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Computing the dimension of dynamically defined sets: E_2 and bounded continued fractions

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Abstract. We present a powerful approach to computing the Hausdorff dimension of certain conformally self-similar sets. We illustrate this method for the dimension $\dim_H(E_2)$ of the set E_2, consisting of those real numbers whose continued fraction expansions contain only the digits 1 or 2. A very striking feature of this method is that the successive approximations converge to $\dim_H(E_2)$ at a super-exponential rate.

0. *Introduction*

Given any number $0 < x < 1$ we can consider its continued fraction expansion

$$x = [a_1, a_2, a_3, \dots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

where $a_1, a_2, a_3, \dots \in \mathbb{N}$. Consider the set E_2 consisting of all points in the interval for which the continued fraction expansion has each a_n equal to either 1 or 2. This forms a Cantor set in the line, contained in the interval $[\frac{1}{2}(\sqrt{3}-1), \sqrt{3}-1]$, of zero Lebesgue measure. Furthermore, it represents sets of numbers with certain diophantine approximability conditions [**Bu1, Bu2**] and its Hausdorff dimension has other number theoretic significance in terms of the Markoff and Lagrange spectra (see §5, [**Cu1, Cu2, CuF**]).

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A number of authors have considered the problem of calculating the Hausdorff dimension $\dim_H(E_2)$ of the set E_2. In 1941, Good showed that $0.5194 \leq \dim_H(E_2) \leq 0.5433$ [**Good**]. In 1982, Bumby [**Bu1, Bu2**] improved these estimates using a thermodynamic approach, showing that $0.5312 \leq \dim_H(E_2) \leq 0.5314$. However, it appears his method is slowly convergent [**Bu2**, p. 2]. Subsequently, using a modified approach and superior computing power, in 1989 Hensley [**Hen1**] showed that $0.53128049 \leq \dim_H(E_2) \leq 0.53128051$. In 1996, he improved this estimate to 0.5312805062772051416 (see [**Hen2**]). In this paper, we shall present an efficient approach to calculating the value of $\dim_H(E_2)$.

More generally, our method is applicable to the computation of the Hausdorff dimension of other dynamically defined sets. One such example arises from Schottky groups. Consider $PSL(2, \mathbb{C})$ acting on $\mathbb{C}$ by linear fractional transformations. A Schottky group $\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$ is a free group contained in $PSL(2, \mathbb{C})$ for which, in particular, the sets $D(\gamma_i) = \{z \in \mathbb{C} : |\gamma_i'(z)| \leq 1\}$ are disjoint (cf. [**Bowen**] and [**PS**]). The limit set $\Lambda \subset \mathbb{C}$ is the set of accumulation points of the sequence $\gamma_{i_1} \ldots \gamma_{i_k}(0)$, say. The value $d = \dim_H \Lambda$ has various important interpretations relating to the corresponding surface of negative curvature. For example, it equals the topological entropy of the geodesic flow restricted to its recurrent set, and is elegantly related to the smallest eigenvalue of the Laplacian [**Sul**]. These applications are beyond the scope of the present paper and we shall return to the analysis of these, and other, examples in a later note. For the present, we shall concentrate on establishing our method in the concrete case of computing the dimension of E_2, and closely related sets.

The approach we use is motivated by related calculations for quantities arising in the study of the linearized Feigenbaum renormalization operator [**AAC**], [**Pol**] and the functional determinant of the Laplacian [**PolR**]. A common feature is an explicit expression as a power series, whose coefficients can readily be computed in terms of closed orbits for an underlying dynamical system and which tend to zero very rapidly, allowing very good approximation by only a few terms.

1. *Hausdorff dimension and periodic orbits*

We briefly recall the definition of Hausdorff dimension.

Definition. Given a subset $E \subset \mathbb{R}^n$ we can consider a cover $\mathcal{U} = \{U_i\}_i$ for E by open sets. For $\delta > 0$ we can define $H^\delta_\epsilon(E) = \inf_{\mathcal{U}} \left\{ \sum_i \operatorname{diam}(U_i)^\delta \right\}$ where the infimum is taken over all open covers $\mathcal{U} = \{U_i\}$ such that $\operatorname{diam}(U_i) \leq \epsilon$. We define $H^\delta(E) = \lim_{\epsilon \to 0} H^\delta_\epsilon(E)$ and, finally,

$$\dim_H(E) = \inf\{\delta : H^\delta(E) = 0\}.$$

For conformal systems, such as those we are considering, the Hausdorff dimension is equal to the box dimension (see [**Fal**]).

The Cantor set E_2 is invariant under the continued fraction transformation $T : [0, 1] \to [0, 1]$ defined by

$$T(x) = \begin{cases} 1/x - [1/x], & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Observe that in this case $T'(x) = -1/x^2$.

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In fact E_2 can be generated using the two contracting maps $T_1, T_2 : [0,1] \to [0,1]$ defined by

$$T_1(x) = \frac{1}{1+x} \quad \text{and} \quad T_2(x) = \frac{1}{2+x}.$$

More precisely, we can write E_2 as the set of limit points of sequences $T_{i_1} \dots T_{i_n}(0)$, say, for $i_1, i_2, \dots \in \{1,2\}$. Note that T_1, T_2 both reverse orientation. The nonlinearity of T_1, T_2 means that E_2 is not exactly self-affine (unlike the classical 'middle third' Cantor set, for example) so its Hausdorff dimension cannot be computed using simple self-similarity arguments. However, it does have a form of asymptotic self-similarity, which gives us a formula for $\dim_H(E_2)$ in terms of pressure.

Definition. Given any continuous function $f : E_2 \to \mathbb{R}$ we define its *pressure* $P(f)$ (with respect to T) as

$$P(f) := \lim_{n\to+\infty} \frac{1}{n} \log \left(\sum_{\substack{T^n x = x \\ x \in E_2}} e^{f(x)+f(Tx)+\cdots+f(T^{n-1}x)} \right).$$

The particular choice $f(x) = -s \log |T'(x)|$ (restricted to E_2) gives the following result.

LEMMA 1. [**Fal, Bed**] *There is a unique solution $0 \le s \le 1$ to $P(-s \log |T'|) = 0$, which occurs precisely at $s = \dim_H(E_2)$.*

Formulae of this general type have a long history. Bowen [**Bowen**] showed that the dimension of quasi-circles could be studied using thermodynamic ideas. Subsequently, Ruelle [**Ru2**] applied the approach to hyperbolic rational maps and McCluskey–Manning [**McM**] studied Smale horseshoes. Our example is covered by the standard theory of 'cookie-cutters' (see, for example, Bedford [**Bed**]).

