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PAPER

sL-approximation spaces capture sL-domains*

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Abstract
In this paper, by means of upper approximation operators in rough set theory, we study representations for sL-domains and its special subclasses. We introduce the concepts of sL-approximation spaces, L-approximation spaces, and bc-approximation spaces, which are special types of CF-approximation spaces. We prove that the collection of CF-closed sets in an sL-approximation space (resp., an L-approximation space, a bc-approximation space) ordered by set-theoretic inclusion is an sL-domain (resp., an L-domain, a bc-domain); conversely, every sL-domain (resp., L-domain, bc-domain) is order-isomorphic to the collection of CF-closed sets of an sL-approximation space (resp., an L-approximation space, a bc-approximation space). Consequently, we establish an equivalence between the category of sL-domains (resp., L-domains) with Scott continuous mappings and that of sL-approximation spaces (resp., L-approximation spaces) with CF-approximable relations.

Keywords: Upper approximation operator; sL-domain; sL-approximation space; L-domain; categorical equivalence

1. Introduction

Domain theory (Gierz et al. 2003) is a branch of mathematics that deals with a specific class of partially ordered sets. The domain structure serves as a denotational model for functional languages, playing a crucial role in theoretical computer science. The research of domain theory can be traced back to the pioneering work of Scott (1972) in the 1970s. The basic feature of domain theory is the interaction between ordered structures and topological structures (Goubault-Larrecq 2013). Recent academic focus has increasingly shifted toward exploring the nexus between domain theory and the mathematical foundations of artificial intelligence. This includes areas such as rough set theory (Lei and Luo 2009; Pawlak 1982; Zou et al. 2018), formal concept analysis (Ganter and Wille 1999; Wang and Li 2022; Zhou et al. 2024), nonclassical logics (Wang and Li 2020; Wang et al. 2021; Wang and Li 2024), and information systems (He and Xu 2019; Spreen et al. 2008; Wang et al. 2022; Wu et al. 2016). The connections between domain theory and various mathematical structures are mainly reflected in the representation of various domains (Wang et al. 2023; Xu and Mao 2008b). The representation of domain structure not only serves to characterize domain structure but also provides specific methods for reconstructions of domains. The simplest ordered structure, a family of sets ordered by set-theoretic inclusion, has inspired scholars to use appropriate families of sets induced by specific mathematical structures to represent various

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domains (Zou et al. 2024). Specifically, the idea that a certain domain structure can be represented by a specific mathematical structure means that a poset P is the aforementioned domain structure if and only if P is order-isomorphic to the family of sets induced by the specific structure described above.

Rough set theory (Pawlak 1982), introduced by Pawlak and further developed by numerous mathematicians, holds fundamental significance in artificial intelligence and in computer science. The key mathematical structures of rough set theory are approximation spaces. A classical approximation space is essentially a pair composed of a finite background set and an equivalence relation. Subsequently, by replacing equivalence relations with binary relations, the notion of approximation spaces was generalized to the so-called generalized approximation spaces, permitting infinite background sets (Khan and Patel 2022; Kondo 2006; Liu and Zhu 2008). The two approximation operators defined by the binary relation of a generalized approximation space are similar to the inner operator and closure operator, which are mutually dual (Yang and Xu 2011). Järvinen (2007) provided the lattice-theoretical background and ordered properties of generalized approximation space. Eklund et al. (2009) showed that partially ordered monads contain appropriate structures for modeling generalized approximation spaces. Lei and Luo (2009) explored representations of complete lattices and algebraic lattices based on approximation operators. Yang and Xu (2009) studied the ordered structures of various families of subsets induced by approximation operators. Zou et al. (2018) characterized continuous posets and Scott closed sets with the help of approximation operators on posets. These studies reveal close links between generalized approximation spaces in rough set theory and ordered structures in domain theory.

Abstract bases, known for their capacity to generate various domains using round ideals (Xu and Mao 2008a), possess a narrow research framework, potentially hindering connections with domain structures and other mathematical structures. To address this limitation, we wanted to extend the class of abstract bases to a broader class, ensuring that every object in the resulting larger class can still yield a continuous domain. This extension adheres to the principle that the resulting class should not be too broad, as it may lose connections with structures rooted in specific mathematical backgrounds, including topology, logic, rough set theory, etc. Notably, an abstract base $(B, \prec)$ is a type of generalized approximation space from the perspective of rough set theory. It is thus natural to wonder whether one can appropriately extend abstract bases to a larger class of approximation spaces. Recently, Wu and Xu (2023) extended an abstract basis to a CF-approximation space and generalized round ideals to CF-closed sets. This extension demonstrated that the class of abstract basis is a proper subclass of CF-approximation spaces. The collection of CF-closed sets of a CF-approximation space can generate a continuous domain and every continuous domain can be reconstructed from a CF-approximation space (may not be an abstract base). This work gave a positive answer to the above problem to some extent.

The concept of L-domains was introduced by Jung (1989,1990a,b). It was proved that the category of L-domains (resp., algebraic L-domains) is a maximal Cartesian closed full subcategory within the category of continuous domains (resp., algebraic domains), which is essential for modeling typed λ-calculus. Recent numerous studies have shown that L-domains are closely related to topology, logic, information system, formal concept analysis, and logic (Spreen 2021; Wang et al., 2021; Wang and Li, 2022, 2023; Wu et al., 2016; Xu and Mao, 2008a, b). Xu (2003) generalized L-domains to sL-domains and characterized sL-domains by function spaces. Lawson and Xu (2003) characterized sL-domains by function spaces in another interest way. This inspires us to seek a set-theoretic characterization of sL-domains without relying on function spaces. Xu and Mao (2008a) explored the representations of L-domains using abstract bases; however, they did not touch the representation of sL-domains. In their work, different from Jung's original concept (Jung 1989), L-domains do not necessarily have a least element. Following the work Xu and Mao (2008a) and Wu and Xu (2023), we will continue to study the connection between domain structures and generalized approximation spaces, aiming to extend Xu and Mao's representation approach from L-domains to sL-domains within the framework of CF-approximation spaces.

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The paper is organized as follows: In Section 2, we recall some basic notions and results in domain theory that will be used later. In Section 3, we introduce the notion of dense CF-approximation subspaces, followed by an investigation into their properties concerning ordered structures. In Section 4, we introduce the concepts of sL-approximation spaces and L-approximation spaces, which are special types of CF-approximation spaces. Subsequently, we prove that every sL-approximation space (resp., L-approximation space, bc-approximation space) can induce an sL-domain (resp., an L-domain, a bc-domain). In Section 5, we show that every sL-domain (resp., L-domain, bc-domain) is order-isomorphic to the collection of CF-closed sets of an sL-approximation space (resp., L-approximation space, bc-approximation space) and establish equivalences between relevant categories, such as the equivalence between category of sL-domains and that of sL-approximation spaces.

2. Preliminaries

In this section, we quickly recall some basic notions and results in domain theory. For notions not explicitly defined herein, the readers may refer to Davey and Priestley (2002),Gierz et al. (2003), and Goubault-Larrecq (2013).

For a set U and $X \subseteq U$, use $\mathcal{P}(U)$ to denote the power set of U, use $\mathcal{P}_{fin}(U)$ to denote the family of all finite subsets of U, and use X^c to denote the complement of X in U. The symbol $F \subseteq_{fin} X$ means that F is a finite subset of X.

Let $(L, \leq)$ be a poset. A *principal ideal* (resp., *principal filter*) of L is a set of the form $\downarrow x = \{y \in L \mid y \leq x\}$ (resp., $\uparrow x = \{y \in L \mid x \leq y\}$). For $A \subseteq L$, we write $\downarrow A = \{y \in L \mid \exists\, x \in A,\ y \leq x\}$ and $\uparrow A = \{y \in L \mid \exists\, x \in A,\ x \leq y\}$. A subset A is a *lower set* (resp., an *upper set*) if $A = \downarrow A$ (resp., $A = \uparrow A$). We say that z is a *lower bound* (resp., an *upper bound*) of A if $A \subseteq \uparrow z$ (resp., $A \subseteq \downarrow z$). A subset B of L is said to be *up-bounded* if B has an upper bound. The *supremum* of A is the least upper bound of A, denoted by $\bigvee A$ or $\sup A$. The *infimum* of A is its greatest lower bound (if it exists), denoted by $\bigwedge A$ or $\inf A$. A nonempty subset D of L is said to be *directed* if every nonempty finite subset of D has an upper bound in D. A poset L is a *directed complete partially ordered set* (dcpo, for short) if every directed subset of L has a supremum.

Recall that in a poset P, we say that x *way-below* y, written $x \ll y$, if for any directed set D having a supremum with $\sup D \geq y$, there exists some $d \in D$ such that $x \leq d$. If $x \ll x$, then x is called a *compact element* of P. The set $\{x \in P \mid x \ll x\}$ is denoted by $K(P)$. The set $\{y \in P \mid x \ll y\}$ will be denoted $\Uparrow x$ and $\{y \in P \mid y \ll x\}$ will be denoted by $\Downarrow x$. A poset P is said to be *continuous* (resp., *algebraic*) if for all $x \in P$, $\Downarrow x$ is directed (resp., $\downarrow x \cap K(P)$ is directed) and $x = \bigvee \Downarrow x$ (resp., $x = \bigvee (\downarrow x \cap K(P))$). If a dcpo P is continuous (resp., algebraic), then P is called a *continuous domain* (resp., *an algebraic domain*). A *semilattice* (resp., *sup-semilattice*) is a poset in which every pair of elements has an infimum (resp., a supremum). A join-semilattice is also called an *upper semilattice*. A *complete lattice* is a poset in which every subset has a supremum (equivalently, has an infimum). A subset B of a poset P is called a *basis* of P if for all $x \in P$, there exists $B_x \subseteq B \cap \Downarrow x$ such that B_x is directed and $\sup B_x = x$. It is well known that a poset P is continuous if and only if it has a basis, and that P is algebraic if and only if $K(P)$ is a basis.

Let L and P be dcpos and let $f : L \longrightarrow P$ be a mapping. Then f is called a *Scott continuous mapping* if for every directed subset $D \subseteq L$, $f(\bigvee D) = \bigvee f(D)$.

Lemma 2.1. (Gierz et al. 2003) *Let P be a poset. Then for all $x, y, u, z \in P$,*

(1) $x \ll y \Rightarrow x \leq y$;
(2) $u \leq x \ll y \leq z \Rightarrow u \ll z$;
(3) if $x \ll y$, $y \ll z$, and $x \vee y$ exists, then $x \vee y \ll z$; and
(4) if P has the least element $\bot$, then $\bot \ll x$.

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Lemma 2.2. (Gierz et al. 2003) *If P is a continuous poset, then the way-below relation $\ll$ has the interpolation property: $x \ll z \Rightarrow \exists y \in P$ such that $x \ll y \ll z$.*

Lemma 2.3. (Wu and Xu 2023, Lemma 4.1) *Let $(L, \leq)$ be a continuous domain and let $B \subseteq L$ be a basis. If $(B, \leq)$ is a sup-semilattice, then $(L, \leq)$ is a sup-semilattice.*

If $A \subseteq \downarrow x \subseteq L$, then we use $\bigvee_x A$ or $\sup_x A$ to denote the supremum of A in sub-poset $(\downarrow x, \leq)$, where the order inherits from L.

