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An area theorem for bounded univalent functions

By PRAN NATH CHICHRA
Punjabi University, Patiala, India

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1. *Introduction.* Let $f(z)$ be regular and univalent in $|z| < 1$ with a power series expansion

$$f(z) = z + a_2 z^2 + \ldots. \tag{1·1}$$

Then it is proved in (2) that

$$\left| \frac{f'(z_1)f'(z_2)}{[f(z_1)-f(z_2)]^2} - \frac{1}{(z_1-z_2)^2} \right| \leqslant \frac{1}{(1-|z_1|^2)(1-|z_2|^2)} \tag{1·2}$$

where z_1 and z_2 are points in $|z| < 1$.

The above inequality is important since, for example, many important distortion theorems may be obtained from it. Letting $z_2 \to z_1$ gives the bound for the Schwarzian derivative, viz.

$$|\{f(z), z\}| \leqslant \frac{6}{(1-|z|^2)^2} \quad (|z| < 1), \tag{1·3}$$

where

$$\{f(z), z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2. \tag{1·4}$$

The inequality (1·3) was first obtained by Nehari (4). In the particular case when $z = 0$, it gives

$$\left| \frac{a_3}{a_1} - \frac{a_2^2}{a_1^2} \right| \leqslant 1, \tag{1·5}$$

which is known as Bieberbach's inequality ((3), p. 213).

The purpose of this paper is to obtain the corresponding inequalities in the case of regular and univalent functions in $|z| < 1$ which are subjected to the condition (*a*) $|f(z)| < 1$ ($|z| < 1$), or (*b*) $\mathrm{Re} f(z) > 0$ ($|z| < 1$). To obtain these we shall require the generalized form of the Bieberbach area theorem which is stated in section 2.

2. Lemma. *Let $f(z)$ be meromorphic univalent in $|z| < 1$ with its only pole at $z = p$ ($|p| < 1$) and with principal part $a_{-1}/(z-p)$ so that $f(z)$ can be expanded in $|z| < 1$ as*

$$f(z) = \frac{a_{-1}}{z-p} + a_0 + a_1 z + a_2 z^2 + \ldots. \tag{2·1}$$

Then we have

$$\sum_{n=0}^{\infty} n|a_n|^2 \leqslant \frac{|a_{-1}|^2}{(1-|p|^2)^2}. \tag{2·2}$$

Proof. In the annulus $|p| < |z| < 1$, we have

$$f(z) = \frac{a_{-1}}{z} \sum_{n=0}^{\infty} \left(\frac{p}{z}\right)^n + \sum_{n=0}^{\infty} a_n z^n. \tag{2·3}$$

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Then from ((1), p. 2, Lemma 1·1), the area of the complement of the domain which is the image of $|z| < r$ ($|p| < r < 1$) is

$$\pi \left| \left[\frac{|a_{-1}|^2 r^2}{(r^2 - |p|^2)^2} - \sum_{n=0}^{\infty} n |a_n|^2 r^{2n} \right] \right|. \tag{2·4}$$

The expression within the square brackets in (2·4) is positive for r near to $|p|$ and $r > |p|$ and therefore for $|p| < r < 1$. Hence (2·2) follows when $r \to 1$.

An immediate consequence of (2·2) is

$$|a_1| \leqslant \frac{|a_{-1}|}{1 - |p|^2}. \tag{2·5}$$

One can see that (2·5) is sharp by taking

$$f(z) = a_{-1}\left(\frac{1}{z} + z\right) + a_0 \tag{2·6}$$

when $p = 0$, and

$$f(z) = a_{-1}\left(\frac{1}{z-p} - \frac{\bar{p}}{p}\frac{z}{1-|p|^2}\right) + a_0 \tag{2·7}$$

when $p \neq 0$. In the case of (2·6), $f(z_1) = f(z_2)$ and $z \neq z$ implies that $z_1 z_2 = 1$; and in the case of (2·7), $f(z_1) = f(z_2)$ and $z_1 \neq z_2$ implies that

$$z_1 = -\frac{p}{\bar{p}}\frac{1 - \bar{p} z_2}{z_2 - p}.$$

Hence in either case, if $|z_2| < 1$ then $|z_1| > 1$ and so $f(z)$ is univalent in $|z| < 1$.

3. Now we shall obtain an inequality corresponding to (1·2) for functions which are regular and univalent in $|z| < 1$ and are subjected to the condition $|f(z)| < 1$ ($|z| < 1$). Let

$$F(z) = \frac{f\left(\frac{z + z_1}{1 + \bar{z}_1 z}\right) - f(z_2)}{1 - f\left(\frac{z + z_1}{1 + \bar{z}_1 z}\right)\overline{f(z_2)}} \quad (|z_1| < 1, |z_2| < 1). \tag{3·1}$$

Then $F(z)$ is regular, univalent and $|F(z)| < 1$ in $|z| < 1$. It vanishes at $z = p$ ($|p| < 1$) where p is given by

$$\frac{p + z_1}{1 + \bar{z}_1 p} = z_2, \quad \text{i.e.} \quad p = \frac{z_2 - z_1}{1 - \bar{z}_1 z_2}. \tag{3·2}$$