Given a finite string $\underline{i} = i_1 \dots i_n \in \{1,2\}^n$ (for simplicity we do not put commas between the i_j) we write $|\underline{i}| = n$. The unique fixed point for the contraction

$$\begin{cases} T_{\underline{i}} := T_{i_1} \cdots T_{i_n} : [0,1] \to [0,1] \\ T_{\underline{i}} : z \mapsto \dfrac{1}{i_1 + \dfrac{1}{i_2 + \cdots \dfrac{1}{i_n + z}}} \end{cases}$$

is the point $x_{\underline{i}}$ with corresponding periodic continued fraction expansion $[\overline{i_1, \dots, i_n}]$. This number is a quadratic surd, i.e. a non-rational root of a quadratic equation with integer entries. The points in the periodic orbit $\{x_{\underline{i}}, Tx_{\underline{i}}, \dots, T^{n-1}x_{\underline{i}}\}$ correspond to quadratic surds with periodic continued fractions $[\overline{i_{k+1}, \dots, i_n, i_1, \dots, i_k}]$, for $k = 1, \dots, n$.

To compute $P(-s \log |T'|)$ we must sum terms of the form $(|(T^{|\underline{i}|})'(x_{\underline{i}})|)^{-s}$. Since $T'(x) = -1/x^2$ then these terms have the following simple form:

$$(|(T^{|\underline{i}|})'(x_{\underline{i}})|)^{-s} = \prod_{r=0}^{|\underline{i}|-1} \frac{1}{T'(T^r x_{\underline{i}})^s} = \prod_{r=0}^{|\underline{i}|-1} [T^r x_{\underline{i}}]^{2s}.$$

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We therefore introduce the notation $w_{\underline{i}} = \prod_{r=0}^{|\underline{i}|-1} T^r x_{\underline{i}}$, the product of all terms in the orbit of $x_{\underline{i}}$. Note that we consider this orbit *with multiplicity* (i.e. we take the product over $|\underline{i}|$ iterates, even if this involves repeats).

Example. The fixed points of T_1 and T_2 are $\frac{1}{2}(\sqrt{5}-1)$ and $\sqrt{2}-1$, respectively, while the string 12 corresponds to the periodic orbit $\{\sqrt{3}-1, \frac{1}{2}(\sqrt{3}-1)\}$. Then, for example, $w_{22} = (\sqrt{2}-1)^2 = 3-2\sqrt{2}$.

We can therefore express pressure as

$$P(-s\log|T'|) = \lim_{n\to\infty} \frac{1}{n} \log\Bigg(\sum_{|\underline{i}|=n} w_{\underline{i}}^{2s}\Bigg).$$

2. *The main result*

In §3 we will introduce a family of linear operators $\mathcal{L}_s$, and functional determinants $\det(I - z\mathcal{L}_s)$ with power series expansion

$$\det(I - z\mathcal{L}_s) = 1 + \sum_{n=1}^{\infty} d_n(s) z^n,$$

where the power series coefficients will be given by

$$d_n(s) = \sum_{n_1+\cdots+n_r=n} \frac{(-1)^r}{r!} \prod_{j=1}^{r} \Bigg(\sum_{|\underline{i}|=n_j} \frac{w_{\underline{i}}^{2s}}{1-(-1)^{|\underline{i}|} w_{\underline{i}}^2}\Bigg), \tag{2.1}$$

and the summation is over all r-tuples $(n_1, \ldots, n_r)$ of positive integers satisfying $n_1 + \cdots + n_r = n$.

Setting $D(s,z) = \det(I - z\mathcal{L}_s)\det(I + z\mathcal{L}_s) = \det(I - z^2\mathcal{L}_s^2)$ we have

$$D(s,z) = 1 + \sum_{n=1}^{\infty} b_n(s) z^n$$

where

$$b_n(s) = \sum_{k=1}^{n-1} (-1)^k d_k(s) d_{n-k}(s). \tag{2.2}$$

Note that the odd terms $b_{2k+1}(s)$ vanish, while the even terms $b_{2k}(s)$ are given by

$$b_{2k}(s) = \sum_{n_1+\cdots+n_r=k} \frac{(-1)^r}{r!} \prod_{j=1}^{r} \Bigg(\sum_{|\underline{i}|=2n_j} \frac{w_{\underline{i}}^{2s}}{1-(-1)^{|\underline{i}|} w_{\underline{i}}^2}\Bigg). \tag{2.3}$$

Our main result in this paper is the following.

THEOREM 1.

(a) *The Hausdorff dimension of E_2 is the largest solution $0 \le s \le 1$ to the equation*

$$D(s,1) = 1 + \sum_{n=1}^{\infty} b_n(s) = 0.$$

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(b) *There exist* $0 < \theta < 1$ *and* $K > 0$ *such that if* s_N *denotes the largest solution* $0 \le s_N \le 1$ *to the equation*

$$1 + \sum_{n=1}^{N} b_n(s) = 0,$$

then $|s_N - \dim_H(E_2)| < K\theta^{N^2}$.

In particular, part (b) of Theorem 1 shows we have a very efficient method of computing $\dim_H(E_2)$. For example, letting $N = 16$ and working with the eight terms $b_2(s), b_4(s), \ldots, b_{16}(s)$ we get a solution s_{16} described in the following result.

COROLLARY 1.1. *The approximation*

$$\dim_H(E_2) \approx s_{16} = 0.53128050627720514162446 86\ldots$$

is accurate to 25 decimal places.

Scheme of the proof of Theorem 1. For part (a) our method will be to introduce a family of transfer operators $\mathcal{L}_s$, whose spectral radius is $e^{P(-s\log|T'|)}$. Since $(-\mathcal{L}_s)^2 = \mathcal{L}_s^2$, then $-\mathcal{L}_s$ also has spectral radius $e^{P(-s\log|T'|)}$, by the spectral radius formula (see [**RS**, p. 192]).

By Lemma 1, the Hausdorff dimension of E_2 is the unique real value of s for which $\mathcal{L}_s$ has spectral radius equal to 1. The operators $\mathcal{L}_s$ will be of Perron–Frobenius type (see Lemma 2), so that in fact $e^{P(-s\log|T'|)}$ is itself an eigenvalue of $\mathcal{L}_s$. In particular, for $s = \dim_H(E_2)$, the value 1 will be the leading eigenvalue of $\mathcal{L}_s$.

In §3 we will show that when restricted to a suitable Banach space of bounded holomorphic functions, the eigenvalues of $\mathcal{L}_s$ are precisely the reciprocals of the zeros z of $\det(I - z\mathcal{L}_s)$. The zeros z of $D(s, z) = \det(I - z\mathcal{L}_s)\det(I + z\mathcal{L}_s)$ are the reciprocals of the union of the eigenvalues of $\mathcal{L}_s$ and $-\mathcal{L}_s$.

Therefore, setting $z = 1$, we see that s will be a solution of

$$D(s, 1) = \det(I - \mathcal{L}_s)\det(I + \mathcal{L}_s) = \det(I - \mathcal{L}_s^2) = 0$$

if and only if 1 is an eigenvalue of either $\mathcal{L}_s$ or $-\mathcal{L}_s$. By the above discussion we know that $s = \dim_H(E_2)$ is one such solution. In fact it is the largest solution, since for $s > \dim_H(E_2)$, the spectral radius of both $\mathcal{L}_s$ and $-\mathcal{L}_s$ is less than 1, since $P(-s\log|T'|)$ is decreasing in s, so that 1 cannot be an eigenvalue of $\mathcal{L}_s$ or $-\mathcal{L}_s$. In §3 we will prove the formulae (2.1) and (2.3) for the power series coefficients of $\det(I - z\mathcal{L}_s)$ and $D(s, z)$.

