We will use $S_0, S_1 \ldots$ to denote letters from $\mathcal{A}_\Phi$. Define a left radius one, right radius one CA $\Phi : W_3^{\mathbb{Z}} \to W_3^{\mathbb{Z}}$ with local rule $\phi : W_3^3 \to W_3$ as

$$\Phi\left(\boxed{a}\boxed{b}\boxed{c}, \boxed{d}\boxed{e}\boxed{f}, \boxed{g}\boxed{h}\boxed{i}\right) = \begin{cases} \boxed{z}\boxed{y}\boxed{g} & \text{if } \begin{array}{ccc} a & b & c \\ & d & e & f \\ & & g & h & i \end{array} \text{ is admissible} \\ \left(\begin{array}{l}\text{where } y{:=}f(d,e,f,g) \\ \text{and } z{:=}f(x,d,e,y), \\ \text{given } x{:=}f(a,b,c,d)\end{array}\right) & \\ \text{arbitrary} & \text{otherwise.} \end{cases}$$

Let the maximal element of $X_{\mathcal{B}}$ be $(\{m_n\}_{n\geq 1}, \{\star_n\}_{n\geq 1})$, and let $\Omega := \overline{\{\Phi^n((\mathbf{x}_{\text{init}})\}_{n\geq 0}}$, where

$$\mathbf{x}_{\text{init}} := \quad \cdots, \boxed{\begin{matrix}\tau^6 *_6 \\ m_6\end{matrix}}\boxed{\begin{matrix}\tau^5 *_5 \\ m_5\end{matrix}}\boxed{\begin{matrix}\tau^4 *_4 \\ m_4\end{matrix}}, \boxed{\begin{matrix}\tau^4 *_4 \\ m_4\end{matrix}}\boxed{\begin{matrix}\tau^3 *_3 \\ m_3\end{matrix}}\boxed{\begin{matrix}\tau^2 *_2 \\ m_2\end{matrix}}, \boxed{\begin{matrix}\tau^2\ 0 \\ \underline{a}\end{matrix}}\boxed{\begin{matrix}\tau^1\ \overline{\boxtimes}_0 \\ \underline{a}\end{matrix}}\boxed{\ }. \boxed{\ }\boxed{\ }\boxed{\ }^{\infty}.$$

LEMMA 4. (Ω, Φ) *is topologically conjugate to* $(X_{\mathcal{B}}, V_{\mathcal{B}})$.

Proof. Let $C_1 : \Omega \to \mathcal{R}$ be the map which takes a sequence $\{S_k\}_{k\in\mathbb{Z}}$ to the ray with steps S_k. Note that if $C_1(\mathbf{x})$ is a fragment from an element $\mathbf{y}$ in $\mathcal{Y}$, then $C_1(\Phi(\mathbf{x}))$ is also a fragment from $\mathcal{Y}$, by Lemma 3 and also by the definition of Φ. Note also (see Figure 3) that $C_1(\mathbf{x}_{\text{init}})$ is a fragment from the space-time diagram of the minimal element

$$\mathbf{a}_{\min} := \quad \cdots \underline{a} \overset{1}{\leftarrow} \underline{a} \overset{1}{\leftarrow} \cdots \underline{a} \overset{1}{\leftarrow} \underline{a} \overset{\overline{\boxtimes}_1}{\leftarrow} \cdot \boxtimes\boxtimes\boxtimes \cdots.$$

Now define $C := F \circ C_1$. By the above remarks, C maps Ω into $X_{\mathcal{B}}$. Note that $C(\mathbf{x}_{\text{init}}) = \mathbf{a}_{\min}$, and that by induction

$$C(\Phi^n(\mathbf{x}_{\text{init}})) = V_{\mathcal{B}}^n(\mathbf{a}_{\min}),$$

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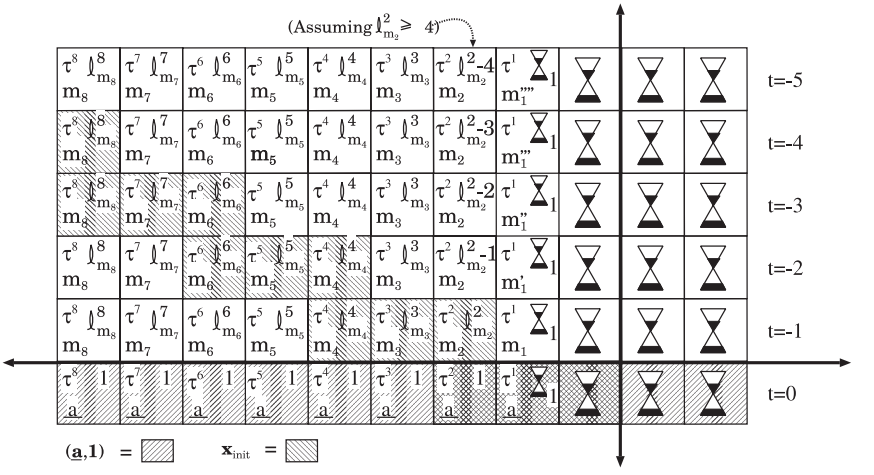


