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Consistent shallow-water equations on the rotating sphere with complete Coriolis force and topography

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Consistent shallow-water equations are derived on the rotating sphere with topography retaining the Coriolis force due to the horizontal component of the planetary angular velocity. Unlike the traditional approximation, this 'non-traditional' approximation captures the increase with height of the solid-body velocity due to planetary rotation. The conservation of energy, angular momentum and potential vorticity are ensured in the system. The caveats in extending the standard shallow-water wisdom to the case of the rotating sphere are exposed. Different derivations of the model are possible, being based, respectively, on (i) Hamilton's principle for primitive equations with a complete Coriolis force, under the hypothesis of columnar motion, (ii) straightforward vertical averaging of the 'non-traditional' primitive equations, and (iii) a time-dependent change of independent variables in the primitive equations written in the curl ('vector-invariant') form, with subsequent application of the columnar motion hypothesis. An intrinsic, coordinate-independent form of the non-traditional equations on the sphere is then given, and used to derive hyperbolicity criteria and Rankine–Hugoniot conditions for weak solutions. The relevance of the model for the Earth's atmosphere and oceans, as well as other planets, is discussed.

Key words: atmospheric flows, shallow water flows, topographic effects

1. Introduction

In the overwhelming majority of studies, large-scale motions in the Earth's atmosphere and ocean are investigated and simulated within the so-called traditional approximation in the primitive equations (Eckart 1960). It consists in neglecting the vertical component of the Coriolis force and the part of its horizontal component due to the vertical velocity. This approximation goes back to Laplace and is consensually adopted, yet intermittently questioned in the literature (see the recent review by Gerkema *et al.* (2008) and references therein). At the present time, one can witness a steadily growing interest in the role of 'non-traditional' effects upon the dynamics of

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the ocean and the atmosphere. In the ocean, they have attracted particular attention in the context of submesoscale flows and internal waves. For example, it was shown recently (Gerkema & Shrira 2005; Colin-de Verdiere 2012) that these effects may significantly influence the propagation and stability properties of internal waves. As the vertical component of the Earth's angular velocity vanishes at the equator, the equatorial region is a place *par excellence* to look for non-traditional effects. Indeed, several recent studies (Raymond 2000; Stewart & Dellar 2011; Hayashi & Itoh 2012) have shown the importance of the non-traditional terms for equatorial circulation, both in the ocean and in the atmosphere.

Most of the above-cited papers have used the reduction of the primitive equations on the rotating sphere to the tangent plane, the simplest case being the f-plane approximation, where latitudinal variations of the Coriolis parameter are neglected. Consistent simultaneous treatment of the non-traditional effects and of the latitudinal variation of the Coriolis parameter on the tangent plane (β-effect) is nontrivial (Grimshaw 1975). A systematic derivation of the non-traditional primitive equations on the beta-plane was provided recently by Dellar (2011). However, in the general circulation context the equations on the whole sphere are indispensable. We should emphasize that the errors due to the neglect of the non-traditional effects in the current general circulation models (GCMs), although small, could accumulate in the long-time simulations of the climate system. The role of these errors is at present not being assessed.

The problem of constructing approximate equations including non-traditional terms while preserving dynamical consistency, in the sense that angular momentum, energy and potential vorticity are conserved, is longstanding. In a seminal paper by White & Bromley (1995) a consistent set of three-dimensional, non-traditional, quasi-hydrostatic equations was proposed. To achieve consistency, it was not sufficient to reincorporate the non-traditional terms into the horizontal momentum and hydrostatic equations; it was necessary that the shallow-atmosphere approximation be abandoned, too. (We should recall that the shallow-atmosphere approximation in the standard hydrostatic primitive equations consists in replacing the radial coordinate by the Earth's radius in the equations of the horizontal motion.) A hierarchy of global atmosphere approximations, from deep non-hydrostatic to shallow hydrostatic, or quasi-hydrostatic, i.e. with the inclusion of vertical Coriolis acceleration, has been developed by White *et al.* (2005).

The standard rotating shallow-water (RSW) equations are intrinsically related to the hydrostatic primitive equations in the traditional approximation, either on the sphere or on the tangent plane. As is well known (see, for example, Zeitlin 2007 and references therein), they can be obtained by vertical averaging or by application of the columnar motion hypothesis from the primitive equations, and are dynamically consistent in the above-defined sense. The RSW equations and their multilayer extensions give an appreciably simple and intuitive prototype model (which, at the same time, retains all essential dynamical properties; see, for example, Vallis 2006) for a plethora of large-scale dynamical phenomena in the ocean and the atmosphere, on the one hand, and provide a testing ground for numerical schemes for GCMs, on the other hand. Indeed, by vertical discretization of the hydrostatic primitive equations, the effective RSW equations for horizontal motions arise in numerical implementations of the dynamical cores of GCMs (Bleck 2002; Lin 2004), which explains the universality of the RSW-based numerical tests for the latter (Williamson *et al.* 1992). (We should remember that the standard RSW equations may be generalized to include horizontal variations of the mean density/temperature of the fluid layer(s), Ripa 1993.) A natural

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question thus arises about the non-traditional extension of the RSW equations. While the answer in the f- and β-plane approximations was given, respectively, quite recently in the papers by Dellar & Salmon (2005) and Dellar (2011), the spherical version of the non-traditional RSW equations is still unknown, to our knowledge.

The importance of having consistent non-traditional RSW equations on the sphere is threefold. First, as explained above, via the vertical discretization, such equations should effectively appear in numerical implementations of the full non-traditional primitive equations on the sphere. Once established, they may then be used for testing numerical strategies for GCMs in the non-traditional approximation. On the other hand, as also mentioned above, the RSW equations allow simple and intuitive modelling of large-scale dynamical phenomena in the atmosphere and the ocean. The non-traditional RSW equations would allow us to study the influence of non-traditional corrections upon these dynamical phenomena by simple means. Finally, the RSW equations are often used for conceptual modelling of the circulation of the atmospheres of the planets, which are *a priori* not shallow (see, for example, Scott & Polvani 2007). Non-traditional RSW equations, incorporating the dependence of the planetary velocity on the altitude (see below), seem to be a more appropriate model in this context.

In what follows we will derive a consistent non-traditional rotating shallow water model on the sphere with topography. Our starting point will be three-dimensional quasi-hydrostatic primitive equations in the Boussinesq approximation. (With more technical effort, the derivation may be extended to the compressible hydrostatic primitive equations.) By three complementary methods, we simplify these equations for a fluid layer which is thin with respect to the planet's radius, and under the approximation of columnar motion, and arrive at a set of shallow-water equations which conserve mass, energy, axial angular momentum and potential vorticity. As compared to the traditional RSW equations, these equations have a number of additional, 'non-traditional' terms. The three derivations (§ 3 in the main text and appendix B) are based, respectively, on the use of Hamilton's (least action) principle under a hypothesis of columnar motion, on the straightforward averaging of the primitive equations written in the conservative form, and on a time-dependent change of independent variables, together with the columnar motion hypothesis, in the primitive equations written in the curl ('vector-invariant', as it is called in the numerical modelling literature) form. We should stress that due attention to the differential geometry should be paid in all calculations. To begin with, the notion of columnar motion should be redefined: the columns on the sphere moving with the same angular velocity, and not linear velocity, as on the plane – see below. Having established the non-traditional shallow-water equations on the rotating sphere, we rewrite them in the intrinsic, coordinate-independent form, convenient for studying their mathematical properties. We thus analyse the hyperbolicity of the derived system, and establish Rankine–Hugoniot jump conditions for weak solutions.

The paper is organized as follows. In § 2 we recall the Boussinesq equations in the quasi-hydrostatic approximation and their properties, including the related variational principle. In § 3 we present a detailed derivation of the non-standard RSW equations by reduction to the columnar motion in the variational (Hamilton's) principle for quasi-hydrostatic Boussinesq equations. We give the intrinsic, coordinate-independent form of the derived system, establish hyperbolicity criteria, and find Rankine–Hugoniot conditions for weak solutions in § 4. Section 5 contains a summary and discussion. The relevant concepts of the differential geometry are recalled in appendix A. Appendix B contains a sketch of two alternative derivations: by a direct

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averaging of the primitive equations written in the conservative form, and by a change of independent variables under the hypothesis of columnar motion in the primitive equations written in the curl form. Intrinsic, coordinate-independent operators and Stokes formulae for the sphere are presented in appendix C.

2. Quasi-hydrostatic Boussinesq equations with a complete Coriolis force in spherical coordinates

2.1. *Full Boussinesq equations in the rotating frame*

The three-dimensional equations of motion for a fluid under the influence of gravity in the Boussinesq approximation in a frame rotating with constant angular velocity $\boldsymbol{\Omega}$ read:

$$\left.\begin{aligned} D_t\boldsymbol{v} + 2\boldsymbol{\Omega}\times\boldsymbol{v} + \nabla\Pi - b\boldsymbol{g} = 0,\\ D_t b = 0,\\ \nabla\cdot\boldsymbol{v} = 0. \end{aligned}\right\} \tag{2.1}$$

Here $\boldsymbol{v}$ is the fluid velocity, $b = \rho/\rho_0$ is the normalized density, with a constant reference value ρ_0, Π is the geopotential with incorporated centrifugal terms, g is gravitational acceleration and $D_t = \partial_t + \boldsymbol{v}\cdot\nabla$ is the advective derivative. By applying the hydrodynamic identity:

$$\boldsymbol{v}\cdot\nabla\boldsymbol{v} = \nabla\frac{\boldsymbol{v}^2}{2} - \boldsymbol{v}\times(\nabla\times\boldsymbol{v}), \tag{2.2}$$

the velocity equations may be rewritten in the curl form:

$$\partial_t\boldsymbol{v} + (\nabla\times\boldsymbol{v})\times\boldsymbol{v} + 2\boldsymbol{\Omega}\times\boldsymbol{v} + \nabla\left(\Pi + \frac{\boldsymbol{v}^2}{2}\right) - b\boldsymbol{g} = 0, \tag{2.3}$$

which is more suitable for demonstration of the Lagrangian conservation of potential vorticity (see White *et al.* 2005):

$$D_t q = 0 \quad \text{with } q = \nabla b\cdot\boldsymbol{\Omega}, \tag{2.4}$$

where $\boldsymbol{\zeta} = \nabla\times\boldsymbol{v}$ is the relative vorticity and $\boldsymbol{\omega} = \boldsymbol{\zeta} + 2\boldsymbol{\Omega}$ is the absolute vorticity.

An important quantity used to analyse the general circulation of the atmosphere and oceans is the axial angular momentum. The mass density of absolute angular momentum is:

$$\boldsymbol{l}_a = \boldsymbol{r}\times(\boldsymbol{v} + \boldsymbol{\Omega}\times\boldsymbol{r}), \tag{2.5}$$

where $\boldsymbol{r}$ is the position vector and $\boldsymbol{v} + \boldsymbol{\Omega}\times\boldsymbol{r} = \boldsymbol{v}^a$ is the absolute velocity. The axial absolute angular momentum density is obtained by projection onto the unit vector $\hat{\boldsymbol{z}}$ in the direction of $\boldsymbol{\Omega}$:

$$l_z = \hat{\boldsymbol{z}}\cdot\boldsymbol{l}_a, \tag{2.6}$$

and its evolution is given by (see White *et al.* 2005),

$$D_t l_z = \hat{\boldsymbol{z}}\cdot[\boldsymbol{r}\times(-\nabla\Pi + b\boldsymbol{g})]. \tag{2.7}$$

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2.2. *Boussinesq equations in spherical coordinates*

Equations (2.1) are written in the abstract vector form, independent of a concrete choice of coordinate system. As applied to a fluid on a sphere they can be expressed in curvilinear spherical coordinates. The equations for the three components of velocity are then rewritten as follows (as usual we are making an assumption of the planet's sphericity and are not considering oblate-spheroidal corrections (see Gates 2004 and White & Wood 2012):

$$D_t v_r - \frac{v_\lambda^2 + v_\phi^2}{r} - 2\Omega \cos\phi v_\lambda + bg = -\partial_r \Pi, \tag{2.8}$$

$$D_t v_\lambda + \frac{v_r v_\lambda - v_\phi v_\lambda \tan\phi}{r} + 2\Omega(-\sin\phi v_\phi + \cos\phi v_r) = -\frac{1}{r\cos\phi}\partial_\lambda \Pi, \tag{2.9}$$

$$D_t v_\phi + \frac{v_r v_\phi + v_\lambda^2 \tan\phi}{r} + 2\Omega \sin\phi v_\lambda = -\frac{1}{r}\partial_\phi \Pi, \tag{2.10}$$

and the buoyancy advection equation is:

$$D_t b = 0, \tag{2.11}$$

where the advection operator is expressed as

$$D_t = \partial_t + v_r\partial_r + \frac{v_\phi}{r}\partial_\phi + \frac{v_\lambda}{r\cos\phi}\partial_\lambda. \tag{2.12}$$

Finally, the incompressibility equation is:

$$\frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\cos\phi}\left(\frac{\partial(\cos\phi v_\phi)}{\partial\phi} + \frac{\partial(v_\lambda)}{\partial\lambda}\right) = 0. \tag{2.13}$$

Here (λ, ϕ, r) are the standard longitude–latitude spherical coordinates: $0 \leqslant \lambda < 2\pi$, $-\pi/2 \leqslant \phi \leqslant \pi/2$, $0 \leqslant r < \infty$ (see figure 1). Note the characteristic derivative-less curvature terms proportional to r^{-1} on the left-hand side of (2.8)–(2.13).

This system of equations is considered in a domain on the surface of a sphere, so the boundary conditions in λ and ϕ are periodic, if applied to the atmosphere, while for the ocean the basin boundary should be imposed. As we are considering the non-dissipative equations, the boundary condition at the bottom is that the latter is a material surface. We will be considering arbitrary topography in what follows. Thus, the radial position of a point at the bottom surface is given by

$$r = r_0 + B(\lambda, \phi), \tag{2.14}$$

where r_0 is the planet's radius and $B(\lambda, \phi)$ is the bottom topography. In what follows, we will be considering a fluid layer between the bottom and a free surface.