Lemma 2.4. (Mao and Xu 2005, Lemma 3.1) *Let P be a poset, $A \subseteq P$ and s, t be two upper bounds of A. If the supremums $\bigvee_s A$ and $\bigvee_t A$ exist and $\bigvee_s A \leq t$, then $\bigvee_s A = \bigvee_t A$.*

Definition 2.5. (Gierz et al. 2003; Xu 2003)

(1) A poset P is called a conditional upper semilattice *(cusl, for short), if every finite up-bounded subset A of P has a supremum.*

(2) A poset P is called an sL-cusl, if for every $p \in P$, $\downarrow p$ is a sup-semilattice.

(3) A poset P is called an L-cusl, if for every $p \in P$, $\downarrow p$ is a sup-semilattice with bottom element.

(4) A continuous domain L is called a continuous sL-domain *(sL-domain, for short), if for every $x \in L$, the set $\downarrow x$ is a sup-semilattice in the order inherited from L.*

(5) A continuous domain L is called a continuous L-domain *(L-domain, for short), if for every $x \in L$, the set $\downarrow x$ is a complete lattice in the order inherited from L.*

(6) A poset P is said to be pointed *if P has a least element.*

(7) A poset P is called a bc-poset, if every up-bounded subset B of P has a supremum. If a bc-poset P is also a continuous domain, then P is called a bc-domain.

It is well known that a bc-domain is a pointed L-domain and a pointed sL-domain is a pointed L-domain. However, a pointed L-domain need not be a bc-domain.

In what follows, we recall some concepts of binary relations, which are also the foundation of rough set theory (Pawlak 1982).

Definition 2.6. (Liu and Zhu 2008; Yang and Xu 2009) *A* generalized approximation space *(GA-space, for short) is a pair (U, R) consisting of a background set U and a binary relation R on U. For a GA-space (U, R), define $R_s, R_p : U \longrightarrow \mathcal{P}(U)$ by*

$$R_s(x) = \{y \in U \mid xRy\};\ R_p(x) = \{y \in U \mid yRx\}$$

for every $x \in U$.

Definition 2.7. (Liu and Zhu 2008; Yang and Xu 2009) *Let (U, R) be a GA-space. Define $\underline{R}, \overline{R} : \mathcal{P}(U) \longrightarrow \mathcal{P}(U)$ by*

$$\underline{R}(A) = \{x \in U \mid\ R_s(x) \subseteq A\};\ \overline{R}(A) = \{x \in U \mid\ R_s(x) \cap A \neq \emptyset\}$$

for every $A \subseteq U$. Operators $\underline{R}$ and $\overline{R}$ are called the lower approximation operator *and* the upper approximation operator *in (U, R), respectively.*

The notion of approximation operators is an important concept in rough set theory. Approximation operators $\underline{R}$ and $\overline{R}$ are dual to each other; that is, for very $A \subseteq U$, $\overline{R}(A^c) = (\underline{R}(A))^c$. Each conclusion about lower approximation operators can be transformed into the counterpart of upper approximation operators. Since upper approximation operators play a key role in this paper, we now list some useful lemmas of upper approximation operators as follows, although they are easy to obtain by definition.

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Lemma 2.8. (Liu and Zhu 2008; Yang and Xu 2009) *Let* (U, R) *be a GA-space.*
(1) For every $\{A_i \mid i \in I\} \subseteq \mathcal{P}(U)$, $\overline{R}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} \overline{R}(A_i)$.
(2) For all $x \in U$, $\overline{R}(\{x\}) = R_p(x)$.

A binary relation $R \subseteq U \times U$ on a set U is said to be *reflexive* if for all $x \in U$, xRx; and R is said to be *transitive* if xRy and yRz imply xRz for all $x, y, z \in U$. A binary relation R is called a *preorder* if it is reflexive and transitive. For a binary relation $R \subseteq U \times U$ and $V \subseteq U$, we use $R|_V$ to denote $R \cap (V \times V)$.

Lemma 2.9. (Wu and Xu 2023) *Let* (U, R) *be a GA-space.*
(1) R is reflexive iff for all $X \subseteq U$, $X \subseteq \overline{R}(X)$.
(2) R is transitive iff for all $X \subseteq U$, $\overline{R}(\overline{R}(X)) \subseteq \overline{R}(X)$.
(3) If R is transitive and $A, B \subseteq U$, *then* $\overline{R}(B) \subseteq \overline{R}(A)$ *whenever* $B \subseteq \overline{R}(A)$.
(4) If R is a preorder, then the operator $\overline{R}$ *is a topological closure operator on U.*

3. Dense CF-approximation subspaces

Wu and Xu (2023) introduced CF-approximation spaces and gave representations of continuous domains by them. In this section, we first recall some key notions and results in Wu and Xu (2023). Moreover, we will introduce the notion of dense CF-approximation subspaces and study the ordered structures of their CF-closed sets.

Definition 3.1. (Wu and Xu 2023) *Let* (U, R) *be a GA-space, let R be a transitive relation and let* $\emptyset \neq \mathcal{F} \subseteq \mathcal{P}_{fin}(U)$. *Then* $(U, R, \mathcal{F})$ *is called a* generalized approximation space with a consistent family of finite subsets *(CF-approximation space, for short) if for every* $F \in \mathcal{F}$, *whenever* $K \subseteq_{fin} \overline{R}(F)$, *there always exists* $G \in \mathcal{F}$ *such that* $K \subseteq \overline{R}(G)$ *and* $G \subseteq \overline{R}(F)$.

Definition 3.2. (Wu and Xu 2023) *Let* $(U, R, \mathcal{F})$ *be a CF-approximation space and* $E \subseteq U$. *Then E is called* a CF-closed set *of* $(U, R, \mathcal{F})$ *if for all* $K \subseteq_{fin} E$, *there always exists* $F \in \mathcal{F}$ *such that* $K \subseteq \overline{R}(F) \subseteq E$ *and* $F \subseteq E$. *The collection of all CF-closed sets of* $(U, R, \mathcal{F})$ *is denoted by* $\mathfrak{C}(U, R, \mathcal{F})$.

Lemma 3.3. (Wu and Xu 2023, Proposition 3.7) *For a CF-approximation space* $(U, R, \mathcal{F})$, *the following statements hold:*
(1) *For every* $F \in \mathcal{F}$, $\overline{R}(F) \in \mathfrak{C}(U, R, \mathcal{F})$.
(2) *If* $E \in \mathfrak{C}(U, R, \mathcal{F})$, $A \subseteq E$, *then* $\overline{R}(A) \subseteq E$.
(3) *If* $\{E_i\}_{i\in I} \subseteq \mathfrak{C}(U, R, \mathcal{F})$ *is a directed family, then* $\bigcup_{i\in I} E_i \in \mathfrak{C}(U, R, \mathcal{F})$.
(4) $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ *is a dcpo.*

Lemma 3.4. (Wu and Xu 2023, Proposition 3.8) *For a CF-approximation space* $(U, R, \mathcal{F})$ *and* $E \subseteq U$, *the following statements are equivalent:*
(1) $E \in \mathfrak{C}(U, R, \mathcal{F})$;
(2) *The family* $\mathcal{A} = \{\overline{R}(F) \mid F \in \mathcal{F}, F \subseteq E\}$ *is directed and* $E = \bigcup \mathcal{A}$;
(3) *There exists a family* $\{F_i\}_{i\in I} \subseteq \mathcal{F}$ *such that* $\{\overline{R}(F_i)\}_{i\in I}$ *is directed, and* $E = \bigcup_{i\in I} \overline{R}(F_i)$;
(4) *There always exists* $F \in \mathcal{F}$ *such that* $K \subseteq \overline{R}(F) \subseteq E$ *whenever* $K \subseteq_{fin} E$.

Example 3.5. (Wu and Xu 2023, Theorem 3.13) *Let* $(L, \leq)$ *be a continuous domain. Write* R_L *for the way-below relation "*$\ll$*" of* $(L, \leq)$ *and write* $\mathcal{F}_L = \{F \subseteq_{fin} L \mid F$ has a top element $c_F\}$. *Then*
(1) $(L, R_L, \mathcal{F}_L)$ *is a CF-approximation space;*
(2) $\mathfrak{C}(L, R_L, \mathcal{F}_L) = \{\downarrow x \mid x \in L\}$.

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We call $(L, R_L, \mathcal{F}_L)$ the *induced CF-approximation space* of the continuous domain L.

Lemma 3.6. (Wu and Xu 2023, Theorem 3.9, Corollary 3.10) *Let $E_1, E_2 \in \mathfrak{C}(U, R, \mathcal{F})$. Then $E_1 \ll E_2$ if and only if there exists $F \in \mathcal{F}$ such that $E_1 \subseteq \overline{R}(F)$ and $F \subseteq E_2$.*

Corollary 3.7. (Wu and Xu 2023, Theorem 3.11) *Let $(U, R, \mathcal{F})$ be a CF-approximation space. Then $\{\overline{R}(F) \mid F \in \mathcal{F}\}$ is a basis of $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$; and thus $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is a continuous domain.*

By Lemma 2.9, if (U, R) is a GA-space with R being a preorder and $\emptyset \neq \mathcal{F} \subseteq \mathcal{P}_{fin}(U)$, then $\overline{R}$ is a topological closure operator and $(U, R, \mathcal{F})$ is a CF-approximation space. So we have the following definition.

Definition 3.8. (Wu and Xu 2023) *Let (U, R) be a GA-space and $\emptyset \neq \mathcal{F} \subseteq \mathcal{P}_{fin}(U)$. The CF-approximation space $(U, R, \mathcal{F})$ is said to be* topological *if R is a preorder.*

Lemma 3.9. (Wu and Xu 2023) *Let $(U, R, \mathcal{F})$ be a topological CF-approximation space. Then $\{\overline{R}(F) \mid F \in \mathcal{F}\} = K(\mathfrak{C}(U, R, \mathcal{F}))$; thus, $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is an algebraic domain.*

Example 3.10. (Wu and Xu 2023) *Let $(L, \leq)$ be an algebraic domain. Write $R_{K(L)} = \leq_{K(L)}$ and write $\mathcal{F}_{K(L)} = \{F \subseteq_{fin} K(L) \mid F$ has a top element $c_F\}$, where $\leq_{K(L)}$ is the restriction of $\leq$ to $K(L)$. Then*

(1) *$(K(L), R_{K(L)}, \mathcal{F}_{K(L)})$ is a topological CF-approximation space;*
(2) $\mathfrak{C}(K(L), R_{K(L)}, \mathcal{F}_{K(L)}) = \{\downarrow x \cap K(L) \mid x \in L\}$.

We call $(K(L), R_{K(L)}, \mathcal{F}_{K(L)})$ the *induced topological CF-approximation space* of the algebraic domain L.

In what follows, we introduce the notions of CF-approximation subspaces and dense CF-approximation subspaces. We will study the connection between CF-approximation spaces and their dense subspaces in terms of ordered structures.

