9

Nodal Domains

In this chapter, we derive general nodal domain theorems of Courant type for Laplacians and p-Laplacians on graphs and hypergraphs.

In Chapters 4 and 8, we have systematically investigated the relation between Laplacian eigenvalues and Cheeger-type constants, that is, the ease or efficiency with which a connected geometric structure can be cut into two pieces. The Cheeger-type constants are defined as the extremal values, the minima, of a corresponding functional. But, we may then also naturally ask for the domains realizing such a minimum, and we may call such a pair of domains and their joint border a *Cheeger cut*. And we can then also ask whether the Laplace operator can help us with that. Naturally, associated with the eigenvalues, we have the eigenfunctions, and these possess *nodal domains*, that is, connected regions on which they do not change sign. One might then suspect that the first nonzero eigenvalue has just two such nodal domains and that they then represent, or at least approximate, a Cheeger cut. While this often is a good intuition, it is not always true for the standard Laplacian Δ, but as we shall see, and perhaps already expect from the fact that for the 1-Laplacian Δ_1, the Cheeger inequality becomes an equality, it turns out to work for that operator.

But the geometry of nodal domains is of geometric interest in any case, and it can also draw upon an independent history, which we shall now briefly sketch.

The Sturm–Liouville oscillation theory determines the number of zeros of eigenfunctions of certain eigenvalue problems for ordinary differential equations. The basic result is that the zeros of the kth eigenfunction of a vibrating string divide the string into exactly k subintervals. As a natural generalization of Sturm–Liouville oscillation theory to higher-dimensional cases, Courant [69] proved a result regarding the zeros of eigenfunctions of certain elliptic differential operators, which is now referred to as Courant's nodal domain theorem. A (strong) *nodal domain* is a maximal connected region on which an

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eigenfunction is either positive or negative. In particular, the eigenfunction must not have zeros on such a nodal domain. Courant then showed that the number of nodal domains for the Dirichlet eigenvalue problem on a connected domain, as described in Section 3.1, is bounded by the degree of the eigenvalue. In particular, an eigenfunction for the first eigenvalue λ_1 has one nodal domain, as it does not change sign, and an eigenfunction for λ_2 has precisely two nodal domains, one on which it is positive and the other on which it is negative.

Analogous results hold on Riemannian manifolds. In particular, often the two nodal domains of the first eigenfunction provide good approximations of the Cheeger cut of the manifold into two large components separated by a small hypersurface. However, this is not always true. In the graph case, sometimes, the two nodal domains of the first eigenfunction provide a cut that is very different from the Cheeger cut. An instructive example where the nodal domain approximation of the Cheeger cut fails is provided by Guattery and Miller [120], and we will present more examples in the first section. Abundant extensions of Courant's nodal domain theorem have been obtained, including the general nodal domain theories on p-Laplacians [71], Riemannian manifolds [58], matrices, and graphs [74].

In this chapter, we investigate various discrete versions of nodal domain theorems, which help us understand the eigenfunctions for discrete structures. Following [106, 210], we shall work within the framework of signed graphs. Theorem 9.2.2 is very general and includes various approaches presented in the literature as special cases. It first appeared in [107], but we present a new proof here. Moreover, Theorem 9.3.1 also unifies many known results, and it includes new statements that are proved here for the first time. Its proof is based on the idea of constructing m-dimensional spaces of functions for comparison in the min–max inequalities, which characterize eigenvalues by Rayleigh quotients. These spaces are obtained by restricting an eigenfunction with m nodal domains to these domains, thereby creating m pairwise orthogonal functions.

9.1 Nodal Domain Theorems on Graphs and Matrices

The fundamental discrete nodal domain theorems for eigenfunctions of *generalized Laplacians*, that is, symmetric matrices with nonpositive off-diagonal entries, were established by Davies, Gladwell, Leydold, and Stadler in 2001 [74]. For detailed historical reviews and further advances in this topic, we refer to [44–46].

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We start by recalling the definition of strong and weak nodal domains for functions on graphs from [74, Definitions 1 and 2]:

Definition 9.1.1 ([74])**.** Let $\Gamma = (V,E)$ be a graph and let $f\colon V \to \mathbb{R}$ be a function. A positive (respectively, negative) *strong nodal domain* of f is a maximal connected induced subgraph of Γ on vertices $v \in V$ with $f(v) > 0$ (respectively, $f(v) < 0$).

A positive (respectively, negative) *weak nodal domain* of f is a maximal connected induced subgraph of Γ on vertices $v \in V$ with $f(v) \geq 0$ (respectively, $f(v) \leq 0$) that contains at least one nonzero vertex.

The main theorem in [74] shows that, on a connected graph, any eigenfunction corresponding to the kth eigenvalue λ_k of the graph Laplacian has at most $k + r - 1$ strong nodal domains, where r is the multiplicity of λ_k and at most k weak nodal domains. Also, their nodal domain theorem was established for general Schrödinger matrices, which include the Laplacian matrices as special cases and are defined as follows. Given an $N\times N$ symmetric matrix $M = (M_{ij})$, the *induced graph* $\Gamma = (V,E)$ is given by

$$V \coloneqq \{v_1,\dots,v_N\} \text{ and } E \coloneqq \{\{v,v'\}\colon M_{vv'} \neq 0 \text{ and } v \neq v'\},$$

and M is called a *compatible matrix* on Γ. A *Schrödinger matrix* on Γ is a compatible matrix with nonpositive off-diagonal entries.

Theorem 9.1.1 ([74])**.** *Let M be a Schrödinger matrix on a simple graph Γ. Let λ_k be the kth eigenvalue of M, and let r be the multiplicity of λ_k. Then, any eigenvector corresponding to λ_k has at most $k + r - 1$ strong nodal domains.*

Moreover, if Γ is connected, then any eigenvector corresponding to λ_k has at most k weak nodal domains.

It is natural to ask if there are nodal domain theorems for compatible matrices on graphs that do not necessarily satisfy the Schrödinger condition. This question was first systematically studied by Mohammadian [210], who proposed the following definitions.

Definition 9.1.2 ([210])**.** Let M be a symmetric matrix, let $\Gamma = (V,E)$ be the corresponding induced graph, and let $f\colon V \to \mathbb{R}$ be a function. Let also $\Gamma^{<}_{M,f}$ be the subgraph of Γ that has vertex set $\Omega \coloneqq \{v \in V\colon f(v) \neq 0\}$ and edge set

$$E^{<} \coloneqq \{\{v,v'\} \in E\colon f(v)M_{vv'}f(v') < 0\}.$$

Similarly, let $\Gamma^{\leq}_{M,f}$ be the graph that has vertex set V and edge set

$$E^{\leq} \coloneqq \{\{v,v'\} \in E\colon f(v)M_{vv'}f(v') \leq 0\}.$$

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A *strong (respectively, weak) nodal domain* of f is a connected component of $\Gamma^{<}_{M,f}$ (respectively, $\Gamma^{\leq}_{M,f}$).

Mohammadian also proved the following nodal domain theorem:

Theorem 9.1.2 ([210])**.** *Let M be a symmetric matrix such that its induced graph Γ is connected. Then, for any kth eigenfunction f_k of M,*

$$c(\Gamma^{\leq}_{M,f_k}) \leq k, \text{ and } c(\Gamma^{<}_{M,f_k}) \leq k + r - 1,$$

where $c(\cdot)$ is the number of connected components, and r is the multiplicity of the kth eigenvalue.

However, Mohammadian's definition of weak nodal domains differs from Definition 9.1.1. Recently, Ge and Liu [106] provided a new definition of weak nodal domain that is compatible with the classical case:

Definition 9.1.3 ([106])**.** Let $\Gamma = (V, E, s)$ be a signed graph, and let $f : V \to \mathbb{R}$ be a function. A sequence $\{v_k\}_{k=1}^{n}$ of vertices is called a *strong nodal domain path* of f, or simply an *S-path* of f, if

$$v_k \sim v_{k+1} \quad \text{and} \quad f(v_k)s(v_k v_{k+1})f(v_{k+1}) > 0,$$

for each $k = 1, 2, \ldots, n-1$.

Similarly, a sequence $\{v_k\}_{k=1}^{n}$ of vertices, for $n \geq 2$, is called a *weak nodal domain path* of f, or simply a *W-path* of f, if for any two vertices v_i and v_j such that $f(v_i) \neq 0$, $f(v_j) \neq 0$, and $f(v_\ell) = 0$ for any $i < \ell < j$, we have that

$$f(v_i)s(v_i v_{i+1}) \cdots s(v_{j-1} v_j)f(v_j) > 0.$$

In particular, according to this definition, every path that contains at most one vertex on which f is nonzero is a weak nodal domain path.