FIGURE 3. The space-time diagram of $\mathbf{a}_{\min}$ and its five predecessors in the adic system. This $\mathbb{Z}^2$-indexed configuration is an admissible element of the subshift of finite type defined using the 2×2 tiles in Figure 1. Finally, the tinted diagonal ray (top left to bottom right) is $\mathbf{x}_{\text{init}}$. If read 'diagonally' from top *right* to bottom *left*, this configuration is a the space-time diagram of $\mathbf{x}_{\text{init}}$ under the CA Φ.

for all $n \in \mathbb{N}$. It can be seen that $C : \{\Phi^n(\mathbf{x}_{\text{init}})\}_{n\geq 0} \to \{V^n_{\mathcal{B}}(\mathbf{a}_{\min})\}_{n\geq 0}$ is a uniformly continuous bijection with a uniformly continuous inverse. Thus C can be extended to a conjugacy between (Ω, Φ) and $(X_{\mathcal{B}}, V_{\mathcal{B}})$. □

Theorem 1 follows directly from Lemma 4.

4. *Examples*

4.1. *Substitution subshifts and odometers.* For any $n \in \mathbb{N}$, let $\mathbb{Z}_n$ be the cyclic group of order n. If $\mathbf{n} := (n_i)_{i=0}^{\infty} \in \mathbb{N}_+^{\mathbb{N}}$ is a sequence of positive integers, then define $\mathcal{Z}_{\mathbf{n}} := \prod_{i=0}^{\infty} \mathbb{Z}_{n_i}$, and let $+$ be the operation of 'addition with carry' on $\mathcal{Z}_{\mathbf{n}}$ (thus, $\mathcal{Z}_{\mathbf{n}}$ is *not* just the direct product of the cyclic groups $\{\mathbb{Z}_{n_i}\}_{i=0}^{\infty}$. Let $\mathbf{1} := (\ldots, 0, 0, 0, 1) \in \mathcal{Z}_{\mathbf{n}}$. The $\mathbf{n}$-*adic odometer* is the homeomorphism $\psi_{\mathbf{n}} : \mathcal{Z}_{\mathbf{n}} \longrightarrow \mathcal{Z}_{\mathbf{n}}$ defined by $\psi_{\mathbf{n}}(\mathbf{x}) = \mathbf{x} + \mathbf{1}$. If for all i, $n_i = N$ for some $N \in \mathbb{N}$, then $\psi_{\mathbf{n}}$ is called the N-*ary odometer* (because then elements of $\mathcal{Z}_{\mathbf{n}}$ behave like the 'N-ary representations' of natural numbers, except that they may have infinitely many non-zero 'digits'). In particular, if $N = 2$, we get the *binary odometer*; if $N = 3$, we get the *ternary odometer*. If the sequence $(n_i)_{i=0}^{\infty}$ is bounded, and if each number which appears in the sequence appears infinitely many times, then $(\mathcal{Z}_{\mathbf{n}}, \psi_{\mathbf{n}})$ is isomorphic to an N-ary odometer, where M is the product of all the numbers which appear in $(n_i)_{i=0}^{\infty}$. For details, and an exposition of odometers, see [**Dow05**].

For any sequence $\mathbf{n} \in \mathbb{N}_+^{\mathbb{N}}$, the dynamical system $(\mathcal{Z}_{\mathbf{n}}, \psi_{\mathbf{n}})$ has a natural adic representation using the ordered Bratteli diagram where $|\mathcal{V}_i| = 1$ and $\mathcal{E}_i = \mathbb{Z}_{n_i}$ (with the obvious ordering, and with 0, not 1, as the minimal element), for every $i \in \mathbb{N}$. For example, the adic representation of the binary odometer appears at the top of Figure 4.