Definition 3.11. *Let $(U, R, \mathcal{F})$ be a CF-approximation space, $V \subseteq U$ and $\mathcal{G} \subseteq \mathcal{F} \cap \mathcal{P}(V)$. Write $R|_V = R \cap (V \times V)$. The triple $(V, R|_V, \mathcal{G})$ is called a* CF-approximation subspace *of $(U, R, \mathcal{F})$ if $(V, R|_V, \mathcal{G})$ itself is a CF-approximation space.*

Lemma 3.12. *Let $(U, R, \mathcal{F})$ be a CF-approximation space, $V \subseteq U$ and $\mathcal{G} \subseteq \mathcal{F} \cap \mathcal{P}(V)$. If for every $K \subseteq_{fin} U$ and $F \in \mathcal{F}$ with $K \subseteq \overline{R}(F)$, there exists $G \in \mathcal{G}$ such that $K \subseteq \overline{R}(G)$ and $G \subseteq \overline{R}(F)$, then $(V, R|_V, \mathcal{G})$ is a CF-approximation subspace of $(U, R, \mathcal{F})$, where $R|_V = R \cap (V \times V)$.*

Proof. It suffices to prove that $(V, R|_V, \mathcal{G})$ is a CF-approximation space. Clearly, $R|_V = R \cap (V \times V)$ is a transitive relation on V. Let $K \subseteq_{fin} V$ and $G \in \mathcal{G}$. If $K \subseteq \overline{R|_V}(G) = \overline{R}(G) \cap V$, then $K \subseteq_{fin} \overline{R}(G)$. It follows from the assumption that there exists $H \in \mathcal{G}$ such that $K \subseteq \overline{R}(H)$ and $H \subseteq \overline{R}(G)$. Noticing that $K, H \in \mathcal{P}(V)$, we have

$K \subseteq \overline{R}(H) \cap V = \overline{R|_V}(H)$, $H \subseteq \overline{R}(G) \cap V = \overline{R|_V}(G)$. Thus, $(V, R|_V, \mathcal{G})$ is a CF-approximation space. □

In view of Lemma 3.12, we give the following definition.

Definition 3.13. *Let $(U, R, \mathcal{F})$ be a CF-approximation space, $V \subseteq U$ and $\mathcal{G} \subseteq \mathcal{F} \cap \mathcal{P}(V)$. If for every $K \subseteq_{fin} U$ and $F \in \mathcal{F}$ with $K \subseteq \overline{R}(F)$, there exists $G \in \mathcal{G}$ such that $K \subseteq \overline{R}(G)$ and $G \subseteq \overline{R}(F)$,*

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then we say the CF-approximation subspace $(V, R|_V, \mathcal{G})$ *is dense in* $(U, R, \mathcal{F})$, *or, a* dense subspace *of* $(U, R, \mathcal{F})$.

Proposition 3.14. *Let* $(V, R|_V, \mathcal{G})$ *be a dense CF-approximation subspace of* $(U, R, \mathcal{F})$. *Then*

$$(\mathfrak{C}(U, R, \mathcal{F}), \subseteq) \cong (\mathfrak{C}(V, R|_V, \mathcal{G}), \subseteq).$$

Proof. We divide the proof into four claims.

Claim 1. We claim that for all $E \in \mathfrak{C}(U, R, \mathcal{F})$, we have $E \cap V \in \mathfrak{C}(V, R|_V, \mathcal{G})$.

Let $E \in \mathfrak{C}(U, R, \mathcal{F})$. If $K \subseteq_{fin} E \cap V$, then $K \subseteq_{fin} E$. It follows from $E \in \mathfrak{C}(U, R, \mathcal{F})$ that there exists $F \in \mathcal{F}$ such that $K \subseteq \overline{R}(F) \subseteq E$ and $F \subseteq E$. By Definition 3.2, there exists $G \in \mathcal{G}$ such that $K \subseteq \overline{R}(G)$ and $G \subseteq \overline{R}(F) \subseteq E$. Noticing that $K, G \in \mathcal{P}(V)$, we have

$$K \subseteq \overline{R}(G) \cap V = \overline{R|_V}(G) \subseteq E \cap V, G \subseteq E \cap V.$$

Thus, $E \cap V \in \mathfrak{C}(V, R|_V, \mathcal{G})$.

Claim 2. We claim that for $E_1, E_2 \in \mathfrak{C}(U, R, \mathcal{F})$, $E_1 \cap V \subseteq E_2 \cap V$ implies $E_1 \subseteq E_2$.

Let $x \in E_1$. Then there exists $F \in \mathcal{F}$ such that $x \in \overline{R}(F) \subseteq E_1$ and $F \subseteq E_1$. Since $(V, R|_V, \mathcal{G})$ is a dense CF-approximation subspace, there exists $G \in \mathcal{G}$ such that $x \in \overline{R}(G)$ and $G \subseteq \overline{R}(F) \subseteq E_1$. Noticing that $G \subseteq V$, we have

$$G \subseteq E_1 \cap V \subseteq E_2 \cap V \subseteq E_2.$$

Thus $x \in \overline{R}(G) \subseteq E_2$. By arbitrariness of $x \in E_1$, we have $E_1 \subseteq E_2$.

Claim 3. We claim that $\mathfrak{C}(V, R|_V, \mathcal{G}) = \{E \cap V \mid E \in \mathfrak{C}(U, R, \mathcal{F})\}$.

Let $S \in \mathfrak{C}(V, R|_V, \mathcal{G})$. Then by Proposition 3.4(2), we conclude that

$$\begin{aligned} S &= \bigcup \{\overline{R|_V}(G) \mid G \in \mathcal{G}, G \subseteq S\} \\ &= \bigcup \{\overline{R}(G) \cap V \mid G \in \mathcal{G}, G \subseteq S\} \\ &= (\bigcup \{\overline{R}(G) \mid G \in \mathcal{G}, G \subseteq S\}) \cap V \end{aligned}$$

and that $(\{\overline{R}(G) \cap V \mid G \in \mathcal{G}, G \subseteq S\}, \subseteq)$ is directed. By Claim 2, we conclude that $(\{\overline{R}(G) \mid G \in \mathcal{G}, G \subseteq S\}, \subseteq)$ is directed. Thus, $\bigcup \{\overline{R}(G) \mid G \in \mathcal{G}, G \subseteq S\} \in \mathfrak{C}(U, R, \mathcal{F})$. By arbitrariness of $S \in \mathfrak{C}(V, R|_V, \mathcal{G})$, we have $\mathfrak{C}(V, R|_V, \mathcal{G}) \subseteq \{E \cap V \mid E \in \mathfrak{C}(U, R, \mathcal{F})\}$. It directly follows from Claim 1 that $\{E \cap V \mid E \in \mathfrak{C}(U, R, \mathcal{F})\} \subseteq \mathfrak{C}(V, R|_V, \mathcal{G})$. Thus,

$$\mathfrak{C}(V, R|_V, \mathcal{G}) = \{E \cap V \mid E \in \mathfrak{C}(U, R, \mathcal{F})\}.$$

Claim 4. We claim that $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq) \cong (\mathfrak{C}(V, R|_V, \mathcal{G}), \subseteq)$.

Define a mapping $f : (\mathfrak{C}(U, R, \mathcal{F}), \subseteq) \longrightarrow (\mathfrak{C}(V, R|_V, \mathcal{G}), \subseteq)$ by:

$$f(E) = E \cap V \in \mathfrak{C}(V, R|_V, \mathcal{G}) \ \ (\forall E \in \mathfrak{C}(U, R, \mathcal{F})).$$

By Claims 2 and 3, we conclude that f is a bijection, f and f^{-1} are order-preserving. So, $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq) \cong (\mathfrak{C}(V, R|_V, \mathcal{G}), \subseteq)$. □

It is easy to see that the topological CF-approximation space $(K(L), R_{K(L)}, \mathcal{F}_{K(L)})$ is a dense subspace of $(L, R_L, \mathcal{F}_L)$ for every algebraic domain $(L, \leq)$. By Lemma 3.9, Example 3.10, and Proposition 3.14, we get a new representation for algebraic domains as follows.

Theorem 3.15. (Representation Theorem for algebraic domains) *A poset* $(L, \leq)$ *is an algebraic domain if and only if there exists a CF-approximation* $(U, R, \mathcal{F})$ *with a topological CF-approximation space as a dense subspace such that* $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq) \cong (L, \leq)$.

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4. sL-approximation spaces

In this section, we will introduce two suitable types of CF-approximation spaces that can generate sL-domains and L-domains, respectively. It is worth noting that a continuous domain is an sL-domain if its principal ideals are sup-semilattices. Therefore, we first consider a CF-approximation subspace induced by a CF-closed set of the original CF-approximation space, where the family of CF-closed sets of the subspace is exactly a principal ideal of that of the original CF-approximation space.

Proposition 4.1. *Let $(U, R, \mathcal{F})$ be a CF-approximation space, $E \in \mathfrak{C}(U, R, \mathcal{F})$ and $\mathcal{F}_E = \{F \in \mathcal{F} \mid F \subseteq E\}$. Then*

(1) *$(E, R|_E, \mathcal{F}_E)$ is a CF-approximation space; and*

(2) *$\mathfrak{C}(E, R|_E, \mathcal{F}_E) = \downarrow E$, where $\downarrow E = \{H \in \mathfrak{C}(U, R, \mathcal{F}) \mid H \subseteq E\}$.*

Proof. (1) Let $F \in \mathcal{F}_E$ and $K \subseteq_{fin} E$. If $K \subseteq \overline{R|_E}(F) = \overline{R}(F) \cap E$, then $K \subseteq_{fin} \overline{R}(F)$. By Definition 3.2, there exists $G \in F$ such that $K \subseteq \overline{R}(G)$ and $G \subseteq \overline{R}(F)$. It follows from $F \in \mathcal{F}_E$ and Lemma 3.3(2) that $F \subseteq E$ and $G \subseteq \overline{R}(F) \subseteq E$. Thus, $G \in \mathcal{F}_E$. Noticing that $K \subseteq E$ and $G \subseteq E$, we have $K \subseteq \overline{R}(G) \cap E = \overline{R|_E}(G)$ and $G \subseteq \overline{R}(F) \cap E = \overline{R|_E}(F)$. Thus, $(E, R|_E, \mathcal{F}_E)$ is a CF-approximation space.

(2) For all $F \in \mathcal{F}_E$, it follows from Lemma 3.3(2) that $\overline{R}(F) \subseteq E$. Thus, $\overline{R|_E}(F) = \overline{R}(F) \cap E = \overline{R}(F)$. Let $H \in \mathfrak{C}(E, R|_E, \mathcal{F}_E)$. Clearly, $H \subseteq E$. If $K \subseteq_{fin} H$, then there exists $F \in \mathcal{F}_E$ such that $K \subseteq \overline{R|_E}(F) = \overline{R}(F) \subseteq H$ and $F \subseteq H$, showing that $H \in \mathfrak{C}(U, R, \mathcal{F})$. Thus, $\mathfrak{C}(E, R|_E, \mathcal{F}_E) \subseteq \downarrow E$. Conversely, let $H \in \mathfrak{C}(U, R, \mathcal{F})$ and $H \subseteq E$. If $K \subseteq_{fin} H$, then there exists $F \in \mathcal{F}$ such that $K \subseteq \overline{R}(F) \subseteq H$ and $F \subseteq H$. It follows from $H \subseteq E$ that $F \in \mathcal{F}_E$ and $\overline{R|_E}(F) = \overline{R}(F)$. Therefore, $K \subseteq \overline{R|_E}(F) \subseteq H$. Thus, $H \in \mathfrak{C}(E, R|_E, \mathcal{F}_E)$. By the arbitrariness of $H \in \downarrow E$, we have $\mathfrak{C}(E, R|_E, \mathcal{F}_E) \supseteq \downarrow E$. Thus, $\mathfrak{C}(E, R|_E, \mathcal{F}_E) = \downarrow E$. □

The CF-approximation space $(E, R|_E, \mathcal{F}_E)$ is called the *induced principal CF-approximation subspace* of the CF-closed set E. By Proposition 4.1, $\mathfrak{C}(E, R|_E, \mathcal{F}_E) = \downarrow E$, exactly a principal ideal of $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$.