It should be immediately stressed that the velocity components v_λ, v_ϕ, v_r in these formulae are the physical ones, most often used in the applications. They are related to the rate of change of spherical coordinates as follows:

$$v_\lambda = r\cos\phi\dot{\lambda}, \quad v_\phi = r\dot{\phi}, \quad v_r = \dot{r}, \tag{2.15}$$

where we use the dot notation for the Lagrangian time derivative. In what follows we will use both the Eulerian and the Lagrangian description of the flow. In the

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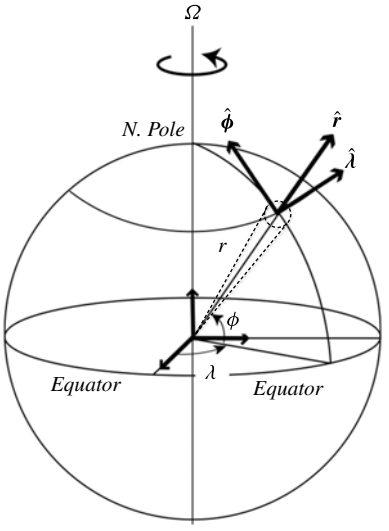


FIGURE 1. Rotating sphere: spherical coordinates.

Eulerian description the coordinates are independent variables and will be referred to as Eulerian coordinates. In the Lagrangian description, coordinates are time-dependent functions of Lagrangian labels. Labels are denoted below by $\boldsymbol{\chi}$ and $\boldsymbol{a}$. In this context, coordinates will be referred to as Lagrangian coordinates and noted with capitals, i.e. $\boldsymbol{X}$ and (Λ, Φ, R). Velocity is obtained from their time derivative, which is Lagrangian by design. In this sense the dot derivative is synonymous with D_t.

As is well known, (see, for example, Korn & Korn 2000) the physical components of any vector correspond to its projections on the unit vectors $\hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{r}}$ of the local orthonormal basis in spherical coordinates (figure 1), which are tangent to coordinate lines and oriented in the direction of increase of the corresponding coordinates:

$$\boldsymbol{v} = \hat{\boldsymbol{\lambda}} v_\lambda + \hat{\boldsymbol{\phi}} v_\phi + \hat{\boldsymbol{r}} v_r. \tag{2.16}$$

In spite of the physical transparency, this basis is less convenient for differential operations with vector fields in curvilinear coordinates than covariant and contravariant bases (see, for example, Korn & Korn 2000) and appendix A. The contravariant and the covariant components of the velocity vector in spherical coordinates are, respectively:

$$v^1 = \dot{\lambda}, \quad v^2 = \dot{\phi}, \quad v^3 = \dot{r} \quad \text{and} \quad v_1 = r^2 \cos^2\phi \dot{\lambda}, \quad v_2 = r^2 \dot{\phi}, \quad v_3 = \dot{r}. \tag{2.17}$$

The equations (2.8)–(2.13) can be straightforwardly rewritten in terms of covariant derivatives introduced in appendix A.

2.3. *Standard approximations*

As was discussed in the Introduction, the full Boussinesq equations of the previous subsection are usually further processed in order to be applied to the description of large-scale atmospheric and oceanic motions. For example, vertical accelerations are normally small, and are hardly relevant at large scales. The standard 'traditional'

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hydrostatic approximation neglects all terms except gravity on the left-hand side of (2.8), and also neglects the part of the horizontal Coriolis acceleration due to v_r in (2.9). Simultaneously, a 'brute force' approximation of the shallow layer is made by replacing r by r_0 in the curvature terms. Potential vorticity in the form

$$q_H = \nabla b \cdot (\boldsymbol{\zeta}_H + 2\boldsymbol{\Omega} \cdot \hat{\boldsymbol{r}}) \tag{2.18}$$

is a Lagrangian invariant of the traditional equations. The $\boldsymbol{\zeta}_H$ in this formula is obtained from the full relative vorticity $\boldsymbol{\zeta}$ by omitting all terms containing the radial component of the relative velocity $\boldsymbol{v}$ and replacing the radial coordinate by the planet's radius: $r \to r_0$ (White *et al.* 2005). The quasi-hydrostatic approximation consists in neglecting the vertical acceleration $D_t v_r$ in (2.8), while maintaining the curvature and the non-traditional Coriolis terms. The potential vorticity in this approximation is (White *et al.* 2005)

$$q_{QH} = \nabla b \cdot (\boldsymbol{\zeta}_{QH} + 2\boldsymbol{\omega}), \tag{2.19}$$

where $\boldsymbol{\zeta}_{QH}$ is obtained from the full relative vorticity $\boldsymbol{\zeta}$ by omitting all terms containing the radial component of the relative velocity $\boldsymbol{v}$. As was shown by Phillips (1966), the 'shallowness' cannot be imposed as above (i.e. by replacing radial position by the planet's radius) in this case without destroying conservation of the axial angular momentum. Below we will be constructing a non-traditional shallow-water approximation, and hence will be retaining non-traditional Coriolis terms. We will therefore proceed with this single simplification (i.e. the neglect of vertical accelerations) for the moment.

2.4. *Variational principle for the Boussinesq equations*

The most straightforward way to establish the variational (Hamilton's, or least action) principle in fluid mechanics is through the Lagrangian interpretation of the equations of motion (see, for example, Salmon 1988). From the Lagrangian viewpoint, the Boussinesq equations are Newton's equations for the fluid parcels labelled by their initial coordinates $\boldsymbol{\chi}$: $\boldsymbol{X}(\boldsymbol{\chi}, t = 0) = \boldsymbol{\chi}$. An additional change of variables $\boldsymbol{\chi} \mapsto \boldsymbol{a}$ to some other set of Lagrangian labels $\boldsymbol{a}$ may be made. The fluid velocity $\boldsymbol{v}$ corresponds to $\dot{\boldsymbol{X}}$. (We will use the shorthand over-dot notation for D_t when it does not lead to confusion.) The non-divergence of velocity means that the elementary volumes are conserved by this mapping, and the advection of b means that the density remains the same for each fluid parcel, although densities of different parcels are different. The former condition in terms of Lagrangian coordinates means that the Jacobian of the mapping $\boldsymbol{\chi} \mapsto \boldsymbol{X}$ is unity, and represents a dynamical constraint, while the latter bears no dynamical content: an initial distribution of buoyancy is transported by fluid parcels to their new locations. The incompressibility constraint is introduced with the help of a Lagrangian multiplier, the pressure, which becomes the geopotential in the present context.

The well-known rules of construction of the action principle for systems of discrete particles (Landau & Lifshitz 1976) say that the Lagrangian is obtained by subtraction of the total kinetic and potential energies. The changes to be made to obtain the variational principle in a rotating system are also known: a vector-potential for the angular velocity $\boldsymbol{\Omega}$ should be introduced (see Landau & Lifshitz 1976 and, for example, Holm & Zeitlin 1998 in the hydrodynamical context), by 'augmenting' the Lagrangian $L\left(\boldsymbol{X}, \dot{\boldsymbol{X}}\right)$:

$$L \to L + \boldsymbol{\mathcal{R}} \cdot \dot{\boldsymbol{X}}, \quad \boldsymbol{\mathcal{R}} : \nabla_X \times \boldsymbol{\mathcal{R}} = 2\boldsymbol{\Omega}, \tag{2.20}$$

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where it is sufficient to choose $\boldsymbol{\mathcal{R}} = \boldsymbol{\Omega} \times \boldsymbol{X}$. Once the action $\mathscr{S}$ is known in terms of the Lagrangian L

$$\mathscr{S} = \int \mathrm{d}t L, \tag{2.21}$$

the equations of motion follow from the principle of least action $\delta\mathscr{S} = 0$:

$$D_t \left[\frac{\delta L}{\delta \dot{\boldsymbol{X}}} \right] - \frac{\delta L}{\delta \boldsymbol{X}} = 0. \tag{2.22}$$

These rules apply as well to the continuous system of fluid parcels – the Lagrangian expressed via Lagrangian density – summation becoming integration over Lagrangian labels: $L = \int \mathrm{d}\boldsymbol{\chi}\mathscr{L}$, and the incompressibility constraint being introduced via a Lagrangian multiplier, the geopotential. As already mentioned, we use spherical Lagrangian coordinates, which allows a simple passage from the Euler to the Lagrangian description and vice versa. We also use spherical Lagrangian labels, which allows their interpretation as initial values of Lagrangian coordinates:

$$\boldsymbol{X}(\boldsymbol{\chi}, t) = (\Lambda(\lambda, \phi, r, t), \Phi(\lambda, \phi, r, t), R(\lambda, \phi, r, t)). \tag{2.23}$$

In this way we obtain the following action functional for the system (2.1):

$$\left.\begin{aligned} \mathscr{S} &= \int \mathrm{d}t \int_{-\pi/2}^{+\pi/2} \cos\phi \mathrm{d}\phi \int_0^{2\pi} \mathrm{d}\lambda \int_0^{\infty} r^2 \mathrm{d}r \mathscr{L}, \\ \mathscr{L} &= \frac{1}{2}(R^2\cos^2\Phi\dot{\Lambda}^2 + R^2\dot{\Phi}^2 + \dot{R}^2) + \Omega R^2\cos^2\Phi\dot{\Lambda} - bgR \\ &\quad + \Pi\left(\frac{R^2\cos\Phi}{r^2\cos\phi}\frac{\partial(\Lambda, \Phi, R)}{\partial(\lambda, \phi, r)} - 1\right). \end{aligned}\right\} \tag{2.24}$$

Here

$$\frac{R^2\cos\Phi}{r^2\cos\phi}\frac{\partial(\Lambda, \Phi, R)}{\partial(\lambda, \phi, r)} = \frac{\partial(R\Lambda\cos\Phi, R\Phi, R)}{\partial(r\lambda\cos\phi, r\phi, r)} \tag{2.25}$$

is the Jacobian of the mapping $(\lambda, \phi, r) \mapsto (\Lambda, \Phi, R)$, and we use the standard notation

$$\frac{\partial(\Lambda, \Phi, R)}{\partial(\lambda, \phi, r)} = \begin{vmatrix} \partial_\lambda\Lambda & \partial_\lambda\Phi & \partial_\lambda R \\ \partial_\phi\Lambda & \partial_\phi\Phi & \partial_\phi R \\ \partial_r\Lambda & \partial_r\Phi & \partial_r R \end{vmatrix}. \tag{2.26}$$

The system here is considered in the whole space. The effects of upper and lower boundaries will be included below in § 3.3. The equations of motion follow, after applying the standard rules of the calculus of variations and integration by parts, and give a Lagrangian version of (2.1), as one can easily check. We will immediately apply the quasi-hydrostatic approximation in (2.24), which will consist in omitting the contribution of the vertical velocity $\dot{R}^2$ in the kinetic energy:

$$\begin{aligned} \mathscr{L} \to \mathscr{L}_{QH} &= \frac{1}{2}(R^2\cos^2\Phi\dot{\Lambda}^2 + R^2\dot{\Phi}^2) + \Omega R^2\cos^2\Phi\dot{\Lambda} - bgR \\ &\quad + \Pi\left(\frac{R^2\cos\Phi}{r^2\cos\phi}\frac{\partial(\Lambda, \Phi, R)}{\partial(\lambda, \phi, r)} - 1\right). \end{aligned} \tag{2.27}$$

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This is the same Lagrangian as used in Dellar (2011), up to a change of independent variables to some set of Cartesian labels (this choice was justified by subsequent passage to the tangent plane, while we are working on the whole sphere and keep spherical Lagrangian labels for consistency). We thus obtain the following quasi-hydrostatic equations of motion:

$$D_t\left[R^2\cos^2\Phi(\dot{\Lambda}+\Omega)\right]+\partial_\Lambda\Pi=0, \tag{2.28}$$

$$D_t\left[R^2\dot{\Phi}\right]+R^2\cos\Phi\sin\Phi(\dot{\Lambda}^2+2\Omega\dot{\Lambda})+\partial_\Phi\Pi=0, \tag{2.29}$$

$$R(\cos^2\Phi\dot{\Lambda}^2+\dot{\Phi}^2)+2\Omega R\cos^2\Phi\dot{\Lambda}-bg-\partial_R\Pi=0, \tag{2.30}$$

$$\frac{R^2\cos\Phi}{r^2\cos\phi}\frac{\partial(\Lambda,\Phi,R)}{\partial(\lambda,\phi,r)}=1, \tag{2.31}$$

$$D_t b=0. \tag{2.32}$$

As is easy to check, upon the standard identifications of the Lagrangian coordinates with the Eulerian coordinates in the Eulerian description of the fluid motion, of the Lagrangian dot derivative with the advective derivative D_t, and with the help of (2.15), these equations reproduce the quasi-hydrostatic version of (2.8)–(2.13).

Canonical momentum densities are defined as usual (Landau & Lifshitz 1976):

$$p_\Lambda=\frac{\delta\mathscr{L}}{\delta\dot{\Lambda}}=R^2\cos^2\Phi\dot{\Lambda}+\Omega R^2\cos^2\Phi,\quad p_\Phi=\frac{\delta\mathscr{L}}{\delta\dot{\Phi}}=R^2\dot{\Phi}. \tag{2.33}$$

It should be emphasized that they coincide with the corresponding covariant components of the absolute velocity. From the Lagrangian density $\mathscr{L}$, the Hamiltonian density

$$\mathscr{H}=p_\Lambda\dot{\Lambda}+p_\Phi\dot{\Phi}-\mathscr{L} \tag{2.34}$$

is constructed, giving the energy density of the system. The conservation of energy follows from the invariance of the action with respect to the time shifts. Potential vorticity conservation follows from the invariance of the action with respect to relabelling of Lagrangian particles (see Salmon 1998) and axial angular momentum conservation follows from the invariance of the action with respect to rotations.

3. Derivation of non-traditional rotating shallow water equations

3.1. *Scales and parameters*

We now introduce characteristic scales and define dimensionless parameters. The first parameter is the aspect ratio h_0/L, where h_0 is a typical layer thickness and L a typical horizontal scale for the flow. Incompressibility implies that $W/U=O(h_0/L)$, where U and W are typical horizontal and vertical velocities, respectively, so that neglecting vertical kinetic energy in the Lagrangian (2.24) is allowed if $h_0/L\ll 1$. Once we assume that this hypothesis holds and use the quasi-hydrostatic Lagrangian (2.27), the aspect ratio disappears from our analysis.

In planar geometry shallowness can only be measured with respect to the flow scale L, but in spherical geometry shallowness with respect to the planetary scale r_0 is defined by the parameter ϵ:

$$\epsilon=\frac{h_0}{r_0}\ll 1. \tag{3.1}$$

We will develop asymptotic expansions in ϵ in what follows.