Part (b) will follow from the fact that for any fixed $0 \le s \le 1$ the terms $b_n(s)$ in the power series expansion for $\det(I - z^2\mathcal{L}_s^2)$ can be bounded in modulus by $c\theta^{n^2}$, where both $0 < \theta < 1$ and $c > 0$ can be chosen independent of s. §3 will be devoted to a proof of this fact. Having proved the decay rate for the $b_n(s)$ we then use the following argument. Let $F(s) = 1 + \sum_{n=1}^{\infty} b_n(s)$ and $F_N(s) = 1 + \sum_{n=1}^{N} b_n(s)$. Let $s_\infty = \dim_H(E_2)$. By the mean value theorem there exists t_N between s_N and s_∞ such that

$$\begin{aligned}(s_N - s_\infty)F_N'(t_N) &= F_N(s_N) - F_N(s_\infty) = -F_N(s_\infty)\\ &= F(s_\infty) - F_N(s_\infty) = \sum_{n=N+1}^{\infty} b_n(s_\infty),\end{aligned}$$

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so that

$$|s_N - s_\infty| \le \left| \frac{1}{F_N'(t_N)} \sum_{n=N+1}^{\infty} b_n(s) \right| \le \frac{1}{|F_N'(t_N)|} \sum_{n=N+1}^{\infty} c\theta^{n^2}$$
$$= \frac{c\theta^{N^2}}{|F_N'(t_N)|} \sum_{r=1}^{\infty} \theta^{2Nr+r^2} \le \frac{c\theta^{N^2+1}}{|F_N'(t_N)|} \sum_{r=1}^{\infty} \theta^{2Nr}$$
$$= \frac{c\theta^{N^2+1}}{|F_N'(t_N)|} \frac{\theta^{2N}}{1-\theta^{2N}} \le \frac{c\theta^{(N+1)^2}}{|F_N'(t_N)|(1-\theta)}.$$

Remark. In fact $\dim_H(E_2)$ is also the largest zero s of $\det(I-\mathcal{L}_s)$, by the above discussion, so that Theorem 1 is also true if we replace $D(s,1)$ by $\det(I-\mathcal{L}_s)$. Note, however, that the odd terms $d_{2k+1}(s)$ in the power series expansion for $\det(I-z\mathcal{L}_s)$ do not vanish. We obtain superior computational results (see §4) by working with $D(s,1)$.

3. *Proof of the main result*

Our main technical device will be to re-write $D(s,z)$ in terms of the trace of a certain operator $\mathcal{L}_s$ defined below. A suitable choice of domain ensures $\mathcal{L}_s$ has good spectral properties, which implies the estimates on the decay rates of $b_n(s)$. Following Mayer [**May1**], we can fix the disk

$$D = \{z \in \mathbb{C} : |z-1| < \tfrac{3}{2}\}.$$

If we consider the family of analytic functions $\mathcal{A}$ on D which have a continuous extension to the closed disk, then this forms a Banach space with respect to the uniform norm $\|f\| = \sup_{x\in D} |f(x)|$.

We can define the *transfer operator* $\mathcal{L}_s : \mathcal{A} \to \mathcal{A}$ (compare with [**May1, Bu1, Bu2**]) by

$$\mathcal{L}_s v(z) = \left(\frac{1}{z+1}\right)^{2s} v\left(\frac{1}{z+1}\right) + \left(\frac{1}{z+2}\right)^{2s} v\left(\frac{1}{z+2}\right).$$

To interpret this definition appropriately we should write

$$\left(\frac{1}{z+1}\right)^{2s} = \exp\left(2s \log\left(\frac{1}{z+1}\right)\right)$$

and

$$\left(\frac{1}{z+2}\right)^{2s} = \exp\left(2s \log\left(\frac{1}{z+2}\right)\right)$$

and observe that for $z \in D$ and $0 \le s \le 1$ the logarithm is well defined.

Note that the functions $(z+1)^{-2s}$ and $(z+2)^{-2s}$ are just the holomorphic extensions of the restrictions of $|T_1'(z)|^s = |T'(T_1 z)|^{-s}$ and $|T_2'(z)|^s = |T'(T_2 z)|^{-s}$ to the unit interval. Thus the operator $\mathcal{L}_s$ is just the Ruelle–Perron–Frobenius operator with weight function $-s\log|T'|$, and we have (see [**PaPol**]) the following.

LEMMA 2. *The value $e^{P(-s\log|T'|)}$ is the unique eigenvalue of maximum modulus for the operator $\mathcal{L}_s$.*

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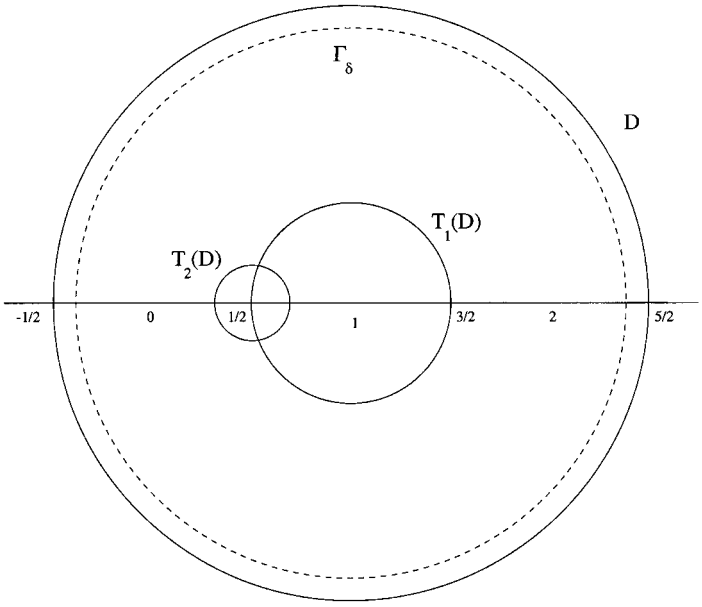


FIGURE 1.

In fact we can go beyond the Perron–Frobenius theory, and derive additional spectral information about $\mathcal{L}_s$. We recall that a *nuclear operator* $L : \mathcal{A} \to \mathcal{A}$ is a linear operator for which there exist functions $f_n \in \mathcal{A}$ with $\|f_n\| = 1$, functionals $l_n \in \mathcal{A}^*$ with $\|l_n\| = 1$, and coefficients $\lambda_n \in \mathbb{C}$ with $\sum_{n=1}^{\infty} |\lambda_n| < \infty$, such that

$$Lv = \sum_{n=1}^{\infty} (\lambda_n l_n(v)) f_n. \tag{3.1}$$

This definition is due to Grothendieck [**Gr**]. In particular, a nuclear operator is compact. We will show that the transfer operator $\mathcal{L}_s : \mathcal{A} \to \mathcal{A}$ is nuclear. Our analysis follows closely that of Mayer in [**May1**].