Definition 4.2. (Rong and Xu 2017) *Let U be a nonempty set, and let $\mathcal{P} \neq \emptyset$ be a family of subsets of U. Let $\mathcal{P}^\sharp$ denote the collection of all unions of nonempty finite subsets of $\mathcal{P}$; that is,*

$$\mathcal{P}^\sharp = \{A \mid \exists\, \emptyset \neq \mathcal{A} \subseteq_{fin} \mathcal{P} \text{ s.t. } A = \cup \mathcal{A}\}.$$

Then, $\mathcal{P}^\sharp$ is called the join saturation *of $\mathcal{P}$.*

Definition 4.3. *Let $(U, R, \mathcal{F})$ be a CF-approximation space, $K \in \mathcal{F}^\sharp \cup \{\emptyset\}$ and $F \in \mathcal{F}$ with $\overline{R}(K) \subseteq \overline{R}(F)$. Define $\mathcal{S}(K, F)$ to be the collection of all $G \in \mathcal{F}$ satisfying the following conditions:*

(sL-1) $\overline{R}(K) \subseteq \overline{R}(G) \subseteq \overline{R}(F)$;

(sL-2) $\forall G' \in \mathcal{F}, \overline{R}(K) \subseteq \overline{R}(G') \subseteq \overline{R}(F) \Longrightarrow \overline{R}(G) \subseteq \overline{R}(G')$.

Write $\mathcal{S}^*(K, F) = \{G \in \mathcal{S}(K, F) \mid G \subseteq \overline{R}(F)\}$.

Remark 4.4. For a family $\mathcal{F}$, $\emptyset \in \mathcal{F} \Leftrightarrow \emptyset \in \mathcal{F}^\sharp \Leftrightarrow \mathcal{F}^\sharp \cup \{\emptyset\} = \mathcal{F}^\sharp$. For a CF-approximation space $(U, R, \mathcal{F})$, if $K \in \mathcal{F}^\sharp \cup \{\emptyset\}$ and $F \in \mathcal{F}$ with $\overline{R}(K) \subseteq \overline{R}(F)$, then $\mathcal{S}^*(K, F) \subseteq \mathcal{S}(K, F)$.

Proposition 4.5. *Let $(U, R, \mathcal{F})$ be a CF-approximation space, $F, F_1, F_2, F_3 \in \mathcal{F}$ and $M, M_1, M_2 \in \mathcal{F}^\sharp \cup \{\emptyset\}$. Then the following statements hold.*

(1) *If $\overline{R}(M) \subseteq \overline{R}(F)$, $G_1 \in \mathcal{S}(M, F)$, and $G_2 \in \mathcal{F}$, then $G_2 \in \mathcal{S}(M, F)$ iff $\overline{R}(G_1) = \overline{R}(G_2)$.*

(2) *If $\overline{R}(M) \subseteq \overline{R}(F_1) \subseteq \overline{R}(F_2)$ and $\mathcal{S}(M, F_1) \neq \emptyset \neq \mathcal{S}(M, F_2)$, then $\mathcal{S}(M, F_1) = \mathcal{S}(M, F_2)$.*

(3) *If $\overline{R}(M_1) \subseteq \overline{R}(M_2) \subseteq \overline{R}(F)$ and $G_i \in \mathcal{S}(M_i, F)$ $(i = 1, 2)$, then $\overline{R}(G_1) \subseteq \overline{R}(G_2)$.*

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(4) *If* $\overline{R}(M) \subseteq \overline{R}(F_1) \cap \overline{R}(F_2)$, $\overline{R}(F_1 \cup F_2) \subseteq \overline{R}(F_3)$, *and* $\mathcal{S}(M, F_i) \neq \emptyset$ $(i = 1, 2, 3)$, *then* $\mathcal{S}(M, F_1) = \mathcal{S}(M, F_2)$.

Proof. (1) If $G_2 \in \mathcal{S}(M, F)$, then by (sL-1), we have $\overline{R}(M) \subseteq \overline{R}(G_i) \subseteq \overline{R}(F)$ $(i = 1, 2)$. It follows from (sL-2) that $\overline{R}(G_1) \subseteq \overline{R}(G_2)$ and $\overline{R}(G_2) \subseteq \overline{R}(G_1)$. Thus, $\overline{R}(G_1) = \overline{R}(G_2)$. Conversely, if $\overline{R}(G_2) = \overline{R}(G_1)$, then it follows from $G_1 \in \mathcal{S}(M, F)$ that G_2 satisfies (sL-1) and (sL-2), showing that $G_2 \in \mathcal{S}(M, F)$.

(2) Let $G_1 \in \mathcal{S}(M, F_1)$ and $G_2 \in \mathcal{S}(M, F_2)$. Then, $\overline{R}(M) \subseteq \overline{R}(G_1) \subseteq \overline{R}(F_1) \subseteq \overline{R}(F_2)$ and $\overline{R}(M) \subseteq \overline{R}(G_2) \subseteq \overline{R}(F_2)$. By (sL-2) and $G_2 \in \mathcal{S}(M, F_2)$, we have $\overline{R}(G_2) \subseteq \overline{R}(G_1) \subseteq \overline{R}(F_1)$. It follows from $\overline{R}(M) \subseteq \overline{R}(G_2) \subseteq \overline{R}(F_1)$, $G_1 \in \mathcal{S}(M, F_1)$ and (sL-2) that $\overline{R}(G_1) \subseteq \overline{R}(G_2)$. Thus, $\overline{R}(G_1) = \overline{R}(G_2)$. By (1), we have $\mathcal{S}(M, F_1) = \mathcal{S}(M, F_2)$.

(3) It follows from $G_2 \in \mathcal{S}(M_2, F)$ that $\overline{R}(M_1) \subseteq \overline{R}(M_2) \subseteq \overline{R}(G_2) \subseteq \overline{R}(F)$. By $G_1 \in \mathcal{S}(M_1, F)$ and (sL-2), we have $\overline{R}(G_1) \subseteq \overline{R}(G_2)$.

(4) It follows from $\overline{R}(F_1 \cup F_2) \subseteq \overline{R}(F_3)$, $\mathcal{S}(M, F_i) \neq \emptyset$ $(i = 1, 2, 3)$ and (2) that $\mathcal{S}(M, F_1) = \mathcal{S}(M, F_3)$ and $\mathcal{S}(M, F_2) = \mathcal{S}(M, F_3)$. Thus, $\mathcal{S}(M, F_1) = \mathcal{S}(M, F_2)$. □

Proposition 4.6. *Let* $(U, R, \mathcal{F})$ *be a CF-approximation space and* $E \in \mathfrak{C}(U, R, \mathcal{F})$. *If* $\mathcal{S}^*(K, F) \neq \emptyset$ *for all* $K \in \mathcal{F}^\sharp$ *and* $F \in \mathcal{F}$ *with* $\overline{R}(K) \subseteq \overline{R}(F)$, *then* $(\mathfrak{C}(E, R|_E, \mathcal{F}_E), \subseteq)$ *is a sup-semilattice.*

Proof. It follows from Proposition 4.1 that $\mathfrak{C}(E, R|_E, \mathcal{F}_E) = {\downarrow}E$ and $\{\overline{R}(F) \mid F \in \mathcal{F}, F \subseteq E\}$ is the basis of $({\downarrow}E, \subseteq)$. To prove $({\downarrow}E, \subseteq)$ is a sup-semilattice, by Lemma 2.3, it suffices to prove $(\{\overline{R}(F) \mid F \in \mathcal{F}, F \subseteq E\}, \subseteq)$ is a sup-semilattice. Let $F_1, F_2 \in \mathcal{F}_E$. Then $F_1 \cup F_2 \subseteq_{fin} E$. By Definition 3.2, there exists $F_3 \in \mathcal{F}$ such that $F_1 \cup F_2 \subseteq \overline{R}(F_3) \subseteq E$ and $F_3 \subseteq E$. Let $G \in \mathcal{S}^*(F_1 \cup F_2, F_3)$. Then, $\overline{R}(F_1) \cup \overline{R}(F_2) = \overline{R}(F_1 \cup F_2) \subseteq \overline{R}(G) \subseteq \overline{R}(F_3)$ and $G \subseteq \overline{R}(F_3) \subseteq E$, showing that $\overline{R}(G)$ is a upper bound of $\overline{R}(F_1)$ and $\overline{R}(F_2)$ in $(\{\overline{R}(F) \mid F \in \mathcal{F}, F \subseteq E\}, \subseteq)$. Let $F_4 \in \mathcal{F}_E$ and $\overline{R}(F_1) \cup \overline{R}(F_2) \subseteq \overline{R}(F_4)$. It follows from $F_3 \cup F_4 \subseteq_{fin} E$ and Definition 3.2 that there exists $F' \in \mathcal{F}$ such that $F_3 \cup F_4 \subseteq \overline{R}(F')$ and $F' \subseteq E$. Thus, $\overline{R}(F_3 \cup F_4) \subseteq \overline{R}(F')$. By Remark 4.4 and Proposition 4.5(4), we have $\mathcal{S}(F_1 \cup F_2, F_4) = \mathcal{S}(F_1 \cup F_2, F_3)$. For $G' \in \mathcal{S}^*(F_1 \cup F_2, F_4)$, by Proposition 4.5(1), $\overline{R}(G) = \overline{R}(G') \subseteq \overline{R}(F_4)$, showing that $\overline{R}(G)$ is the least upper bound of $\overline{R}(F_1)$ and $\overline{R}(F_2)$ in $(\{\overline{R}(F) \mid F \in \mathcal{F}, F \subseteq E\}, \subseteq)$. This shows that $(\{\overline{R}(F) \mid F \in \mathcal{F}, F \subseteq E\}, \subseteq)$ is a sup-semilattice. By Lemma 2.3, ${\downarrow}E = \{H \in \mathfrak{C}(U, R, \mathcal{F}) \mid H \subseteq E\}$ is a sup-semilattice. □

Corollary 4.7. *Let* $(U, R, \mathcal{F})$ *be a CF-approximation space. If* $\mathcal{S}^*(K, F) \neq \emptyset$ *for all* $K \in \mathcal{F}^\sharp$ *and* $F \in \mathcal{F}$ *with* $\overline{R}(K) \subseteq \overline{R}(F)$, *then* $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ *is an sL-domain.*

Proof. By Corollary 3.7, we have that $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is a continuous domain. By Propositions 4.1 and 4.6, we have that $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is an sL-domain. □

Based on Corollary 4.7, we give the definition of ultra sL-approximation spaces.