Now, fix a signed graph $\Gamma = (V, E, s)$, let $f : V \to \mathbb{R}$ be a function, and let $\Omega := \{v \in V : f(v) \neq 0\}$. We define equivalence relations $\overset{S}{\sim}$ and $\overset{W}{\sim}$ on Ω, as follows. Given $v, v' \in \Omega$, let $v \overset{S}{\sim} v'$ (respectively, $v \overset{W}{\sim} v'$) if $v = v'$ or there exists an S-path (respectively, a W-path) connecting v and v'. Let also $\{S_i\}_{i=1}^{p}$ (respectively, $\{W_i\}_{i=1}^{q}$) be the equivalence classes of the relation $\overset{S}{\sim}$ (respectively, $\overset{W}{\sim}$) on Ω. The induced subgraph of an equivalence class S_i is called a *strong nodal domain* of f. Similarly, the induced subgraph of a set

$$W_i^0 := W_i \cup \{v \in V : \text{there exists a W-path from } v \text{ to some vertex in } W_i\}$$

is called a *weak nodal domain* of f. We also let $\mathfrak{S}(f)$ (respectively, $\mathfrak{W}(f)$) be the number of strong (respectively, weak) nodal domains of f.

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Theorem 9.1.3 ([106]). *Let M be a symmetric matrix, let $\Gamma = (V,E)$ be its induced signed graph, and let λ_k be its kth eigenvalue. If f_k is an eigenfunction corresponding to λ_k, then*

$$\mathfrak{S}(f_k) \le k + r - 1, \text{ and } \mathfrak{W}(f_k) \le k + c - 1, \tag{9.1.1}$$

where r is the multiplicity of λ_k and c is the number of connected components of Γ. In particular, if Γ is connected, then $\mathfrak{W}(f_k) \le k$.

The *proof* will be contained in that of Theorem 9.2.2, which is a more general result.

We conclude this section with some examples that show how different the nodal domains of the second eigenfunction and the Cheeger cuts can be from each other. In particular, on the following two pages, we use a gray line for representing the Cheeger cut, and we use a gray dotted line for the cut that is determined by the two nodal domains of the first nontrivial eigenfunction.

The *ladder graph* on $2n$ vertices consists of two paths with vertices $v_1,\dots,v_n$ and $v_{2n},\dots,v_{n+1}$, respectively, together with an edge for each pair (v_j, v_{2n+1-j}). As a generalization, we define the *broken ladder graph* $BL(n,k)$ as the graph obtained from the ladder graph on $2n$ vertices by removing k rungs from one side, for a given $k \in \{0,\dots,n-1\}$. In particular, $BL(n,0)$ is the original ladder graph, while for $k > 0$, the edges between the first k pairs (v_j, v_{2n+1-j}) are removed. (See also illustrations below.) Let $\lambda_2(BL(n,k))$ be the second eigenvalue (therefore, the first nonzero eigenvalue) of the normalized Laplacian of $BL(n,k)$, and let $h(BL(n,k))$ be the Cheeger constant of $BL(n,k)$. For $n = 8$ and $k = 0,1,\dots,6$, we have

$$\begin{aligned}
\lambda_2(BL(8,0)) &= 0.060171,\\
\lambda_2(BL(8,1)) &= 0.0660208,\\
\lambda_2(BL(8,2)) &= 0.0720744,\\
\lambda_2(BL(8,3)) &= 0.0764965,\\
\lambda_2(BL(8,4)) &= 0.0637421,\\
\lambda_2(BL(8,5)) &= 0.0424294,\\
\lambda_2(BL(8,6)) &= 0.0301588,
\end{aligned}$$

and these are all simple eigenvalues. Moreover,

$$\begin{aligned}
h(BL(8,0)) &= 1/11,\\
h(BL(8,1)) &= 1/10,\\
h(BL(8,2)) &= 1/9,\\
h(BL(8,3)) &= 1/8,
\end{aligned}$$

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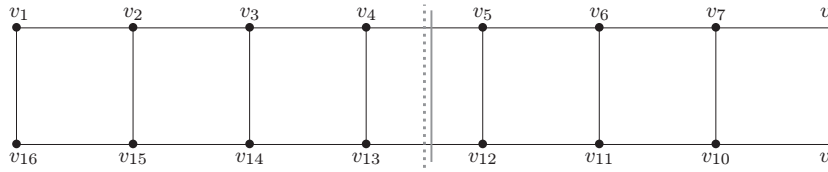


Figure 9.1 Ladder graph $BL(8,0)$.

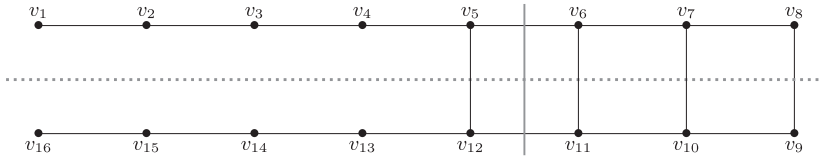


Figure 9.2 Ladder graph $BL(8,4)$.

$$h(BL(8,4)) = 1/8,$$
$$h(BL(8,5)) = 1/9,$$
$$h(BL(8,6)) = 1/11.$$

Of course, they all fall into the range determined in Theorem 4.3.1. For the ladder graph $BL(8,0)$ on 16 nodes shown in Figure 9.1, one eigenvector of $\lambda_2(BL(8,0))$, with entries rounded to two decimal entries, is

$$(0.30, 0.32, 0.22, 0.08, -0.08, -0.22, -0.32, -0.30,$$
$$-0.30, -0.32, -0.22, -0.08, 0.08, 0.22, 0.32, 0.30).$$

Its two nodal domains,

$$\{v_1, v_2, v_3, v_4, v_{13}, v_{14}, v_{15}, v_{16}\} \quad \text{and} \quad \{v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\},$$

give the exact Cheeger cut.

The same holds for $BL(8,1)$, $BL(8,2)$, and $BL(8,3)$. However, things start to change from the ladder graph $BL(8,4)$ shown in Figure 9.2. In fact, an eigenvector corresponding to $\lambda_2(BL(8,4))$, with entries rounded to three decimal entries, is

$$(0.337, 0.446, 0.359, 0.226, 0.078, 0.022, 0.006, 0.001,$$
$$-0.001, -0.006, -0.022, -0.078, -0.226, -0.359, -0.446, -0.337).$$

Its two nodal domains, $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ and $\{v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}$, yield a cut that is far from the Cheeger cut, which is given by

$$\{v_1, v_2, v_3, v_4, v_5, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\} \quad \text{and} \quad \{v_6, v_7, v_8, v_9, v_{10}, v_{11}\}$$

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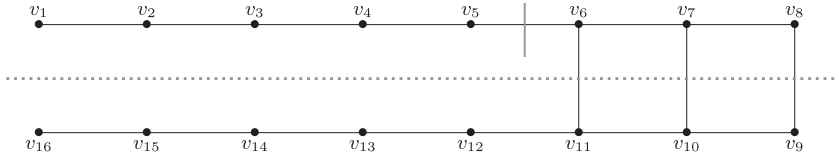


Figure 9.3 Ladder graph $BL(8,5)$.

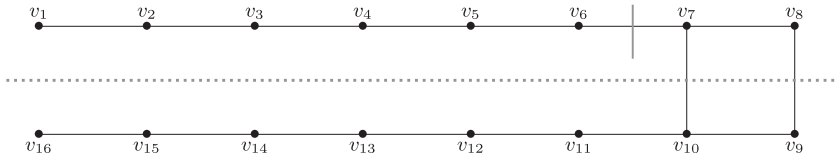


Figure 9.4 Ladder graph $BL(8,6)$.

Moreover, an eigenvector corresponding to $\lambda_2(BL(8,5))$ on the graph of Figure 9.3, with entries rounded to three decimal entries, is

$$(0.305, 0.412, 0.359, 0.275, 0.168, 0.057, 0.016, 0.004,$$
$$-0.004, -0.016, -0.057, -0.168, -0.275, -0.359, -0.412, -0.305).$$

Thus, the two nodal domains, $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ and $\{v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}$, do not yield the Cheeger cut, which is given by

$$\{v_1, v_2, v_3, v_4, v_5\} \quad \text{and} \quad \{v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}.$$

Similarly, an eigenvector corresponding to $\lambda_2(BL(8,6))$ on the graph of Figure 9.4, with entries rounded to two decimal entries, is

$$(0.28, 0.38, 0.35, 0.29, 0.22, 0.13, 0.04, 0.01,$$
$$-0.01, -0.04, -0.13, -0.22, -0.29, -0.35, -0.38, -0.28).$$

Its two nodal domains, $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ and $\{v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}$, again do not yield the right Cheeger cut, given by

$$\{v_1, v_2, v_3, v_4, v_5, v_6\} \quad \text{and} \quad \{v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}.$$

The broken ladder graph $BL(2k, k)$ was introduced by Guattery and Miller in [120] to show that the first nontrivial eigenfunction of the unnormalized Laplacian does not approximate the unnormalized Cheeger cut. Our preceding computations demonstrate that the same phenomenon still occurs for the normalized Laplacian and the standard Cheeger cut. For the 1-Laplacian, in contrast, the cut obtained from the nodal domains of the second eigenfunction coincides with the Cheeger cut, as we shall see in Theorem 9.2.1.

