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TILING WITH PUNCTURED INTERVALS

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Abstract. It was shown by Gruslys, Leader and Tan that any finite subset of $\mathbb{Z}^n$ tiles $\mathbb{Z}^d$ for some d. The first non-trivial case is the punctured interval, which consists of the interval $\{-k, \dots, k\} \subset \mathbb{Z}$ with its middle point removed: they showed that this tiles $\mathbb{Z}^d$ for $d = 2k^2$, and they asked if the dimension needed tends to infinity with k. In this note we answer this question: we show that, perhaps surprisingly, every punctured interval tiles $\mathbb{Z}^4$.

§1. *Introduction.* A *tile* is a finite non-empty subset of $\mathbb{Z}^n$ for some n. We say that a tile T *tiles* $\mathbb{Z}^d$ if $\mathbb{Z}^d$ can be partitioned into copies of T, that is, subsets that are translations, rotations or reflections, or any combination of these, of T.

For example, the tile $\mathtt{X.X} = \{-1, 1\} \subset \mathbb{Z}$ tiles $\mathbb{Z}$. The tile $\mathtt{XX.XX} = \{-2, -1, 1, 2\} \subset \mathbb{Z}$ does not tile $\mathbb{Z}$, but we can also regard it as a tile in $\mathbb{Z}^2$, and indeed it tiles $\mathbb{Z}^2$, as shown, for example, in [8].

Chalcraft [12, 13] conjectured that, for any tile $T \subset \mathbb{Z}^n$, there is some dimension d for which T tiles $\mathbb{Z}^d$. This was proved by Gruslys *et al* [8]. The first non-trivial case is the *punctured interval* $T = \underbrace{\mathtt{XXXXX}}_{k}.\underbrace{\mathtt{XXXXX}}_{k}$. The authors of [8] showed that T tiles $\mathbb{Z}^d$ for $d = 2k^2$, but they were unable to prove that the smallest required dimension d was quadratic in k, or even that $d \to \infty$ as $k \to \infty$. They therefore asked the following question.

Question 1 (Gruslys *et al* [8]). Let T be the punctured interval $\underbrace{\mathtt{XXXXX}}_{k}.\underbrace{\mathtt{XXXXX}}_{k}$, and let d be the least number such that T tiles $\mathbb{Z}^d$. Does $d \to \infty$ as $k \to \infty$?

In this paper we will show that, rather unexpectedly, d does not tend to infinity.

THEOREM 2. *Let T be the punctured interval $\underbrace{\mathtt{XXXXX}}_{k}.\underbrace{\mathtt{XXXXX}}_{k}$. Then T tiles $\mathbb{Z}^4$. Furthermore, if k is odd or congruent to* 4 *modulo* 8*, then T tiles $\mathbb{Z}^3$.*

We have already noted that $\mathtt{X.X}$ tiles $\mathbb{Z}$, and $\mathtt{XX.XX}$ tiles $\mathbb{Z}^2$ but not $\mathbb{Z}$. It can be shown via case analysis that, for $k \geqslant 3$, the tile T does not tile $\mathbb{Z}^2$. However, this proof is tedious and provides little insight, and since it is not the focus of this paper, we omit it. For odd $k \geqslant 3$ and for $k \equiv 4 \pmod 8$, 3 is therefore the least d

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such that T tiles $\mathbb{Z}^d$. For the remaining cases, namely $k \equiv 0, 2, 6 \pmod 8$, $k \geqslant 6$, we do not know whether the least possible d is 3 or 4.

In this paper, we will first prove the result for odd k. This will introduce some key ideas, which we will develop to prove the result for general k, and then to improve the dimension from 4 to 3 for $k \equiv 4 \pmod 8$.

Finally, we give some background. Tilings of $\mathbb{Z}^2$ by polyominoes (edge-connected tiles in $\mathbb{Z}^2$) have been thoroughly investigated. For example, Golomb [6] showed that results of Berger [2] implied that there is no algorithm which decides whether copies of a given finite set of polyominoes tile $\mathbb{Z}^2$. It is unknown whether the same is true for tilings by a single polyomino. For tilings of $\mathbb{Z}$ by sets of general one-dimensional tiles, such an algorithm does exist, as demonstrated by Adler and Holroyd [1]. Kisisel [11] introduced an ingenious technique for proving that certain tiles do not tile $\mathbb{Z}^2$ without having to resort to case analysis.