First observe that the closure of the image disks

$$T_1(D) = \{z : |z - 8/7| < 6/7\} \quad \text{and} \quad T_2(D) = \{z : |z - 4/9| < 2/9\}$$

are still within the original disk D (see Figure 1).

Consider the curve

$$\Gamma = \{z \in \mathbb{C} : |z - 10/9| = 1\}.$$

Note that Γ does not intersect $T_1(D)$ or $T_2(D)$, and ξ^{2s} is well defined and analytic in the region enclosed by Γ. Using Cauchy's theorem to write

$$(T_j z)^{2s} v(T_j z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\xi^{2s} v(\xi)}{\xi - T_j z} \, d\xi$$

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for $j = 1, 2$, we have

$$\begin{aligned}\mathcal{L}_s v(z) &= \frac{1}{2\pi i}\int_\Gamma \xi^{2s} v(\xi)\left(\frac{1}{\xi - T_1 z} + \frac{1}{\xi - T_2 z}\right) d\xi \\ &= \sum_{n=0}^{\infty}\left(\left(T_1 z - \frac{10}{9}\right)^n + \left(T_2 z - \frac{10}{9}\right)^n\right)\left(\frac{1}{2\pi i}\int_\Gamma \frac{\xi^{2s} v(\xi)}{\left(\xi - \frac{10}{9}\right)^{n+1}} d\xi\right),\end{aligned} \tag{3.2}$$

where for the last line we have just used the trivial summation of the geometric series

$$\sum_{n=0}^{\infty}\left(\frac{T_j z - \frac{10}{9}}{\xi - \frac{10}{9}}\right)^n = \left(1 - \frac{T_j z - \frac{10}{9}}{\xi - \frac{10}{9}}\right)^{-1}.$$

The expansion in (3.2) is valid since Γ was chosen to be outside the two image discs $T_1(D)$, $T_2(D)$ (see Figure 1), so $|(T_j z - \frac{10}{9})/(\xi - \frac{10}{9})| < 1$ for $z \in D$, $\xi \in \Gamma$.

Now let us write

$$g_{j,n}(z) := \left(T_j z - \frac{10}{9}\right)^n \quad \text{and} \quad m_n(v) := \frac{1}{2\pi i}\int_\Gamma \frac{\xi^{2s} v(\xi)}{\left(\xi - \frac{10}{9}\right)^{n+1}} d\xi.$$

Note that the functions $g_{j,n}$ belong to $\mathcal{A}$, and the functionals m_n are linear in v. Introducing the normalizations

$$f_{2n+j} = g_{j,n}/\|g_{j,n}\|, \quad l_{2n+j} = m_n/\|m_n\| \tag{3.3}$$

for $n \geq 0$ and $j = 1, 2$, we can express $\mathcal{L}_s$ as

$$\mathcal{L}_s v(z) = \sum_{n=1}^{\infty}(\lambda_n l_n(v)) f_n(z),$$

where $\lambda_{2n+j} = \|g_{j,n}\|.\|m_n\|$. Moreover, we have the following exponential bound for the coefficients λ_n.

LEMMA 3. *$\lambda_n \leq \frac{45}{16\pi}\gamma^n$ where $\gamma = \sqrt{\frac{8}{9}}$.*

Proof. Now $\lambda_{2n+j} = \|g_{j,n}\|.\|m_n\|$, so we will estimate the norms $\|g_{j,n}\|$ and $\|m_n\|$. For $\xi \in \Gamma$ we have $|\xi - \frac{10}{9}| = 1$, which also means we can bound $|\xi^{2s}| \leq |\frac{19}{9}|^{2s} < 5$ for $0 \leq s \leq 1$. Thus $\|m_n\| \leq \frac{5}{2\pi}$. For $z \in D$ we have $|T_1 z - \frac{10}{9}|, |T_2 z - \frac{10}{9}| \leq \frac{8}{9}$ so that $\|g_{j,n}\| \leq \left(\frac{8}{9}\right)^n$. Thus $\lambda_{2n+j} \leq \frac{5}{2\pi}\left(\frac{8}{9}\right)^n = \frac{45}{16\pi}\gamma^{2n+2}$, for $j = 1, 2$, and the result follows. □

In particular, this means that $\sum \lambda_n < \infty$, so we have proved the following.

COROLLARY 3.1. *The transfer operator $\mathcal{L}_s : \mathcal{A} \to \mathcal{A}$ is nuclear.*

The immediate benefit of Corollary 3.1 is that nuclear operators are trace class. That is, we can define the *trace* $\mathrm{tr}(\mathcal{L}_s)$ to be the sum of its eigenvalues, counted with multiplicities. Following [**Gr, Ru1, May1**] and [**Fried**] we then define the *Fredholm determinant* of $(I - z\mathcal{L}_s) : \mathcal{A} \to \mathcal{A}$ by

$$\det(I - z\mathcal{L}_s) = \exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}\,\mathrm{tr}(\mathcal{L}_s^n)\right) \tag{3.4}$$

for $z \in \mathbb{C}$.

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Nuclear operators enjoy many of the properties of finite-dimensional matrices. In particular, we have the following.

COROLLARY 3.2. [**Gr**] *The zeros z of $\det(I - z\mathcal{L}_s)$, counted with multiplicities, are precisely the non-zero eigenvalues of $\mathcal{L}_s$, again counted with (algebraic) multiplicities.*

We have the following trace formula.

LEMMA 4.

$$\operatorname{tr}(\mathcal{L}_s^n) = \sum_{|\underline{i}|=n} \frac{w_{\underline{i}}^{2s}}{1 - (-1)^{|\underline{i}|} w_{\underline{i}}^2}. \tag{3.5}$$

Proof. Given any string $\underline{i} = i_1 \dots i_n$, recall that $T_{\underline{i}} = T_{i_1} \cdots T_{i_n}$ denotes the associated local inverse to T^n, and $x_{\underline{i}}$ denotes the fixed point of $T_{\underline{i}}$. We have that $T'_{\underline{i}}(x_{\underline{i}}) = (-1)^{|\underline{i}|} w_{\underline{i}}^2$.