Now construct the function

$$\Phi(z) = \frac{F(z)}{(1 + e^{i\gamma} F(z))^2} \quad (0 \leqslant \gamma < 2\pi). \tag{3·3}$$

Evidently $\Phi(z)$ is regular in $|z| < 1$ and vanishes at $z = p$ where p is defined in (3·2). Also $\Phi(z)$ is univalent in $|z| < 1$, for otherwise $\Phi(z_1) = \Phi(z_2)$ with $z_1 \neq z_2$ implies either $F(z_1) = F(z_2)$ with $z_1 \neq z_2$ or $F(z_1) F(z_2) = e^{-2i\gamma}$. The first case is not possible for $F(z)$ is univalent in $|z| < 1$. The second possibility is ruled out because $|F(z)| < 1$ in $|z| < 1$.

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Thus $1/\Phi(z)$ is meromorphic univalent with its simple pole at $z = p$, and therefore in $|z| < 1$ it has an expansion of the form (2·1), i.e.

$$\frac{1}{\Phi(z)} = \frac{b_{-1}}{z-p} + b_0 + b_1 z + b_2 z^2 + \ldots, \tag{3·4}$$

where
$$b_{-1} = \lim_{z \to p} \frac{z-p}{\Phi(z)} = \frac{1}{\Phi'(p)}$$

and
$$b_1 = \lim_{z \to 0} \left[\frac{\frac{1}{\Phi(z)} - \frac{b_{-1}}{z-p} - b_0}{z} \right] = -\frac{\Phi'(0)}{(\Phi(0))^2} + \frac{1}{p^2 \Phi'(p)}.$$

Now using (2·5) we have

$$|b_1| \leqslant \frac{|b_{-1}|}{1-|p|^2}, \tag{3·5}$$

i.e.
$$\left| \left[\frac{\Phi'(0)\,\Phi'(p)}{(\Phi(0))^2} - \frac{1}{p^2} \right] \right| \leqslant \frac{1}{1-|p|^2}. \tag{3·5'}$$

Differentiating (3·1) and (3·3) we have

$$F'(z) = \frac{(1-|z_1|^2)(1-|f(z_2)|^2) f'\left(\dfrac{z+z_1}{1+\bar{z}_1 z}\right)}{(1+\bar{z}_1 z)^2 \left[1 - \overline{f(z_2)} f\left(\dfrac{1+\bar{z}_1 z}{z+z_1}\right)\right]^2}$$

and
$$\Phi'(z) = \frac{(1-e^{i\gamma} F(z))\, F'(z)}{(1+e^{i\gamma} F(z))^3}.$$

Using (3·1), (3·3) and the above, and noticing that $F(p) = 0$, we obtain from (3·5'), after simplification,

$$\left| \left\{ \frac{f'(z_1) f'(z_2)}{(f(z_1) - f(z_2))^2} - \frac{1}{(z_1 - z_2)} \right\} - e^{2i\gamma} \left\{ \frac{f'(z_1) f'(z_2)}{(1 - f(z_1)\overline{f(z_2)})^2} \right\} \right| \leqslant \frac{1}{(1-|z_1|^2)(1-|z_2|)^2}. \tag{3·6}$$

Choose γ such that

$$2\gamma = \pi + \arg \left\{ \frac{f'(z_1) f'(z_2)}{(f(z_1) - f(z_2))^2} - \frac{1}{(z_1 - z_2)^2} \right\} - \arg \left\{ \frac{f'(z_1) f'(z_2)}{(1 - f(z_1)\overline{f(z_2)})^2} \right\}.$$

Then (3·6) gives

$$\left| \frac{f'(z_1) f'(z_2)}{(f(z_1) - f(z_2))^2} - \frac{1}{(z_1 - z_2)^2} \right| \leqslant \frac{1}{(1-|z_1|^2)(1-|z_2|^2)} - \left| \frac{f'(z_1) f'(z_2)}{(1 - f(z_1)\overline{f(z_2)})^2} \right|, \tag{3·7}$$

which is the analogue of (1·2) for functions which are regular, univalent and bounded in modulus by unity in $|z| < 1$. In fact we can obtain (1·2) from (3·7) as a corollary in the following manner.

In the above discussion we have assumed that $|f(z)| < 1$ ($|z| < 1$). Suppose now only that $f(z)$ is regular and univalent in $|z| < 1$. Given z_1, z_2 in $|z| < 1$ consider $\rho < 1$ but near enough to 1 to ensure that z_1/ρ, z_2/ρ also lie in $|z| < 1$. Let $M \geqslant M(\rho, f)$ and

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consider $[f(\rho z)]/M$. In $|z| < 1$ this function is regular, univalent and less than 1 in modulus and so we have, from (3·7),

$$\left|\frac{f'(z_1)f'(z_2)}{(f(z_1)-f(z_2))^2}-\frac{1}{(z_1-z_2)^2}\right| \leqslant \frac{\rho^2}{(\rho^2-|z_1|^2)(\rho^2-|z_2|^2)}-\left|\frac{f'(z_1)f'(z_2)}{\left(M-\frac{f(z_1)\overline{f(z_2)}}{M}\right)^2}\right|. \tag{3·8}$$

If in (3·8) we first of all let $M\to\infty$ and then let $\rho\to 1$ we obtain (1·2).