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Another global small parameter, the ratio of the centrifugal to gravitational accelerations on a sphere rotating with angular velocity Ω, is present in the system:

$$\gamma = \frac{r_0 \Omega^2}{g}. \tag{3.2}$$

Apart from the global parameters ϵ and γ, other parameters arise, characterizing a given type of the fluid motion. Thus, a planetary Rossby number for the motions with a typical velocity U on the rotating sphere is:

$$\mu = \frac{U}{\Omega r_0}. \tag{3.3}$$

If $\mu \ll 1$, this means that the typical fluid velocity is small compared to the velocity due to planetary rotation, which is generally true for large-scale atmospheric and oceanic motions, such as on the Earth. We should stress that this (mild) condition does not necessarily mean that the Rossby number of a given, let us say, synoptic system of typical scale L is small, as generally $L \ll r_0$. Accordingly, the parameter ϵ introduced in (3.1) may be thought of as a product of two parameters: $\epsilon = (h_0/L)(L/r_0)$, although we will not introduce these latter. The textbook derivations of the shallow-water equations are based on the hypothesis of smallness of h_0/L, which is equivalent to the hypothesis of smallness of ϵ for planetary-scale motions. Below we prefer to use the columnar motion hypothesis, which will allow us to treat even the situations with $\epsilon = O(1)$.

Although the scale of the overall thickness of the fluid layer h is given by h_0, it does not define a typical amplitude of the variations $h - h_0$ of h. A corresponding dimensionless parameter

$$\alpha = \frac{\|h - h_0\|}{h_0} \tag{3.4}$$

should be introduced in the asymptotic analysis of the shallow water system (Reznik, Zeitlin & BenJelloul 2001). As usual, the relationship between the small parameters should be fixed in order to render the asymptotic procedure unambiguous. We will work in what follows under the hypothesis

$$\frac{\epsilon \alpha}{\gamma \mu} = O(1), \tag{3.5}$$

although we do not introduce α explicitly below, postponing a discussion of its role to § 5.

One of the important features of the non-traditional RSW equations that we seek to establish is the possibility of including an arbitrary bottom topography. Topography introduces new parameters, such as its typical height. In what follows, we will allow the typical height of the topography to be of order h_0.

3.2. *Columnar motion on the sphere*

Let us recall that the essence of the shallow-water dynamics on the plane is the columnar character of the corresponding fluid motion. Indeed, a standard way of deriving shallow-water equations is to vertically average the full primitive equations, to assume hydrostatic balance, and to apply the mean-field approximation, i.e. to neglect vertical variations of the horizontal velocity. The same method works for deriving the

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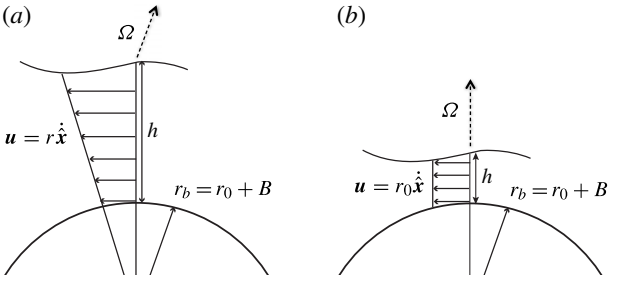


FIGURE 2. Columnar motion on the sphere: constant angular (*a*) versus constant linear (*b*) velocity of the column. Here $\boldsymbol{u}$ denotes the vector of velocity tangent to the sphere.

non-traditional RSW equations on the plane, which generalize the standard ones to the case where the angular velocity of the global rotation is oblique with respect to the fluid columns, as was shown by Dellar & Salmon (2005). However, the situation is more delicate on the rotating sphere, where obliqueness of the angular velocity of the overall rotation combines with the curvature of the surface. In the traditional approximation the obliqueness of the global angular velocity is neglected, as well as the variation of the planetary velocity with height, by virtue of the shallowness of the fluid layer. The motion is, thus, considered columnar in the 'flat' sense (see figure 2*b*) and the approximation turns out to be consistent. If the traditional approximation is to be relaxed, the notion of columnar motion should be revisited. Indeed, due to the curvature, the fluid 'columns' on the sphere are, in fact, solid angles. Hence, strictly speaking, the hypothesis of columnar motion, i.e. height-independence, should be applied to the angular velocity, rather than the physical velocity, leading to the dependence of this latter on altitude (see figure 2*a*). One could naively think that in the shallow-layer approximation this effect is negligible, but, as was already mentioned, the 'brute-force' application of this approximation, i.e. replacement of the radial position by the constant planet's radius, leads to problems with the conservation of the axial angular momentum (Phillips 1966). For example, if one applies the straightforward averaging approach and mean-field approximation to the physical velocities, one gets the RSW equations, which do not conserve axial angular momentum (see § B.1 below). In what follows, we will thus apply the hypothesis of columnar motion in the spherical sense, i.e. considering that the angular velocities of the fluid element in the form of a solid angle are independent of the radial position.

3.3. *Columnar motion reduction in the variational principle*

The most systematic way to obtain various approximations for (perfect) fluid motion is to do this directly in the variational principle (see, for example, Salmon 1998). The non-traditional RSW equations on the tangent plane were obtained in this way (Dellar & Salmon 2005; Dellar 2011). We thus start with the variational principle established for Boussinesq equations in the quasi-hydrostatic limit in § 2.4 and systematically apply the columnar motion approximation. The hypothesis of columnar motion, as applied to the Lagrangian coordinates Λ and Φ, means that

$$\Lambda = \Lambda(\lambda, \phi, t), \quad \Phi = \Phi(\lambda, \phi, t). \tag{3.6}$$

Let us use it first in the incompressibility constraint (2.31):

$$\frac{R^2\cos\Phi}{r^2\cos\phi}\frac{\partial(\Lambda,\Phi,R)}{\partial(\lambda,\phi,r)} = \frac{R^2\partial_r R\cos\Phi}{r^2\cos\phi}\frac{\partial(\Lambda,\Phi)}{\partial(\lambda,\phi)} = \frac{R^2\partial_r R}{r^2}\frac{\partial(\Lambda\cos\Phi,\Phi)}{\partial(\lambda\cos\phi,\phi)} = 1. \tag{3.7}$$

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This relation can be rewritten as

$$R^2 \mathrm{d}RJ = r^2 \mathrm{d}r, \tag{3.8}$$

where we introduced the shorthand notation:

$$J = J(\Phi, \Lambda; \phi, \lambda) = \frac{\partial(\Lambda \cos\Phi, \Phi)}{\partial(\lambda \cos\phi, \phi)}. \tag{3.9}$$

Due to the independence of the angular coordinates from r, the relation (3.8) can be immediately integrated:

$$\frac{R^3}{3} = \frac{r^3}{3} J^{-1} + \mathscr{F}(\lambda, \phi, t), \tag{3.10}$$

where $\mathscr{F}$ is an arbitrary function ('integration constant'). From now on we will be considering a fluid layer of variable thickness H between two material surfaces: a bottom, with an arbitrary topography B, and the top, which is a free surface. In Lagrangian terms:

$$R_b = r_0 + B(\Lambda, \Phi), \quad R_t = r_0 + B(\Lambda, \Phi) + H(\Lambda, \Phi, t). \tag{3.11}$$

Under Lagrangian mapping they are images of $r_b = r_0 + B(\lambda, \phi)$ and $r_t = r_0 + B + H_I(\lambda, \phi)$, respectively, where $H = H_I$ at $t = 0$. By an additional change of independent variables the initial height of the fluid layer may be rendered uniform (see, for example, Zeitlin, Medvedev & Plougonven 2003), so without loss of generality we may consider $r_b = r_0$ and $r_t = r_0 + h_0$, with constant h_0 (by doing this we lose the interpretation of λ, ϕ, r as the initial position of a fluid parcel; we trade here transparency for technical convenience). With these assignments $\mathscr{F}$ and J can be eliminated from (3.10) in favour of $r_0 + B(\Lambda, \Phi)$ and $H(\Lambda, \Phi, t)$:

$$R^3 = (r^3 - r_0^3)\frac{(r_0 + H + B)^3 - (r_0 + B)^3}{(r_0 + h_0)^3 - r_0^3} + (r_0 + B)^3. \tag{3.12}$$

The Jacobian J is thus expressed as follows:

$$J^{-1} = \frac{(r_0 + H + B)^3 - (r_0 + B)^3}{(r_0 + h_0)^3 - r_0^3}. \tag{3.13}$$

We can immediately exploit the relations (3.12) and (3.13) in order to establish the continuity equation for columnar elements. By rewriting them in the form:

$$\frac{(r_0 + H + B)^3 - (r_0 + B)^3}{3} = \frac{(r_0 + h_0)^3 - r_0^3}{3} J^{-1}, \tag{3.14}$$

differentiating in time, and re-using (3.14) we get:

$$D_t\left[\frac{(r_0 + H + B)^3 - (r_0 + B)^3}{3}\right] + \frac{(r_0 + H + B)^3 - (r_0 + B)^3}{3}\frac{\dot{J}}{J} = 0. \tag{3.15}$$

From the definition of J (3.9) it follows that

$$\frac{\dot{J}}{J} = \frac{\partial(\dot{\Lambda}\cos\Phi, \Phi)}{\partial(\Lambda\cos\Phi, \Phi)} + \frac{\partial(\Lambda\cos\Phi, \dot{\Phi})}{\partial(\Lambda\cos\Phi, \Phi)} - \tan\Phi\frac{\partial(\dot{\Phi}\Lambda\cos\Phi, \Phi)}{\partial(\Lambda\cos\Phi, \Phi)}. \tag{3.16}$$

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Upon the standard correspondence between Lagrangian and Eulerian descriptions, one recognizes on the right-hand side of this formula the divergence of the (physical) angular velocity vector $\hat{\boldsymbol{\lambda}}\cos\Phi\dot{\Lambda}+\hat{\boldsymbol{\phi}}\dot{\Phi}$ in Lagrangian spherical coordinates. Denoting $\boldsymbol{u}$ as the physical tangent velocity calculated at some fixed r, $\boldsymbol{\nabla}_{\boldsymbol{x}}$ as the tangent gradient operator along the sphere of the same radius, and $\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot(\ldots)$ as the divergence in the tangent plane, we obtain the Eulerian counterpart of (3.15):

$$D_t h + h\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\boldsymbol{u}=0, \tag{3.17}$$

where $D_t=\partial/\partial t+\boldsymbol{u}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}}$ and

$$h=\frac{(r_0+H+B)^3-(r_0+B)^3}{3r_0^2} \tag{3.18}$$

is the pseudo-thickness of the layer, which coincides with the true layer thickness H at order ϵ. (Note that the expression for h is characteristic for columns – solid angles.)

We will now consider the variational principle (2.24) in the quasi-hydrostatic approximation (2.27) applied to the fluid layer between the bottom and the free surface. The relation (3.8) allows us to make a change of the radial integration variable $r\to R$. The reduced Lagrangian, where we omit the already used incompressibility constraint, thus becomes:

$$\begin{aligned} L_r \;=\; & \int_{-(\pi/2)}^{\pi/2}\cos\phi\,\mathrm{d}\phi\int_0^{2\pi}\mathrm{d}\lambda\, J\int_{R_b}^{R_t}R^2\mathrm{d}R \\ & \times\left[\tfrac{1}{2}(R^2\cos^2\Phi\,\dot{\Lambda}^2+R^2\dot{\Phi}^2)+\Omega R^2\cos^2\Phi\,\dot{\Lambda}-bgR\right]. \end{aligned} \tag{3.19}$$

The R-integration is trivial, and by replacing b by its vertically averaged value $\bar{b}$ we get:

$$\begin{aligned} L_r \;=\; & \int_{-(\pi/2)}^{\pi/2}\cos\phi\,\mathrm{d}\phi\int_0^{2\pi}\mathrm{d}\lambda z\, J\left[\left(\frac{1}{2}(\cos^2\Phi\,\dot{\Lambda}^2+\dot{\Phi}^2)+\Omega\cos^2\Phi\,\dot{\Lambda}\right)\right. \\ & \left.\times\left(\frac{R_t^5-R_b^5}{5}\right)-\bar{b}g\left(\frac{R_t^4-R_b^4}{4}\right)\right]. \end{aligned} \tag{3.20}$$

Here the first and second terms in the square brackets represent the kinetic and potential energy densities, respectively. Substituting the expressions for R_t, R_b and J in terms of h and B from (3.11), (3.13) in this formula we arrive at the reduced Lagrangian, which, upon applying the variational procedure (2.21) and (2.22), gives the equations of deep columnar motion on the sphere, together with (3.17). They may be of interest in their own right; however, our goal is to establish shallow water equations. For this, as explained in § 3.1, we introduce the small parameters μ,ϵ,γ, and rescale the time and the radial coordinate as follows:

$$t\to(\mu\Omega)^{-1}t,\quad R\to r_0(1+\epsilon z). \tag{3.21}$$

In the remainder of this section we use the dimensionless H, h and B:

$$H\to\epsilon r_0 H,\quad h\to\epsilon r_0 h,\quad B\to\epsilon r_0 B \tag{3.22}$$

unless explicitly stated otherwise. Upon this scaling, different terms in the truncated Lagrangian (3.19) scale differently: the kinetic energy terms containing the square

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of the angular velocity are proportional to μ^2, the Coriolis term containing Ω is proportional to μ, and the potential energy term containing g is proportional to γ^{-1}. The dimensionless Jacobian (3.13) is expressed as

$$J^{-1} = \frac{(1+\epsilon\,(B+H))^3 - (1+\epsilon B)^3}{(1+\epsilon)^3 - 1}, \tag{3.23}$$

and the dimensionless pseudo-thickness (3.18) as

$$h = \frac{(1+\epsilon)^3 - 1}{3\epsilon} J^{-1}. \tag{3.24}$$

Up to an inessential constant, the dimensionless kinetic-energy part of (3.20) is

$$\frac{(1+\epsilon(H+B))^5 - (1+\epsilon B)^5}{5\epsilon} \frac{1}{h} \left[\dot{\Lambda} \cos^2 \Phi + \frac{\mu}{2} \left(\dot{\Lambda}^2 \cos^2 \Phi + \dot{\Phi}^2 \right) \right]. \tag{3.25}$$

So far, no approximation has been made. We will express the kinetic and potential energy densities as functions of h and will use the local mass conservation (3.17) in the variational calculations of the next subsection. Retaining the terms $O(1)$, $O(\mu)$, $O(\epsilon)$ and omitting higher-order terms, the dimensionless kinetic energy density in (3.20) is

$$(1+\epsilon(h+2B))\, \dot{\Lambda} \cos^2 \Phi + \frac{\mu}{2} \left(\dot{\Lambda}^2 \cos^2 \Phi + \dot{\Phi}^2 \right). \tag{3.26}$$

As to the potential energy density, it may be expanded in powers of ϵ or not, depending on the dynamical regime of interest. Since our main focus is on the role of the full Coriolis force, we simply retain the leading term in potential energy, i.e.

$$-\frac{\epsilon}{\gamma\mu} \frac{h+2B}{2} \tag{3.27}$$

where we supposed, for simplicity, that $\bar{b}$ is constant, which we took equal to unity without loss of generality. (The generalization to horizontally varying $\bar{b}(\lambda, \phi)$ is straightforward and leads to Ripa's non-traditional equations (Ripa 1993), which are not presented.) Retaining the terms (3.26) and (3.27) and omitting higher-order terms, the dimensionless reduced Lagrangian is given by

$$\begin{aligned} L_{RSW} = & \int_{-(\pi/2)}^{\pi/2} \cos\phi \mathrm{d}\phi \int_0^{2\pi} \mathrm{d}\lambda \left[\mu \left(\frac{1}{2}\dot{\Phi}^2 + \frac{1}{2} \cos^2 \Phi \dot{\Lambda}^2 \right) \right. \\ & \left. + (1 + \underline{\epsilon(h+2B)}) \cos^2 \Phi \dot{\Lambda} - \frac{\epsilon}{\gamma\mu} \frac{h+2B}{2} \right]. \end{aligned} \tag{3.28}$$

The independent dynamical variables in the Lagrangian (3.28) are Λ and Φ. The variable h is not independent, and is expressed in terms of Λ and Φ through (3.24) and (3.9). We underline in (3.28) the term that leads to the non-traditional corrections to the rotating shallow-water equations on the sphere – see below. Under our assumptions (see § 3.2), the last, potential energy, term in (3.28) is $O(1)$. Another $O(1)$ term is the second term, originating from the Coriolis term (containing Ω), in (3.20). The remaining kinetic energy term is $O(\mu)$, and the correction to the Coriolis term is $O(\epsilon)$. Several dynamical regimes can be envisaged depending on the ratio

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of μ and ϵ. In the essentially shallow situation, all $O(\epsilon)$ terms are to be omitted – corresponding in spirit to the traditional approximation such as described in § 2.4. This indeed will give the traditional RSW system (see below). If non-shallow corrections are to be taken into account, a relation between μ and ϵ should be prescribed. For $\epsilon \ll \mu \ll 1$, in the leading order only the linear in ϵ correction to the Coriolis term is to be retained, which is the regime we adopt. Although we will not develop it, an essentially deep columnar motion regime is possible, where $\epsilon = O(1)$ and all terms in (3.20) are to be kept, which results in rather exotic, but still consistent equations of motion. In the sequel no further approximation, with respect to those leading to (3.28), is made and all calculations are exact.