Definition 4.8. *Let* $(U, R, \mathcal{F})$ *be a CF-approximation space and let* $\mathcal{F}^\sharp$ *be the join saturation of* $\mathcal{F}$. *If* $\mathcal{S}^*(M, F) \neq \emptyset$ *for all* $M \in \mathcal{F}^\sharp$ *and* $F \in \mathcal{F}$ *with* $\overline{R}(M) \subseteq \overline{R}(F)$; *that is to say, there exists* $G \in \mathcal{F}$ *that satisfies the following conditions for all* $M \in \mathcal{F}^\sharp$ *and* $F \in \mathcal{F}$ *with* $\overline{R}(M) \subseteq \overline{R}(F)$:

(sL-1′) $\overline{R}(M) \subseteq \overline{R}(G)$, $G \subseteq \overline{R}(F)$; *and*

(sL-2) $\forall G' \in \mathcal{F}, \overline{R}(M) \subseteq \overline{R}(G') \subseteq \overline{R}(F) \Longrightarrow \overline{R}(G) \subseteq \overline{R}(G')$,

then $(U, R, \mathcal{F})$ *is called an* ultra sL-approximation space.

By Corollary 4.7, for an ultra sL-approximation space $(U, R, \mathcal{F})$, one has that $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is an sL-domain.

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The conditions of ultra sL-approximation spaces are too strong to be satisfied by all induced CF-approximation spaces of sL-domains (see Example 5.2 in the next section). So, we now consider weaker conditions than those of ultra sL-approximation spaces which can also guarantee that the collection of CF-closed sets forms an sL-domain.

By shifting the precondition "$\overline{R}(M) \subseteq \overline{R}(F)$" in Definition 4.8 to "$M \subseteq \overline{R}(F)$" and shifting the condition (sL-1′) in Definition 4.8 to (sL-1), we give the following definition.

Definition 4.9. *Let $(U, R, \mathcal{F})$ be a CF-approximation space. If $\mathcal{S}(M, F) \neq \emptyset$ for all $M \in \mathcal{F}^\sharp$ and $F \in \mathcal{F}$ with $M \subseteq \overline{R}(F)$; that is to say, there exists $G \in \mathcal{F}$ that satisfies the following conditions for all $M \in \mathcal{F}^\sharp$ and $F \in \mathcal{F}$ with $M \subseteq \overline{R}(F)$:*

(sL-1) $\overline{R}(M) \subseteq \overline{R}(G) \subseteq \overline{R}(F)$; and

(sL-2) $\forall G' \in \mathcal{F}, \overline{R}(M) \subseteq \overline{R}(G') \subseteq \overline{R}(F) \Longrightarrow \overline{R}(G) \subseteq \overline{R}(G')$,

then $(U, R, \mathcal{F})$ is called an sL-approximation space.

It is clear that (sL-1′) implies (sL-1), that $M \subseteq \overline{R}(F)$ implies $\overline{R}(M) \subseteq \overline{R}(F)$ and that $\mathcal{S}^*(M, F) \neq \emptyset$ implies $\mathcal{S}(M, F) \neq \emptyset$. So, we immediately have the following proposition.

Proposition 4.10. *All ultra sL-approximation spaces are sL-approximation spaces.*

Next, we explore properties of sL-approximation spaces.

Proposition 4.11. *Let $(U, R, \mathcal{F})$ be an sL-approximation space, $F_1, F_2 \in \mathcal{F}$, $M, M_1, M_2 \in \mathcal{F}^\sharp$, and $E \in \mathfrak{C}(U, R, \mathcal{F})$. Then the following statements hold.*

(1) *If $F_1 \cup F_2 \subseteq E$* and *$M \subseteq \overline{R}(F_1) \cap \overline{R}(F_2)$, then $\mathcal{S}(M, F_1) = \mathcal{S}(M, F_2)$.*

(2) *If $F_1 \cup F_2 \subseteq E$, $M_1 \subseteq \overline{R}(F_1)$, $M_2 \subseteq \overline{R}(F_2)$, $\overline{R}(M_1) \subseteq \overline{R}(M_2)$, and $G_i \in \mathcal{S}(M_i, F_i)$ $(i = 1, 2)$, then $\overline{R}(G_1) \subseteq \overline{R}(G_2)$.*

Proof. (1) By $F_1 \cup F_2 \subseteq_{fin} E \in \mathfrak{C}(U, R, \mathcal{F})$ and Definition 3.2, there exists $F_3 \in \mathcal{F}$ such that $F_1 \cup F_2 \subseteq \overline{R}(F_3) \subseteq E$. It follows from Lemma 2.9(3) that $\overline{R}(F_1) \cup \overline{R}(F_2) \subseteq \overline{R}(F_3)$. Thus, $M \subseteq \overline{R}(F_1) \subseteq \overline{R}(F_3)$ and $M \subseteq \overline{R}(F_2) \subseteq \overline{R}(F_3)$. So, $\mathcal{S}(M, F_i) \neq \emptyset$ $(i = 1, 2, 3)$. By Proposition 4.5(2), we have $\mathcal{S}(M, F_1) = \mathcal{S}(M, F_3)$ and $\mathcal{S}(M, F_2) = \mathcal{S}(M, F_3)$. Thus, $\mathcal{S}(M, F_1) = \mathcal{S}(M, F_2)$.

(2) Clearly, there exists $F_3 \in \mathcal{F}$ such that $F_1 \cup F_2 \subseteq \overline{R}(F_3) \subseteq E$. So, we have $M_1 \subseteq \overline{R}(F_1) \subseteq \overline{R}(F_3)$ and $M_2 \subseteq \overline{R}(F_2) \subseteq \overline{R}(F_3)$ and $\mathcal{S}(M_i, F_i) \neq \emptyset \neq \mathcal{S}(M_i, F_3)$ $(i = 1, 2)$. By Proposition 4.5(2), we have $\mathcal{S}(M_1, F_1) = \mathcal{S}(M_1, F_3)$ and $\mathcal{S}(M_2, F_2) = \mathcal{S}(M_2, F_3)$. It follows from $G_i \in \mathcal{S}(M_i, F_i)$ $(i = 1, 2)$ that $G_1 \in \mathcal{S}(M_1, F_3)$ and $G_2 \in \mathcal{S}(M_2, F_3)$. By $\overline{R}(M_1) \subseteq \overline{R}(M_2)$ and Proposition 4.5(3), we have $\overline{R}(G_1) \subseteq \overline{R}(G_2)$. □

Theorem 4.12. *If $(U, R, \mathcal{F})$ is an sL-approximation space, then the dcpo $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is an sL-domain.*

Proof. Clearly, $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is a continuous domain. So it suffices to show that for every $E \in (\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$, the set $\downarrow E = \{H \mid H \in \mathfrak{C}(U, R, \mathcal{F}), H \subseteq E\}$ is a sup-semilattice ordered by set inclusion. For any $H_1, H_2 \in \downarrow E$, assume that $H_1 \cup H_2 \neq \emptyset$ without losing generality. Construct

$$\mathcal{H} = \bigcup \{\mathcal{S}(F_1 \cup F_2, F_3) \mid F_1, F_2, F_3 \in \mathcal{F}, F_i \subseteq H_i \ (i = 1, 2), F_3 \subseteq E, F_1 \cup F_2 \subseteq \overline{R}(F_3)\}$$

and

$$H = \bigcup \{\overline{R}(G) \mid G \in \mathcal{H}\}.$$

To show H is the least upper bound of H_1 and H_2 in $\downarrow E$, we divide the proof into several steps.

Step 1. It follows directly from Lemma 3.3(2) that $H \subseteq E$.

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Step 2. Assert that $H_1 \cup H_2 \subseteq H$.

Let $x \in H_1 \cup H_2$. Assume that $x \in H_1$ without losing generality. Then by Definition 3.2, there exists $F_1 \in \mathcal{F}$ such that $x \in \overline{R}(F_1)$ and $F_1 \subseteq H_1$. Select $F_2 \in \mathcal{F}$ with $F_2 \subseteq H_2$. Then $F_1 \cup F_2 \subseteq H_1 \cup H_2 \subseteq E$. It follows from $E \in \mathfrak{C}(U, R, \mathcal{F})$ that there exists $F_3 \in \mathcal{F}$ such that $F_1 \cup F_2 \subseteq \overline{R}(F_3)$ and $F_3 \subseteq E$. By the definition of sL-approximation spaces, we have $\mathcal{S}(F_1 \cup F_2, F_3) \neq \emptyset$. For $G \in \mathcal{S}(F_1 \cup F_2, F_3) \subseteq \mathcal{H}$, we have $x \in \overline{R}(F_1 \cup F_2) = \overline{R}(F_1) \cup \overline{R}(F_2) \subseteq \overline{R}(G) \subseteq H$. By arbitrariness of $x \in H_1 \cup H_2$, we have $H_1 \cup H_2 \subseteq H$.

Step 3. Assert that $H \in \mathfrak{C}(U, R, \mathcal{F})$.

It suffices to show the family $\{\overline{R}(G) \mid G \in \mathcal{H}\}$ is directed. Clearly, $\{\overline{R}(G) \mid G \in \mathcal{H}\} \neq \emptyset$. Let $G_1 \in \mathcal{H}$. Then there exist $F_{11}, F_{12}, F_{13} \in \mathcal{F}$ such that

$$F_{11} \subseteq H_1, F_{12} \subseteq H_2, F_{13} \subseteq E, F_{11} \cup F_{12} \subseteq \overline{R}(F_{13}) \text{ and } G_1 \in \mathcal{S}(F_{11} \cup F_{12}, F_{13}).$$

Let $G_2 \in \mathcal{H}$. Then there exist $F_{21}, F_{22}, F_{23} \in \mathcal{F}$ such that

$$F_{21} \subseteq H_1, F_{22} \subseteq H_2, F_{23} \subseteq E, F_{21} \cup F_{22} \subseteq \overline{R}(F_{23}) \text{ and } G_2 \in \mathcal{S}(F_{21} \cup F_{22}, F_{23}).$$

It follows from $F_{11} \cup F_{21} \subseteq_{fin} H_1 \in \mathfrak{C}(U, R, \mathcal{F})$ that there exists $F_1 \in \mathcal{F}$ such that $F_{11} \cup F_{21} \subseteq \overline{R}(F_1)$ and $F_1 \subseteq H_1$. Similarly, there exists $F_2 \in \mathcal{F}$ such that $F_{12} \cup F_{22} \subseteq \overline{R}(F_2)$ and $F_2 \subseteq H_2$. Thus, $F_{11} \cup F_{12} \subseteq \overline{R}(F_1 \cup F_2)$ and $F_{21} \cup F_{22} \subseteq \overline{R}(F_1 \cup F_2)$. Since $F_1 \cup F_2 \subseteq H_1 \cup H_2 \subseteq E$, there exists $F_3 \in \mathcal{F}$ such that $F_1 \cup F_2 \subseteq \overline{R}(F_3)$ and $F_3 \subseteq E$. Let $G_3 \in \mathcal{S}(F_1 \cup F_2, F_3) \subseteq \mathcal{H}$. Noticing that $F_3 \cup F_{13} \subseteq_{fin} E$ and $F_{11} \cup F_{12} \subseteq \overline{R}(F_1 \cup F_2)$, by Proposition 4.11(2), we have $\overline{R}(G_1) \subseteq \overline{R}(G_3)$. Similarly, $\overline{R}(G_2) \subseteq \overline{R}(G_3)$. Thus, $(\{\overline{R}(G) \mid G \in \mathcal{H}\}, \subseteq)$ is directed. It follows from Lemma 3.4(3) that $H = \bigcup\{\overline{R}(G) \mid G \in \mathcal{H}\} \in \mathfrak{C}(U, R, \mathcal{F})$.