A Bratteli diagram $\mathcal{B}$ is *stationary* if $\boldsymbol{\tau}^n := \boldsymbol{\tau}$ for all $n \in \mathbb{N}$, and there is exactly one edge from each vertex in $\mathcal{V}_1$ to $\boxtimes$. Suppose that $\mathcal{B}$ is simple and stationary, and that every minimal edge in $\mathcal{E}_n$ has the same source in $\mathcal{E}_{n-1}$. The resulting adic system

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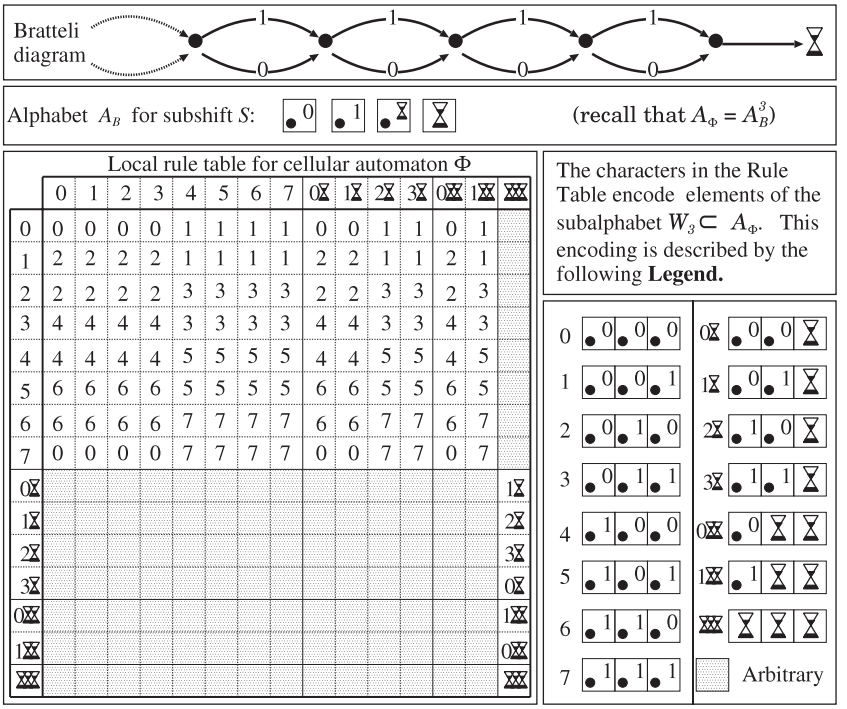


FIGURE 4. The rule table for a CA embedding the binary odometer. Grey cells in the rule table are arbitrary; these values are never encountered when ϕ acts on an 'odometer' configuration, so it doesn't matter what we put here. Also note that we have only defined ϕ on elements of the sub-alphabet W_3; the definition of ϕ on the rest of $\mathcal{A}_\Phi$ is also arbitrary.

$(X_\mathcal{B}, V_\mathcal{B})$ is also called *stationary*. Clearly, any stationary adic system is linearly recurrent. The family of stationary adic systems is (up to conjugacy) the union of the family of *aperiodic primitive substitution systems* and the family of N-ary odometers (see [**For97**, Theorem 17] or [**DHS99**, Proposition 20]). For example, the adic system generated by the Bratteli diagram in Figure 2 is conjugate to the substitution subshift generated by the substitution $\boldsymbol{\tau}(a) = aab$, $\boldsymbol{\tau}(b) = ab$. (For an introduction to substitution systems, see [**Fog02**] or [**DHS99**].) Thus, Theorem 1 yields the following result.

COROLLARY 5.

(a) *Any aperiodic primitive substitution system can be embedded in a one-dimensional cellular automaton.*

(b) *Any stationary odometer can be embedded in a one-dimensional cellular automaton.*

(Corollary 5(b) was earlier proved in [**CPY07**], but the CA in [**CPY07**] were *left-permutative*, whereas the CA produced by Theorem 1 are not.) Examining the construction in Theorem 1, we observe that the CA (X, Φ) which embeds a stationary odometer has left radius 0. Thus, if $\mathcal{A}_\Phi = \{a_1, a_2, \ldots, a_K\}$, then it is possible to describe the local rule ϕ using a $K \times K$ table, where entry (j, k) corresponds to $\phi(a_j, a_k)$. Figures 4 and 5 show the rule tables for the CA which embed the binary and ternary odometers.