3.4. *Derivation of the non-traditional rotating shallow-water equations*

By requiring stationarity of the action

$$\delta \int L_{RSW}\left[\Lambda,\, \dot{\Lambda},\, \Phi,\, \dot{\Phi}\right] \mathrm{d}t = 0 \tag{3.29}$$

we get:

$$\int \cos\phi \mathrm{d}\phi \mathrm{d}\lambda \mathrm{d}t \left[\mu(\dot{\Phi}\delta\dot{\Phi} + \cos^2\Phi\dot{\Lambda}\delta\dot{\Lambda} - \dot{\Lambda}^2\cos\Phi\sin\Phi\delta\Phi) + \cos^2\Phi\delta\dot{\Lambda}\right.$$
$$- 2\dot{\Lambda}\sin\Phi\cos\Phi\delta\Phi - \frac{\epsilon}{\gamma\mu}\left(\frac{h}{2} + \delta B\right) + \epsilon\left(\cos^2\Phi\dot{\Lambda}(\delta h + 2\delta B) + (h + 2B)\right.$$
$$\left.\left.\times\,(\cos^2\Phi\delta\dot{\Lambda} - 2\cos\Phi\sin\Phi\dot{\Lambda}\delta\Phi)\right)\right] = 0. \tag{3.30}$$

Integrating by parts in time we get:

$$\int \cos\phi \mathrm{d}\phi \mathrm{d}\lambda \mathrm{d}t \left[\mu\left[-(\ddot{\Phi} + \cos\Phi\sin\Phi\dot{\Lambda}^2)\delta\Phi - (\cos^2\Phi\ddot{\Lambda} - 2\cos\Phi\sin\Phi\dot{\Lambda}\dot{\Phi})\delta\Lambda\right]\right.$$
$$+ 2\cos\Phi\sin\Phi(\dot{\Phi}\delta\Lambda - \dot{\Lambda}\delta\Phi) - \frac{\epsilon}{\gamma\mu}\left(\frac{\delta h}{2} + \delta B\right) - \epsilon\cos^2\Phi(\dot{h} + 2\dot{B})\delta\Lambda$$
$$+ \epsilon\left[\cos^2\Phi\dot{\Lambda}(\delta h + 2\delta B) - 2\cos\Phi\sin\Phi\dot{\Lambda}(h + 2B)\delta\Phi\right.$$
$$\left.\left. + 2\dot{\Phi}\cos\Phi\sin\Phi(h + 2B)\delta\Lambda\right]\right] = 0. \tag{3.31}$$

In order to calculate δh and δB we use

$$\delta h = -h\frac{\delta J}{J} \tag{3.32}$$

(see (3.13), (3.18)), where $\delta J/J$ is, in fact, the covariant divergence of the vector with components $(\delta\Lambda, \delta\Phi)$ in spherical coordinates (Λ, Φ) – see (3.16) and appendix A – and also

$$\delta B = \delta\Phi\partial_\Phi B + \delta\Lambda\partial_\Lambda B, \tag{3.33}$$

as the bottom topography is a material line. With the help of these formulae, the contributions containing δh and δB in (3.30) may be integrated by parts by transforming the integration from the variables (λ, ϕ) to (Λ, Φ) with the help of the inverse relationship (3.24) between h and the Jacobian J. Using the fact that the standard integral theorems hold in curvilinear coordinates, if expressed in terms

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of covariant derivatives, as well as the fact that the covariant derivative of a scalar coincides with the ordinary derivative, we get for any F:

$$\int \cos\phi \mathrm{d}\phi\mathrm{d}\lambda\mathrm{d}t\, F\delta h = \int \cos\phi \mathrm{d}\phi\mathrm{d}\lambda\mathrm{d}t \left[\frac{1}{h}\left[\dot{\Lambda}\partial_\Lambda(Fh^2)\delta\Lambda + \dot{\Phi}\partial_\Phi(Fh^2)\delta\Phi\right]\right]. \quad (3.34)$$

We also use (3.32) and (3.33) written for the time derivatives and thus get for any F:

$$\int \cos\phi \mathrm{d}\phi\mathrm{d}\lambda\mathrm{d}t\, F(\dot{h}+2\dot{B}) = \int \cos\phi \mathrm{d}\phi\mathrm{d}\lambda\mathrm{d}t \left[-\frac{hF}{\cos\Phi}(\partial_\Lambda(\cos\Phi\dot{\Lambda}) + \partial_\Phi(\cos\Phi\dot{\Phi})) + 2F(\dot{\Lambda}\partial_\Lambda B + \dot{\Phi}\partial_\Phi B)\right]. \quad (3.35)$$

Using these formulae in (3.31) and gathering the terms with Λ and Φ, respectively, we arrive at the equations:

$$\mu(\ddot{\Lambda}\cos^2\Phi - 2\dot{\Lambda}\dot{\Phi}\cos\Phi\sin\Phi) - 2\dot{\Phi}\sin\Phi\cos\Phi = -\frac{\epsilon}{\gamma\mu}\partial_\Lambda(h+B) + \underline{\epsilon\left[2\cos\Phi\sin\Phi(h+2B)\dot{\Phi} + \frac{1}{h}\partial_\Lambda(\cos^2\Phi\dot{\Lambda}h^2) + \cos\Phi\left[h\partial_\Lambda(\cos\Phi\dot{\Lambda}) + h\partial_\Phi(\cos\Phi\dot{\Phi}) - 2\cos\Phi\dot{\Phi}\partial_\Phi B\right]\right]}, \quad (3.36)$$

$$\mu(\ddot{\Phi} + \dot{\Lambda}^2\cos\Phi\sin\Phi) + 2\dot{\Lambda}\sin\Phi\cos\Phi = -\frac{\epsilon}{\gamma\mu}\partial_\Phi(h+B) + \underline{\epsilon\left[-2\cos\Phi\sin\Phi(h+2B)\dot{\Lambda} + \cos^2\Phi\dot{\Lambda}\partial_\Phi h + \frac{1}{h}\partial_\Phi(\cos^2\Phi\dot{\Lambda}h^2) + 2\cos^2\Phi\dot{\Lambda}\partial_\Phi B\right]}. \quad (3.37)$$

The derivatives with respect to Lagrangian coordinates in this formulae are understood in the sense

$$\partial_\Lambda(\ldots) = \frac{\partial(\ldots,\Phi)}{\partial(\Lambda,\Phi)}; \quad \partial_\Phi(\ldots) = \frac{\partial(\Lambda,\ldots)}{\partial(\Lambda,\Phi)}, \quad (3.38)$$

and may be transformed to derivatives with respect to λ, ϕ with the help of the Jacobian J. Together with (3.17), equations (3.36) and (3.37) constitute the non-traditional RSW equations on the sphere. The underlined terms in (3.36) and (3.37) represent the non-traditional corrections we were looking for. They result from a modification of hydrostatic equilibrium by quasi-hydrostatic terms, non-traditional components of the Coriolis force, and the $O(\epsilon)$ correction to the standard Coriolis force.

It can be immediately shown (see also § 4 below), either from the invariance of the action with respect to rotations around $\hat{z}$ (shifts in Λ), via the standard Noether theorem, or by a direct computation, that the axial momentum with the mass density

$$l_z = r_0^2\cos^2\Phi\left(\dot{\Lambda} + \Omega\left(1 + \frac{h+2B}{r_0}\right)\right), \quad (3.39)$$

where we restored dimensions, is conserved. This is the consistent shallow-water reduction of the angular momentum (2.5). The material conservation of potential

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vorticity can be established from the particle relabelling symmetry of the action (see Salmon 1998), yet we will give a simpler derivation below in § B.2. We only notice here that the conjugate momenta (see (2.34)) for the variables Λ, Φ, which are at the same time the covariant components of the angular velocity (see (2.34)), are:

$$p_\Lambda = r_0^2 \cos^2\Phi \left(\dot{\Lambda} + \Omega \left(1 + \frac{h+2B}{r_0} \right) \right), \quad p_\Phi = r_0^2 \dot{\Phi}, \tag{3.40}$$

as follows from (3.28) with restored dimensions, and the fact that potential vorticity is the two-dimensional curl of the vector (p_Λ, p_ϕ) normalized by h.

We can rewrite (3.36) and (3.37) by introducing physical velocity components *at the bottom*:

$$\boldsymbol{u} = u\,\hat{\boldsymbol{\lambda}} + v\,\hat{\boldsymbol{\phi}} : u = r_0 \cos\phi \dot{\lambda}, \quad v = r_0 \dot{\phi}. \tag{3.41}$$

It should be stressed that there is an ambiguity in the choice of radial distance (a value between r_0 and $r_0 + B + h$) while transforming angular velocities into physical velocities. This is one of the problems with a naive application of the columnar motion hypothesis. We make the simplest choice, which is consistent – see the discussion in § 5.

By restoring dimensions and using dimensional h and B we thus get:

$$\begin{aligned}
&\partial_t u + \frac{u}{r_0 \cos\phi}\partial_\lambda u + \frac{v}{r_0}\partial_\phi u - \frac{uv\tan\phi}{r_0} - 2\Omega \sin\phi v \\
&\quad = -\frac{g}{r_0 \cos\phi}\partial_\lambda (h+B) \\
&\qquad + \underline{\frac{2\Omega}{r_0}\left(v \sin\phi (h+2B) + h\partial_\lambda u + u\partial_\lambda h - v\cos\phi\partial_\phi B + \frac{h}{2}\partial_\phi (v\cos\phi) \right)},
\end{aligned} \tag{3.42}$$

$$\begin{aligned}
&\partial_t v + \frac{u}{r_0 \cos\phi}\partial_\lambda v + \frac{v}{r_0}\partial_\phi v + \frac{u^2\tan\phi}{r_0} + 2\Omega \sin\phi u \\
&\quad = -\frac{g}{r_0}\partial_\phi (h+B) \\
&\qquad + \underline{\frac{2\Omega}{r_0}\left(-u \sin\phi (h+2B) + u\cos\phi\partial_\phi (h+B) + \frac{h}{2}\partial_\phi\,(u\cos\phi) \right)}
\end{aligned} \tag{3.43}$$

$$\partial_t h + \frac{1}{r_0 \cos\phi}\left(\partial_\lambda (hu) + \partial_\phi (hv\cos\phi) \right) = 0, \tag{3.44}$$

where, again, non-traditional contributions are underlined. If they are omitted, the traditional RSW equations on the sphere are recovered. Traditional equations are usually written in terms of the the *position* of the free surface $\tilde{H}$. Here, non-traditional equations are written in terms of the pseudo-thickness h. The formulation in terms of $\tilde{H}$ follows from (3.42)–(3.44) by the replacement $h = \tilde{H} - B$.

In the f-plane approximation, where the (3.42)–(3.44) are considered at fixed reference latitude $\phi = \phi_0$ in the limit $r_0 \to \infty$, with the tangent-plane replacements $(1/(r_0\cos\phi_0))\partial_\lambda \to \partial_x$, $(1/r_0)\partial_\phi \to \partial_y$, equations (19*a*,*b*) of Dellar & Salmon (2005) are recovered from (3.42) and (3.43).

There are two variants of non-traditional β-plane equations in the literature. Grimshaw (1975) considered non-traditional equations by retaining the beta-effect only in the vertical component Ω_z of the overall rotation. Dellar (2011) extended this

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result to the case where the horizontal component Ω_y varies with latitude and the vertical component Ω_z varies with z. The variation of Ω_z with z is captured by the non-traditional extension of the standard Coriolis term in (3.42) and (3.43), while the meridional variation of Ω_y and Ω_z are also recovered from our equations in the tangent-plane limit, where trigonometric factors are expanded to first order about the reference latitude.

We have, thus, derived the non-traditional shallow-water equations on the rotating sphere from the variational principle. Alternative derivations by vertical averaging, and by using the equations of motion in curl form, are presented in appendix B. We would like to stress that the standard vertical averaging, which straightforwardly leads to the traditional shallow-water equations, gives an erroneous set of equations violating the conservation of angular momentum if non-traditional corrections are taken into account – see § B.1. This difficulty explains our preference for using the variational principle, where the conservation laws are automatically accounted for. We should also emphasize that the difference between the traditional and non-traditional systems of RSW equations comes from a single new term in the Lagrangian (3.28), which is underlined. Three other terms being the standard kinetic and potential energy terms and the Coriolis term in the rotating frame, it would be tempting to try to establish the form of the non-traditional terms from Newton's equations of motion. While the structure of the first 'non-shallow' correction to the Coriolis term is rather obvious, the problem is to establish the precise values of the coefficients in front of h and B. As we have seen, they come from the expansion of both the Lagrangian and the Jacobian of the Lagrangian mapping. They may be fixed by imposing the conservation of the axial angular momentum (3.39), although the problem in this way is displaced towards the definition of the latter, which would be consistent with the approximation.