Step 4. Assert that H is the least upper bound of H_1 and H_2 in $\downarrow E$.

Let $H' \in \downarrow E$ and $H_1 \cup H_2 \subseteq H'$. For all $x \in H$, there exist $F_i \in \mathcal{F}$ $(i = 1, 2, 3)$ such that

$$F_i \subseteq H_i \ (i = 1, 2), F_3 \subseteq E, F_1 \cup F_2 \subseteq \overline{R}(F_3), G \in \mathcal{S}(F_1 \cup F_2, F_3), x \in \overline{R}(G).$$

It follows from $F_1 \cup F_2 \subseteq_{fin} H_1 \cup H_2 \subseteq H'$ and $H' \in \mathfrak{C}(U, R, \mathcal{F})$ that there exists $F_4 \in \mathcal{F}$ such that $F_1 \cup F_2 \subseteq \overline{R}(F_4) \subseteq H'$ and $F_4 \subseteq H'$. Noticing that $F_3 \cup F_4 \subseteq E$, by Proposition 4.11(1), we have $G \in \mathcal{S}(F_1 \cup F_2, F_3) = \mathcal{S}(F_1 \cup F_2, F_4)$. Thus, $x \in \overline{R}(G) \subseteq \overline{R}(F_4) \subseteq H'$. By arbitrariness of $x \in H$, we have $H \subseteq H'$, showing that H is the least upper bound of H_1 and H_2 in $\downarrow E$.

To sum up, $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is an sL-domain. □

The reader may wonder whether dense subspaces of sL-approximation spaces are sL-approximation spaces. For topological CF-approximation spaces, we have the following affirmative answer.

Proposition 4.13. *Let $(U, R, \mathcal{F})$ be a topological sL-approximation space and let $(V, R|_V, \mathcal{G})$ be a dense CF-approximation subspace of $(U, R, \mathcal{F})$. Then $(V, R|_V, \mathcal{G})$ is an sL-approximation space.*

Proof. For all $M \in \mathcal{G}^\sharp$ and $F \in \mathcal{G}$ with $M \subseteq \overline{R|_V}(F)$, we conclude that $M \subseteq \overline{R}(F)$ and $\overline{R}(M) \subseteq \overline{R}(F)$ hold in $(U, R, \mathcal{F})$. Since $(U, R, \mathcal{F})$ is an sL-approximation space, we conclude that $\mathcal{S}(M, F) \neq \emptyset$. Let $G_1 \in \mathcal{S}(M, F)$. Then by (sL-1) and R being a preorder, we have $M \subseteq \overline{R}(M) \subseteq \overline{R}(G_1) \subseteq \overline{R}(F)$. By the density of $(V, R|_V, \mathcal{G})$, there exists a $G \in \mathcal{G}$ such that $M \subseteq \overline{R}(G) \subseteq \overline{R}(G_1) \subseteq \overline{R}(F)$. Since G_1 satisfies (sL-2), we conclude that $\overline{R}(G_1) \subseteq \overline{R}(G)$ and thus $\overline{R}(G_1) = \overline{R}(G)$. By Proposition 4.5(1) that $G \in \mathcal{S}(M, F)$.

Next, we show that G satisfies (sL-1) and (sL-2) in $(V, R|_V, \mathcal{G})$. Clearly, (sL-1) $\overline{R|_V}(M) \subseteq \overline{R|_V}(G) \subseteq \overline{R|_V}(F)$ holds in $(V, R|_V, \mathcal{G})$. To show (sL-2), we need show for all $G' \in \mathcal{G}$, $\overline{R|_V}(M) \subseteq \overline{R|_V}(G') \subseteq \overline{R|_V}(F)$ implies that $\overline{R|_V}(G) \subseteq \overline{R|_V}(G')$ also holds in $(V, R|_V, \mathcal{G})$. It follows from $\overline{R|_V}(M) \subseteq \overline{R|_V}(G') = \overline{R}(G') \cap V$ and R being a preorder that $M \subseteq \overline{R}(G')$ and $\overline{R}(M) \subseteq \overline{R}(G')$. It follows from $G' \subseteq \overline{R|_V}(G') \subseteq \overline{R|_V}(F) = \overline{R}(F) \cap V$ that $G' \subseteq \overline{R}(F)$. Thus, we conclude that $\overline{R}(M) \subseteq \overline{R}(G') \subseteq \overline{R}(F)$. By G satisfying (sL-2) in $(U, R, \mathcal{F})$, we have $\overline{R}(G) \subseteq \overline{R}(G')$ and $\overline{R|_V}(G) \subseteq \overline{R|_V}(G')$,

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showing that (sL-2) holds in $(V, R|_V, \mathcal{G})$. So, by Definition 4.9, $(V, R|_V, \mathcal{G})$ is an sL-approximation space. □

Next, we consider a special kind of sL-approximation spaces – L-approximation spaces.

Definition 4.14. *Let $(U, R, \mathcal{F})$ be a CF-approximation space. If $\mathcal{S}(M, F) \neq \emptyset$ for all $M \in \mathcal{F}^\sharp \cup \{\emptyset\}$ with $M \subseteq \overline{R}(F)$; that is to say, there exists $G \in \mathcal{F}$ that satisfies the following conditions for all $M \in \mathcal{F}^\sharp \cup \{\emptyset\}$ with $M \subseteq \overline{R}(F)$:*
(sL-1) $\overline{R}(M) \subseteq \overline{R}(G) \subseteq \overline{R}(F)$; and
(sL-2) $\forall G' \in \mathcal{F}, \overline{R}(M) \subseteq \overline{R}(G') \subseteq \overline{R}(F) \Longrightarrow \overline{R}(G) \subseteq \overline{R}(G')$,
then $(U, R, \mathcal{F})$ is called an L-approximation space.

Remark 4.15. Clearly, by shifting the precondition "$M \in \mathcal{F}^\sharp$" in Definition 4.9 of sL-approximation spaces to "$M \in \mathcal{F}^\sharp \cup \{\emptyset\}$," we get the concept of L-approximation spaces. Thus, all L-approximation spaces are sL-approximation spaces. For an sL-approximation space $(U, R, \mathcal{F})$, if $\emptyset \in \mathcal{F}$, then $(U, R, \mathcal{F})$ is an L-approximation space automatically.

Theorem 4.16. *For an L-approximation space $(U, R, \mathcal{F})$, the dcpo $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is an L-domain.*

Proof. It follows from Theorem 4.12 that $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is an sL-domain. To show $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is an L-domain, it suffices to prove for all $E \in (\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$, the principal ideal $\downarrow E$ has a least element. It follows from $E \in \mathfrak{C}(U, R, \mathcal{F})$ and $\emptyset \subseteq_{fin} E$ that there exists $F \in \mathcal{F}$ such that $F \subseteq E$ and $\emptyset \subseteq \overline{R}(F)$. Let $G \in \mathcal{S}(\emptyset, F)$, we next show $\overline{R}(G)$ is precisely the least element of $(\downarrow E, \subseteq)$. Let $H \in \downarrow E$. Then there exists $F_1 \in \mathcal{F}$ such that $F_1 \subseteq H \subseteq E$. It follows from $F \cup F_1 \subseteq_{fin} E$ and Proposition 4.11(1) that $\mathcal{S}(\emptyset, F) = \mathcal{S}(\emptyset, F_1)$. Thus, $G \in \mathcal{S}(\emptyset, F_1)$ and $\overline{R}(G) \subseteq \overline{R}(F_1) \subseteq H$, showing that $\overline{R}(G)$ is the least element of $\downarrow E$. By arbitrariness of $E \in \mathfrak{C}(U, R, \mathcal{F})$, we conclude that $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is an L-domain. □

Theorem 4.17. *For a topological sL-approximation space (resp., topological L-approximation space) $(U, R, \mathcal{F})$, the dcpo $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is an algebraic sL-domain (resp., algegbraic L-domain).*

Proof. It follows from Lemma 3.9 that $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is an algebraic domain. By Theorem 4.12 (resp., Theorem 4.16), we know that $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is an sL-domain (resp., L-domain), and hence an algebraic sL-domain (resp., L-domain). □

Remark 4.18. Let $(U, R, \mathcal{F})$ be a topological CF-approximation space. For all $M \in \mathcal{F}^\sharp, F \in \mathcal{F}$, by Lemma 2.9(4), we conclude that $M \subseteq \overline{R}(F) \Leftrightarrow \overline{R}(M) \subseteq \overline{R}(F)$, (sL-1)$\Leftrightarrow$ (sL-1′) and $\mathcal{S}(M, F) = \mathcal{S}^*(M, F)$. So, $(U, R, \mathcal{F})$ is a topological ultra sL-approximation space if and only if $(U, R, \mathcal{F})$ is a topological sL-approximation space.

5. Representations of sL-domains

In this section, by means with sL-approximation spaces (resp., L-approximation spaces, bc-approximation spaces), we shall discuss representations for sL-domains (resp., L-domains, bc-domains).

Theorem 5.1. *Let $(L, R_L, \mathcal{F}_L)$ be the CF-approximation space induced by an sL-domain $(L, \leq)$. Then $(L, R_L, \mathcal{F}_L)$ is an sL-approximation space.*

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Proof. To prove that $(L, R_L, \mathcal{F}_L)$ is an sL-approximation space, let $M \in (\mathcal{F}_L)^\sharp$, $F \in \mathcal{F}_L$, and $M \subseteq \overline{R_L}(F)$. We show that $\mathcal{S}(M, F) \neq \emptyset$. We use c_F to denote the greatest element in F. Thus $M \subseteq_{fin} \overline{R_L}(F) = \downarrow c_F$. Since L is an sL-domain, we conclude that $\bigvee_{c_F} M$ exists. Set $G = \{\bigvee_{c_F} M\} \in \mathcal{F}_L$. Clearly, $\overline{R_L}(G) = \Downarrow \bigvee_{c_F} M \subseteq \downarrow c_F = \overline{R_L}(F)$ and G satisfies (sL-1). To check that G satisfies (sL-2), let $H \in \mathcal{F}_L$ and $\overline{R_L}(M) \subseteq \overline{R_L}(H) \subseteq \overline{R_L}(F)$. It follows from Lemma 2.8 that for all $m \in M$, $\downarrow m \subseteq \downarrow c_H \subseteq \downarrow c_F$. Since L is a continuous domain, we have $M \subseteq \downarrow c_H \subseteq \downarrow c_F$. By Lemma 2.4, we have $\bigvee_{c_H} M = \bigvee_{c_F} M \leq c_H$. Thus, $\overline{R_L}(G) = \Downarrow \bigvee_{c_F} M \subseteq \downarrow c_H = \overline{R_L}(H)$, showing that G satisfies (sL-2). Thus, $G \in \mathcal{S}(M, F) \neq \emptyset$. This shows that $(L, R_L, \mathcal{F}_L)$ is an sL-approximation space. □