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9.2 Nonlinear Nodal Domain Theorems on Graphs

In order to derive nodal domain properties for p-Laplacians on graphs or signed graphs, we need to generalize the concepts from Sections 7.1 and 7.2.

Let $\Gamma = (V, E)$ be a graph with the following additional structure:

- Two positive measures $w\colon E \to \mathbb{R}^+$ and $\mu\colon V \to \mathbb{R}^+$;
- Real coefficients $\kappa\colon V \to \mathbb{R}$; and
- A sign function $s\colon E \to \{-1, 1\}$.

For $p > 1$, the *signed p-Laplacian* $\Delta_p^s\colon C(V) \to C(V)$ is defined by

$$\Delta_p^s f(v) := \sum_{v' \in V} w_{vv'} |f(v) - s(vv')f(v')|^{p-2}(f(v) - s(vv')f(v')) + \kappa_v |f(v)|^{p-2} f(v),$$

for $v \in V$ and $f \in C(V)$. Moreover, the *eigenvalue problem* for Δ_p^s, when $p > 1$, consists in finding $\lambda \in \mathbb{R}$ and $f \neq 0$ such that

$$\Delta_p^s f(v) = \lambda \mu_v |f(v)|^{p-2} f(v), \quad \forall v \in V.$$

The *signed* 1*-Laplacian*[1] Δ_1^s is the set-valued map defined by

$$\Delta_1^s f(v) := \left\{ \sum_{v' \in V} w_{vv'} z_{vv'} + \kappa_v z_v \;\middle|\; \begin{array}{l} z_{vv'} \in \operatorname{Sgn}(f(v) - s(vv')f(v')), \\ z_{vv'} = -s(vv') z_{v'v}, \\ z_v \in \operatorname{Sgn}(f(v)) \end{array} \right\},$$

where

$$\operatorname{Sgn}(t) := \begin{cases} \{1\} & \text{if } t > 0, \\ [-1, 1] & \text{if } t = 0, \\ \{-1\} & \text{if } t < 0. \end{cases}$$

The *eigenvalue problem* for Δ_1^s consists in finding $\lambda \in \mathbb{R}$ and $f \neq 0$ such that

$$\Delta_1^s f(v) \bigcap \lambda \mu_v \operatorname{Sgn}(f(v)) \neq \emptyset, \quad \forall v \in V.$$

By applying the Lusternik–Schnirelman theory, we can define a sequence of min–max eigenvalues for Δ_p^s as follows:

$$\lambda_k(\Delta_p^s) := \inf_{\operatorname{genus}(S) \geq k} \sup_{f \in S} \frac{\sum_{\{v,v'\} \in E} w_{vv'} |f(v) - s(vv')f(v')|^p + \kappa_v |f(v)|^p}{\sum_{v \in V} \mu_v |f(v)|^p},$$

[1] The idea to consider a signed version of the 1-Laplacian comes from a short discussion with Delio Mugnolo at the workshop [131]. Mugnolo asked the third author: Can the graph 1-Laplacian characterize the bipartiteness of a graph? Now we know the answer is yes, but we need to consider the signed version of the 1-Laplacian with a negative signature.

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for $k \geq 1$, where genus(S) denotes the Krasnoselskii genus of the centrally symmetric compact subset S in $C(V) \cong \mathbb{R}^V$ (see Definition 2.6.2). We also denote by Gen_k the collection of all centrally symmetric compact subsets with Krasnoselskii genus larger than or equal to k.

Remark 9.2.1. Any symmetric matrix can be represented by Δ_2^s, for some suitable parameters $w_{vv'}$, $s(vv')$, and κ_v. Also, if $\kappa \equiv 0$, then Δ_p^s is the classical p-Laplacian on signed graphs. If $s \equiv +1$, then Δ_p^s is the p-Schrödinger operator on graphs [76]. Moreover, if we let $\kappa \equiv 0$ and $w \equiv 1$, and we let μ be the weighted degree function on V, then we obtain an equivalent formulation of the normalized p-Laplacian that has been discussed in Chapter 7.

We now prove some preliminary results on positive and negative nodal domains for the p-Laplacians on simple graphs (i.e., graphs with $s \equiv +1$).

Proposition 9.2.1. *Let Γ be a connected (simple) graph, and let $p > 1$. Then, any eigenfunction corresponding to the second eigenvalue $\lambda_2(\Delta_p)$ of the p-Laplacian on Γ has exactly one positive weak nodal domain and one negative weak nodal domain.*

Proof By Proposition 7.1.2, we have that

$$\sum_{v \in V} \deg v \cdot \phi_p(f(v)) = 0.$$

This implies that $\{v \in V : f(v) > 0\} \neq \emptyset$ and $\{v \in V : f(v) < 0\} \neq \emptyset$. Thus, f has at least one positive and one negative weak nodal domain. But as we shall see in Theorem 9.2.2, any eigenfunction corresponding to $\lambda_2(\Delta_p)$ has at most two weak nodal domains. Hence, the claim follows. □

Example 9.2.1. In the graph with 10 vertices that is depicted in Figure 9.5, the function

$$f := \mathbf{1}_{\{5,6,7,8,9,10\}} - \mathbf{1}_{\{1,2,3,4\}}$$

is an eigenfunction corresponding to $\lambda_2(\Delta_1)$, and it has exactly three weak nodal domains.

Thus, Proposition 9.2.1 does not hold for $p = 1$. Instead, the following property holds.

Proposition 9.2.2. *Let Γ be a connected (simple) graph. Then, any eigenfunction f corresponding to $\lambda_2(\Delta_1)$ has exactly one positive weak nodal domain or exactly one negative weak nodal domain. Moreover, if f has at least two positive strong nodal domains and two negative strong nodal domains, then the number of weak nodal domains of f is exactly two.*

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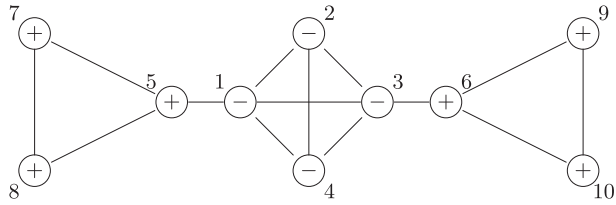


Figure 9.5 An example graph discussed in Example 9.2.1.

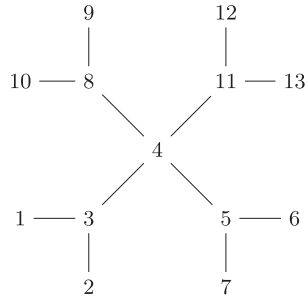


Figure 9.6 An example graph discussed in Example 9.2.2 where the second 1-Laplacian eigenvalue has variational multiplicity three and the hypotheses of Proposition 9.2.2 are satisfied.

Proof Assume without loss of generality, that $\{f > 0\} \neq \emptyset$. If $\{f < 0\} = \emptyset$, then f has exactly one positive weak nodal domain. If $\{f < 0\} \neq \emptyset$, then by Theorem 7.2.3, either $\{f \geq 0\}$ or $\{f \leq 0\}$ induces a connected subgraph on Γ, which implies that f has either exactly one positive weak nodal domain or exactly one negative weak nodal domain.

Now, assume that f has two positive strong nodal domains and two negative strong nodal domains. Then, both $\{f > 0\}$ and $\{f < 0\}$ induce disconnected subgraphs, and by Theorem 7.2.3, both $\{f \geq 0\}$ and $\{f \leq 0\}$ induce connected subgraphs, which implies that f has exactly one positive weak nodal domain and exactly one negative weak nodal domain. □

Example 9.2.2. In the graph with 13 vertices that is depicted in Figure 9.6, the function

$$f := \mathbf{1}_{\{1,2,3,5,6,7\}} - \mathbf{1}_{\{8,9,10,11,12,13\}}$$

is an eigenfunction corresponding to $\lambda_2(\Delta_1)$. It has four strong nodal domains (two positive ones and two negative ones), and it has exactly two weak nodal domains.

While we have seen above that the nodal domains of $\lambda_2(\Delta)$ can be very different from a Cheeger cut, we shall now show that for Δ_1, the nodal domains

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of $\lambda_2(\Delta_1)$ do yield a Cheeger cut (with some technical provisos that will of course be spelled out in detail). This is of course in line with the fact that this eigenvalue equals the Cheeger constant itself.

We shall denote the complement of a vertex set A by A^c. Thus, for every nonempty $A \subset V$ with nonempty complement, (A, A^c) is a cut of Γ.