4. Non-traditional RSW equations in intrinsic form. Hyperbolicity and Rankine–Hugoniot conditions

In this section we write down the intrinsic coordinate-independent formulation of the non-traditional RSW system on the sphere (3.42)–(3.44) obtained in the previous section. We recall that $\boldsymbol{u}$ denotes the physical velocity tangent to a sphere of radius r_0 in the rotating frame, and $\boldsymbol{x}$ denotes a position on this sphere. All necessary concepts are explained in appendix C. From the intrinsic formulation we will deduce the integral form of the equations, the eigenvalues of the linearized system, which serve to establish criteria of hyperbolicity, and the Rankine–Hugoniot jump conditions for weak solutions.

4.1. *Non-traditional equations in intrinsic form*

With the help of intrinsic operators on the sphere, defined in appendix C, we can reformulate the non-traditional RSW system (3.42)–(3.44) in a coordinate-independent way. One can check that the equations of (3.42)–(3.44) are the components of the following system,

$$\partial_t h + \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot (h\boldsymbol{u}) = 0, \tag{4.1}$$

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} \boldsymbol{u} + |\boldsymbol{u}|^2 \frac{\boldsymbol{x}}{r_0^2} + \boldsymbol{\nabla}_{\boldsymbol{x}} \left(g(h+B) + h\boldsymbol{u} \cdot \left(\frac{\boldsymbol{x}}{r_0} \times \boldsymbol{\Omega} \right) \right)$$
$$= \left(-2\frac{\boldsymbol{x}}{r_0} \cdot \boldsymbol{\Omega} + \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot ((h+2B)\pi_{\boldsymbol{x}} \boldsymbol{\Omega}) \right) \frac{\boldsymbol{x}}{r_0} \times \boldsymbol{u} - \frac{\boldsymbol{x}}{r_0} \times \boldsymbol{\Omega}\, \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot (h\boldsymbol{u}), \tag{4.2}$$

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where $\boldsymbol{u}$ is the three-dimensional velocity vector tangent to the sphere and defined in (3.41), and $\pi_{\boldsymbol{x}}\boldsymbol{\Omega}$ is the projection of the constant angular velocity $\boldsymbol{\Omega}$ onto the tangent plane at $\boldsymbol{x}$. As before, we use the notation $\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot(\ldots)$ for the divergence in the tangent plane (see (3.17) and (C 2)).

Equation (4.2) is indeed very close to that of Dellar & Salmon (2005). To recover the traditional equation, one has to remove from (4.2) the term with $h\boldsymbol{u}\cdot((\boldsymbol{x}/r_0)\times\boldsymbol{\Omega})$, the term with $\pi_{\boldsymbol{x}}\boldsymbol{\Omega}$ and the term with $\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot(h\boldsymbol{u})$. If one wants to include the (traditional) centrifugal force, one has to add $\boldsymbol{\nabla}_{\boldsymbol{x}}((\boldsymbol{\Omega}\cdot\boldsymbol{x})^2/2)$ to the left-hand side of (4.2). Note that, in contrast to the traditional situation, including the centrifugal term is not equivalent to modifying the topography B.

The energy equation is obtained by multiplying (4.1) by $|\boldsymbol{u}|^2/2+g(h+B)$, (4.2) by $h\boldsymbol{u}$, and adding the results. With the help of (C 8) this yields

$$\partial_t\left(h\frac{|\boldsymbol{u}|^2}{2}+g\left(\frac{h^2}{2}+hB\right)\right)+\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\left(\left(\frac{|\boldsymbol{u}|^2}{2}+g(h+B)+h\boldsymbol{u}\cdot\left(\frac{\boldsymbol{x}}{r_0}\times\boldsymbol{\Omega}\right)\right)h\boldsymbol{u}\right)=0. \tag{4.3}$$

It should be emphasized that, in the presence of dissipation, the equality should be replaced by (negative) inequality in this formula.

The vorticity equation can be obtained by replacing $\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot(h\boldsymbol{u})$ in (4.2) by $-\partial_t h$ and using (C 8), which gives

$$\begin{aligned}&\partial_t\left(\frac{\boldsymbol{l}_a}{r_0}\right)+\boldsymbol{\nabla}_{\boldsymbol{x}}\left(\frac{|\boldsymbol{u}|^2}{2}+g(h+B)+h\boldsymbol{u}\cdot\left(\frac{\boldsymbol{x}}{r_0}\times\boldsymbol{\Omega}\right)\right)\\&\quad+\left(\operatorname{curl}_{\boldsymbol{x}}\boldsymbol{u}+2\frac{\boldsymbol{x}}{r_0}\cdot\boldsymbol{\Omega}-\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot((h+2B)\pi_{\boldsymbol{x}}\boldsymbol{\Omega})\right)\frac{\boldsymbol{x}}{r_0}\times\boldsymbol{u}=0,\end{aligned} \tag{4.4}$$

where

$$\frac{\boldsymbol{l}_a}{r_0}=\boldsymbol{u}+\left(1+\frac{h+2B}{r_0}\right)\boldsymbol{\Omega}\times\boldsymbol{x} \tag{4.5}$$

and we defined the scalar field $\operatorname{curl}_{\boldsymbol{x}}\boldsymbol{u}=-\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot((\boldsymbol{x}/r_0)\times\boldsymbol{u})$, which is the projection of the curl of velocity onto the local normal to the sphere. The absolute vorticity is written, correspondingly:

$$\omega=\operatorname{curl}_{\boldsymbol{x}}\frac{\boldsymbol{l}_a}{r_0}=\operatorname{curl}_{\boldsymbol{x}}\boldsymbol{u}+2\frac{\boldsymbol{x}}{r_0}\cdot\boldsymbol{\Omega}-\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot((h+2B)\pi_{\boldsymbol{x}}\boldsymbol{\Omega}), \tag{4.6}$$

and by taking the curl of (4.4) we get

$$\partial_t\omega+\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot(\omega\boldsymbol{u})=0. \tag{4.7}$$

Next, one can get the momentum equation by multiplying (4.4) (which we transform with the help of (C 8)) by h, and by using (4.1), which gives, after some computations:

$$\begin{aligned}&\partial_t\left(h\frac{\boldsymbol{l}_a}{r_0}\right)+\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\left(h\frac{\boldsymbol{l}_a}{r_0}\otimes\boldsymbol{u}\right)+\boldsymbol{\nabla}_{\boldsymbol{x}}\left(\frac{h^2}{2}\left(g+2\boldsymbol{u}\cdot\left(\frac{\boldsymbol{x}}{r_0}\times\boldsymbol{\Omega}\right)\right)\right)\\&\quad+\left(g+2\boldsymbol{u}\cdot\left(\frac{\boldsymbol{x}}{r_0}\times\boldsymbol{\Omega}\right)\right)h\boldsymbol{\nabla}_{\boldsymbol{x}}B+h\left(|\boldsymbol{u}|^2+\left(1+\frac{h+2B}{r_0}\right)\boldsymbol{u}\cdot(\boldsymbol{\Omega}\times\boldsymbol{x})\right)\frac{\boldsymbol{x}}{r_0^2}\\&=-h\left(1+\frac{h+2B}{r_0}\right)\frac{\boldsymbol{\Omega}\cdot\boldsymbol{x}}{r_0^2}\boldsymbol{x}\times\boldsymbol{u}\end{aligned} \tag{4.8}$$

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where $\nabla_x \cdot (\boldsymbol{v} \otimes \boldsymbol{u})$ is defined in (C 4). Using the identity $\boldsymbol{x} \times \nabla_x \cdot (\boldsymbol{v} \otimes \boldsymbol{u}) = \nabla_x \cdot ((\boldsymbol{x} \times \boldsymbol{v}) \otimes \boldsymbol{u}) - \boldsymbol{u} \times \boldsymbol{v}$ for the tangent vector field $\boldsymbol{u}$, and an arbitrary vector field $\boldsymbol{v}$, one deduces the angular momentum budget

$$\partial_t (h \boldsymbol{L}_a) + \nabla_x \cdot (h \boldsymbol{L}_a \otimes \boldsymbol{u}) + \frac{\boldsymbol{x}}{r_0} \times \nabla_x \left(\frac{h^2}{2} \left(g + 2\boldsymbol{u} \cdot \left(\frac{\boldsymbol{x}}{r_0} \times \boldsymbol{\Omega} \right) \right) \right) + \left(g + 2\boldsymbol{u} \cdot \left(\frac{\boldsymbol{x}}{r_0} \times \boldsymbol{\Omega} \right) \right) h \frac{\boldsymbol{x}}{r_0} \times \nabla_x B = h \left(1 + \frac{h + 2B}{r_0} \right) \boldsymbol{\Omega} \times \left(\boldsymbol{u} \times \frac{\boldsymbol{x}}{r_0} \right) \tag{4.9}$$

where

$$\boldsymbol{L}_a = \frac{\boldsymbol{x}}{r_0} \times \frac{\boldsymbol{l}_a}{r_0}. \tag{4.10}$$

The traditional version of (4.9) is obtained by replacing all of the factors $1 + ((h + 2B)/r_0)$ by 1, and by removing the two terms with $\boldsymbol{u} \cdot ((\boldsymbol{x}/r_0) \times \boldsymbol{\Omega})$. As follows from (4.9), the volume density of the reduced axial angular momentum (4.10) is locally conserved. The integral balances of the conserved quantities follow then from the Stokes formulae (C 9)–(C 12). The mass and energy equations are

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathscr{D}} h \, \mathrm{d}\boldsymbol{x} + \int_{\partial \mathscr{D}} h \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}\ell = 0, \tag{4.11}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathscr{D}} \left(h \frac{|\boldsymbol{u}|^2}{2} + g \left(\frac{h^2}{2} + hB \right) \right) \mathrm{d}\boldsymbol{x} + \int_{\partial \mathscr{D}} \left(\frac{|\boldsymbol{u}|^2}{2} + g(h + B) + h\boldsymbol{u} \cdot \left(\frac{\boldsymbol{x}}{r_0} \times \boldsymbol{\Omega} \right) \right) h \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}\ell = 0. \tag{4.12}$$

These quantities are strictly conserved, while momentum and angular momentum balance equations lead to volume sources, namely for the angular momentum

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathscr{D}} h \boldsymbol{L}_a \, \mathrm{d}\boldsymbol{x} + \int_{\partial \mathscr{D}} \boldsymbol{L}_a h \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}\ell + \int_{\partial \mathscr{D}} \frac{h^2}{2} \left(g + 2\boldsymbol{u} \cdot \left(\frac{\boldsymbol{x}}{r_0} \times \boldsymbol{\Omega} \right) \right) \frac{\boldsymbol{x}}{r_0} \times \boldsymbol{v} \, \mathrm{d}\ell = - \int_{\mathscr{D}} \left(g + 2\boldsymbol{u} \cdot \left(\frac{\boldsymbol{x}}{r_0} \times \boldsymbol{\Omega} \right) \right) h \frac{\boldsymbol{x}}{r_0} \times \nabla_x B \, \mathrm{d}\boldsymbol{x} + \int_{\mathscr{D}} h \left(1 + \frac{h + 2B}{r_0} \right) \boldsymbol{\Omega} \times \left(\boldsymbol{u} \times \frac{\boldsymbol{x}}{r_0} \right) \mathrm{d}\boldsymbol{x}. \tag{4.13}$$

Note that the angular momentum is nevertheless strictly conserved in the direction $\boldsymbol{\Omega}$ in the case of zonally-symmetric B, because the right-hand side of (4.13) is orthogonal to $\boldsymbol{\Omega}$. The identities (4.11)–(4.13) will be useful for numerical implementation in finite volume methods.

4.2. *Hyperbolicity and Rankine–Hugoniot jump conditions for weak solutions*

Consider a solution to the non-traditional rotating shallow-water system in the weak sense for the conservative equations (4.1) and (4.8) (or equivalently (4.9)). This means that the integral balance, obtained by multiplying the equation by a smooth test function and formally integrating by parts in time and space, holds.

Assume that the solution jumps through a space–time hypersurface (B being nevertheless continuous), with normal given at point $(t, \boldsymbol{x})$ by $(\nu_t, \boldsymbol{\nu})$ with $|\boldsymbol{\nu}| = 1$,

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$\boldsymbol{v}\cdot\boldsymbol{x}=0$ (i.e. for a discontinuity moving with velocity $\boldsymbol{W}$, $v_t=-\boldsymbol{W}\cdot\boldsymbol{v}$). Using space–time Stokes formulae similar to (C 9)–(C 12) we deduce the Rankine–Hugoniot conditions at the discontinuity. They are obtained by comparing the integral over a domain including a neighbourhood around a portion of the hypersurface with the sum of the two integrals at both sides of the hypersurface. The conclusion is that the boundary integrals obtained on both sides of the discontinuity are equal. We deduce the following Rankine–Hugoniot conditions, where $[\ldots]$ denotes, as usual, the jump of any quantity across the hypersurface.

(i) For the mass equation (4.1),

$$[h]v_t+[h\,\boldsymbol{u}\cdot\boldsymbol{v}]=0. \tag{4.14}$$

(ii) For the momentum equation (4.8) (or equivalently (4.9)),

$$\left[h\left(\boldsymbol{u}+\left(1+\frac{h+2B}{r_0}\right)\boldsymbol{\Omega}\times\boldsymbol{x}\right)\right]v_t+\left[h\left(\boldsymbol{u}+\left(1+\frac{h+2B}{r_0}\right)\boldsymbol{\Omega}\times\boldsymbol{x}\right)\boldsymbol{u}\cdot\boldsymbol{v}+\frac{h^2}{2}\left(g+2\boldsymbol{u}\cdot\left(\frac{\boldsymbol{x}}{r_0}\times\boldsymbol{\Omega}\right)\right)\boldsymbol{v}\right]=0. \tag{4.15}$$

This can be written also, using (4.14), as

$$[h\boldsymbol{u}]\,v_t+[h\boldsymbol{u}\,(\boldsymbol{u}\cdot\boldsymbol{v})]+\boldsymbol{\Omega}\times\frac{\boldsymbol{x}}{r_0}\left(\left[h^2\right]v_t+\left[h^2\,\boldsymbol{u}\cdot\boldsymbol{v}\right]\right)+\left[\frac{h^2}{2}\left(g+2\boldsymbol{u}\cdot\left(\frac{\boldsymbol{x}}{r_0}\times\boldsymbol{\Omega}\right)\right)\right]\boldsymbol{v}=0. \tag{4.16}$$

(iii) For the energy equation (4.3),

$$\left[h\frac{|\boldsymbol{u}|^2}{2}+g\left(\frac{h^2}{2}+hB\right)\right]v_t+\left[\left(\frac{|\boldsymbol{u}|^2}{2}+g(h+B)+h\boldsymbol{u}\cdot\left(\frac{\boldsymbol{x}}{r_0}\times\boldsymbol{\Omega}\right)\right)h\,\boldsymbol{u}\cdot\boldsymbol{v}\right]\leqslant 0, \tag{4.17}$$

where B can, in fact, be removed because of (4.14). In (4.17), we have to capture jumps according to the orientation of $\boldsymbol{v}$. The inequality instead of equality results from dissipation induced by shocks.