Define a linear operator $\mathcal{L}_{\underline{i}} : \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{L}_{\underline{i}} v(z) = (T_{\underline{i}} z)^{2s} v(T_{\underline{i}} z).$$

This allows us to write $\mathcal{L}_s^n = \sum_{|\underline{i}|=n} \mathcal{L}_{\underline{i}}$ and since the trace is additive we have $\operatorname{tr}(\mathcal{L}_s^n) = \sum_{|\underline{i}|=n} \operatorname{tr}(\mathcal{L}_{\underline{i}})$, so it will suffice to compute $\operatorname{tr}(\mathcal{L}_{\underline{i}})$ for all $|\underline{i}| = n$. Now if α is an eigenvalue of $\mathcal{L}_{\underline{i}}$, then the eigenvalue equation $\mathcal{L}_{\underline{i}} v(z) = \alpha v(z)$ evaluated at the point $z = x_{\underline{i}}$ gives us $w_{\underline{i}}^{2s} v(x_{\underline{i}}) = \alpha v(x_{\underline{i}})$. So provided the eigenvector v does not vanish at $x_{\underline{i}}$, then $\alpha = w_{\underline{i}}^{2s}$. If, however, $v(x_{\underline{i}}) = 0$, then differentiating the eigenvalue equation and evaluating at the point $z = x_{\underline{i}}$ gives

$$(-1)^{|\underline{i}|} w_{\underline{i}}^2 w_{\underline{i}}^{2s} v'(x_{\underline{i}}) = \alpha v'(x_{\underline{i}}),$$

so that if $v'(x_{\underline{i}}) \neq 0$ then $\alpha = (-1)^{|\underline{i}|} w_{\underline{i}}^2 w_{\underline{i}}^{2s}$.

Repeating this argument shows that any eigenvalue of $\mathcal{L}_{\underline{i}}$ must be of the form $(-1)^{k|\underline{i}|} w_{\underline{i}}^{2(s+k)}$, for some integer $k \geq 0$. In fact *all* of these values are indeed eigenvalues (see [**Kam**, Lemma 2]), and each is of algebraic multiplicity one (see [**May2**, Lemma 7.10]), so we have

$$\operatorname{tr}(\mathcal{L}_{\underline{i}}) = \sum_{k=0}^{\infty} (-1)^{k|\underline{i}|} w_{\underline{i}}^{2(s+k)} = \frac{w_{\underline{i}}^{2s}}{1 - (-1)^{|\underline{i}|} w_{\underline{i}}^2},$$

and we deduce that

$$\operatorname{tr}(\mathcal{L}_s^n) = \sum_{|\underline{i}|=n} \operatorname{tr}(\mathcal{L}_{\underline{i}}) = \sum_{|\underline{i}|=n} \frac{w_{\underline{i}}^{2s}}{1 - (-1)^{|\underline{i}|} w_{\underline{i}}^2},$$

completing the proof of the lemma. □

Substituting the trace formula (3.5) into the expression (3.4), and using the power series expansion for the exponential function, we calculate the power series coefficients $d_n(s)$ of $\det(I - z\mathcal{L}_s)$ to be as in (2.1). The power series coefficients $b_n(s)$ for $D(s, z)$ are then given by (2.2) and (2.3), and the scheme of proof outlined in §2 completes the proof of Theorem 1(a).

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It remains to prove part (b) of Theorem 1. By the outline proof of Theorem 1 given in §2, we know it suffices to show the coefficients $b_n(s)$ in the power series expansion for $D(s,z)$ decay at a rate $O(\theta^{n^2})$, for some $0 < \theta < 1$. The first step towards this is the formal expression (see [**Gr**]) for the Fredholm determinant,

$$\det(I - z\mathcal{L}_s) = \begin{vmatrix} 1 - z\lambda_1 l_1(f_1) & -z\lambda_1 l_1(f_2) & -z\lambda_1 l_1(f_3) & \dots \\ -z\lambda_2 l_2(f_1) & 1 - z\lambda_2 l_2(f_2) & -z\lambda_2 l_2(f_3) & \dots \\ -z\lambda_3 l_3(f_1) & -z\lambda_3 l_3(f_2) & 1 - z\lambda_3 l_3(f_3) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix},$$

which leads to the following alternative characterization of its power series coefficients.

LEMMA 5. [**Gr**], [**Fried**, p. 506] *We can write* $\det(I - z\mathcal{L}_s) = 1 + \sum_{n=1}^{\infty} d_n(s) z^n$ *where*

$$d_n(s) = (-1)^n \sum_{i_1 < \dots < i_n} \lambda_{i_1} \dots \lambda_{i_n} \operatorname{Det}(l_{i_j}(f_{i_k}))_{j,k=1}^{n} \tag{3.6}$$

and $\operatorname{Det}[l_{i_j}(f_{i_k})]_{j,k=1}^{n}$ *denotes the determinant of the* $(n \times n)$ *matrix with entries* $l_{i_j}(f_{i_k})$, $1 \le j, k \le n$, *where the functionals* l_{i_j} *and functions* f_{i_k} *are as in (3.3).*

Proof of Theorem 1(b). To estimate the speed at which the terms $b_n(s)$ tend to zero we will use expression (3.6) to prove the analogous fact about the coefficients $d_n(s)$. We first recall Hadamard's theorem [**Had**] that an $(n \times n)$ matrix with entries of modulus at most 1 has determinant with modulus bounded by $n^{n/2}$.

We next note that for any $0 < \gamma < 1$ we have

$$\sum_{k_1 < \dots < k_n} \gamma^{k_1 + \dots + k_n} = \frac{\gamma^{n(n+1)/2}}{(1-\gamma)(1-\gamma^2)\cdots(1-\gamma^n)}. \tag{3.7}$$

To see this we just compute

$$\begin{aligned} \sum_{k_1 < \dots < k_n} \gamma^{k_1 + \dots + k_n} &= \sum_{k_1=0}^{\infty} \sum_{k_2=k_1+1}^{\infty} \cdots \sum_{k_n=k_{n-1}+1}^{\infty} \gamma^{k_1 + \dots + k_n} \\ &= \sum_{k_1=0}^{\infty} \sum_{k_2=k_1+1}^{\infty} \cdots \sum_{k_{n-1}=k_{n-2}+1}^{\infty} \gamma^{k_1 + \dots + k_{n-2} + 2k_{n-1}} \left(\frac{\gamma}{1-\gamma} \right) \\ &= \sum_{k_1=0}^{\infty} \sum_{k_2=k_1}^{\infty} \cdots \sum_{k_{n-2}=k_{n-3}+1}^{\infty} \gamma^{k_1 + \dots + k_{n-3} + 3k_{n-2}} \left(\frac{\gamma^{2+1}}{(1-\gamma)(1-\gamma^2)} \right) \\ &= \cdots \\ &= \frac{\gamma^{n + \dots + 2 + 1}}{(1-\gamma)\cdots(1-\gamma^n)} \end{aligned}$$

from which (3.7) immediately follows.

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In particular, if we choose $\gamma = \sqrt{\frac{8}{9}}$ as in Lemma 3 then we know $\lambda_{i_j} \leq \frac{45}{16\pi}\gamma^{i_j+1}$. So by (3.7) we can now estimate

$$|d_n(s)| \leq \left(\frac{45}{16\pi}\right)^n \frac{\gamma^{1+n(n+1)/2}}{(1-\gamma)(1-\gamma^2)\cdots(1-\gamma^n)} n^{n/2}$$
$$\leq C\left(\frac{45}{16\pi}\right)^n \left(\frac{8}{9}\right)^{n(n+1)/4} n^{n/2},$$

where $C = \gamma \prod_{r=1}^{\infty}(1-\gamma^r)^{-1} > 0$.

In particular, we see that the coefficients $d_n(s) = \mathrm{O}(\theta^{n^2})$, where $\theta = \sqrt{\gamma} = \left(\frac{8}{9}\right)^{1/4}$.