Example 5.2. *Let $L = [0, 10]$ be a closed interval with the ordinary order of number. Clearly, L is an sL-domain. By Theorem 5.1, the induced CF-approximation space $(L, R_L, \mathcal{F}_L)$ is an sL-approximation space. However, $(L, R_L, \mathcal{F}_L)$ is not an ultra sL-approximation space. To see this, assume $(L, R_L, \mathcal{F}_L)$ is an ultra sL-approximation space. Then for $M = \{2, 4\} \in \mathcal{F}_L^\sharp$ and $F = \{1, 3, 4\} \in \mathcal{F}_L$, we have $\overline{R_L}(M) = [0, 4) \subseteq \overline{R_L}(F) = [0, 4)$. By the assumption, there exists $G \in \mathcal{F}_L$ that satisfies (sL-1′), that is, $\overline{R_L}(M) = [0, 4) \subseteq \overline{R_L}(G) = [0, c_G)$ and $G \subseteq \overline{R_L}(F) = [0, 4)$. Therefore, $c_G \geq 4$ and $c_G < 4$, which leads to a contradiction. This shows that the sL-approximation space $(L, R_L, \mathcal{F}_L)$ is not an ultra sL-approximation space.*

Theorem 5.3. (Representation Theorem for sL-domains) *For a poset $(L, \leq)$, the following statements are equivalent.*

(1) *L is an sL-domain;*

(2) *There exists an sL-approximation space $(U, R, \mathcal{F})$ such that $(L, \leq) \cong (\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$.*

Proof. (1) ⇒ (2): If L is an sL-domain, then by Theorem 5.1, the induced CF-approximation space $(L, R_L, \mathcal{F}_L)$ is an sL-approximation space. Define $f : L \to \mathfrak{C}(L, R_L, \mathcal{F}_L)$ such that for all $x \in L$, $f(x) = \downarrow x$. Then it follows from Example 3.5(2) and the continuity of L that f is an order-isomorphism.

(2) ⇒ (1): It follows directly from Theorem 4.12. □

Theorem 5.4. (Representation Theorem for L-domains) *A poset $(L, \leq)$ is an L-domain iff there exists an L-approximation space $(U, R, \mathcal{F})$ such that $(L, \leq) \cong (\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$.*

Proof. ⇐: Follows directly from Theorem 4.16.

⇒: Let L be an L-domain. It suffices to prove that $(L, R_L, \mathcal{F}_L)$ is an L-approximation space. It follows from Theorem 5.1 that $(L, R_L, \mathcal{F}_L)$ is an sL-approximation space. To prove that $(L, R_L, \mathcal{F}_L)$ is an L-approximation space, let $M = \emptyset$ and $F \in \mathcal{F}_L^\sharp$. We show that $\mathcal{S}(\emptyset, F) \neq \emptyset$. Since L is an L-domain, we conclude that $\downarrow c_F$ has the least element denoted by $\bot_F$. Set $G = \{\bot_F\} \in \mathcal{F}_L$. Clearly, $\overline{R_L}(\emptyset) = \emptyset \subseteq \overline{R_L}(G) = \Downarrow \bot_F \subseteq \downarrow c_F = \overline{R_L}(F)$, showing that G satisfies (sL-1). If $G_1 \in \mathcal{F}$ and $\overline{R_L}(\emptyset) = \emptyset \subseteq \overline{R_L}(G_1) = \downarrow c_{G_1} \subseteq \downarrow c_F = \overline{R_L}(F)$, then by the continuity of L, we have $c_{G_1} \leq c_F$. Since $\bot_F$ is the least element in $\downarrow c_F$, we have $\bot_F \leq c_{G_1}$. Therefore $\overline{R_L}(G) = \Downarrow \bot_F \subseteq \downarrow c_{G_1} = \overline{R_L}(G_1)$, showing that G satisfies (sL-2). Thus, $(L, R_L, \mathcal{F}_L)$ is an L-approximation space. □

Definition 5.5. *Let $(U, R, \mathcal{F})$ be a CF-approximation space. If for all $M \in \mathcal{F}^\sharp \cup \{\emptyset\}$ and $F \in \mathcal{F}$ with $M \subseteq \overline{R}(F)$, there exists $G \in \mathcal{F}$ that satisfies the following conditions:*

(sL-1) $\overline{R}(M) \subseteq \overline{R}(G) \subseteq \overline{R}(F)$;

(sL-2′) $\forall G' \in \mathcal{F}, \overline{R}(M) \subseteq \overline{R}(G') \Longrightarrow \overline{R}(G) \subseteq \overline{R}(G')$,

then $(U, R, \mathcal{F})$ is called a bc-approximation space.

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Remark 5.6. Clearly, (sL-2′) implies (sL-2). So all bc-approximation spaces are L-approximation spaces. For all $M \in \mathcal{F}^\sharp \cup \{\emptyset\}$ and $F \in \mathcal{F}$ with $M \subseteq \overline{R}(F)$, we use $\Sigma(M, F)$ to denote all $G \in \mathcal{F}$ that satisfies (sL-1) and (sL-2′). Obviously, $\Sigma(M, F) \subseteq \mathcal{S}(M, F)$.

Proposition 5.7. *Let $(U, R, \mathcal{F})$ be a bc-approximation space, $F_1, F_2 \in \mathcal{F}$ and $M, M_1, M_2 \in \mathcal{F}^\sharp \cup \{\emptyset\}$. Then the following statements hold.*

(1) *If $M \subseteq \overline{R}(F)$, $G_1 \in \Sigma(M, F)$, and $G_2 \in \mathcal{F}$, then $G_2 \in \Sigma(M, F)$ iff $\overline{R}(G_1) = \overline{R}(G_2)$.*

(2) *If $M \subseteq \overline{R}(F_1) \cap \overline{R}(F_2)$, then $\Sigma(M, F_1) = \Sigma(M, F_2)$.*

(3) *If $M_1 \subseteq \overline{R}(F_1)$, $\overline{R}(M_1) \subseteq \overline{R}(M_2)$, $M_2 \subseteq \overline{R}(F_2)$, $G_1 \in \Sigma(M_1, F_1)$, and $G_2 \in \Sigma(M_2, F_2)$, then $\overline{R}(G_1) \subseteq \overline{R}(G_2)$.*

Proof. (1) Similar to the proof of Proposition 4.5(1).

(2) Let $G_1 \in \Sigma(M, F_1)$ and $G_2 \in \Sigma(M, F_2)$. By (sL-1), we have $\overline{R}(M) \subseteq \overline{R}(G_1) \subseteq \overline{R}(F_1)$ and $\overline{R}(M) \subseteq \overline{R}(G_2) \subseteq \overline{R}(F_2)$. By (sL-2′), we have $\overline{R}(G_1) \subseteq \overline{R}(G_2)$ and $\overline{R}(G_2) \subseteq \overline{R}(G_1)$. Thus, $\overline{R}(G_1) = \overline{R}(G_2)$. By (1), we have $\Sigma(M, F_1) = \Sigma(M, F_2)$.

(3) It follows from $G_2 \in \Sigma(M_2, F_2)$ and $\overline{R}(M_1) \subseteq \overline{R}(M_2)$ that $\overline{R}(M_1) \subseteq \overline{R}(M_2) \subseteq \overline{R}(G_2)$. By $G_1 \in \Sigma(M_1, F_1)$ and (sL-2′) that $\overline{R}(G_1) \subseteq \overline{R}(G_2)$. □

Theorem 5.8. *Let $(U, R, \mathcal{F})$ be a bc-approximation space. Then $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is a bc-domain.*

Proof. Clearly, $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is a continuous domain. So it suffices to show $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ has the least element and any up-bounded two-element family $\{H_1, H_2\} \subseteq (\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ has the supremum in $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$.

Let $F \in \mathcal{F}$ and $G \in \Sigma(\emptyset, F) \neq \emptyset$. It follows from Proposition 5.7(2) that $\overline{R}(G)$ is the least element in $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$.

Let E be an upper bound of $\{H_1, H_2\}$. Construct

$$\mathcal{H} = \bigcup \{\Sigma(F_1 \cup F_2, F_3) \in \mathcal{F} \mid F_i \in \mathcal{F}\ (i = 1, 2, 3), F_i \subseteq H_i\ (i = 1, 2), F_3 \subseteq E, F_1 \cup F_2 \subseteq \overline{R}(F_3)\}$$

and

$$H = \bigcup \{\overline{R}(G) \mid G \in \mathcal{H}\}.$$

We divide the proof of H being the supremum of $\{H_1, H_2\}$ in $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ into several steps.

Step 1. It follows directly from Lemma 3.3(2) that $H \subseteq E$.

Step 2. Similar to the proof of Step 2 in Theorem 4.12, we have $H_1 \cup H_2 \subseteq H$.

Step 3. Similar to the proof of Step 3 in Theorem 4.12, we have $H \in \mathfrak{C}(U, R, \mathcal{F})$.

Step 4. Assert that H is the least upper bound of H_1 and H_2 in $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$.

Let $H' \in \mathfrak{C}(U, R, \mathcal{F})$ and $H_1 \cup H_2 \subseteq H'$. For all $x \in H$, there exist $F_i \in \mathcal{F}$ $(i = 1, 2, 3)$ such that

$$F_i \subseteq H_i\ (i = 1, 2), F_3 \subseteq E, F_1 \cup F_2 \subseteq \overline{R}(F_3), G \in \Sigma(F_1 \cup F_2, F_3), \text{and} x \in \overline{R}(G).$$

It follows from $F_1 \cup F_2 \subseteq_{fin} H_1 \cup H_2 \subseteq H' \in \mathfrak{C}(U, R, \mathcal{F})$ that there exists $F_4 \in \mathcal{F}$ such that $F_1 \cup F_2 \subseteq \overline{R}(F_4) \subseteq H'$ and $F_4 \subseteq H'$. By Proposition 5.7(2), we have $G \in \Sigma(F_1 \cup F_2, F_3) = \Sigma(F_1 \cup F_2, F_4)$. Thus, $x \in \overline{R}(G) \subseteq \overline{R}(F_4) \subseteq H'$. By arbitrariness of $x \in H$, we have $H \subseteq H'$, showing that H is the least upper bound of H_1 and H_2 in $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$.

To sum up, we see that $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is a bc-domain. □

Theorem 5.9. (Representation Theorem for bc-domains) *A poset $(L, \leq)$ is a bc-domain if and only if there exists some bc-approximation space such that $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq) \cong (L, \leq)$.*

Proof. ⇐: Follows from Theorem 5.8.