A similar problem is to consider whether a tile T tiles certain finite regions, such as cuboids. There is a significant body of research on tilings of rectangles in $\mathbb{Z}^2$ by polyominoes (see, for example, Conway and Lagarias [3] and Dahlke [4]). Friedman [5] has collected some results on tilings of rectangles by small one-dimensional tiles. More recently, Gruslys *et al* [9] and Tomon [14] considered the related problem of partitioning the Boolean lattice into copies of a poset, and similarly Gruslys [7] and Gruslys and Letzter [10] have worked on the problem of partitioning the hypercube into copies of a graph.

§2. *Preliminaries and the odd case.* We begin with the case of k odd. This is technically much simpler than the general case, and allows us to demonstrate some of the main ideas in the proof of Theorem 2 in a less complicated setting.

THEOREM 3. *Let T be the punctured interval $\underbrace{\mathrm{XXXXX}}_{k}.\underbrace{\mathrm{XXXXX}}_{k}$, with k odd. Then T tiles $\mathbb{Z}^3$.*

Throughout this section, T is fixed, and $k \geqslant 3$. We will not yet assume that k is odd, because the tools that we are about to develop will be relevant to the general case too.

We start with an important definition from [8]: a *string* is a one-dimensional infinite line in $\mathbb{Z}^d$ with every $(k+1)$th point removed. Crucially, a string is a disjoint union of copies of T.

We cannot tile $\mathbb{Z}^d$ with strings, as each string intersects $[k+1]^d$ in either 0 or k points, and $(k+1)^d$ is not divisible by k. However, we could try to tile $\mathbb{Z}^d$ by using strings in $d-1$ of the d possible directions, leaving holes that can be filled with copies of T in the final direction. We therefore consider $\mathbb{Z}^d$ as consisting of slices equivalent to $\mathbb{Z}^{d-1}$, each of which will be partially tiled by strings: that is, a subset of each $\mathbb{Z}^{d-1}$ slice will be tiled by strings.

Given a partial tiling of the discrete torus $\mathbb{Z}_{k+1}^{d-1} = (\mathbb{Z}/(k+1)\mathbb{Z})^{d-1}$ by lines with one point removed, we can lift it to $\mathbb{Z}^{d-1}$ by filling $\mathbb{Z}^{d-1}$ with repeating copies of $\mathbb{Z}_{k+1}^{d-1}$ (regarded as $[k+1]^{d-1}$). This produces a partial tiling of $\mathbb{Z}^{d-1}$

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by strings. We will restrict our attention to these tilings at first, as they are easy to work with.

We will call a set $X \subset \mathbb{Z}_{k+1}^{d-1}$ a *hole* in $\mathbb{Z}_{k+1}^{d-1}$ if $\mathbb{Z}_{k+1}^{d-1} \setminus X$ can be tiled with strings. One particularly useful case of this is when $d = 3$ and X either has exactly one point in each row of $\mathbb{Z}_{k+1}^2$ or exactly one point in each column of $\mathbb{Z}_{k+1}^2$. Then X is clearly a hole, since a string in $\mathbb{Z}_{k+1}^2$ is just a row or column minus a point.

The following result will allow us to fill the gaps in the final direction, assuming we have chosen the partial tilings of the $\mathbb{Z}^{d-1}$ slices carefully.

LEMMA 4. *Let $S \subset \mathbb{Z}^d$, $|S| = 3$. Then there exists $Y \subset S \times \mathbb{Z}$ such that T tiles Y, and for every $n \in \mathbb{Z}$, $|Y \cap (S \times \{n\})| = 2$.*

Proof. Let $S = \{x_1, x_2, x_3\}$. For $i = 1, 2, 3$, place a copy of T beginning at $\{x_i\} \times \{n\}$ for every $n \equiv ik \pmod{3k}$. The union Y of these tiles has the required property.