The same is true for the coefficients $b_n(s)$ of the determinant $D(s, z)$, by the following argument. Writing

$$\mathcal{L}_s^2 v = \mathcal{L}_s(\mathcal{L}_s v) = \sum_{n=1}^{\infty} \lambda_n l_n \left(\sum_{m=1}^{\infty} \lambda_m l_m(v) f_m \right) f_n = \sum_{n=1}^{\infty} (\mu_n l_n(v)) f_n,$$

say, we see that

$$|\mu_n| \leq \lambda_n \sum_{r=1}^{\infty} \lambda_r \leq \frac{45\gamma^{n+1}}{16\pi(1-\gamma)},$$

and applying the same estimates as above gives that $b_n(s) = O(\theta^{n^2})$, as required. This completes the proof of Theorem 1(b). □

4. *Some calculations*

We now present some details of our calculation of $\dim_H(E_2)$. Of course there is nothing special about the digits 1 and 2 in the definition of E_2. More generally we could restrict the continued fraction coefficients to any finite alphabet $\{n_1, \ldots, n_r\}$, and consider the resulting Cantor set $E_{\{n_1,\ldots,n_r\}}$. With the appropriate contraction maps $T_i(x) = 1/(n_i + x)$ we would generate the corresponding periodic orbits, define the Fredholm determinant accordingly, and compute its zeros to obtain the Hausdorff dimension of $E_{\{n_1,\ldots,n_r\}}$. This method works equally well if we further forbid a finite number of strings from our finite alphabet (so that the underlying dynamics is a subshift of finite type), see [**PaPol**].

The fundamental objects in our calculations are the strings $\underline{i} = i_1 \ldots i_n$ and their corresponding weights $w_{\underline{i}}$. An elementary yet crucial property of these weights is that they are multiplicative. That is, if $\underline{i}' = (\underline{i})^r$ is the r-fold concatenation of the string $\underline{i}$ then $w_{\underline{i}'} = w_{\underline{i}}^r$.

To illustrate the speed at which the weights $w_{\underline{i}}$ tend to zero as $|\underline{i}|$ grows we list the strings up to period five in Table 1. We only give one string per cyclic permutation class, and do not list those which are concatenation powers of smaller strings (since the weights of these can be computed using the multiplicative property).

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TABLE 1. Weights associated to strings.

$\underline{i}$	Orbit	$w_{\underline{i}}$
1	$\{\frac{1}{2}(\sqrt{5}-1)\}$	$\frac{1}{2}(\sqrt{5}-1)$
2	$\{\sqrt{2}-1\}$	$\sqrt{2}-1$
12	$\{\sqrt{3}-1, \frac{1}{2}(\sqrt{3}-1)\}$	$2-\sqrt{3}$
112	$\{\frac{1}{2}(\sqrt{10}-2), \frac{1}{3}(\sqrt{10}-1), \frac{1}{3}(\sqrt{10}-2)\}$	$\sqrt{10}-3$
122	$\{\frac{1}{6}(\sqrt{85}-5), \frac{1}{10}(\sqrt{85}-5), \frac{1}{6}(\sqrt{85}-7)\}$	$\frac{1}{2}(\sqrt{85}-9)$
1112	$\{\frac{1}{3}(2\sqrt{6}-3), \frac{2}{5}(\sqrt{6}-1), \frac{1}{2}(\sqrt{6}-1), \frac{1}{5}(2\sqrt{6}-3)\}$	$5-2\sqrt{6}$
1122	$\{\frac{1}{10}(\sqrt{221}-9), \frac{1}{14}(\sqrt{221}-5), \frac{1}{14}(\sqrt{221}-9), \frac{1}{10}(\sqrt{221}-11)\}$	$\frac{1}{2}(15-\sqrt{221})$
1222	$\{\frac{2}{7}(\sqrt{30}-3), \frac{1}{6}(\sqrt{30}-3), \frac{2}{7}(\sqrt{30}-4), \frac{1}{4}(\sqrt{30}-4)\}$	$11-2\sqrt{30}$
11112	$\{\frac{1}{5}(\sqrt{65}-5), \frac{1}{8}(\sqrt{65}-3), \frac{1}{7}(\sqrt{65}-4), \frac{1}{7}(\sqrt{65}-3), \frac{1}{8}(\sqrt{65}-5)\}$	$\sqrt{65}-8$
11122	$\{\frac{1}{8}(\sqrt{145}-7), \frac{1}{12}(\sqrt{145}-5), \frac{1}{10}(\sqrt{145}-5), \frac{1}{12}(\sqrt{145}-7), \frac{1}{8}(\sqrt{145}-9)\}$	$\sqrt{145}-12$
11212	$\{\frac{1}{14}(\sqrt{533}-15), \frac{1}{22}(\sqrt{533}-7), \frac{1}{22}(\sqrt{533}-15), \frac{1}{14}(\sqrt{533}-13), \frac{1}{26}(\sqrt{533}-13)\}$	$\frac{1}{2}(\sqrt{533}-23)$
11222	$\{\frac{1}{12}(5\sqrt{13}-11), \frac{1}{17}(5\sqrt{13}-6), \frac{1}{17}(5\sqrt{13}-11), \frac{1}{12}(5\sqrt{13}-13), \frac{1}{13}(5\sqrt{13}-13)\}$	$5\sqrt{13}-18$
12122	$\{\frac{1}{11}(\sqrt{290}-9), \frac{1}{19}(\sqrt{290}-10), \frac{1}{10}(\sqrt{290}-10), \frac{1}{19}(\sqrt{290}-9), \frac{1}{11}(\sqrt{290}-13)\}$	$\sqrt{290}-13$
12222	$\{\frac{1}{34}(\sqrt{2813}-29), \frac{1}{58}(\sqrt{2813}-29), \frac{1}{34}(\sqrt{2813}-39), \frac{1}{38}(\sqrt{2813}-37), \frac{1}{38}(\sqrt{2813}-39)\}$	$\frac{1}{2}(\sqrt{2813}-53)$

We now explicitly compute the first few terms of $D(s,z)$. We have

$$\begin{aligned} D(s,z) &= \det(I - z\mathcal{L}_s)\det(I + z\mathcal{L}_s) \\ &= \exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}\sum_{|\underline{i}|=n}\left(\frac{w_{\underline{i}}^{2s}}{1-(-1)^{|\underline{i}|}w_{\underline{i}}^2} + \frac{(-1)^n w_{\underline{i}}^{2s}}{1-(-1)^{|\underline{i}|}w_{\underline{i}}^2}\right)\right) \\ &= \exp\left(-\sum_{n=1}^{\infty} C_n(s) z^n\right) \end{aligned}$$

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TABLE 2.