Theorem 9.2.1. *For any eigenfunction f of a (simple) graph Γ corresponding to $\lambda_2(\Delta_1)$, and for any strong nodal domain A of f, (A, A^c) is a Cheeger cut of Γ.*

Proof Let $\lambda_2 := \lambda_2(\Delta_1)$ and, without loss of generality, assume that A is a positive strong nodal domain of f. Then, by the definition of a nodal domain,

$$\mathrm{Sgn}(f(v) - f(v')) \subset \mathrm{Sgn}(\mathbf{1}_A(v) - \mathbf{1}_A(v')),$$

for any $\{v, v'\} \in E$, and $\mathrm{Sgn}(f(v)) \subset \mathrm{Sgn}(\mathbf{1}_A(v))$, for any $v \in V$.

Thus, $\Delta_1 f(v) \subset \Delta_1 \mathbf{1}_A(v)$ and therefore,

$$\Delta_1 \mathbf{1}_A(v) \bigcap \lambda_2 \deg v \cdot \mathrm{Sgn}(\mathbf{1}_A(v)) \supset \Delta_1 f(v) \bigcap \lambda_2 \deg v \cdot \mathrm{Sgn}(f(v)) \neq \emptyset,$$

for all $v \in V$, which implies that $\mathbf{1}_A$ is also an eigenfunction corresponding to λ_2. As a consequence, $R_1(\mathbf{1}_A) = R_1(f) = \lambda_2$. Moreover, by Theorem 7.2.1, $\lambda_2 = h$. Also, one can check that $R_1(\mathbf{1}_A) = |\partial A|/\mathrm{vol}\,(A)$. Next, we show that $\mathrm{vol}\,(A) \leq \mathrm{vol}\,(A^c)$. To prove this, we can assume that Γ is connected, as otherwise we can simply consider the connected component containing A. Since $(\lambda_2, \mathbf{1}_A)$ is an eigenpair, by the definition of the 1-Laplace eigen-equation, there exists $z_{vv'} \in \mathrm{Sgn}(f(v) - f(v'))$ with $z_{vv'} = -z_{v'v}$, such that

$$\begin{cases} \sum_{v' \sim v} z_{vv'} = \lambda_2 \deg v, & \forall v \in A, \\ |\sum_{v' \sim v} z_{vv'}| \leq \lambda_2 \deg v, & \forall v \in A^c. \end{cases}$$

Therefore,

$$\begin{aligned} \lambda_2 \mathrm{vol}\,(A) &= \lambda_2 \sum_{v \in A} \deg v \\ &= \sum_{v \in A} \sum_{v' \sim v} z_{vv'} \\ &= \sum_{\{v, v'\} \in E} (z_{vv'} + z_{v'v}) - \sum_{v \in A^c} \sum_{v' \sim v} z_{vv'} \\ &= -\sum_{v \in A^c} \sum_{v' \sim v} z_{vv'} \\ &\leq \sum_{v \in A^c} \lambda_2 \deg v \\ &= \lambda_2 \mathrm{vol}\,(A^c), \end{aligned}$$

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which implies that $\mathrm{vol}\,(A) \leq \mathrm{vol}\,(A^c)$, since $\lambda_2 > 0$ for any connected graph. Hence, we have that $\mathrm{vol}\,(A) = \min\{\mathrm{vol}\,(A), \mathrm{vol}\,(A^c)\}$, and thus

$$\frac{|\partial A|}{\min\{\mathrm{vol}\,(A), \mathrm{vol}\,(A^c)\}} = \frac{|\partial A|}{\mathrm{vol}\,(A)} = R_1(\mathbf{1}_A) = \lambda_2 = h,$$

which implies that (A, A^c) is a Cheeger cut of Γ. □

By revisiting the proof of Theorem 9.2.1, we can also prove

Lemma 9.2.1. *Let f be an eigenfunction of a (simple) graph Γ corresponding to $\lambda_2(\Delta_1)$. Then,*

$$\mathrm{vol}\,\{f > 0\} \leq \mathrm{vol}\,\{f < 0\} + \mathrm{vol}\,\{f = 0\}$$

and

$$\mathrm{vol}\,\{f < 0\} \leq \mathrm{vol}\,\{f > 0\} + \mathrm{vol}\,\{f = 0\}.$$

Note that, in the statement of Lemma 9.2.1, we cannot replace $\{f > 0\}$ and $\{f < 0\}$ by a positive and a negative strong nodal domain, respectively. For example, the graph in Example 9.2.1 is such that $\{5,7,8\}$ and $\{6,9,10\}$ are the two positive strong nodal domains, while $\{1,2,3,4\}$ is the negative strong nodal domain of the eigenfunction

$$f = \mathbf{1}_{\{5,6,7,8,9,10\}} - \mathbf{1}_{\{1,2,3,4\}}$$

corresponding to $\lambda_2(\Delta_1)$. In this case, $\mathrm{vol}\,\{5,7,8\} = \mathrm{vol}\,\{6,9,10\} < \mathrm{vol}\,\{1,2,3,4\}$.

The preceding suggests that when f not only has a positive and a negative domain but also a nontrivial zero set, that zero set could be assigned to either of these domains, leading to two different Cheeger cuts. In fact, again by the proof of Proposition 9.2.1, if $\{f > 0\} \neq \emptyset$, then $(\{f > 0\}, \{f \leq 0\})$ is a Cheeger cut, and similarly, when $\{f < 0\} \neq \emptyset$, then $(\{f < 0\}, \{f \geq 0\})$ is a Cheeger cut. We conclude

Corollary 9.2.1. *If an eigenfunction for $\lambda_2(\Delta_1)$ has zeros and it changes sign, then Γ has at least two different Cheeger cuts.*

However, we should also note that either $\{f > 0\}$ or $\{f < 0\}$ might be empty. For example, for the broken ladder $BL(8,4)$, the Cheeger cut

$$(\{1,2,3,4,5,12,13,14,15,16\}, \{6,7,8,9,10,11\})$$

is unique, while the eigenfunction

$$f = \mathbf{1}_{\{6,7,8,9,10,11\}}$$

corresponding to $\lambda_2(\Delta_1)$ has no points that take negative values.

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We now turn to the nodal domains of eigenfunctions for higher-order eigenvalues. We need the following preliminary lemma (see [5, 157, 266]).

Lemma 9.2.2. *Let $p \geq 1$, and let $t, s, a, b \in \mathbb{R}$. Then,*

$$\begin{cases} |ta + sb|^p \geq (|t|^p a + |s|^p b)|a+b|^{p-2}(a+b), & \text{if } ab \leq 0, \\ |ta + sb|^p \leq (|t|^p a + |s|^p b)|a+b|^{p-2}(a+b), & \text{if } ab \geq 0. \end{cases} \tag{9.2.1}$$

For $p > 1$, the equality holds if and only if either $ab = 0$ or $t = s$.

In the particular case of $p = 1$, we further have that, for any $z \in \operatorname{Sgn}(a+b)$,

$$\begin{cases} |ta + sb| \geq (|t|a + |s|b)z, & \text{if } ab \leq 0, \\ |ta + sb| \leq (|t|a + |s|b)z, & \text{if } ab \geq 0. \end{cases}$$

For a signed graph (Γ, s) and a function $f\colon V \to \mathbb{R}$, we let $\overline{\mathfrak{S}}(f)$ (respectively, $\overline{\mathfrak{W}}(f)$) denote the number of strong (respectively, weak) nodal domains of f in the *opposite signed graph* $(\Gamma, -s)$ (see Definition 4.6.2).

Theorem 9.2.2. *Let $\Gamma = (V, E)$ be a signed graph on N nodes. For any eigenfunction f corresponding to the kth min–max eigenvalue $\lambda_k(\Delta_p^s)$, we have*

$$\mathfrak{W}(f) \leq \mathfrak{S}(f) \leq k + r - 1 \quad \text{and} \quad \overline{\mathfrak{W}}(f) \leq \overline{\mathfrak{S}}(f) \leq N - k + r,$$

where r is the multiplicity of $\lambda_k(\Delta_p^s)$. If $p > 1$, we further have

$$\mathfrak{W}(f) \leq k + c - 1 \quad \text{and} \quad \overline{\mathfrak{W}}(f) \leq N - k + c,$$

where c is the number of connected components of Γ.

Proof We split the proof into multiple steps.

- Let $\lambda_k := \lambda_k(\Delta_p^s)$. Assume that f has m strong nodal domains and denote them by $V_1, \ldots, V_m$. The idea of this step is that we shall consider the m-dimensional function space spanned by the restrictions of f to $V_1, \ldots, V_m$, and we shall then use the variational properties of the eigenvalues. For this purpose, we consider the linear function space X spanned by $f|_{V_1}, \ldots, f|_{V_m}$, where

$$f|_{V_i}(v) := \begin{cases} f(v), & \text{if } v \in V_i, \\ 0, & \text{if } v \notin V_i. \end{cases}$$

  Since $V_1, \ldots, V_m$ are pairwise disjoint, we have that $\dim X = m$. Moreover, each edge $\{v, v'\}$ intersects at most two nodal domains. Thus, by the definition of nodal domains, for $i \neq j$ and $\{v, v'\} \in E$ with $v \in V_i$ and $v' \in V_j$, we have

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$$
\begin{aligned}
&\left(f|_{V_i}(v) - s(vv')f|_{V_i}(v')\right) \cdot \left(f|_{V_j}(v) - s(vv')f|_{V_j}(v')\right) \\
&= -f|_{V_i}(v)s(vv')f|_{V_j}(v') \\
&\geq 0,
\end{aligned}
$$

since f has opposite signs on adjacent nodal domains connected by positive edges and the same signs on adjacent nodal domains connected by negative edges.