For $n \equiv 0, k+1, \ldots, 2k-1 \pmod{3k}$, $Y \cap (S \times \{n\}) = \{x_1, x_3\} \times \{n\}$.
For $n \equiv k, 2k+1, \ldots, 3k-1 \pmod{3k}$, $Y \cap (S \times \{n\}) = \{x_1, x_2\} \times \{n\}$.
For $n \equiv 2k, 1, \ldots, k-1 \pmod{3k}$, $Y \cap (S \times \{n\}) = \{x_2, x_3\} \times \{n\}$. □

We will now prove Theorem 3. We know that if $X \subset \mathbb{Z}_{k+1}^2$ has one point in each row or column then X is a hole of size $k+1$. Since $k+1$ is even, we can try to choose X_n in each slice $\mathbb{Z}_{k+1}^2 \times \{n\}$ so that $\bigcup_{n\in\mathbb{Z}} X_n$ is the disjoint union of $(k+1)/2$ sets Y_i of the form in Lemma 4.

We can do this as follows.

For $n \equiv 0, k+1, \ldots, 2k-1 \pmod{3k}$, let $X_n = \{(0,0), (1,1), \ldots, (k-1, k-1), (k,k)\}$.
For $n \equiv k, 2k+1, \ldots, 3k-1 \pmod{3k}$, let $X_n = \{(0,0), (0,1), (2,2), (2,3), \ldots, (k-1, k-1), (k-1, k)\}$.
For $n \equiv 2k, 1, \ldots, k-1 \pmod{3k}$, let $X_n = \{(0,1), (1,1), (2,3), (3,3), \ldots, (k-1, k), (k,k)\}$.
Then let $X = \bigcup_{n\in\mathbb{Z}} (X_n \times \{n\}) \subset \mathbb{Z}_{k+1}^2 \times \mathbb{Z}$.

Each X_n is a hole, so we can tile $(\mathbb{Z}_{k+1}^2 \times \mathbb{Z}) \setminus X$ with strings. Also, X is the disjoint union of sets of the form Y from Lemma 4: for $0 \leqslant i \leqslant (k-1)/2$, let $S_i = \{(2i, 2i), (2i, 2i+1), (2i+1, 2i+1)\}$. Then $X \cap (S_i \times \mathbb{Z})$ is precisely the set Y generated from S_i in the proof of Lemma 4. Hence T tiles X.

Since $(\mathbb{Z}_{k+1}^2 \times \mathbb{Z}) \setminus X$ can be tiled with strings, we can partially tile $\mathbb{Z}^3$ with strings, leaving a copy of X empty in each copy of $\mathbb{Z}_{k+1}^2 \times \mathbb{Z}$. We can tile all of these copies of X with T, so T tiles $\mathbb{Z}^3$, completing the proof of Theorem 3.

§3. *The general case.* We now move on to general k.

THEOREM 5. *Let T be the tile* $\underbrace{\text{XXXXX}}_{k}.\underbrace{\text{XXXXX}}_{k}$. *Then T tiles $\mathbb{Z}^4$.*

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We will assume throughout that T is fixed and $k \geqslant 3$.

For even k, the construction used to prove Theorem 3 does not work, as all holes in $\mathbb{Z}^2_{k+1}$ have size $(k+1)^2 - mk$ for some m, and this is always odd, so we cannot use Lemma 4. The same is true if we replace 2 with a larger dimension, or if, as in [**8**], we use strings in which every $(2k+1)$th point, rather than every $(k+1)$th point, is removed. We will therefore need a new idea.

Instead of using strings in $d-1$ out of d directions, we could only use them in $d-2$ directions and fill the gaps with copies of T in the two remaining directions. We will show that this approach works in the case $d = 2$, giving a tiling of $\mathbb{Z}^4$. The strategy will be to produce a partial tiling of each $\mathbb{Z}^3$ slice and use the construction from Lemma 4 to fill the gaps with tiles in the fourth direction.

We will again build partial tilings of $\mathbb{Z}^2$, and therefore of higher dimensions, from partial tilings of the discrete torus $\mathbb{Z}^2_{k+1}$. The following result is a special case of one proved in [**8**].