N	Value of s_N
2	0.573 649 751 993 150 441 880 511 191 052 754 537 358 022 798 8451
4	0.530 668 590 837 578 693 240 136 089 281 967 263 833 813 933 2442
6	0.531 281 500 411 940 509 424 637 807 279 905 311 402 876 386 9533
8	0.531 280 506 053 282 407 361 734 294 731 191 695 946 929 538 9062
10	0.531 280 506 277 212 243 672 274 089 624 535 093 716 259 857 8572
12	0.531 280 506 277 205 141 592 435 787 861 805 662 052 800 294 6593
14	0.531 280 506 277 205 141 624 468 668 037 201 957 416 201 902 3770
16	0.531 280 506 277 205 141 624 468 647 368 469 870 184 503 421 8984

where

$$C_n(s) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \dfrac{2}{n} \displaystyle\sum_{|\underline{i}|=n} \frac{w_{\underline{i}}^{2s}}{1 - w_{\underline{i}}^2}, & \text{if } n \text{ is even.} \end{cases}$$

We now use the series expansion of the exponential function to compute the coefficients $b_n(s)$ in the power series $\det(I - z^2\mathcal{L}_s^2) = 1 + \sum_{n=1}^{\infty} b_n(s) z^n$.

If n is odd then $b_n(s)$ is zero, since all the odd terms $C_i(s)$ are zero. If n is even then we can express $b_n(s)$ in terms of $C_i(s)$, $i \le n$. Writing $C_i = C_i(s)$ the first few non-zero coefficients $b_n(s)$ are as follows:

$$\begin{aligned} b_2(s) &= -C_2 \\ b_4(s) &= -C_4 + \tfrac{1}{2}C_2^2 \\ b_6(s) &= -C_6 + C_2C_4 - \tfrac{1}{6}C_2^3 \\ b_8(s) &= -C_8 + \tfrac{1}{2}C_4^2 + C_2C_6 - \tfrac{1}{2}C_2^2C_4 + \tfrac{1}{24}C_2^4. \end{aligned}$$

Our aim is to estimate the Hausdorff dimension of E_2. By part (b) of Theorem 1 we know the solutions s_N to $1 + \sum_{n=1}^{N} b_n(s_N) = 0$ converge to $\dim_H(E_2)$ at a super-exponential rate, as illustrated by Table 2.

In particular, if we solve the degree-16 truncation $1 + \sum_{n=1}^{16} b_n(s) = 0$, then we derive the approximation

$$\dim_H(E_2) = 0.531\,280\,506\,277\,205\,141\,624\,4686$$

which is correct to the 25 decimal places given.

As noted previously, our algorithm will efficiently compute the Hausdorff dimension of other Cantor sets consisting of numbers whose continued fraction expansions have a constrained alphabet. We list a few calculations in Table 3. This should be compared with Table 1 in [**Hen2**, p. 16].

All dimensions are correct to the given accuracy, and we do not round up the final decimal place. To obtain the dimension of $E_2 = E_{1,2}$ we used a degree-16 truncation.

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TABLE 3.

Alphabet	Dimension
{1, 2}	0.531 280 506 277 205 141 624 4686
{1, 3}	0.454 489 077 6618
{1, 4}	0.411 182 724 774 79
{2, 3}	0.337 436 780 806 063 636 304 494 910 386 8456
{2, 4}	0.306 312 768 052 784 030 277 9083
{3, 4}	0.263 737 482 897 426 558 759 863 384 275 45
{10, 11}	0.146 921 235 390 783 463 311 108 628 515 904 073 067 083 129 676 7553
{100, 10 000}	0.052 246 592 638 658 878 652 588 416 300 508 181 012 676 284 431 6811

To obtain the dimension† of $E_{2,3}$ we used a degree-14 truncation. All other values were obtained using a degree-12 truncation. In fact this allowed us to compute the dimension of $E_{10,11}$ and $E_{100,10\,000}$ to a precision of (at least) 61 and 122 decimal places respectively, though for ease of presentation we do not give these values in full.

The speed of convergence of our solutions is essentially controlled by the smallest derivative at a fixed point of T lying in $E_{\{n_1,\dots,n_r\}}$. For the set $E_{100,10\,000}$, for example, this smallest derivative is roughly $-10\,002$, so we have extremely fast convergence. The degree-2 truncation already gives the same estimate as in [**Hen2**], while the degree-12 truncation gives an accuracy of (at least) 122 decimal places.

5. *The Markoff and Lagrange spectra*

We now consider some number-theoretic applications of the improved estimates on Hausdorff dimension arising from our method. These applications are closely related to Bumby's original motivation for computing $\dim_H(E_2)$.

Given an irrational number α, it is well known (see Hardy and Wright [**HW**, Ch. 11]) that there exists $C > 0$ such that infinitely many rationals p/q satisfy $|\alpha - p/q| < C/q^2$. A natural question is to ask how small C can be chosen. We define

$$C(\alpha) = \limsup_{q\to\infty} \left[\min_{p\in\mathbb{N}} |q^2\alpha - pq| \right].$$

By convention (see [**CuF**]) we take reciprocals of these numbers, and define the *Lagrange spectrum* to be the set $\mathbb{L} = \{1/C(\alpha) : 0 < \alpha < 1\}$.

A closely related notion is that of the *Markoff spectrum* $\mathbb{M}$. Consider quadratic forms $f(x, y) = ax^2 + bxy + cy^2$ (with $a, b, c \in \mathbb{Z}$) for which $d(f) := b^2 - 4ac > 0$. If we denote $m(f) = \inf |f(x, y)|$, then $\mathbb{M}$ is defined to be the set of all possible values of $\sqrt{d(f)}/m(f)$.

In fact both spectra can also be defined in terms of doubly infinite sequences of positive integers. Given $a = (a_n)_{n\in\mathbb{Z}}$ we define

$$\lambda_i(a) = [a_i, a_{i+1}, \dots] + [0, a_{i-1}, a_{i-2}, \dots]$$

† Note our value for the set $E_{2,3}$ disagrees with Hensley's (which is presumably a misprint, the last three digits of his value for $E_{2,3}$ being the same as the (correct) value for $E_{1,4}$ which appears beneath it in Table 1 [**Hen2**]).

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where as usual $[c_0, c_1, \dots]$ denotes the continued fraction whose partial quotients are $c_0, c_1, \dots$. We then define

$$L(a) = \limsup_{|i|\to\infty} \lambda_i(a), \quad M(a) = \sup_{i\in\mathbb{Z}} \lambda_i(a). \tag{5.1}$$

Then the Lagrange and Markoff spectra are defined, respectively, as the set of all values of $L(a)$ or of $M(a)$.

It is clear from this definition that the Lagrange spectrum is a subset of the Markoff spectrum, and in fact it can be shown (see [**CuF**]) that it is a proper subset. Both spectra are closed subsets of $\mathbb{R}$. In fact both spectra contain the interval $[\mu, +\infty)$, where $\mu \approx 4.527\,829\,566$ (in fact this μ is optimal and is known exactly, a result due to Freiman [**Fre**], see [**CuF**, p. 55]).