Consider now a small perturbation to a smooth solution of the non-traditional RSW system (4.1) and (4.2). If one wants to characterize the linear stability of such a perturbation, in the sense of well-posedness, one has to write down a linearized system taking into account only the differential part of the equations. Assume that the perturbation propagates locally only in one tangent direction $\boldsymbol{v}$ at $\boldsymbol{x}$, with $|\boldsymbol{v}|=1$, $\boldsymbol{v}\cdot\boldsymbol{x}=0$. Then the differential terms of the linearized system correspond exactly to the jumps appearing in the Rankine–Hugoniot conditions. Therefore, using (4.14), (4.16) and transforming them to the variables h, $\boldsymbol{u}$, one gets the following characteristic matrix in terms of the variables h, $\boldsymbol{u}\cdot\boldsymbol{v}$, $\boldsymbol{u}\cdot(\boldsymbol{x}/r_0\times\boldsymbol{v})$ (intermediate calculations are omitted):

$$\begin{pmatrix}\boldsymbol{u}\cdot\boldsymbol{v} & h & 0\\ g+2\boldsymbol{u}\cdot\left(\dfrac{\boldsymbol{x}}{r_0}\times\boldsymbol{\Omega}\right) & \boldsymbol{u}\cdot\boldsymbol{v}+2h\left(\dfrac{\boldsymbol{x}}{r_0}\times\boldsymbol{\Omega}\right)\cdot\boldsymbol{v} & h\,\boldsymbol{\Omega}\cdot\boldsymbol{v}\\ 0 & h\,\boldsymbol{\Omega}\cdot\boldsymbol{v} & \boldsymbol{u}\cdot\boldsymbol{v}\end{pmatrix}. \tag{4.18}$$

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Its eigenvalues are $\boldsymbol{u}\cdot\boldsymbol{v}$ and

$$\boldsymbol{u}\cdot\boldsymbol{v}+h\left(\frac{\boldsymbol{x}}{r_0}\times\boldsymbol{\Omega}\right)\cdot\boldsymbol{v}\pm\left(h^2\left|\frac{\boldsymbol{x}}{r_0}\times\boldsymbol{\Omega}\right|^2+h\left(g+2\boldsymbol{u}\cdot\left(\frac{\boldsymbol{x}}{r_0}\times\boldsymbol{\Omega}\right)\right)\right)^{1/2}. \tag{4.19}$$

Therefore, the system is not everywhere hyperbolic, but it is strictly hyperbolic under the condition that $|\boldsymbol{u}|<g/(2|\boldsymbol{\Omega}|)$, which means $\gamma\mu<1/2$ in terms of the dimensionless parameters introduced in § 3.1. This condition is always satisfied within the domain of validity of our non-traditional RSW system. Note that the matrix (4.18) is identical to that obtained by Dellar & Salmon (2005) in the planar approximation.

The eigenvalue $\boldsymbol{u}\cdot\boldsymbol{v}$ has the associated eigenvector $(-h\boldsymbol{\Omega}\cdot\boldsymbol{v},\,0,\,g+2\boldsymbol{u}\cdot((\boldsymbol{x}/r_0)\times\boldsymbol{\Omega}))$. Since the components correspond to the variables h, $\boldsymbol{u}\cdot\boldsymbol{v}$, $\boldsymbol{u}\cdot(\boldsymbol{x}/r_0\times\boldsymbol{v})$, one deduces that the gradient of the eigenvalue $\boldsymbol{u}\cdot\boldsymbol{v}$ with respect to these variables is orthogonal to the eigenvector, which means that the characteristic field associated with this eigenvalue is linearly degenerate. It indicates that the system admits contact discontinuities associated with the eigenvalue $\boldsymbol{u}\cdot\boldsymbol{v}$. These are non-dissipative discontinuities characterized by the Rankine–Hugoniot jump relations (4.14)–(4.17), which simplify in this case ($v_t=-\boldsymbol{u}\cdot\boldsymbol{v}$) to

$$[\boldsymbol{u}\cdot\boldsymbol{v}]=0,\qquad\left[\frac{h^2}{2}\left(g+2\boldsymbol{u}\cdot\left(\frac{\boldsymbol{x}}{r_0}\times\boldsymbol{\Omega}\right)\right)\right]=0. \tag{4.20}$$

In contrast to the traditional situation, h has a jump across such a contact discontinuity, unless $\boldsymbol{v}$ and $(\boldsymbol{x}/r_0)\times\boldsymbol{\Omega}$ are collinear. This jump can be interpreted as follows: there is no advection of momentum through a contact discontinuity, hence the vertically integrated pressure forces on each side of the discontinuity must balance. The non-traditional contribution to quasi-hydrostatic balance is discontinuous if horizontal velocity is. This induces a discontinuity in the pressure field, which must be compensated by a discontinuity of layer thickness to maintain the balance.

5. Summary and discussion

To summarize, we established the non-traditional rotating shallow-water equations on the sphere in the presence of arbitrary topography. We cross-checked the derivation by three different methods, giving the same result under the same approximations. Precisely, the approximations we use are small ratio of centrifugal to gravitational accelerations γ, small global Rossby number μ, small dimensionless thickness of the layer ϵ, and small typical dimensionless amplitude α of the deviations of the layer surface, such that the combination $((\epsilon\alpha)/(\gamma\mu))$ is of the order unity. In the zeroth order in ϵ the traditional RSW equations on the sphere arise, while in the limit of small ϵ and μ the quasigeostrophic equilibrium between the Coriolis and pressure forces follows. Thus, the non-traditional approximation is, in fact, a 'deeper' shallow-water approximation, with respect to the traditional one.

In the 'flat' limit the obtained equations coincide with non-traditional RSW equations on the f- and β-plane known in the literature, respectively, when the Coriolis parameter is maintained constant or dependence on latitude is retained in the leading order. The equations conserve potential vorticity and the axial component of the angular momentum if the topography is axisymmetric. Together with the correct tangent plane limits, this proves the consistency of the equations. It should be

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stressed that a by-product of the derivation is the set of equations for *deep* columnar motion on the rotating sphere, which may be of importance for certain planetary atmospheres. In any case, all of our three methods allow the straightforward inclusion of higher-order corrections in 'shallowness' parameter ϵ, as well as a generalization to a horizontally varying density.

Finally, we gave an intrinsic, coordinate-independent formulation of the non-traditional RSW system. We showed that within the domain of validity of the model, corresponding to the above-described values of parameters, the system is hyperbolic. As is well known, hyperbolic systems admit weak solutions with discontinuities, which correspond physically to fronts formed by the breaking waves. We established Rankine–Hugoniot conditions for such solutions which generalize the standard hydraulic jump conditions to the shallow-water system on the sphere. They are based upon a plausible physical conjecture of energy decrease across the jump. Indeed, our shallow-water equations may be interpreted (and derived, see § B.1) as vertically averaged primitive equations. Weak solutions and related fronts, therefore, are proxies for the various wave-breaking processes and related zones of enhanced mixing and dissipation, which can arise in the full system. Energy decrease in such zones is the simplest consistent assumption. Mathematically, it provides an entropy inequality necessary for construction of consistent numerical schemes. We should stress that momentum conservation, which is equivalent to angular momentum conservation, is the basis of the derivation of jump conditions. Efficient finite-volume schemes for numerical modelling of non-traditional shallow-water equations on the sphere can be built on this basis, which is one of our principal motivations.

We now check the relevance of the above-described approximations for the Earth's atmosphere and oceans, and for the atmospheres of other planets. For crude estimates, if one takes typical values of parameters for the synoptic motions in the Earth's atmosphere: $U \sim 10$ m s^{-1}, $h_0 \sim 10$ km, and for the mesoscale motions in the ocean: $U \sim 0.1$ m s^{-1}, $h_0 \sim 3$ km, together with $\Omega = 7.3 \times 10^{-5}$ s^{-1} and $r_0 \approx 6400$ km for the Earth (see, for example, Pedlosky 1979), this gives

$$\gamma \approx 3.5 \times 10^{-3}, \tag{5.1}$$

and we arrive at the following estimates:

$$\text{Atmosphere}: \quad \epsilon \approx 1.6 \times 10^{-3}, \quad \mu \approx 2.1 \times 10^{-2}, \tag{5.2}$$

$$\text{Ocean}: \quad \epsilon \approx 5 \times 10^{-4}, \quad \mu \approx 2.1 \times 10^{-4}. \tag{5.3}$$

This means, first, that our approximations are consistent for both the Earth's atmosphere and oceans and, second, that for the ocean the traditional RSW approximation with omission of the $O(\epsilon)$ terms, while retaining $O(\mu)$ terms, is less justified than for the atmosphere. In addition, the parameter $(\epsilon/(\mu\gamma))$ appearing in the dimensionless non-traditional RSW equations turns out to be large both in the atmosphere and in the ocean. This means that the parameter α should be small, for consistency. Then, the non-traditional terms containing derivatives of the position of the free surface $h + B$ can be omitted in this regime in (3.42)–(3.44), and $h + B$ can be replaced by its non-perturbed value $h_0 + B$. Further simplifications can be made depending on the smallness of the topography and/or its steepness. We should also mention that higher-order corrections in ϵ are comparable with the corrections due to the deviation of the Earth's form from sphericity and the passage to oblate-spheroidal curvilinear coordinates is then necessary (Gates 2004; White & Wood 2012); however, our procedure may be repeated in these coordinates as well.

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Concerning other planets, the relevant parameters are available at the NASA website http://nssdc.gsfc.nasa.gov/planetary/planetfact.html. A great uncertainty in the estimates, as those above, comes from the values of the wind speeds, and of the typical depth of the atmosphere, especially for giant planets. With reasonable guesses for the latter we arrive at values of ϵ that are not much different from the Earth's values. For Jupiter and Saturn we thus get estimates for μ roughly of the same order as for the Earth, while the values of μ are much greater for Venus and Mars due to the slow rotation and small radius, respectively. The close to unity value of μ for Venus means that deep columnar motion corrections to the kinetic energy terms in the reduced Lagrangian could be relevant in this case. Otherwise, the non-traditional approximation, as we defined it, is as well justified for the other planets as for the Earth, although the crude estimates above can, and should, be refined.

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Appendix A. Relevant concepts from differential geometry and spherical coordinates

We recall that if one considers a transformation from the Cartesian coordinates $\boldsymbol{x}(x_1, x_2, x_3)$ to spherical coordinates $\boldsymbol{\xi}(\xi_1 = \lambda, \xi_2 = \phi, \xi_3 = r)$, the transformation matrix is defined as

$$E = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}, \quad e^i_j = \frac{\partial x^i}{\partial \xi^j}, \quad i, j = 1, 2, 3. \tag{A 1}$$

The Jacobian of the transformation is $J = |E|$, the covariant basis is defined by the vectors $\boldsymbol{e}_i = e^j_i$ and the dual contravariant basis by $\boldsymbol{e}^i = (1/J)\boldsymbol{e}_j \times \boldsymbol{e}_k$, where i, j, k are 1, 2, 3 or their cyclic permutation. The relationships between the velocity vector $\boldsymbol{v}$ and its covariant v_i and contravariant components v^i are

$$v_i = \boldsymbol{v} \cdot \boldsymbol{e}_i \quad \text{and} \quad v^i = \boldsymbol{v} \cdot \boldsymbol{e}^i. \tag{A 2}$$

The covariant and contravariant basis vectors are not normalized:

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = g_{ij}, \quad \text{while } \boldsymbol{e}_i \cdot \boldsymbol{e}^j = \delta^j_i, \tag{A 3}$$

where δ^j_i, the Kronecker delta, is the unit matrix, and g_{ij} is the metric tensor, which has the following form in spherical coordinates (λ, ϕ, r):

$$g_{ij} = \mathrm{diag}(r^2 \cos^2 \phi, r^2, 1). \tag{A 4}$$

The inverse of g_{ij} is g^{ij}, and $\det(g_{ij})$ is denoted by g.

The physical basis is formed by the normalized counterparts of $\boldsymbol{e}_i$:

$$\boldsymbol{i}_i = \frac{1}{\sqrt{g_{ii}}}\boldsymbol{e}_i = \pm\sqrt{g_{ii}}\boldsymbol{e}^i. \tag{A 5}$$

The relations between the physical $(\hat{F}_i)$, covariant (F_i) and contravariant (F^i) components of a vector $\boldsymbol{F}$ are as follows:

$$\hat{F}_i = \sqrt{g_{ii}}F^i = \pm\frac{1}{\sqrt{g_{ii}}}F_i. \tag{A 6}$$

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The contravariant and covariant components of a vector product $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ are given by

$$c_i = \sqrt{g}\epsilon_{ijk}a^j b^k, \quad c^i = \frac{1}{\sqrt{g}}\epsilon^{ijk}a_j b_k, \tag{A 7}$$

where ϵ_{ijk} is an absolutely antisymmetric invariant unit tensor, with $\epsilon_{123} = \epsilon^{123} = 1$, and summation over repeated indices is understood everywhere if not otherwise explicitly stated.

The three-index Christoffel symbols are defined as follows:

$$\Gamma^k_{ij} = \frac{g^{kl}}{2}(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}). \tag{A 8}$$

They are used to construct the covariant derivative of a vector:

$$D_i F^k = \partial_i F^k + \Gamma^k_{ij} F^j, \quad D_i F_k = \partial_i F_k - \Gamma^j_{ik} F_j. \tag{A 9}$$

The covariant derivative of a scalar coincides with the ordinary derivative $D_i F = \partial_i F$. The covariant derivative obeys all the usual properties of the derivatives. The covariant divergence and covariant curl of a vector are defined, respectively, as follows:

$$D_i F^i = \partial_i F^i + \Gamma^k_{ik} F^i = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}F^i), \tag{A 10}$$

$$\frac{1}{\sqrt{g}} D_i F_j \epsilon^{ijk} \equiv \frac{1}{2\sqrt{g}}(D_i F_j - D_j F_i) = \frac{1}{2\sqrt{g}}(\partial_i F_j - \partial_j F_i). \tag{A 11}$$

With these definitions the integral theorems of vector analysis (Gauss theorem and Stokes theorem) hold in curvilinear coordinates.