PROPOSITION 6. *If $x \in \mathbb{Z}^2_{k+1}$, then $\mathbb{Z}^2_{k+1} \setminus \{x\}$ can be tiled with strings.*

Proof. Let $x = (x_1, x_2)$, where the first coordinate is horizontal and the second vertical. Since a string is a row or column minus one point, we can place a string $(\{n\} \times \mathbb{Z}_{k+1}) \setminus \{(n, x_2)\}$ in each column, leaving only the row $\mathbb{Z}_{k+1} \times \{x_2\}$ empty. Placing the string $(\mathbb{Z}_{k+1} \times \{x_2\}) \setminus \{x\}$ in this row completes the tiling of $\mathbb{Z}^2_{k+1} \setminus \{x\}$. □

The sets S of size 3 that we will use in Lemma 4 will have two points, say x_1 and x_2, in one $\mathbb{Z}^2_{k+1}$ layer and one point, say x_3, in another layer. Every layer will contain points from exactly one such set S. Let Y be the set constructed from S in the proof of Lemma 4. In a given slice $\mathbb{Z}^3 \times \{n\}$, there are therefore two cases:

(1) $Y \cap (S \times \{n\}) = \{x_1, x_3\} \times \{n\}$ or $\{x_2, x_3\} \times \{n\}$;
(2) $Y \cap (S \times \{n\}) = \{x_1, x_2\} \times \{n\}$.

In case (1), each $\mathbb{Z}^2_{k+1}$ layer contains exactly one point of Y; T then tiles the rest of the layer by Proposition 6.

In case (2), some of the layers contain two points of Y, and some of the layers contain no points. Holes of size 0 and 2 do not exist, so we will need copies of T in the third direction to fill some gaps (where Y consists of copies of T in the fourth direction). The following lemma provides us with a way to do this.

LEMMA 7. *Let $A \subset \mathbb{Z}^d$, $|S| = 3k$. Then there exists $B \subset S \times \mathbb{Z}$ such that T tiles B, and*

$$|B \cap (S \times \{n\})| = \begin{cases} k+1 & \text{if } n \equiv 1, \ldots, k \pmod{2k}, \\ k-1 & \text{if } n \equiv k+1, \ldots, 2k \pmod{2k}. \end{cases}$$

Proof. Let $A = \{a_1, \ldots, a_{3k}\}$. Then we have the following.

For $i = 1, \ldots, k$, place a copy of T beginning at $\{a_i\} \times \{n\}$ for every $n \equiv i \pmod{6k}$.

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For $i = k+1, \ldots, 2k$, place a copy of T beginning at $\{a_i\} \times \{n\}$ for every $n \equiv i + k \pmod{6k}$.
For $i = 2k+1, \ldots, 3k$, place a copy of T beginning at $\{a_i\} \times \{n\}$ for every $n \equiv i + 2k \pmod{6k}$.
We now observe that the union B of these tiles has the required property.
For $n \equiv 1, \ldots, k \pmod{6k}$, $B \cap (A \times \{n\}) = \{a_{2k+n}, \ldots, a_{3k}, a_1, \ldots, a_n\}$ (size $k+1$).
For $n \equiv k+1, \ldots, 2k \pmod{6k}$, $B \cap (A \times \{n\}) = \{a_1, \ldots, a_k\} \setminus \{a_{n-k}\}$ (size $k-1$).
For $n \equiv 2k+1, \ldots, 3k \pmod{6k}$, $B \cap (A \times \{n\}) = \{a_{n-2k}, \ldots, a_{n-k}\}$ (size $k+1$).
For $n \equiv 3k+1, \ldots, 4k \pmod{6k}$, $B \cap (A \times \{n\}) = \{a_{k+1}, \ldots, a_{2k}\} \setminus \{a_{n-2k}\}$ (size $k-1$).
For $n \equiv 4k+1, \ldots, 5k \pmod{6k}$, $B \cap (A \times \{n\}) = \{a_{n-3k}, \ldots, a_{n-2k}\}$ (size $k+1$).
For $n \equiv 5k+1, \ldots, 6k \pmod{6k}$, $B \cap (A \times \{n\}) = \{a_{2k+1}, \ldots, a_{3k}\} \setminus \{a_{n-3k}\}$ (size $k-1$). □