The portions of both spectra below the value 3 are known exactly, and here too they coincide. Markoff [**Mar**] characterized this portion as the set of numbers $(1/z)\sqrt{9z^2-4}$, where $x^2 + y^2 + z^2 = 3xyz$, and $x, y \le z$ are positive integers. Such triples (x, y, z) are known as *Markoff triples*. It follows that the smallest value in both spectra is $\sqrt{5}$ (in fact this is Hurwitz's theorem). The next smallest values (in ascending order) are $\sqrt{8}$, $\sqrt{221}/5$, $\sqrt{1517}/13$ (see [**CuF**, p. 10] for a more complete list). Since this portion is countable, and its only accumulation point is the value 3 itself, we see that $\dim_H(M \cap [0, 3]) = \dim_H(L \cap [0, 3]) = 0$.

It is in the interval $(3, \mu)$ where the two spectra differ. The largest known number in $\mathbb{M}$ but not in $\mathbb{L}$ is $\beta \approx 3.293$ (the number is known exactly, see [**CuF**, p. 35]). Since the number μ above is optimal, there is a *gap* (μ_-, μ) in both spectra, for some number μ_-. That is, both spectra do not intersect this interval. Various other gaps have been identified, the first (historically) being the interval $(\sqrt{12}, \sqrt{13})$ (see Perron [**Per**]). Note that if $a = (a_n)_{n\in\mathbb{Z}}$ has at least one entry greater than 2 then $M(a) \ge \sqrt{13}$, and indeed $M(a) = \sqrt{13}$ if and only if $a = (3, 3, 3, \dots)$. However, if a has entries only 1's and 2's then $M(a) \le \sqrt{12}$, so we have Perron's result.

This is the first indication of why the Cantor set E_2 might be interesting in this context. Related Cantor sets are important in the problem of finding the Lebesgue measure and Hausdorff dimension of various portions of the spectra. Let us define $\mathbb{L}_t = \mathbb{L} \cap [0, t]$ and $\mathbb{M}_t = \mathbb{M} \cap [0, t]$.

If we consider $\Lambda_1 \subset E_2$ to be those numbers whose continued fraction expansions do *not* have consecutive triples $(i_k i_{k+1} i_{k+2}) = (121)$ then $\mathbb{M}_{\sqrt{10}} = \mathbb{M} \cap [0, \sqrt{10}] \subset \Lambda_1 + \Lambda_1$ [**CuF**, p. 65]. In particular, $\dim_H(\mathbb{M}_{\sqrt{10}}) \le 2\dim_H(\Lambda_1)$, and we can use Theorem 1 to estimate the value of $\dim_H(\Lambda_1)$.

Similarly, if we consider $\Lambda_2 \subset E_2$ to be those numbers whose continued fraction expansions do *not* have consecutive quadruples $(i_k i_{k+1} i_{k+2} i_{k+3}) = (1212)$ then $\mathbb{M}_{\sqrt{689}/8} = \mathbb{M} \cap [0, \sqrt{689}/8] \subset \Lambda_2 + \Lambda_2$ and $\dim_H(\mathbb{M}_{\sqrt{10}}) \le 2\dim_H(\Lambda_1)$ [**CuF**, p. 75].

Using degree-16 truncated equations we can estimate†

$$\dim_H(\Lambda_1) \approx 0.406\,075\,287\,8114, \quad \dim_H(\Lambda_2) \approx 0.485\,825\,9763.$$

† Our estimate for $\dim_H(\Lambda_2)$ disagrees with that of Bumby in [**Bu1**], who claims the value is between 0.469 and 0.470. The reader may convince himself of the validity of our result by solving a low-order truncated version of $1 + \sum b_n(s) = 0$. For example, the degree-4 truncation (which only contains terms corresponding to the two fixed points and the three period-4 orbits) gives the approximation 0.4843.

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Therefore, we have the following result.

THEOREM 2. *We can estimate*

$$\dim_H(\mathbb{L}_{\sqrt{10}}) \le \dim_H(\mathbb{M}_{\sqrt{10}}) \lesssim 0.812\,150\,575\,6228$$
$$\dim_H(\mathbb{L}_{\sqrt{689}/8}) \le \dim_H(\mathbb{M}_{\sqrt{689}/8}) \lesssim 0.971\,651\,9526.$$

Both estimates are correct to the number of decimal places given, and the final digit is not rounded up.

In particular, the above result implies that $\mathbb{M}_{\sqrt{689}/8}$ has zero Lebesgue measure. Defining

$$t_+ = \inf\{t : \mathbb{M}_t \text{ has positive Lebesgue measure}\},$$

we see that $t_+ > \sqrt{689}/8$. In fact this lower bound was improved by Bumby [**Bu1**] who showed $t_+ > 3.334\,37$, again by choosing a set $\Lambda \subset E_2$ corresponding to some subshift of finite type, and showing that $\dim_H(\Lambda) < 1/2$ (see [**Bu1**, p. 206] for full details). The increased efficiency of our method should make it possible to improve this lower bound on t_+ by judicious choice of a set Λ with Hausdorff dimension closer to $1/2$.

6. *Limit sets for Schottky groups*

We end by briefly outlining how our method of calculation can be applied to the more geometric setting of Schottky groups. Consider n pairs of closed disks $D_1, \ldots, D_n, D_{n+1}, \ldots, D_{2n}$ in the complex plane which are pairwise disjoint. For $i = 1, \ldots, n$ we can choose linear fractional transformations $T_i : \mathbb{C} \to \mathbb{C}$ which map the interior of D_i to the exterior of D_{n+i} (and consequently $T_i^{-1} : \mathbb{C} \to \mathbb{C}$ maps the interior of D_{n+i} to the exterior of D_i).

We call the group $\Gamma = \langle T_1, \ldots, T_n \rangle$ of linear fractional transformations a Schottky group. This is isomorphic to a free group on n generators. We define the limit set $\Lambda \subset \mathbb{C}$ associated to Γ to be the limit points of the set $\Gamma(0)$. The set Λ is a Cantor set in $\mathbb{C}$.

Let $\mathcal{A}$ denote the family of *real* analytic functions $f : \cup_{i=1}^{2n} \operatorname{int}(D_i) \to \mathbb{C}$. We define a transfer operator $L_s : \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{L}_s h(z) = \sum_{i \ne j+n \bmod 2n} \frac{1}{|T_i'(T_i^{-1}z)|^s} h(T_i^{-1}z), \quad \text{if } z \in D_j.$$

We need to work with real analytic functions to be assured that the terms $1/|T_i'(T_i^{-1}z)|^s$ are well defined.

To apply the theory we have described in the preceding sections, we may consider the complexification of real analytic functions $f : \cup_{i=1}^{2n} \operatorname{int}(D_i) \to \mathbb{C}$. This means that we consider those functions which extend as *complex analytic* functions $f : \tilde{D} \to \mathbb{C}$, where $\tilde{D} \supset \cup_{i=1}^{2n} \operatorname{int}(D_i)$ is an open set in $\mathbb{C}^2$. The transfer operators act in a natural way on the Banach space of analytic functions on $\tilde{D}$, which have a continuous extension to its closure. These are again nuclear operators.

We will pursue these and other related themes in a forthcoming article.

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