Appendix B. Alternative derivations of the non-traditional RSW equations

In this appendix we sketch two alternative derivations of the non-traditional RSW equations on the sphere.

B.1. *Straightforward vertical averaging of the primitive equations*

As discussed in the main text, the notion of columnar motion should be revisited on the sphere, and the hypothesis must be applied to the angular instead of the linear velocities. For example, the shallow-water continuity equation is obtained on the plane by vertical averaging of the three-dimensional continuity equation, or of the divergence equation in the case of an incompressible fluid. The form of the divergence equation in spherical coordinates (2.13) shows that when the traditional shallow-layer replacement $r \to r_0$ is abandoned, the natural vertical integration is not over dr, but over r^2dr. In other words, while in the planar geometry the vertical integration is over an elementary cylinder, in spherical geometry it is over an elementary cone (solid angle). However, the vertical integration of equations expressed in term of physical components of the velocity vector with the r^2dr measure is not the key to getting the right set of equations: one ends up with equations of a similar structure to (3.42)–(3.44), but with incorrect coefficients, leading to non-conservation of the axial angular momentum. In what follows, we will see that actually considering the equations expressed in terms of contravariant components and integrating over an elementary cone will lead to the dynamically consistent equations.

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By integrating (2.13) between the bottom r_b and arbitrary material surface r inside the fluid layer and using the Leibnitz formula for differentiation of the integrals with variable limits, we get an expression for the vertical velocity $v_r(r)$:

$$r^2 v_r(r) - r_b^2 v_r(r_b) = -\frac{1}{\cos\phi}\left[\partial_\phi\left(\cos\phi\int_{r_b}^{r} r\mathrm{d}r\, v_\phi\right) + \partial_\lambda\int_{r_b}^{r} r\mathrm{d}r\, v_\lambda\right]. \tag{B 1}$$

Using the definition of the radial velocity as the advective derivative of r, we get:

$$D_t\left(\frac{r^3 - r_b^3}{3}\right) + \frac{1}{\cos\phi}\left[\partial_\phi\left(\cos\phi\int_{r_b}^{r} r\mathrm{d}r\, v_\phi\right) + \partial_\lambda\int_{r_b}^{r} r\mathrm{d}r\, v_\lambda\right] = 0. \tag{B 2}$$

We replace v_ϕ, v_λ by their expressions in terms of angular velocities $\dot\phi$, $\dot\lambda$: $(v_\phi, v_\lambda) = (r\dot\phi,\ r\cos\phi\dot\lambda)$. (Here the dots are used to denote the angular velocities – not to be confused with Lagrangian derivatives. We recall that these angular velocities are precisely the components of the contravariant velocity (see (2.17)).) Assuming angular velocities to be constant through the layer (columnar motion), which is equivalent to replacing them by their radial averages (denoted by bars) and neglecting corrections (mean-field approximation), we arrive at the counterpart of (3.16):

$$\left(\partial_t + \bar{\dot\lambda}\partial_\lambda + \bar{\dot\phi}\partial_\phi\right)\left(\frac{r^3 - r_b^3}{3}\right) + \frac{1}{\cos\phi}\left[\partial_\phi(\cos\phi\bar{\dot\phi}) + \partial_\lambda(\cos\phi\bar{\dot\lambda})\right]\left(\frac{r^3 - r_b^3}{3}\right) = 0. \tag{B 3}$$

Under the same assumptions, (B 1) gives

$$r^2 v_r(r) - r_b^2 v_r(r_b) = -\frac{1}{\cos\phi}\left[\partial_\phi(\cos\phi\bar{\dot\phi}) + \partial_\lambda(\cos\phi\bar{\dot\lambda})\right]\left(\frac{r^3 - r_b^3}{3}\right). \tag{B 4}$$

We use the same definitions, with Euler–Lagrange correspondence, as in (3.11) for upper and lower boundaries. We can immediately rescale them (keeping the same notation for dimensionless thickness and topography) with the help of the parameter ϵ:

$$r_b = r_0(1 + \epsilon B(\lambda, \phi)), \quad r_t = r_0(1 + \epsilon(H(\lambda, \phi, t) + B(\lambda, \phi))), \tag{B 5}$$

and recover, at leading order in ϵ, the traditional continuity equation of the RSW on the sphere:

$$(\partial_t + \dot\lambda\partial_\lambda + \dot\phi\partial_\phi)h + \frac{1}{\cos\phi}\left[\partial_\phi(\cos\phi\dot\phi) + \partial_\lambda(\cos\phi\dot\lambda)\right]h = 0, \tag{B 6}$$

where we omitted bars over the mean angular velocities in order to simplify notation.

A consistent way to average the momentum equations for the horizontal velocity components is to rewrite them first in the contravariant form, and then integrate over the volume of the layer. The advantage of this approach is that, first, the contravariant components of the velocity are precisely the angular velocities (see (2.17)), which will then be subject to the columnar motion hypothesis, and that, second, the standard integral theorems (Gauss and Stokes theorems) are unambiguously valid in curvilinear coordinates only when covariant derivatives are used. The covariant derivatives take care of the curvature terms in (2.9) and (2.10), which would otherwise pose

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problems upon integration and application of the integral theorems. We thus rewrite the primitive equations in terms of the contravariant components:

$$\partial_t v^i + v^k D_k v^i + \frac{1}{\sqrt{g}}\epsilon^{ijk}\Omega_j v_k = -g^{ik}D_k\Pi + bg^i, \tag{B 7}$$

where Ω_j are the covariant components of the angular velocity vector, and g^j are the contravariant components of the gravitational acceleration. Combining these equations with the covariant form of the continuity equation $D_i v^i = 0$ we obtain the conservative form of (B 7):

$$\partial_t v^i + D_k(v^k v^i) + \frac{1}{\sqrt{g}}\epsilon^{ijk}\Omega_j v_k = -g^{ik}D_k\Pi + bg^i. \tag{B 8}$$

We then integrate these equations over the volume of the fluid layer between two material surfaces, and exploit the fact that the integral theorems hold in curvilinear coordinates with the replacement of ordinary derivatives by the covariant ones. In the quasi-hydrostatic approximation, the equation of vertical motion does not contain the time derivative of the vertical velocity. In terms of contravariant components of velocity, this equation reads (see (2.30) for its Lagrangian counterpart):

$$r(\cos^2\phi\dot{\lambda}^2 + \dot{\phi}^2) + 2\Omega r\cos^2\phi\dot{\lambda} - bg - \partial_r\Pi = 0, \tag{B 9}$$

and may be straightforwardly integrated from the free surface r_t downward, giving the geopotential Π at any radial distance as

$$\Pi(r) = \Pi(r_t) + \bar{b}g(r_t - r) - (\cos^2\phi\dot{\lambda}^2 + \dot{\phi}^2 + 2\Omega\cos^2\phi\dot{\lambda})\left(\frac{r_t^2 - r^2}{2}\right), \tag{B 10}$$

where the value of the pressure at the free surface is taken to be zero without loss of generality.

We now integrate (B 8) for $v^{1,2}$ over the volume with the measure $r^2\,\mathrm{d}r\cos\phi\,\mathrm{d}\phi\,\mathrm{d}\lambda$, where the integration limits in r are (r_b, r_t). Here we adopt the same hypotheses as in § 3, namely: (i) constant angular velocities are assumed throughout the layer, (ii) only zero-order in ϵ terms are to be retained while integrating the terms with derivatives on the left-hand side, (iii) the first-order term in ϵ is to be retained in the Coriolis term, and the expression of Π from the quasi-hydrostatic balance (B 10) is to be used with the first-order term in ϵ to be retained in the non-standard Coriolis term in the quasi-hydrostatic balance. The explicit form of (B 8) for $v^{1,2}$ is

$$\partial_t\dot{\lambda} + D_k(v^k\dot{\lambda}) - 2\Omega\left(\dot{\phi}\tan\phi - \frac{v_r}{r}\right) = -\frac{1}{r^2\cos^2\phi}\partial_\lambda\Pi, \tag{B 11}$$

$$\partial_t\dot{\phi} + D_k(v^k\dot{\phi}) + 2\Omega\dot{\lambda}\cos\phi\sin\phi = -\frac{1}{r^2}\partial_\phi\Pi, \tag{B 12}$$

where, again, the dotted variables mean corresponding angular velocities. The integration of the derivative terms on the left-hand side via the properties of covariant derivatives (see appendix A) gives the advective horizontal derivative of the respective angular velocities multiplied by h and integrated over the solid angle. After accomplishing integration by parts, this latter integration, whose sole role is to ensure the validity of the standard integral theorems, may be omitted. The integration of the remaining terms, where the expressions (B 4) and (B 9) for the vertical velocity and geopotential as functions of r are to be substituted, is straightforward. The whole integration procedure, after the passage to the physical components of velocity (3.41) and the use of the continuity equation, reproduces (3.42)–(3.44).

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B.2. *Columnar motion reduction and change of independent variables in the primitive equations in the curl form*

B.2.1. *Curl form of equations in time-dependent curvilinear coordinates*

Our starting point in this subsection will be the primitive equations written in the curl form (2.3). In the arbitrary curvilinear coordinate system $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3) \mapsto \boldsymbol{r}(\boldsymbol{\xi}, t)$, the equations are written as

$$\partial_t \begin{pmatrix} v_1^a \\ v_2^a \\ v_3^a \end{pmatrix} + \left(\boldsymbol{\nabla}_{\boldsymbol{\xi}} \times \begin{pmatrix} v_1^a \\ v_2^a \\ v_3^a \end{pmatrix} \right) \times \begin{pmatrix} v^1 - w^1 \\ v^2 - w^2 \\ v^3 - w^3 \end{pmatrix} + \boldsymbol{\nabla}_{\boldsymbol{\xi}} \mathscr{B} = 0, \tag{B 13}$$

where $\boldsymbol{\nabla}_{\boldsymbol{\xi}}$ is the vector notation for ∂_{ξ_i} and the Bernoulli function $\mathscr{B}$ is given by

$$\mathscr{B} = \left(\frac{(r\dot{\hat{\boldsymbol{r}}})^2}{2} + \Pi - gr - \boldsymbol{v}^a \cdot \boldsymbol{w} \right), \tag{B 14}$$

where, as in § 3.2, we supposed a constant b equal to unity. We denote here by $\left(v_1^a, v_2^a, v_3^a\right)$ the covariant components of the absolute velocity $\boldsymbol{v}^a$ tangent to the sphere, by $\left(v^1, v^2, v^3\right)$ the contravariant components of the full three-dimensional relative velocity $\boldsymbol{v} = \dot{\boldsymbol{r}}$, and by $\left(w^1, w^2, w^3\right)$ the contravariant components of $\boldsymbol{w}$, where $\boldsymbol{w} = \partial_t \boldsymbol{r}(\boldsymbol{\xi}, t)$ describes the motion of the curvilinear system itself.

The advantage of (B 13) is that, in spite of being written in curvilinear coordinates, the metric factors in the vector products (see appendix A) cancel due to the combined use of covariant and contravariant components.

We consider a mapping of spherical coordinates onto themselves with a time-dependent radial coordinate: $\boldsymbol{\xi} = (\lambda, \phi, \xi_3) \mapsto \boldsymbol{r} = (\lambda, \phi, r(\lambda, \phi, \xi_3, t))$, Then the covariant basis is

$$\boldsymbol{e}_1 = r \cos\phi \hat{\boldsymbol{\lambda}} + \partial_1 r \hat{\boldsymbol{r}}, \quad \boldsymbol{e}_2 = r\hat{\boldsymbol{\phi}} + \partial_2 r \hat{\boldsymbol{r}}, \quad \boldsymbol{e}_3 = \partial_3 r \hat{\boldsymbol{r}}. \tag{B 15}$$

The related Jacobian is then $J = \det(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) = r^2 \cos\phi \partial_3 r$ and the vectors of the dual contravariant basis are

$$\boldsymbol{e}^1 = \frac{1}{r\cos\phi}\hat{\boldsymbol{\lambda}}, \quad \boldsymbol{e}^2 = \frac{1}{r}\hat{\boldsymbol{\phi}}, \quad \boldsymbol{e}^3 = \frac{-\partial_1 r \hat{\boldsymbol{\lambda}} - \cos\phi \partial_2 r \hat{\boldsymbol{\phi}} + r\cos\phi \hat{\boldsymbol{r}}}{r \partial_3 r \cos\phi}. \tag{B 16}$$

The explicit expressions of covariant and contravariant components of velocity are:

$$\left(v_1^a, v_2^a, v_3^a\right) = \left(r^2 \cos^2\phi(\dot{\lambda} + \Omega), r^2\dot{\phi}, 0\right), \tag{B 17}$$

$$\left(v^1 - w^1, v^2 - w^2, v^3 - w^3\right) = \left(\dot{\lambda}, \dot{\phi}, \frac{-\dot{\lambda}\partial_1 r - \dot{\phi}\partial_2 r + u_r - \partial_t r}{\partial_3 r}\right) \tag{B 18}$$

and $\boldsymbol{v}^a \cdot \boldsymbol{w} = 0$.