The reasoning behind this lemma is that there exist sets $X \subset \mathbb{Z}_{k+1}^2 \times \mathbb{Z}$ that are missing exactly $k+1$ points in every $\mathbb{Z}_{k+1}^2$ layer and can be tiled with strings. If we take $d = 2$ in Lemma 7, we would like to choose such a set X and a set $A \subset \mathbb{Z}_{k+1}^2$ (abusing notation slightly, as $\mathbb{Z}_{k+1}^2$ is not actually a subset of $\mathbb{Z}^2$) such that the resulting B in Lemma 7 is disjoint from X. Then $(\mathbb{Z}_{k+1}^2 \times \mathbb{Z}) \setminus (B \cup X)$ contains either two or zero points in each $\mathbb{Z}_{k+1}^2$ layer, which is what we wanted.

In order for this construction to work, we need the set $B \cap (A \times \{n\})$ to be a hole whenever it has size $k+1$, and to be a subset of a hole of size $k+1$ whenever it has size $k-1$, so that we actually can tile the required points with strings. By observing the forms of the sets $B \cap (A \times \{n\})$ in the proof of Lemma 7, we see that it is sufficient to choose the a_n such that for all n, $\{a_n, \ldots, a_{n+k}\}$ is a hole. Here we regard the indices n of the points a_n of A as integers modulo $3k$, so $a_{3k+1} = a_1$ and so on. The following proposition says that we can do this.

PROPOSITION 8. *There exists a set $A = \{a_1, \ldots, a_{3k}\} \subset \mathbb{Z}_{k+1}^2$ such that for all n, $\{a_n, \ldots, a_{n+k}\}$ contains either one point in every row or one point in every column. Here the indices are regarded as integers modulo $3k$.*

Proof. For $n = 1, \ldots, k+1$, let $a_n = (n-1, n-1)$.
For $n = k+2, \ldots, 2k-1$, let $a_n = (n-k-2, n-k-1)$.
For $n = 2k, 2k+1, 2k+2$, let $a_n = (n-k-2, n-2k)$.
For $n = 2k+3, \ldots, 3k$, let $a_n = (n-2k-3, n-2k)$.
Note that all the a_n are distinct. Let us regard the first coordinate as horizontal and the second as vertical.
Then, for $n = 1, \ldots, 2k$, $\{a_n, \ldots, a_{n+k}\}$ contains one point in every column.
For $n = 2k+1, \ldots, 3k$, $\{a_n, \ldots, a_{n+k}\}$ contains one point in every row. □

From now on, a_n refers to the points defined in the above proof. This proposition is the motivation for choosing the value $6k$ in the proof of Lemma 7.

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We can now prove Theorem 5. We will need three distinct partial tilings of $\mathbb{Z}^3$ slices, corresponding to the three cases in the proof of Lemma 4 with $d = 3$. The repeating unit in each of these partial tilings will have size $(k+1) \times (k+1) \times 6k$, so we will work in $\mathbb{Z}^2_{k+1} \times \mathbb{Z}_{6k}$.

We start by choosing the sets S as in Lemma 4. These will be as follows.

For $n = 1, \dots, k$, $S_n = \{(0, 0, n), (a_n, n + k), (a_{k+1}, n + k)\}$.
For $n = k + 1, \dots, 2k$, $S_n = \{(0, 0, n + k), (a_n, n + 2k), (a_{2k+1}, n + 2k)\}$.
For $n = 2k + 1, \dots, 3k$, $S_n = \{(0, 0, n + 2k), (a_n, n + 3k), (a_1, n + 3k)\}$.
We will refer to the points in S_n as $x_{n,1}, x_{n,2}, x_{n,3}$ in the order given.

We can construct a set $Y_n \subset \mathbb{Z}^4$ from each S_n using the construction in the proof of Lemma 4. Let $Y = \bigcup_{1 \leqslant n \leqslant 3k} Y_n$. For a given $m \in \mathbb{Z}$, there are two possibilities for the structure of $Y \cap (\mathbb{Z}^2_{k+1} \times \mathbb{Z}_{6k} \times \{m\})$.