For the case of $p > 1$, let $g := \sum_{i=1}^{m} t_i f|_{V_i}$ and consider the function ϕ_p introduced in (7.1.4). Then,

$$
\begin{aligned}
&\sum_{\{v,v'\}\in E} w_{vv'}\,|g(v) - s(vv')g(v')|^p + \sum_{v\in V} \kappa_v |g(v)|^p \\
&= \sum_{\{v,v'\}\in E} w_{vv'} \left|\sum_{i=1}^{m} t_i \left(f|_{V_i}(v) - s(vv')f|_{V_i}(v')\right)\right|^p + \sum_{i=1}^{m} |t_i|^p \sum_{v\in V_i} \kappa_v |f(v)|^p \\
&\leq \sum_{v\sim v'} w_{vv'} \sum_{i=1}^{m} |t_i|^p \left(f|_{V_i}(v) - s(vv')f|_{V_i}(v')\right) \phi_p(f(v) - s(vv')f(v')) \\
&\quad + \sum_{i=1}^{m} |t_i|^p \sum_{v\in V_i} \kappa_v |f(v)|^p \\
&= \sum_{i=1}^{m} |t_i|^p \sum_{v\in V_i} f(v) \left(\sum_{v'\in V} w_{vv'} \phi_p(f(v) - s(vv')f(v'))\right) + \sum_{i=1}^{m} |t_i|^p \sum_{v\in V_i} \kappa_v |f(v)|^p \\
&= \sum_{i=1}^{m} |t_i|^p \sum_{v\in V_i} f(v) \left(\sum_{v'\in V} w_{vv'} \phi_p(f(v) - s(vv')f(v')) + \kappa_v |f(v)|^{p-2} f(v)\right) \\
&= \sum_{i=1}^{m} |t_i|^p \sum_{v\in V_i} f(v) \cdot \Delta_p^s f(v) \\
&= \sum_{i=1}^{m} |t_i|^p \sum_{v\in V_i} f(v) \cdot \lambda_k \mu_v |f(v)|^{p-2} f(v) \\
&= \lambda_k \sum_{i=1}^{m} |t_i|^p \sum_{v\in V_i} \mu_v |f(v)|^p \\
&= \lambda_k \sum_{v\in V} \mu_v |g(v)|^p,
\end{aligned}
$$

where the inequality follows by taking $a = f|_{V_i}(v) - s(vv')f|_{V_i}(v')$ and $b = f|_{V_j}(v) - s(vv')f|_{V_j}(v')$ in Lemma 9.2.2.

For $g \in X \setminus 0$, the preceding inequality implies that

$$
R_p(g) := \frac{\sum_{\{v,v'\}\in E} w_{vv'} |g(x) - s(vv')g(v')|^p + \kappa_v |g(v)|^p}{\sum_{v\in V} \mu_v |g(v)|^p} \leq \lambda_k .
$$

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Moreover, by the definition of the variational eigenvalues, we have

$$\lambda_m = \inf_{X' \in \mathrm{Gen}_m} \sup_{g' \in X' \setminus 0} R_p(g') \le \sup_{g \in X \setminus 0} R_p(g) \le \lambda_k < \lambda_{k+r},$$

which implies that $m \le k + r - 1$.

For the case of $p = 1$, we have

$$\begin{aligned}
&\sum_{\{v,v'\} \in E} w_{vv'} \left| g(v) - s(vv')g(v') \right| + \sum_{v \in V} \kappa_v |g(v)| \\
&= \sum_{\{v,v'\} \in E} w_{vv'} \left| \sum_{i=1}^{m} t_i \left(f|_{V_i}(v) - s(vv') f|_{V_i}(v') \right) \right| + \sum_{i=1}^{m} |t_i| \sum_{v \in V_i} \kappa_v |f(v)| \\
&\le \sum_{\{v,v'\} \in E} w_{vv'} \sum_{i=1}^{m} |t_i| \left(f|_{V_i}(v) - s(vv') f|_{V_i}(v') \right) z_{vv'} + \sum_{i=1}^{m} |t_i| \sum_{v \in V_i} \kappa_v |f(v)| \\
&= \sum_{i=1}^{m} |t_i| \sum_{v \in V_i} f(v) \left(\sum_{v' \in V} w_{vv'} z_{vv'} \right) + \sum_{i=1}^{m} |t_i| \sum_{v \in V_i} \kappa_v |f(v)| \\
&= \sum_{i=1}^{m} |t_i| \sum_{v \in V_i} f(v) \left(\sum_{v' \in V} w_{vv'} z_{vv'} + \kappa_v z_v \right) \\
&\in \sum_{i=1}^{m} |t_i| \sum_{v \in V_i} f(v) \cdot \Delta_1^s f(v) \cap \lambda_k \mu_v \mathrm{Sgn}(f(v)) \\
&\subset \sum_{i=1}^{m} |t_i| \sum_{v \in V_i} f(v) \cdot \lambda_k \mu_v \mathrm{Sgn}(f(v)) \\
&= \lambda_k \sum_{i=1}^{m} |t_i| \sum_{v \in V_i} \mu_v |f(v)| \\
&= \lambda_k \sum_{v \in V} \mu_v |g(v)|,
\end{aligned}$$

where the inequality follows by taking

$$a = f|_{V_i}(v) - s(vv') f|_{V_i}(v') \quad \text{and} \quad b = f|_{V_j}(v) - s(vv') f|_{V_j}(v')$$

in Lemma 9.2.2, while the inclusion relation follows by the fact that

$$\begin{aligned}
&\sum_{v' \in V} w_{vv'} z_{vv'} + \kappa_v z_v \in \Delta_1^s f(v) \cap \lambda_k \mu_v \mathrm{Sgn}(f(v)), \\
&z_{vv'} \in \mathrm{Sgn}(f(v) - s(vv') f(v')), \\
&z_{vv'} = -s(vv') z_{v'v}, \text{ and} \\
&z_v \in \mathrm{Sgn}(f(v)).
\end{aligned}$$

The rest of the proof is the same as above. This shows that $\mathfrak{S}(f) \le k + r - 1$.

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- We shall now derive the corresponding inequality for $\overline{\mathfrak{S}}(f)$, which is the number of strong nodal domains of f in the opposite signed graph $(\Gamma, -s)$. Assume now that f has m strong nodal domains on $(\Gamma, -s)$, where m is not necessarily the same as in the previous step and denote them by $V_1, \dots, V_m$. Let X now be the linear function space spanned by $f|_{V_1}, \dots, f|_{V_m}$. Then, similar to the previous step, one can prove that $R_p(g) \geq \lambda_k$ for any $g \in X \setminus \{0\}$. By the intersection property of Krasnoselskii's $\mathbb{Z}_2$-genus (see Lemma 2.6.3), we have that

$$X' \cap X \setminus \{0\} \neq \emptyset, \quad \text{for any} \quad X' \in \mathrm{Gen}_{N-m+1}.$$

  Therefore,

$$\begin{aligned}
\lambda_{N-m+1} &= \inf_{X' \in \mathrm{Gen}_{N-m+1}} \sup_{g' \in X' \setminus 0} R_p(g') \\
&\geq \inf_{X' \in \mathrm{Gen}_{N-m+1}} \inf_{g' \in X' \cap X \setminus 0} R_p(g') \\
&\geq \inf_{g \in X \setminus 0} R_p(g) \\
&\geq \lambda_k \\
&> \lambda_{k-r},
\end{aligned}$$

  which implies that $N - m + 1 \geq k - r + 1$, or equivalently, $m \leq N - k + r$. This shows that $\overline{\mathfrak{S}}(f) \leq N - k + r$.
- The inequalities $\mathfrak{W}(f) \leq \mathfrak{S}(f)$ and $\overline{\mathfrak{W}}(f) \leq \overline{\mathfrak{S}}(f)$ trivially follow from the definition of strong and weak nodal domains.
- Assume now that f has m weak nodal domains and denote them by $U_1, \dots, U_m$. Then, $m \leq k + r - 1$. Let also $V_1, \dots, V_c$ denote the c connected components of the vertex set of the graph, that is, $\bigsqcup_{l=1}^{c} V_l = V$. Then, $w_{vv'} = 0$ whenever $v \in V_l$, $v' \in V_{l'}$ and $l \neq l'$. Therefore, for any $i \in \{1, \dots, m\}$, there exists a unique $l \in \{1, \dots, c\}$ such that $U_i \subset V_l$. For $l = 1, \dots, c$, let now

$$I_l = \{i \in \{1, \dots, m\} \colon U_i \subset V_l\}$$

  be the index set corresponding to V_l. Then, $\bigsqcup_{l=1}^{c} I_l = \{1, \dots, m\}$. Assume, by contradiction, that $m \geq k + c$. Let X be the linear function space spanned by $f|_{U_1}, \dots, f|_{U_m}$, and let Z be the linear function space spanned by

$$f|_{\cup_{i \in I_1} U_i}, \dots, f|_{\cup_{i \in I_c} U_i}.$$

  Then, Z is a linear subspace of X, $\dim X = m$, and $\dim Z = c$. Moreover, by the above steps, $R_p(g') \leq \lambda_k$, for any $g' \in X \setminus 0$. By the definition of variational eigenvalues, it then follows that