B.2.2. *Reduction to the shallow-water equations*

As above, we restrict the flow to move in columns and apply the same approximations as above while expanding in the small parameters μ and ϵ. In order to determine the vertical velocity v_r we integrate as before (see (B 4)) the continuity equation between the bottom $r = r_0 + B(\lambda, \phi)$ and some material surface

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$r(\xi_3)$, with the same result, modulo a change of independent variables. We thus deduce that $v^3 - w^3 = 0$, in accordance with the continuity equation and the fact that topography B is a material surface. As in the previous subsection we vertically integrate the quasi-hydrostatic relation between a material surface $r(\xi_3)$ and the top of the layer at $\xi = 1$, with the same result (see (B 10)) modulo a change of independent variables. We substitute all thus-obtained expressions in (B 13) and average vertically from $\xi_3 = 0$ to $\xi_3 = 1$, which is equivalent to evaluating all quantities at the mid-layer $\xi_3 = 1/2$. Retaining the dependence on r only in the absolute velocity part of the kinetic energy, according to our hypotheses, we get an approximate expression for the Bernoulli function:

$$\mathscr{B}_a = \frac{(r_0\dot{\hat{\boldsymbol{r}}})^2}{2} + g(h+B) + r_0 h(\boldsymbol{\Omega} \times \dot{\hat{\boldsymbol{r}}}) \cdot \hat{\boldsymbol{r}}. \tag{B 19}$$

Covariant components of the horizontal absolute velocity in this approximation are

$$v_1^a = r_0^2 \cos^2\phi \left(\dot{\lambda} + \Omega\left(1 + \frac{h+2B}{r_0}\right)\right), \quad v_2^a = r_0^2\dot{\phi}, \tag{B 20}$$

and the contravariant components of the relative velocity are, as before $v^1 = \dot{\lambda}$, $v^2 = \dot{\phi}$. We should emphasize that the variables (v_1^a, v_2^a) are, respectively, axial and zonal angular momentum densities, and their conjugate variables (v^1, v^2) are the components of relative angular velocity, as was already established in § 3.3, see (3.40). We thus obtain the curl form of the non-traditional RSW equations (3.42) and (3.43):

$$\partial_t \begin{pmatrix} v_1^a \\ v_2^a \\ 0 \end{pmatrix} + \left(\boldsymbol{\nabla}_{\boldsymbol{\xi}} \times \begin{pmatrix} v_1^a \\ v_2^a \\ 0 \end{pmatrix}\right) \times \begin{pmatrix} v^1 \\ v^2 \\ 0 \end{pmatrix} + \boldsymbol{\nabla}_{\boldsymbol{\xi}} \mathscr{B}_a = 0. \tag{B 21}$$

We rewrite (B 21) by replacing $\boldsymbol{\nabla}_{\boldsymbol{\xi}}$ by the horizontal gradient operator $\boldsymbol{\nabla}_{\boldsymbol{x}}$ defined in § 3.3 and calculated at the sphere of radius r_0, and obtain the final form of the non-traditional RSW equations on the sphere spanned by the unit vectors $\hat{\boldsymbol{\lambda}}$, $\hat{\boldsymbol{\phi}}$:

$$\partial_t \boldsymbol{l}_a + (\boldsymbol{\nabla}_{\boldsymbol{x}} \times \boldsymbol{l}_a) \times \boldsymbol{u} + r_0 \boldsymbol{\nabla}_{\boldsymbol{x}} \mathscr{B}_a = 0, \tag{B 22}$$

where $\boldsymbol{u}$ is defined in (3.41) and $\boldsymbol{x} \times \boldsymbol{l}_a$ is the mass density of the absolute angular momentum of the non-traditional shallow-water system:

$$\boldsymbol{l}_a = r_0 \left(\boldsymbol{u} + \boldsymbol{\Omega} \times \boldsymbol{x}\left(1 + \frac{h+2B}{r_0}\right)\right). \tag{B 23}$$

Its third component coincides with (3.39). These are the same non-traditional RSW equations as obtained in the main text (4.4). Taking the curl of (B 22), projecting onto the vector $\hat{\boldsymbol{r}}$ and combining with the continuity equation, the material conservation of potential vorticity

$$q = \frac{\boldsymbol{x} \cdot (\boldsymbol{\nabla}_{\boldsymbol{x}} \times \boldsymbol{l}_a)}{h} \tag{B 24}$$

follows – namely $\dot{q} = 0$.

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Appendix C. Intrinsic operators and Stokes formulae on the sphere

C.1. *Intrinsic operators on the sphere*

Let us first recall the intrinsic differential operators on a sphere. Consider the sphere of radius r_0 centred at the origin in the three-dimensional Euclidean space. A point on this sphere will be denoted by $\boldsymbol{x}$, with $|\boldsymbol{x}| = r_0$. At each point $\boldsymbol{x}$ there is a tangent plane to the sphere, with the unit normal vector $\hat{\boldsymbol{x}} = \boldsymbol{x}/r_0$. The sphere is a Riemannian submanifold of the three-dimensional Euclidean space. We therefore have the following well-defined notions, valid for general Riemannian submanifolds.

(i) A tangent vector field $\boldsymbol{u}(\boldsymbol{x})$ is a vector field defined on the manifold, with values in the ambient Euclidean space, which lies in the tangent space at each position (i.e. $\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{x} = 0$ for all $\boldsymbol{x}$ in the case of the sphere).

(ii) If $P(\boldsymbol{x})$ is a scalar field, its gradient $\nabla_{\boldsymbol{x}} P$ is a tangent vector field, defined by the identity $\mathrm{d}P = \nabla_{\boldsymbol{x}} P \cdot \mathrm{d}\boldsymbol{x}$. If we have local coordinates ξ_i, the gradient can be computed by writing $\partial P/\partial \xi_i = \nabla_{\boldsymbol{x}} P \cdot \partial \boldsymbol{x}/\partial \xi_i$. We have the formula

$$\nabla_{\boldsymbol{x}}(PQ) = P\nabla_{\boldsymbol{x}} Q + Q\nabla_{\boldsymbol{x}} P. \tag{C 1}$$

(iii) The volume element is defined as follows. Taking coordinates ξ_i, a vector $\mathrm{d}\boldsymbol{x}$ in the tangent space can be written $\mathrm{d}\boldsymbol{x} = \sum_i \partial \boldsymbol{x}/\partial \xi_i \, \mathrm{d}\xi_i$ (the scalars $\mathrm{d}\xi_i$ are the contravariant components of $\mathrm{d}\boldsymbol{x}$ relative to the coordinate system). Then the scalar product of two vectors $\mathrm{d}\boldsymbol{x}^1$, $\mathrm{d}\boldsymbol{x}^2$ in the tangent space can be written as $\mathrm{d}\boldsymbol{x}^1 \cdot \mathrm{d}\boldsymbol{x}^2 = \sum_{ij} \mathrm{d}\xi_i^1 b_{ij}(\boldsymbol{x}) \mathrm{d}\xi_j^2$, where $b(\boldsymbol{x})$ is a symmetric positive definite matrix. We can thus define $q(\boldsymbol{x}) = (\det b(\boldsymbol{x}))^{1/2}$, and the volume element is given by $\mathrm{d}^N\boldsymbol{x} = q\mathrm{d}\xi_1 \ldots \mathrm{d}\xi_N$. One can check that the result is independent of the choice of the coordinate system.

(iv) Given a tangent vector field $\boldsymbol{u}(\boldsymbol{x})$, it can be written in the coordinates (ξ_i) as $\boldsymbol{u}(\boldsymbol{x}) = \sum_i \partial \boldsymbol{x}/\partial \xi_i \, \boldsymbol{u}_i(\boldsymbol{x})$ (contravariant decomposition). Then its divergence is the scalar field

$$\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} = \frac{1}{q} \sum_i \frac{\partial (q\boldsymbol{u}_i)}{\partial \xi_i}. \tag{C 2}$$

The result does not depend on the choice of the coordinate system. We have the formula, for a scalar field P,

$$\nabla_{\boldsymbol{x}} \cdot (P\boldsymbol{u}) = P\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} + \boldsymbol{u} \cdot \nabla_{\boldsymbol{x}} P. \tag{C 3}$$

(v) Given a tangent vector field $\boldsymbol{u}(\boldsymbol{x})$ and a scalar field $P(\boldsymbol{x})$, we define the scalar field $(\boldsymbol{u} \cdot \nabla_{\boldsymbol{x}})P \equiv \boldsymbol{u} \cdot (\nabla_{\boldsymbol{x}} P) = \mathrm{d}P \; \boldsymbol{u}$.

(vi) Given a tangent vector field $\boldsymbol{u}(\boldsymbol{x})$ and a vector field $\boldsymbol{v}(\boldsymbol{x})$ with values in the ambient Euclidean space, we define the vector field $(\boldsymbol{u} \cdot \nabla_{\boldsymbol{x}})\boldsymbol{v}$ by its ith component in a fixed basis, given by $(\boldsymbol{u} \cdot \nabla_{\boldsymbol{x}})\boldsymbol{v}_i$. Equivalently, $(\boldsymbol{u} \cdot \nabla_{\boldsymbol{x}})\boldsymbol{v} = \mathrm{d}\boldsymbol{v} \, \boldsymbol{u}$. We can define also $\nabla_{\boldsymbol{x}} \cdot (\boldsymbol{v} \otimes \boldsymbol{u})$ by its ith component in a fixed basis, given by $\nabla_{\boldsymbol{x}} \cdot (\boldsymbol{v}_i \boldsymbol{u})$, or equivalently

$$\nabla_{\boldsymbol{x}} \cdot (\boldsymbol{v} \otimes \boldsymbol{u}) = \boldsymbol{v}\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla_{\boldsymbol{x}})\boldsymbol{v}. \tag{C 4}$$

The previous definitions are valid for any Riemannian submanifold. For our case of the two-dimensional sphere in the three-dimensional space, we have the following.

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(i) For a tangent vector field $\boldsymbol{u}(\boldsymbol{x})$, $\boldsymbol{x}\times\boldsymbol{u}$ is also a tangent vector field, where $\times$ is the usual three-dimensional cross product. Then we define the scalar field

$$\mathrm{curl}_{\boldsymbol{x}}\boldsymbol{u} = -\nabla_{\boldsymbol{x}}\cdot\left(\frac{\boldsymbol{x}}{r_0}\times\boldsymbol{u}\right). \tag{C 5}$$

We have $\mathrm{curl}_{\boldsymbol{x}}\nabla_{\boldsymbol{x}}P = 0$ for all scalar fields P.

(ii) For a three-dimensional vector field $\boldsymbol{v}(\boldsymbol{x})$, we define its projection on the tangent plane

$$\pi_{\boldsymbol{x}}\boldsymbol{v} = \boldsymbol{v} - \left(\boldsymbol{v}\cdot\frac{\boldsymbol{x}}{r_0}\right)\frac{\boldsymbol{x}}{r_0}. \tag{C 6}$$

We have $\boldsymbol{x}\times(\boldsymbol{x}\times\boldsymbol{v}) = -r_0^2\,\pi_{\boldsymbol{x}}\boldsymbol{v}$.

(iii) If $\boldsymbol{\Omega}$ is a constant vector in the three-dimensional space,

$$\nabla_{\boldsymbol{x}}\cdot(\pi_{\boldsymbol{x}}\boldsymbol{\Omega}) = -2\,\frac{\boldsymbol{x}\cdot\boldsymbol{\Omega}}{r_0^2}. \tag{C 7}$$

This formula, in fact, reflects the constant curvature of the sphere.

(iv) If $\boldsymbol{u}(\boldsymbol{x})$ is a tangent vector field, then

$$\boldsymbol{u}\cdot\nabla_{\boldsymbol{x}}\boldsymbol{u} + |\boldsymbol{u}|^2\frac{\boldsymbol{x}}{r_0^2} = \pi_{\boldsymbol{x}}\,(\boldsymbol{u}\cdot\nabla_{\boldsymbol{x}}\boldsymbol{u}) = (\mathrm{curl}_{\boldsymbol{x}}\boldsymbol{u})\frac{\boldsymbol{x}}{r_0}\times\boldsymbol{u} + \nabla_{\boldsymbol{x}}\frac{|\boldsymbol{u}|^2}{2}. \tag{C 8}$$

The first equality can be obtained by writing $\boldsymbol{x}\cdot(\boldsymbol{u}\cdot\nabla_{\boldsymbol{x}}\boldsymbol{u}) = \boldsymbol{u}\cdot\nabla_{\boldsymbol{x}}(\boldsymbol{x}\cdot\boldsymbol{u}) - \boldsymbol{u}\cdot(\boldsymbol{u}\cdot\nabla_{\boldsymbol{x}}\boldsymbol{x}) = -|\boldsymbol{u}|^2$.

C.2. *Stokes formulae on the sphere*

Consider a subdomain $\mathscr{D}$ of the two-dimensional sphere, having an unitary external normal $\boldsymbol{\nu}$ to the boundary. Note that for $\boldsymbol{x}\in\partial\mathscr{D}$, $\boldsymbol{\nu}(\boldsymbol{x})$ is normal to the boundary, but tangent to the sphere, $\boldsymbol{x}\cdot\boldsymbol{\nu}(\boldsymbol{x}) = 0$. Then we have the following integration formulae.

(i) If $\boldsymbol{u}(\boldsymbol{x})$ is a tangent vector field,

$$\int_{\mathscr{D}}\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} = \int_{\partial\mathscr{D}}\boldsymbol{u}\cdot\boldsymbol{\nu}\,\mathrm{d}\ell, \tag{C 9}$$

where $\mathrm{d}\boldsymbol{x}$ is the area element and $\mathrm{d}\ell$ is the length element on the boundary.

(ii) If $\boldsymbol{u}(\boldsymbol{x})$ is a tangent vector field and $\boldsymbol{v}(\boldsymbol{x})$ a three-dimensional vector field,

$$\int_{\mathscr{D}}\nabla_{\boldsymbol{x}}\cdot(\boldsymbol{v}\otimes\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} = \int_{\partial\mathscr{D}}\boldsymbol{v}\,(\boldsymbol{u}\cdot\boldsymbol{\nu})\,\mathrm{d}\ell, \tag{C 10}$$

where the two sides are three-dimensional vectors.

(iii) For a scalar field $P(\boldsymbol{x})$,

$$\int_{\mathscr{D}}\left(\nabla_{\boldsymbol{x}}P - 2P\frac{\boldsymbol{x}}{r_0^2}\right)\mathrm{d}\boldsymbol{x} = \int_{\partial\mathscr{D}}P\boldsymbol{\nu}\,\mathrm{d}\ell, \tag{C 11}$$

where the two sides are three-dimensional vectors. This formula is obtained by taking the scalar product with an arbitrary constant vector $\boldsymbol{\Omega}$. Since $\boldsymbol{\Omega}\cdot\nabla_{\boldsymbol{x}}P = (\pi_{\boldsymbol{x}}\boldsymbol{\Omega})\cdot\nabla_{\boldsymbol{x}}P = \nabla_{\boldsymbol{x}}\cdot(P\,\pi_{\boldsymbol{x}}\boldsymbol{\Omega}) - P\nabla_{\boldsymbol{x}}\cdot(\pi_{\boldsymbol{x}}\boldsymbol{\Omega})$, (C 11) follows immediately from (C 9) and (C 7). Note that taking $P = 1$ in (C 11) yields a relation between the integral of the normal on the boundary and the centre of mass of $\mathscr{D}$.

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(iv) For a scalar field $P(\boldsymbol{x})$,

$$\int_{\mathscr{D}} \boldsymbol{x} \times \nabla_x P \, \mathrm{d}\boldsymbol{x} = \int_{\partial\mathscr{D}} P \boldsymbol{x} \times \boldsymbol{v} \, \mathrm{d}\ell. \tag{C 12}$$

This can be obtained again by taking the scalar product with an arbitrary constant vector $\boldsymbol{\Omega}$, and using (C 9) and $\nabla_x \cdot (\boldsymbol{x} \times \boldsymbol{\Omega}) = -r_0 \mathrm{curl} \pi_x \boldsymbol{\Omega} = -r_0 \mathrm{curl} \nabla (\boldsymbol{\Omega} \cdot \boldsymbol{x}) = 0$.

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