When $z_2\to z_1$ (3·7) gives, on replacing z_1 by z in the resulting equation,

$$|\{f(z),z\}| \leqslant \frac{6}{(1-|z|^2)^2}-\frac{6|f'(z)|^2}{(1-|f(z)|^2)^2} \quad (|z|<1), \tag{3·9}$$

where $\{f(z),z\}$ denotes the Schwarzian derivative defined in (1·4).

If $f(z)$ has power series expansion in the unit circle of the form

$$f(z) = a_1 z + a_2 z^2 + \ldots, \tag{3·10}$$

then taking $z = 0$ in (3·9) we obtain

$$\left|\frac{a_3}{a_1}-\frac{a_2^2}{a_1^2}\right| \leqslant 1-|a_1|^2, \tag{3·11}$$

which is the analogue of Bieberbach's inequality (1·5) in our present context.

The inequalities (3·9) and (3·11) were earlier obtained by Singh (5).

Case of equality. Equality holds in (3·7) when it holds in (3·5). From the discussion at the end of section 2, it follows that for equality we must have

$$\frac{(1+e^{i\gamma}F(z))^2}{F(z)} = \frac{1}{\Phi(z)} = b_{-1}\left[\frac{1}{z-p}-\frac{\overline{p}}{p}\frac{z}{(1-|p|^2)}\right]+b_0 \tag{3·12}$$

where b_0, b_{-1} and γ have to be chosen properly. After obtaining $F(z)$ from the above we can find $f(z)$ from (3·1).

In particular, when $p = 0$, i.e. when $z_1 = z_2$, we see that equality holds for functions $w = f(z)$ given by

$$\frac{f(z)}{(1+e^{i\gamma}f(z))^2} = \frac{a_1 z}{(1+e^{i\gamma}z)^2} \quad (0\leqslant\gamma<2\pi) \tag{3·13}$$

and

$$\frac{f(z)}{1+e^{i\gamma}(f(z))^2} = \frac{a_1 z}{1+e^{i\gamma}z^2} \quad (0\leqslant\gamma<2\pi). \tag{3·14}$$

The functions (3·13) map $|z| < 1$ onto $|w| < 1$ with a slit pointing towards the origin and with the free end at $e^{-i\gamma}$. If λ be the distance from the origin of the free end of the slit, then λ is given by $\lambda/(1+\lambda)^2 = \frac{1}{4}a_1$. The functions (3·14) map $|z| < 1$ onto $|w| < 1$ with two slits pointing towards the origin and with the free ends at $\pm i e^{-i\gamma}$ respectively. If λ be the distance from the origin of the free ends of the slits, then λ is given by $\lambda/(1+\lambda^2) = \frac{1}{2}a_1$.

4. Now we shall use (3·7) to obtain an analogue of (1·2) for functions which are regular, univalent and have positive real part in the unit circle.

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Let $f(z)$ be regular univalent and satisfy $\mathrm{Re} f(z) > 0 (|z| < 1)$. Then it can be written as

$$f(z) = \frac{1+g(z)}{1-g(z)} \tag{4·1}$$

where $|g(z)| < 1 (|z| < 1)$ ((3), p. 169). Now (4·1) gives

$$g(z) = \frac{f(z)-1}{f(z)+1}. \tag{4·2}$$

Since the linear transformation of a univalent function leads again to a univalent function $g(z)$ is regular and univalent in the unit circle. As $g(z)$ is regular univalent and satisfies $|g(z)| < 1 (|z| < 1)$ (3·7) gives

$$\left| \frac{g'(z_1)\,g'(z_2)}{(g(z_1)-g(z_2))^2} - \frac{1}{(z_1-z_2)^2} \right| \leqslant \frac{1}{(1-|z_1|^2)(1-|z_2|^2)} - \left| \frac{g'(z_1)\,g'(z_2)}{(1-g(z_1)\,\overline{g(z_2)})^2} \right|.$$

Now using (4·2), we get from the above,

$$\left| \frac{f'(z_1)f'(z_2)}{(f(z_1)-f(z_2))^2} - \frac{1}{(z_1-z_2)^2} \right| \leqslant \frac{1}{(1-|z_1|^2)(1-|z_2|^2)} - \left| \frac{f'(z_1)f'(z_2)}{(f(z_1)+\overline{f(z_2)})^2} \right|. \tag{4·3}$$

This corresponds to (1·2) when $f(z)$ is regular, univalent and satisfies the condition $\mathrm{Re} f(z) > 0$ in $|z| < 1$.

When $z_2 \to z_1$, (4·3) yields

$$|\{f(z), z\}| \leqslant \frac{6}{(1-|z|^2)^2} - \frac{3}{2}\frac{|f'(z)|^2}{(\mathrm{Re} f(z))^2} \tag{4·4}$$

with equality for $f(z) = 1+z/1-z$.

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