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$$\max_{g' \in Y \setminus \{\mathbf{0}\}} R_p(g') = \lambda_k,$$

for any linear subspace $Y \subset X \setminus (Z \setminus 0)$ with $\dim Y \geq k$. Now, fix such a space Y. Then, the basic property of variational eigenvalues implies that there exists an eigenfunction

$$g = \sum_{i=1}^{m} t_i f|_{U_i} \in Y \setminus 0,$$

with $R_p(g) = \lambda_k$. This implies that, in this case, we have equality in Lemma 9.2.2. As a consequence, for any U_i and U_j that are connected by an edge, we have that one of the following holds:

(a) $t_i = t_j$, or

(b) There exist $v \in U_i$ and $v' \in U_j$ such that $f(v) = 0 \neq f(v')$ and $\{v, v'\} \in E$.

In this case, since f and g are eigenfunctions, we have that

$$\sum_{v' \in V} w_{vv'} |s(vv')f(v')|^{p-2}(s(vv')f(v')) = 0,$$

$$\sum_{v' \in V} w_{vv'} |s(vv')g(v')|^{p-2}(s(vv')g(v')) = 0.$$

Since every $v' \sim v$ has to satisfy $v' \in U_i$ or $v' \in U_j$, from these equalities, we obtain

$$\begin{aligned}
&\left(|t_i|^{p-2}t_i - |t_j|^{p-2}t_j\right) \sum_{v' \in U_j} w_{vv'} |f(v')|^{p-2}(s(vv')f(v')) \\
&= |t_i|^{p-2}t_i \sum_{v' \in V} w_{vv'} \phi_p(s(vv')f(v')) - \sum_{v' \in V} w_{vv'} \phi_p(s(vv')g(v')) \\
&= 0.
\end{aligned}$$

By the definition of a weak nodal domain path, we have that, for any $v', v'' \in U_j$,

$$(s(vv')f(v')) \cdot (s(vv'')f(v'')) = f(v')s(v'v)s(vv'')f(v'') \geq 0.$$

This implies that

$$\sum_{v' \in U_j} w_{vv'} |f(v')|^{p-2}(s(vv')f(v')) \neq 0.$$

Thus, $|t_i|^{p-2}t_i - |t_j|^{p-2}t_j = 0$, which yields $t_i = t_j$.

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Hence, $t_i = t_j$ whenever U_i and U_j are connected by an edge. Therefore, in each connected component V_l, we have that $t_i = t_j$ whenever $i, j \in I_l$, for $l = 1, \ldots, m$. But this implies that $g \in Z \setminus 0$, which is a contradiction, since $g \in Y$. This proves that $\mathfrak{W}(f) \leq k + c - 1$.

- Assume now that f has m weak nodal domains $U_1, \ldots, U_m$ on $(\Gamma, -s)$. Assume also, by contradiction, that $m \geq N - k + c + 1$. Let X be the linear function space spanned by $f|_{U_1}, \ldots, f|_{U_m}$, and let Z be the linear function space spanned by

$$f|_{\cup_{i \in I_1} U_i}, \ldots, f|_{\cup_{i \in I_c} U_i}.$$

Then, Z is a linear subspace of X, $\dim X = m$, and $\dim Z = c$. Furthermore, as in the previous steps, we have that $R_p(g') \geq \lambda_k$, for any $g' \in X \setminus \{0\}$. The intersection property of Krasnoselskii's $\mathbb{Z}_2$-genus (Lemma 2.6.3) implies that

$$\min_{g' \in Y \setminus \{\mathbf{0}\}} R_p(g') = \lambda_k,$$

for any linear subspace $Y \subset X \setminus (Z \setminus 0)$ such that $\dim Y \geq N - k + 1$. Fix such a space Y. Then the properties of variational eigenvalues imply that there exists an eigenfunction

$$g = \sum_{i=1}^{m} t_i f|_{U_i} \in Y \setminus 0,$$

with $R_p(g) = \lambda_k$. This implies that, also in this case, we must have equality in Lemma 9.2.2. The rest of the proof is the same as in the previous step. In particular, we arrive to a contradiction. This shows that $\overline{\mathfrak{W}}(f) \leq N - k + c$.

□

Theorem 9.2.2 unifies many results that give upper bounds to the number of nodal domains for p-Laplacians on graphs and signed graphs, including Theorem 4.1 in [106], Theorem 5.4 in [157] for signed graphs, Theorems 3.4 and 3.5 in [266], as well as the inequality

$$\mathfrak{N}(f) \leq \min\{\mathfrak{S}(f), \overline{\mathfrak{S}}(f)\} \leq \min\{k + r - 1, N - k + r\},$$

where $\mathfrak{N}(f)$ is the number of connected components of the support of f (see Theorem 5.3 in [157] and Theorem 2.2 in [160] and Theorem 9.3.1). We would like to remark here that in [160], the authors also introduce the concept of up and down nodal domains for general nonlinear eigenproblems. In this general setting, the strong nodal count $\mathfrak{S}(f)$ (resp., $\overline{\mathfrak{S}}(f)$) actually equals the number of down (resp., up) nodal domains.

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Remark 9.2.2. If f is an eigenfunction corresponding to an eigenvalue λ of Δ_1^s, then for any nodal domain A of f, both $f|_A$ and $\mathbf{1}_{A\cap\{f>0\}} - \mathbf{1}_{A\cap\{f<0\}}$ are also eigenfunctions corresponding to λ.

9.3 Nodal Domain Theorems on Hypergraphs

In this section, we work with concepts of positive and negative nodal domains similar to that in Definition 9.1.1, originally introduced for oriented hypergraphs in [219]. The results provided here are mainly taken from [157]. However, the first two statements in Theorem 9.3.1 are presented here for the first time. We give several preliminary definitions before presenting analogues of nodal domain theorems for general hypergraphs.

Definition 9.3.1 ([157, 219])**.** Let $\Gamma = (V, E, \varphi)$ be an oriented hypergraph. Given a function $f\colon V \to \mathbb{R}$, let $\operatorname{supp}_{\pm}(f) := \{v \in V\colon \pm f(v) > 0\}$. A *positive strong nodal domain* of f is a connected component of the induced sub-hypergraph of Γ on $\operatorname{supp}_{+}(f)$. A *negative strong nodal domain* of f is a connected component of the induced sub-hypergraph of Γ on $\operatorname{supp}_{-}(f)$.

A positive (respectively, negative) *weak nodal domain* of f is a maximal connected induced sub-hypergraph of Γ on $\{v \in V\colon f(v) \geq 0\}$ (respectively, $\{v \in V\colon f(v) \leq 0\}$), that contains at least one nonzero vertex.

In the sequel, we shall need to work with functions that have the same sign on all the inputs of any oriented hyperedge and also the same sign on all outputs. For a simple graph, this is a trivial condition, as each oriented edge has only a single input and a single output, but for general hypergraphs, this is a nontrivial condition. We shall need this condition both for weak signs (≥ 0, ≤ 0) and strong signs (> 0, < 0).

Definition 9.3.2. A function $f\colon V \to \mathbb{R}$ is *sign-separable* if, for any $e \in E$, $f(v)f(v') \geq 0$ for all v and v' which are co-oriented in e. That is, the set e_{in} of inputs of e (respectively, the set e_{out} of outputs of e) is contained either in $\{v \in V\colon f(v) \geq 0\}$ or in $\{v \in V\colon f(v) \leq 0\}$.

Definition 9.3.3. A function $f\colon V \to \mathbb{R}$ is *strongly sign-separable* if, for any $e \in E$, the set e_{in} (respectively, the set e_{out}) is contained in either of $\{v \in V\colon f(v) > 0\}$, or $\{v \in V\colon f(v) < 0\}$, or $\{v \in V\colon f(v) = 0\}$.

Clearly, any strongly sign-separable function is also sign-separable. Moreover, if Γ is a simple graph, then any $f\colon V \to \mathbb{R}$ is strongly sign-separable.