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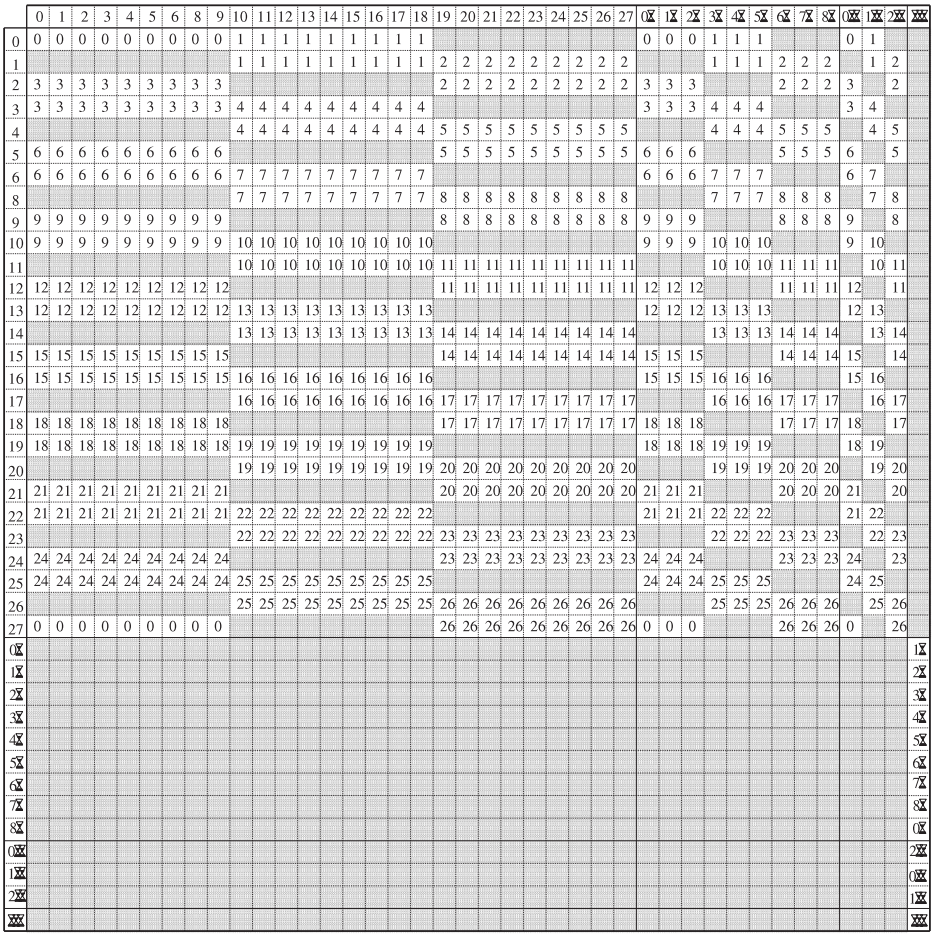


FIGURE 5. The rule table for a CA embedding the ternary odometer. As in Figure 4, the shaded areas can be filled in arbitrarily, and we have only defined the rule on the sub-alphabet $W_3 \subset \mathcal{A}_\Phi$.

4.2. *Toeplitz subshifts.* If $\mathcal{A}$ is a finite alphabet, and $X = \mathcal{A}^{\mathbb{Z}}$, a *Toeplitz sequence* is a σ-aperiodic element $\mathbf{x} \in X$ such that for each $m \in \mathbb{Z}$, there is some p such that $x_{m+pn} = x_m$ for each $n \in \mathbb{Z}$. The *Toeplitz subshift* associated with $\mathbf{x}$ (also sometimes called the *Toeplitz flow*) is the subshift $(\mathcal{O}_\sigma(\mathbf{a}), \sigma)$. See [**Dow05**].

Given a Bratteli diagram $\mathcal{B}$ whose vertices are labelled from $\mathcal{A}$, we say that $\mathcal{B}$ has the *equal path number property* if for each $n \geq 1$, the number of paths from $\boxtimes$ to any $a \in \mathcal{V}_n$ is constant. A dynamical system (Y, T) with Y a metric space is *expansive* if there exists some $\epsilon > 0$ such that for each $\mathbf{x} \neq \mathbf{y}$, there is some integer n such that $d(T^n(\mathbf{x}), T^n(\mathbf{y})) \geq \epsilon$. It is shown in [**GJ00**, Theorem 8] that the family of expansive adic systems associated with properly ordered Bratteli diagrams having the equal path number property is (up to conjugacy) the family of Toeplitz flows.