(1) The set $Y \cap (\mathbb{Z}^2_{k+1} \times \mathbb{Z}_{6k} \times \{m\})$ consists of pairs of the form $\{x_{n,1}, x_{n,2}\}$ or $\{x_{n,1}, x_{n,3}\}$. Then it contains exactly one point in each $\mathbb{Z}^2_{k+1}$ layer. We can therefore tile $(\mathbb{Z}^2_{k+1} \times \mathbb{Z}_{6k} \times \{m\}) \setminus Y$ entirely with strings, by Proposition 6.

(2) The set $Y \cap (\mathbb{Z}^2_{k+1} \times \mathbb{Z}_{6k} \times \{m\})$ consists of pairs of the form $\{x_{n,2}, x_{n,3}\}$. Then it contains either 2 or 0 points in each $\mathbb{Z}^2_{k+1}$ layer.
If $A = \{a_1, \dots, a_{3k}\}$, and B is the set constructed from A in the proof of Lemma 7, then, by the choice of the S_n, the sets B and $Y \cap (\mathbb{Z}^2_{k+1} \times \mathbb{Z}_{6k} \times \{m\})$ are disjoint. Furthermore, if C is the union of these two sets, then, for every n, $C \cap (\mathbb{Z}^2_{k+1} \times \{n\} \times \{m\}) = \{a_r, \dots, a_{r+k}\}$ for some r, and by Proposition 8, this contains either one point in every row or one point in every column and is therefore a hole.
Since T tiles B, it also tiles $(\mathbb{Z}^2_{k+1} \times \mathbb{Z}_{6k} \times \{m\}) \setminus Y$.

Also, T tiles Y by Lemma 4. Hence T tiles $\mathbb{Z}^2_{k+1} \times \mathbb{Z}_{6k} \times \mathbb{Z}$, and therefore also $\mathbb{Z}^4$, completing the proof of Theorem 5.

§4. *The 4 modulo 8 case.* To finish the proof of Theorem 2, all that remains is to prove the following.

THEOREM 9. *Let T be the tile* $\underbrace{\text{XXXXX}}_{k}.\underbrace{\text{XXXXX}}_{k}$, *with $k \equiv 4 \pmod 8$. Then T tiles $\mathbb{Z}^3$.*

We will prove this by constructing partial tilings of each $\mathbb{Z}^2$ slice and filling in the gaps using the construction from the proof of Lemma 4. We will define three subsets X_1, X_2, X_3 of $\mathbb{Z}^2$ and show that T tiles each of them. However, two of these tilings will not make use of strings.

Let $S_1 = \{(x, x+n(k+1)) \mid n \in \mathbb{Z}, x \equiv 2n, 2n+1, 2n+2, 2n+3 \pmod 8\}$.
Let $S_2 = \{(x, x+n(k+1)) \mid n \in \mathbb{Z}, x \equiv 2n+4, 2n+5, 2n+6, 2n+7 \pmod 8\}$.
Let $S_3 = \{(x, x + n(k + 1) + 1) \mid n \in \mathbb{Z}, x \equiv 2n + 2, 2n + 3, 2n + 4, 2n + 5 \pmod 8\}$.
Let $X_1 = \mathbb{Z}^2 \setminus (S_2 \cup S_3)$, $X_2 = \mathbb{Z}^2 \setminus (S_1 \cup S_3)$, $X_3 = \mathbb{Z}^2 \setminus (S_1 \cup S_2)$.
Let the first coordinate be horizontal and the second vertical.

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X_3 is $\mathbb{Z}^2$ with every $(k+1)$th diagonal removed, so each row (or column) is Z with every $(k+1)$th point removed, that is, a string. Hence T tiles X_3.

We will show that X_1 can be tiled with vertical copies of T and X_2 can be tiled with horizontal copies of T.