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Theorem 9.3.1. *Let $\Gamma = (V, E, \varphi)$ be an oriented hypergraph on N vertices, and let $p \geq 1$. Let f be an eigenfunction corresponding to the kth min–max eigenvalue λ_k of Δ_p, and let r be the multiplicity of λ_k.*

1. *If f is sign-separable, then $\mathfrak{S}(f) \leq k + r - 1$.*
2. *If $p > 1$, Γ is connected and f is strongly sign-separable, then $\mathfrak{W}(f) \leq k$.*
3. *If Γ has only inputs, then $\mathfrak{S}(f) \leq N - k + r$.*
4. *If $p > 1$, Γ is connected and has only inputs, then $\mathfrak{W}(f) \leq N - k + 1$.*
5. $\mathfrak{N}(f) \leq \min\{k + r - 1, N - k + r\}$.

Proof 1. Claim: If f is sign-separable, then $\mathfrak{S}(f) \leq k + r - 1$.

Assume the contrary, by contradiction. Then, f is a sign-separable eigenfunction of λ_k that has at least $k+r$ nodal domains, denoted by $V_1, \ldots, V_{k+r}$. Assume, without loss of generality, that

$$\lambda_k = \lambda_{k+1} = \ldots = \lambda_{k+r-1} < \lambda_{k+r}.$$

Moreover, let X be the space of linear functions spanned by $f|_{V_1}, \ldots, f|_{V_{k+r}}$. Since $V_1, \ldots, V_{k+r}$ are pairwise disjoint, this implies that $\dim X = k+r$. Hence, given $g \in X \setminus \{\mathbf{0}\}$, there exists $(t_1, \ldots, t_{k+r}) \neq \mathbf{0}$ such that

$$g = \sum_{i=1}^{k+r} t_i f|_{V_i}.$$

By definition of positive and negative nodal domains, each hyperedge e intersects at most one positive nodal domain and at most one negative nodal domain. Moreover, since f is sign-separable, for $l \neq l'$ and $e \in E$,

$$(\delta f|_{V_l}(e)) \left(\delta f|_{V_{l'}}(e)\right) \geq 0.$$

For $p > 1$, we have

$$\begin{aligned}
&\sum_{e\in E}\left|\sum_{v\in e_{in}} g(v) - \sum_{v'\in e_{out}} g(v')\right|^p = \sum_{e\in E}\left|\sum_{v\in e_{in}}\sum_{l=1}^{k+r} t_l f|_{V_l}(v) - \sum_{v'\in e_{out}}\sum_{l=1}^{k+r} t_l f|_{V_l}(v')\right|^p \\
&= \sum_{e\in E}\left|\sum_{l=1}^{k+r} t_l \delta f|_{V_l}(e)\right|^p \leq \sum_{e\in E}\sum_{l=1}^{k+r} |t_l|^p \delta f|_{V_l}(e)\,|\delta f(e)|^{p-2}\,\delta f(e) \\
&= \sum_{l=1}^{k+r} |t_l|^p \sum_{v\in V_l} f(v) \sum_{e\ni v} |\delta f(e)|^{p-2} \left(\sum_{v'\in e\,:\, o_e(v,v')=-1} f(v') - \sum_{v'\in e\,:\, o_e(v,v')=1} f(v')\right) \\
&= \sum_{l=1}^{k+r} |t_l|^p \sum_{v\in V_l} f(v)\lambda_k \deg v \cdot |f(v)|^{p-2} f(v) \\
&= \lambda_k \sum_{l=1}^{k+r} |t_l|^p \sum_{v\in V_l} \deg v \cdot |f(v)|^p = \lambda_k \sum_{v\in V} \deg v \cdot |g(v)|^p,
\end{aligned}$$

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where the inequality follows by taking $a = \sum_{v\in e} f|_{V_l}(v)$ and $b = \sum_{v\in e} f|_{V_l'}(v)$ in Lemma 9.2.2.

Similarly, for $p = 1$, we have

$$\begin{aligned}
\sum_{e\in E}\left|\sum_{v\in e} g(v)\right| &= \sum_{e\in E}\left|\sum_{l=1}^{k+r} t_l \delta f|_{V_l}(e)\right| \\
&\le \sum_{e\in E}\sum_{l=1}^{k+r} |t_l|\cdot \delta f|_{V_l}(e) z_e \\
&= \sum_{l=1}^{k+r} |t_l| \sum_{v\in V_l} f(v)\left(\sum_{e_{in}\ni v} z_e - \sum_{e_{out}\ni v} z_e\right) \\
&= \sum_{l=1}^{k+r} |t_l| \sum_{v\in V_l} f(v)\lambda_k \deg v \cdot z_v \\
&= \lambda_k \sum_{l=1}^{k+r} |t_l| \sum_{v\in V_l} \deg v \cdot |f(v)| \\
&= \lambda_k \sum_{v\in V} \deg v \cdot |g(v)|.
\end{aligned}$$

Then we have that $R_p(g) \le \lambda_k$, where R_p is the Rayleigh quotient corresponding to Δ_p, which has been discussed, for oriented hypergraphs, in Section 7.4. By the min–max principle for Δ_p, we have

$$\lambda_{k+r} = \min_{X'\in \mathrm{Gen}_{k+r}} \max_{g'\in X'\setminus 0} R_p(g') \le \max_{g\in X\setminus 0} R_p(g) = \lambda_k,$$

but this is a contradiction, since $\lambda_k < \lambda_{k+r}$. This proves the first claim.

2. Claim: If $p > 1$ and f is strongly sign-separable, then $\mathfrak{W}(f) \le k$.

   If the multiplicity of λ_k is $r = 1$, then this claim follows directly from the first one. Therefore, it suffices to focus on the case where $r \ge 2$.

   Assume the contrary, by contradiction. Then, f is a strong sign-separable eigenfunction of λ_k, which has multiplicity $r \ge 2$, and f has at least $k + 1$ weak nodal domains, which we denote by $V_1, \ldots, V_{k+1}$. Assume that

$$\lambda_{k-1} < \lambda_k = \lambda_{k+1} = \ldots = \lambda_{k+r-1} < \lambda_{k+r},$$

and let X be the space of functions spanned by $f|_{V_1}, \ldots, f|_{V_{k+1}}$. As in the proof of the first claim, this implies that $R_p(g) \le \lambda_k$, for all $g \in X \setminus \mathbf{0}$, and therefore

$$\max_{g\in Y\setminus\{\mathbf{0}\}} R_p(g) = \lambda_k,$$

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for any linear subspace $Y \subset X \setminus \{f\}$ such that $\dim Y \geq k$. Now, fix such a space Y. Then, there exists an eigenfunction

$$g = \sum_{i=1}^{k+1} t_i f|_{V_i} \in Y$$

such that $R_p(g) = \lambda_k$. By Lemma 9.2.2, for any V_l and $V_{l'}$ that are connected by a path, we have one of the following:

(a) $t_l = t_{l'}$, or

(b) There exist $v \in V_l$, $e \in E$, and $v' \in V_{l'}$ such that $f(v) = 0 \neq f(v')$ and $e \supset \{v, v'\}$.

In this case, since f and g are eigenfunctions, we have that

$$\sum_{e \ni v} |\delta f(v,e)|^{p-2}\, \delta f(v,e) = 0 \quad \text{and} \quad \sum_{e \ni v} |\delta g(v,e)|^{p-2}\, \delta g(v,e) = 0,$$

where

$$\delta f(v,e) \coloneqq \sum_{v' \in e\colon o_e(v,v')=-1} f(v') - \sum_{v' \in e\colon o_e(v,v')=1} f(v').$$

Since every e such that $v \in e$ has to satisfy $e \subset V_l$ or $e \subset V_{l'}$, from these equalities, we obtain

$$\begin{aligned}&\left(|t_l|^{p-2} t_l - |t_{l'}|^{p-2} t_{l'}\right) \sum_{e \ni v, e \subset V_{l'}} \left|\delta f|_{V_{l'}}(v,e)\right|^{p-2} \delta f|_{V_{l'}}(v,e) \\ &= \phi_p(t_l) \sum_{e \ni v} \phi_p(\delta f(v,e)) - \sum_{e \ni v} \phi_p(\delta g(v,e)) = 0.\end{aligned}$$

The strongly sign-separable property implies that

$$\sum_{e \ni v,\ e \subset V_{l'}} \left|\delta f|_{V_{l'}}(v,e)\right|^{p-2} \delta f|_{V_{l'}}(v,e) \neq 0.$$

Thus, $|t_l|^{p-2} t_l - |t_{l'}|^{p-2} t_{l'} = 0$, which yields $t_l = t_{l'}$.

Hence, $t_l = t_{l'}$ whenever V_l and $V_{l'}$ are connected by a path. Since Γ is connected, $V_1, \ldots, V_{k+1}$ are all connected by a path. Therefore, $t_1 = \ldots = t_{k+1}$. This implies that $g \in \operatorname{span}(f)$, which is a contradiction.