Inspection of [**GJ00**, proof of Theorem 8] reveals that condition (LR2) is satisfied. To find which Toeplitz systems satisfy condition (LR1), we look at how one generates Toeplitz sequences, which is described in [**Dow05**]. Fix an alphabet $\mathcal{A}$. Pick an integer $s_1 \geq 2$, and

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pick a subset $F_1 \subset [0, s_1)$ and an $\mathcal{A}$-valued labelling of F_1, say $A_1 \in \mathcal{A}^{F_1}$. For each $n \in \mathbb{Z}$, let $\mathbf{x}_{ns_1+F_1} = A_1$. Let $\mathrm{Per}_{s_1}(\mathbf{x}) := F_1 + s_1\mathbb{Z}$. Given s_k, choose s_{k+1} such that s_{k+1} is a proper multiple of s_k. Choose $F_{k+1} \subset [0, s_{k+1} - 1)\backslash \mathrm{Per}_{s_k}(\mathbf{x})$, and choose $A_{k+1} \in \mathcal{A}^{F_{k+1}}$. Let $x_{ns_{k+1}+F_{K+1}} = A_{k+1}$ for all integers n. For any integer n, the sets $[ns_k, (n+1)s_k)$ are *k-intervals*, and the coordinates in a k-interval not filled in during the first k steps in the construction are called *k holes*. The choices above must be made so that eventually all of $\mathbf{x}$ is filled in. Also, after completing this procedure, we redo the construction, at each stage k redefining F_k so that all essentially s_k-periodic parts of $\mathbf{x}$ are filled in.

COROLLARY 6. *Let $(\mathcal{O}_\sigma(\mathbf{a}), \sigma)$ be a Toeplitz subshift, and let $\{s_k\}_{k=1}^\infty$ be the sequence defined above. Suppose there is some $K > 0$ such that for each $i \in \mathbb{N}$, $s_{i+1}/s_i \leq K$. Then $(\mathcal{O}_\sigma(\mathbf{a}), \sigma)$ can be embedded in a one-dimensional cellular automaton.*

Proof. If $(X_\mathcal{B}, V_\mathcal{B})$ is the adic representation of $(\mathcal{O}_\sigma(\mathbf{a}), \sigma)$ from [**GJ00**, Theorem 8], then $(X_\mathcal{B}, V_\mathcal{B})$ satisfies (LR2) by construction. We must show that it also satisfies (LR1). Vertices in $\mathcal{V}_i$ correspond to the words $\{x_{ns_i+1}, \ldots x_{(n+1)s_i-1} \mid n \in \mathbb{Z}\}$; there are finitely many such words. If $s_{i+1}/s_i \leq K$, this means that each word corresponding to a vertex in $\mathcal{V}_{i+1}$ is a concatenation of at most K words represented by vertices in $\mathcal{V}_i$. Thus, there at most K edges with a specified vertex as range, so (LR2) is satisfied. Now apply Theorem 1. □

4.3. *Sturmian sequences.* In [**Dur03**, Proposition 5.1], it is shown that a *Sturmian subshift* (see [**Fog02**]) is isomorphic to a linearly recurrent adic system if and only if the coefficients of the continued fraction expansion of α are bounded. It follows that this class of Sturmian subshifts can also be embedded in the CA described in Theorem 1.

5. *Odometers in the Glider–Wall cellular automaton*

Corollary 5(b) only covers stationary odometers. We now give a different construction which can embed any odometer.

The *Glider–Wall* cellular automaton Φ (see [**Mil88**]) is defined on the alphabet $\mathcal{A} = \{\emptyset, L, R, W\}$ ('empty', 'left gliding', 'right gliding' and 'wall', respectively). We give here only a partial definition of its local rule $\phi : \mathcal{A}^3 \to \mathcal{A}$. Using '$*$' to represent any letter: $\phi(*, W, *) = W$, $\phi(W, L, \emptyset) = R$, $\phi(\emptyset, R, W) = L$, $\phi(*, \emptyset, L) = L$, $\phi(R, \emptyset, *) = R$, and $\phi(*, R, \emptyset) = \phi(\emptyset, L, *) = \emptyset$. (Note that these equations do not define the rule on all of $\mathcal{A}^3$. However, this much is sufficient for our purposes.) A physical interpretation of Φ is that Ls glide to the left through empty space, until they hit a wall, when they change into an R and glide to the right through empty space; and vice versa for particles R.

THEOREM 7. *Let $\mathbf{n} = (n_i)_{i=1}^\infty$ be a sequence of positive integers. If n_1 is even, then the Glider–Wall CA embeds the* **n***-adic odometer.*

Proof. Choose $\mathbf{x} := \cdots U_k\, U_{k-1} \cdots U_1 \cdot W\, W\, W \cdots$ where the word $U_k := W L \emptyset\emptyset \cdots W$ has length $|W_k| = \frac{1}{2}\prod_{i=1}^k n_i$. Then columns of the Φ-space-time diagram of $\mathbf{x}$ under the U_k portion of $\mathbf{x}$ will have periodicity $\prod_{i=1}^k n_i$, and a conjugacy between (Ω, Φ) and the given odometer can be established. □

Question. Are there similar CA which embed other (non-linearly recurrent) adic systems?

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