Note that $(x, x+n(k+1)) + (2, k+3) = (x+2, (x+2)+(n+1)(k+1))$. Also, if $x \equiv 2n+r \pmod 8$, then $x+2 \equiv 2(n+1)+r \pmod 8$. Hence, by the definitions of S_2 and S_3, we see that X_1 is invariant under translation by $(2, k+3)$. To show that vertical copies of T tile X_1, it therefore suffices to show that T tiles the columns $X_1 \cap (\{0\} \times \mathbb{Z})$ and $X_1 \cap (\{1\} \times \mathbb{Z})$.

But in fact, if $(0, y) \in S_2$, then $0 \equiv 2n+4$ or $2n+6 \pmod 8$, so $1 \equiv 2n+5$ or $2n+7 \pmod 8$, so also $(1, y+1) \in S_2$. The converse also holds, and the same is true for S_3. Thus we only need to check the case $x = 0$:

$(0, n(k+1)) \in S_2$ for $n \equiv 1, 2, 5, 6 \pmod 8$, that is, $n \equiv 1, 2 \pmod 4$;

$(0, n(k+1)+1) \in S_3$ for $n \equiv 2, 3, 6, 7 \pmod 8$, that is, $n \equiv 2, 3 \pmod 4$.

Therefore $(0, y) \notin X_1$ for $y \equiv k+1, 2(k+1), 2(k+1)+1, 3(k+1)+1 \pmod{4(k+1)}$, so copies of T beginning at positions 1 and $2(k+1)+2$ modulo $4(k+1)$ tile $X_1 \cap (\{0\} \times \mathbb{Z})$.

Hence T tiles X_1.

Note that $(x, x+n(k+1))+(k+2, 1) = (x+k+2, (x+k+2)+(n-1)(k+1))$. Since $k \equiv 4 \pmod 8$, if $x \equiv 2n+r \pmod 8$ then $x+k+2 \equiv 2(n-1)+r \pmod 8$. Hence X_2 is invariant under translation by $(k+2, 1)$, by the definitions of S_1 and S_3. To show that horizontal copies of T tile X_2, it is therefore enough to show that T tiles the row $X_2 \cap (\mathbb{Z} \times \{0\})$.

We can express S_1 as $\{(y-n(k+1), y) \mid y \equiv -n, 1-n, 2-n, 3-n \pmod 8\}$.

Similarly $S_3 = \{(y-n(k+1)-1, y) \mid y \equiv 3-n, 4-n, 5-n, 6-n \pmod 8\}$.

Therefore $(-n(k+1), 0) \in S_1$ for $n \equiv 0, 1, 2, 3 \pmod 8$, and $(-n(k+1)-1, 0) \in S_3$ for $n \equiv 3, 4, 5, 6 \pmod 8$.

Hence $(x, 0) \notin X_2$ for $x \equiv 0, 2(k+1)-1, 3(k+1)-1, 4(k+1)-1, 5(k+1)-1, 5(k+1), 6(k+1), 7(k+1) \pmod{8(k+1)}$, so copies of T beginning at positions $k+1, 3(k+1), 5(k+1)+1, 7(k+1)+1$ modulo $8(k+1)$ tile $X_2 \cap (\mathbb{Z} \times \{0\})$.

Hence T tiles X_2.

The set $S_1 \cup S_2 \cup S_3$ can be partitioned into sets of the form $S = \{x_1, x_2, x_3\}$, where $x_1 = (x, y) \in S_1$, $x_2 = (x+4, y+4) \in S_2$, $x_3 = (x+2, y+3) \in S_3$. Then $|S| = 3$, so we can construct the corresponding set $Y \subset \mathbb{Z}^3$ as in Lemma 4. Now, given $n \in \mathbb{Z}$, $(S \times \{n\}) \setminus Y = \{x_i\}$ for some $i \in \{1, 2, 3\}$. Then $Y \cap (X_i \times \{n\}) = \emptyset$. If we do this for all such sets S, and let U be the (disjoint) union of the resulting sets Y, then $U \cap (X_i \times \{n\}) = \emptyset$, and $\mathbb{Z}^2 \times \{n\} \subset U \cup (X_i \times \{n\})$. Recall that T tiles each Y and therefore U.

We can do this for every n, choosing a partial tiling X_i for the corresponding $\mathbb{Z}^2$ layer. Together with U, these form a tiling of $\mathbb{Z}^3$ by T. This completes the proof of Theorem 9, and therefore also the proof of Theorem 2.