3. Claim: If Γ has only inputs, then $\mathfrak{S}(f) \leq N - k + r$.

Assume the contrary, by contradiction. Then, f is an eigenfunction of λ_k, that has multiplicity r, and f has at least $N - k + r + 1$ nodal domains, which we denote by $V_1, \ldots, V_{N-k+r+1}$. Let X be the linear function space, which is spanned by

$$f|_{V_1}, \ldots, f|_{V_{N-k+r+1}}.$$

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Then, since $V_1, \ldots, V_{N-k+r+1}$ are pairwise disjoint, $\dim X = N - k + r + 1$. Hence, given $g \in X \setminus 0$, there exists $(t_1, \ldots, t_{N-k+r+1}) \neq \mathbf{0}$ such that

$$g = \sum_{i=1}^{N-k+r+1} t_i f|_{V_i}.$$

By definition of positive and negative nodal domains, each hyperedge e intersects at most one positive nodal domain and at most one negative nodal domain. Thus, for $l \neq l'$ and $e \in E$, we have

$$\left(\sum_{v \in e_{in}} f|_{V_l}(v)\right) \cdot \left(\sum_{v \in e_{in}} f|_{V_{l'}}(v)\right) \leq 0.$$

Now, with a little abuse of notation, let $e = e_{in}$. For $p > 1$, we have that

$$\begin{aligned}
\sum_{e \in E} \left|\sum_{v \in e} g(v)\right|^p &= \sum_{e \in E} \left|\sum_{v \in e} \sum_{l=1}^{N-k+r+1} t_l f|_{V_l}(v)\right|^p \\
&= \sum_{e \in E} \left|\sum_{l=1}^{N-k+r+1} t_l \left(\sum_{v \in e} f|_{V_l}(v)\right)\right|^p \\
&\geq \sum_{e \in E} \sum_{l=1}^{N-k+r+1} |t_l|^p \left(\sum_{v \in e} f|_{V_l}(v)\right) \left|\sum_{v \in e} f(v)\right|^{p-2} \sum_{v \in e} f(v) \\
&= \sum_{l=1}^{N-k+r+1} |t_l|^p \sum_{v \in V_l} f(v) \left(\sum_{e \in E \colon e \ni v} \left|\sum_{v' \in e} f(v')\right|^{p-2} \left(\sum_{v' \in e} f(v')\right)\right) \\
&= \sum_{l=1}^{N-k+r+1} |t_l|^p \sum_{v \in V_l} f(v) \lambda_k \deg v \cdot |f(v)|^{p-2} f(v) \\
&= \lambda_k \sum_{l=1}^{N-k+r+1} |t_l|^p \sum_{v \in V_l} \deg v \cdot |f(v)|^p \\
&= \lambda_k \sum_{v \in V} \deg v \cdot |g(v)|^p,
\end{aligned}$$

where the inequality follows by taking $a = \sum_{v \in e} f|_{V_l}(v)$ and $b = \sum_{v \in e} f|_{V_l'}(v)$ in Lemma 9.2.2. Similarly, for $p = 1$, we have

$$\begin{aligned}
\sum_{e \in E} \left|\sum_{v \in e} g(v)\right| &= \sum_{e \in E} \left|\sum_{l=1}^{N-k+r+1} t_l \left(\sum_{v \in e} f|_{V_l}(v)\right)\right| \\
&\geq \sum_{e \in E} \sum_{l=1}^{N-k+r+1} |t_l| \left(\sum_{v \in e} f|_{V_l}(v)\right) z_e
\end{aligned}$$

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$$
\begin{aligned}
&= \sum_{l=1}^{N-k+r+1} |t_l| \sum_{v \in V_l} f(v) \left(\sum_{e \in E \colon e \ni v} z_e \right) \\
&= \sum_{l=1}^{N-k+r+1} |t_l| \sum_{v \in V_l} f(v) \lambda_k \deg v \cdot z_v \\
&= \lambda_k \sum_{l=1}^{N-k+r+1} |t_l| \sum_{v \in V_l} \deg v \cdot |f(v)| \\
&= \lambda_k \sum_{v \in V} \deg v \cdot |g(v)|,
\end{aligned}
$$

where $z_e \in \operatorname{Sgn}(\sum_{v \in e} f(v))$ and $z_v \in \operatorname{Sgn}(f(v))$. By Lemma 9.2.2, it follows that $R_p(g) \geq \lambda_k$.

By the intersection property of the $\mathbb{Z}_2$-genus (Lemma 2.6.3), $X' \cap X \setminus \{0\} \neq \emptyset$, for any $X' \in \operatorname{Gen}_{k-r}$. Therefore,

$$
\begin{aligned}
\lambda_{k-r} &= \inf_{X' \in \operatorname{Gen}_{k-r}} \sup_{g' \in X' \setminus 0} R_p(g') \\
&\geq \inf_{X' \in \operatorname{Gen}_{k-r}} \sup_{g' \in X' \cap X \setminus 0} R_p(g') \\
&\geq \inf_{X' \in \operatorname{Gen}_{k-r}} \inf_{g' \in X' \cap X \setminus 0} R_p(g') \\
&\geq \inf_{X' \in \operatorname{Gen}_{k-r}} \inf_{g' \in X \setminus 0} R_p(g') \\
&= \inf_{g \in X \setminus 0} R_p(g) \\
&\geq \lambda_k.
\end{aligned}
$$

Together with $\lambda_{k-r} \leq \ldots \leq \lambda_{k-1} \leq \lambda_k$, this implies that

$$\lambda_{k-r} = \ldots = \lambda_{k-1} = \lambda_k,$$

therefore, the multiplicity of λ_k is at least $r+1$. This is a contradiction.

4. Claim: If $p > 1$, Γ is connected and has only inputs, then $\mathfrak{W}(f) \leq N-k+1$.

   Assume the contrary, by contradiction. Then, f is an eigenfunction of λ_k that has multiplicity $r \geq 2$, and f has at least $N-k+2$ nodal domains, which we denote by $V_1, \ldots, V_{N-k+2}$. Let X be the linear function space spanned by $f|_{V_1}, \ldots, f|_{V_{N-k+2}}$. Since $V_1, \ldots, V_{N-k+2}$ are pairwise disjoint, we have that $\dim X = N-k+2$. Also, for any linear subspace $Y \subset X \setminus \{f\}$ such that $\dim Y = N-k+1$, as in the proof of the third claim, one can show that

$$\min_{g \in Y \setminus 0} R_p(g) = \lambda_k.$$

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Now, given $g \in Y \setminus \{\mathbf{0}\}$, there exists $(t_1, \ldots, t_{N-k+2}) \neq \mathbf{0}$ such that

$$g = \sum_{i=1}^{N-k+2} t_i f|_{V_i}.$$

By definition of nonpositive and non-negative nodal domains, each hyperedge e intersects at most one nonpositive nodal domain and at most one non-negative nodal domain. By following the same reasoning as in the proof of the second claim, we can show that $t_1 = \ldots = t_{N-k+2}$. Hence, $g = t_1 f$, which leads to a contradiction, since $f \notin Y$ while $g \in Y$.

5. Claim: $\mathfrak{N}(f) \leq \min\{k + r - 1, N - k + r\}$.

   Assume that the support of f has m connected components and denote them by $V_1, \ldots, V_m$. Let then X be the linear function space spanned by $f|_{V_1}, \ldots, f|_{V_m}$. As for the first and third claims, one can check that $R_p(g) = \lambda_k$ for any $g \in X \setminus 0$. By the min–max principle for Δ_p,

$$\lambda_m = \inf_{X' \in \mathrm{Gen}_m} \sup_{g' \in X' \setminus 0} R_p(g') \leq \sup_{g \in X \setminus 0} R_p(g) = \lambda_k \leq \ldots \leq \lambda_{k+r-1} < \lambda_{k+r},$$

which implies that $m \leq k + r - 1$. Also, by the intersection property of Krasnoselskii's $\mathbb{Z}_2$-genus (Lemma 2.6.3), $X' \cap X \setminus \{0\} \neq \emptyset$ for any $X' \in \mathrm{Gen}_{N-m+1}$. Therefore,

$$\begin{aligned}
\lambda_{N-m+1} &= \inf_{X' \in \mathrm{Gen}_{N-m+1}} \sup_{g' \in X' \setminus 0} R_p(g') \\
&\geq \inf_{X' \in \mathrm{Gen}_{N-m+1}} \inf_{g' \in X' \cap X \setminus 0} R_p(g') \\
&\geq \inf_{g \in X \setminus 0} R_p(g) \\
&\geq \lambda_k \\
&\geq \ldots \\
&\geq \lambda_{k-r+1} \\
&> \lambda_{k-r},
\end{aligned}$$

which yields $N - m + 1 \geq k - r + 1$, or equivalently, $m \leq N - k + r$.

□

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