§5. *Open problems.* Theorem 2, together with the result that a punctured interval $T = \underbrace{\text{XXXXX}}_{k}.\underbrace{\text{XXXXX}}_{k}$ does not tile $\mathbb{Z}^2$ for $k \geqslant 3$, determines the smallest

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dimension d such that T tiles $\mathbb{Z}^d$ in the cases k odd and $k \equiv 4 \pmod 8$. However, for other values of k, we do not know whether the smallest such dimension d is 3 or 4.

Question 10. Let T be the punctured interval $\underbrace{\text{XXXXX}}_{k}.\underbrace{\text{XXXXX}}_{k}$, where $k \equiv 0, 2, 6 \pmod 8$, $k \geqslant 6$. Does T tile $\mathbb{Z}^3$?

It is also natural to consider more general tiles. The next non-trivial case is that of an interval with a non-central point removed. One might wonder if there is an analogue of Theorem 2 for these tiles.

Question 11. Does there exist a number d such that, for any tile T consisting of an interval in $\mathbb{Z}$ with one point removed, T tiles $\mathbb{Z}^d$?

For general one-dimensional tiles, Gruslys *et al* [**8**] conjectured that there is a bound on the dimension in terms of the size of the tile.

CONJECTURE 12 (Gruslys *et al* [**8**]). For any positive integer t, there exists a number d such that any tile $T \subset \mathbb{Z}$ with $|T| \leqslant t$ tiles $\mathbb{Z}^d$.

This conjecture remains unresolved. The authors of [**8**] showed that if d always exists then $d \to \infty$ as $t \to \infty$, by exhibiting a tile of size $3d - 1$ that does not tile $\mathbb{Z}^d$. This gives a simple lower bound on d; better bounds would be of great interest.

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References

1. A. Adler and F. C. Holroyd, Some results on one-dimensional tilings. *Geom. Dedicata* **10** (1981), 49–58.
2. R. Berger, The undecidability of the domino problem. *Mem. Amer. Math. Soc.* **66** (1966), 1–72.
3. J. H. Conway and J. C. Lagarias, Tiling with polyominoes and combinatorial group theory. *J. Combin. Theory* Ser. A **53** (1990), 183–208.
4. K. Dahkle, Tiling rectangles with polyominoes. http://eklhad.net/polyomino/index.html (retrieved 7 October 2018).
5. E. Friedman, Problem of the Month (February 1999). https://www2.stetson.edu/~efriedma/mathmagic/0299.html (retrieved 7 October 2018).
6. S. W. Golomb, Tiling with sets of polyominoes. *J. Combin. Theory* **9** (1970), 60–71.
7. V. Gruslys, Decomposing the vertex set of a hypercube into isomorphic subgraphs. *Preprint*, 2016, arXiv:1611.02021.
8. V. Gruslys, I. Leader and T. S. Tan, Tiling with arbitrary tiles. *Proc. Lond. Math. Soc. (3)* **112** (2016), 1019–1039.
9. V. Gruslys, I. Leader and I. Tomon, Partitioning the Boolean lattice into copies of a poset. *J. Combin. Theory Ser. A* **161** (2019), 81–98.
10. V. Gruslys and S. Letzter, Almost partitioning the hypercube into copies of a graph. *Preprint*, 2016, arXiv:1612.04603.
11. A. U. O. Kisisel, Polyomino convolutions and tiling problems. *J. Combin. Theory* Ser. A **95** (2001), 373–380.

https://doi.org/10.1112/S0025579318000384 Published online by Cambridge University Press

12. The Math Forum, Two tiling problems. http://mathforum.org/kb/message.jspa?messageID=6223965 (retrieved 7 October 2018).
13. MathOverflow, Does every polyomino tile $\mathbb{R}^n$ for some n? https://mathoverflow.net/questions/49915/does-every-polyomino-tile-rn-for-some-n (retrieved 7 October 2018).
14. I. Tomon, Almost tiling of the Boolean lattice with copies of a poset. *Electron. J. Combin.* **25** (2018), #P1.38.

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