⇒: It suffices to prove that for a bc-domain L, $(L, R_L, \mathcal{F}_L)$ is a bc-approximation space. Let $M \in (\mathcal{F}_L)^\sharp \cup \{\emptyset\}$, $F \in \mathcal{F}_L$ with $M \subseteq \overline{R_L}(F)$. We use c_F to denote the greatest element in F. If

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$M = \emptyset$, let $\bot$ be the least element of L and $G = \{\bot\} \in \mathcal{F}_L$, then $G \in \Sigma(\emptyset, F)$. If $M \neq \emptyset$, then $M \subseteq_{fin} \overline{R_L}(F) = \mathop{\downarrow} c_F$. Since L is a bc-domain, we conclude that $\bigvee M$ exists. Set $G = \{\bigvee M\} \in \mathcal{F}_L$. Clearly, $G \in \Sigma(M, F)$, that is G satisfies (sL-1) and (sL-2′), as desired. □

Proposition 5.10. *A topological CF-approximation space* $(U, R, \mathcal{F})$ *is an sL-approximation space if and only if* $(\{\overline{R}(F) \mid F \in \mathcal{F}\}, \subseteq)$ *is an sL-cusl.*

Proof. $\Longrightarrow$: Let $F, F_1, F_2 \in \mathcal{F}$ and $\overline{R}(F_1) \cup \overline{R}(F_2) \subseteq \overline{R}(F)$. By Lemmas 2.8 and 2.9(4), we have $F_1 \cup F_2 \subseteq \overline{R}(F_1 \cup F_2) \subseteq \overline{R}(F)$. It follows from $F_1 \cup F_2 \in \mathcal{F}^\sharp$ that there exists $G \in \mathcal{S}(F_1 \cup F_1, F)$ such that G satisfies (sL-1) and (sL-2). By (sL-1), we conclude that $\overline{R}(G)$ is an upper bound of $\overline{R}(F_1), \overline{R}(F_2)$ in

$$\downarrow\overline{R}(F) = \{\overline{R}(G) \mid G \in \mathcal{F},\ \overline{R}(G) \subseteq \overline{R}(F)\}.$$

By (sL-2), we conclude that $\overline{R}(G)$ is the least upper bound of $\overline{R}(F_1), \overline{R}(F_2)$ in $\downarrow\overline{R}(F)$, showing that $\downarrow\overline{R}(F)$ is a sup-semilattice. Thus, $(\{\overline{R}(F) \mid F \in \mathcal{F}\}, \subseteq)$ is an sL-cusl.

$\Longleftarrow$: Let $M = \bigcup_{i=1}^n F_i \in \mathcal{F}^\sharp$ and $F \in \mathcal{F}$ with $M \subseteq \overline{R}(F)$. By Lemma 2.8, we have $\overline{R}(M) = \bigcup_{i=1}^n \overline{R}(F_i)$. Thus, $\{\overline{R}(F_i) \mid i \leq n\} \subseteq \downarrow\overline{R}(F)$. It follows from $(\{\overline{R}(G) \mid G \in \mathcal{F}\}, \subseteq)$ is an sL-cusl that there exists $G \in \mathcal{F}$ such that $\overline{R}(G)$ is the least upper bound of $\{\overline{R}(F_i) \mid i \leq n\}$ in $\downarrow\overline{R}(F)$. It is easy to check that G satisfies (sL-1) and (sL-2) and $G \in \mathcal{S}(M, F) \neq \emptyset$. Thus $(U, R, \mathcal{F})$ is an sL-approximation space. □

Theorem 5.11. (Representation Theorem for algebraic (s)L-domains) *A poset* $(L, \leq)$ *is an algebraic sL-domain (resp., algebraic L-domain) if and only if there exists some topological sL-approximation space (resp., topological L-approximation space)* $(U, R, \mathcal{F})$ *such that* $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq) \cong (L, \leq)$.

Proof. Notice that the upper approximation operator in a topological GA-space is a closure operator. So the proof here is similar to that of Wu and Xu (2024) (Theorems 4.2, 4.3). □

The above representation theorems demand much in some sense. For example, to represent an sL-domain, by Theorem 5.3, one needs use a CF-approximation space which itself is an sL-approximation space. However, we will see that it would be sufficient if the CF-approximation space has a dense sL-approximation subspace in the sense of Definition 3.13.

Theorem 5.12. (Representation Theorem by using dense subspaces) *Let* $(L, \leq)$ *be a poset. Then the following statements are equivalent.*

(1) *L is an sL-domain (resp., L-domain, bc-domains).*

(2) *There exists a dense CF-approximation subspace* $(V, Q, \mathcal{G})$ *of an sL-approximation space* $(U, R, \mathcal{F})$ *(resp., L-approximation space, bc-approximation subspace) such that* $(\mathfrak{C}(V, Q, \mathcal{G}), \subseteq) \cong (L, \leq)$.

(3) *There exists a CF-approximation space* $(U, R, \mathcal{F})$ *which has a dense sL-approximation subspace* $(V, Q, \mathcal{G})$ *(resp., L-approximation subspace, bc-approximation subspace) such that* $(L, \leq) \cong (\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$.

Proof. It directly follows from Theorems 5.3, 5.4, 5.9, Proposition 3.14 and that every CF-approximation space is a dense CF-approximation subspace of itself. □

In what follows, we first recall the definition of CF-approximable relations in Wu and Xu (2023) and then establish categorical equivalences of related categories. For some basic notions and results in category theory, please refer to Barr and Wells (1990).

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Definition 5.13. (Wu and Xu 2023) *Let* $(U_1, R_1, \mathcal{F}_1)$ *and* $(U_2, R_2, \mathcal{F}_2)$ *be two CF-approximation spaces and let* $\Theta \subseteq \mathcal{F}_1 \times \mathcal{F}_2$ *be a binary relation. If*

(1) *for all* $F \in \mathcal{F}_1$, *there exists* $G \in \mathcal{F}_2$ *such that* $F \,\Theta\, G$;

(2) *for all* $F, F' \in \mathcal{F}_1$, $G \in \mathcal{F}_2$, *if* $F \subseteq \overline{R_1}(F')$, $F \,\Theta\, G$, *then* $F' \,\Theta\, G$;

(3) *for all* $F \in \mathcal{F}_1$, $G, G' \in \mathcal{F}_2$, *if* $F \,\Theta\, G$, $G' \subseteq \overline{R_2}(G)$, *then* $F \,\Theta\, G'$;

(4) *for all* $F \in \mathcal{F}_1$, $G \in \mathcal{F}_2$, *if* $F \,\Theta\, G$, *then there are* $F' \in \mathcal{F}_1$, $G' \in \mathcal{F}_2$ *such that* $F' \subseteq \overline{R_1}(F)$, $G \subseteq \overline{R_2}(G')$, *and* $F' \,\Theta\, G'$; *and*

(5) *for all* $F \in \mathcal{F}_1$, $G_1, G_2 \in \mathcal{F}_2$, *if* $F \,\Theta\, G_1$ *and* $F \,\Theta\, G_2$, *then there exists* $G_3 \in \mathcal{F}_2$ *such that* $G_1 \cup G_2 \subseteq \overline{R_2}(G_3)$ *and* $F \,\Theta\, G_3$,

then Θ *is called a* CF-approximable relation *from* $(U_1, R_1, \mathcal{F}_1)$ *to* $(U_2, R_2, \mathcal{F}_2)$.

We denote the category of sL-approximation spaces (resp., L-approximation spaces, bc-approximation spaces, topological sL-approximation spaces, and topological L-approximation spaces) by **sL-APPS** (resp., **L-APPS**, **bc-APPS**, **TopsL-APPS**, and **TopL-APPS**), which is a full subcategory of the category of CF-approximation spaces with CF-approximable relations as morphisms. We use **sL-DOM** (resp., **L-DOM**, **bc-DOM**, **AsL-DOM**, and **AL-DOM**) to denote the category of sL-domains (resp., L-domains, bc-domains algebraic sL-domains, and algebraic L-domains) with Scott continuous mappings. By Wu and Xu (2023, Theorem 5.9), we know that the category of CF-approximation spaces is equivalent to that of continuous domains. Thus, we have the following results.

Theorem 5.14. *The category* ***sL-APPS*** *(resp.,* ***L-APPS***, ***bc-APPS***, ***TopsL-APPS***, ***TopL-APPS****) is equivalent to* ***sL-DOM*** *(resp.,* ***L-DOM***, ***bc-DOM***, ***AsL-DOM***, ***AL-DOM****).*

6. Concluding remarks

It is well known that representing mathematical structures through various families of subsets of other structures has proven effective in both practical applications and theoretical studies. By utilizing upper approximation operators from rough set theory, we introduced sL-approximation spaces, L-approximation spaces, and bc-approximation spaces and provided several representation theorems using the families of CF-closed sets in these spaces for sL-domains, L-domains, and bc-domains. We have shown that the collection of CF-closed sets in an sL-approximation space (resp., an L-approximation space, a bc-approximation space) equipped with the set-theoretic inclusion order, forms an sL-domain (resp., an L-domain, a bc-domain). Furthermore, every sL-domain (resp., L-domain, bc-domain) is order-isomorphic to the collection of CF-closed sets of an sL-approximation space (resp., an L-approximation space, a bc-approximation space) with respect to the set-theoretic inclusion order. CF-approximation spaces and sL-domains are generalizations of abstract bases and L-domains, respectively. Therefore, our work can be seen as a extension of Xu and Mao (2008a). However, Xu and Mao did not explore the deeper relationship between the L-domains and abstract bases from the method of category theory. Hence, our work offers a broader perspective in terms of both the research objects and the methods employed. The work presented in this paper is grounded in both rough set theory and domain theory. The categorical equivalence established here may further synthesize these two independent branches of mathematics in theoretical computer science.

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References

Barr, M. and Wells, C. (1990). *Category Theory for Computing Science*, 3rd ed., Englewood Cliffs, Prentice Hall.

Davey, B. A. and Priestley, H. A. (2002). *Introduction to Lattices and Order*, Cambridge, Cambridge University Press.

Eklund, P., Galán, M. A. and Gähler, W. (2009). Partially ordered monads for monadic topologies, rough sets and Kleene algebras, *Electronic Notes in Theoretical Computer Science* **225** 67–81.

Ganter, B. and Wille, R. (1999). *Formal Concept Analysis*, Berlin, Springer-Verlag.

Gierz, G., Hofmann, K. H., Keimel, K., Lawson, J. D., Mislove, M. and Scott, D. S. (2003). *Continuous Lattices and Domains*, Cambridge, Cambridge University Press.

Goubault-Larrecq, J. (2013). *Non-Hausdorff Topology and Domain Theory*, Cambridge, Cambridge University Press.

He, Q. and Xu, L. (2019). Weak algebraic information systems and a new equivalent category of DOM of domains, *Theoretical Computer Science* **763** 1–19.

Järvinen, J. (2007). Lattice theory for rough sets. In: PetersJ. F. et al. (ed.) *Transactions on Rough Sets VI*, Lecture Notes in Computer Science, Berlin, Springer-Verlag, 400–498.

Jung, A. (1989). Cartesian closed categories of domains. In: *CWI Tracts*, vol. **66**, Amesterdam, Centrum voor Wiskunde en Informatica.

Jung, A. (1990a). The clssification of continuous domains. In: Proceedings, Fifth Annual IEEE Symposium on Logic in Computer Science, Philadelphia, IEEE Computer Society Press, 35–40.

Jung, A. (1990b). Cartesian closed categories of algebraic cpos. *Theoretical Computer Science* **70** (2) 233–250.

Khan, M. A. and Patel, V. S. (2022). A formal study of a generalized rough set model based on subset approximation structure. *International Journal of Approximate Reasoning* **140** 52–74.

Kondo, M. (2006). On the structure of generalized rough sets. *Information Sciences* **176** (5) 589–600.

Lawson, J. and Xu, L. (2003). When does the class $[\mathcal{A} \to \mathcal{B}]$ consist of continuous domains. *Topology and its Applications* **130** 91–97.

Lei, Y. and Luo, M. (2009). Rough concept lattices and domains. *Annals of Pure and Applied Logic* **159** (3) 333–340.

Liu, G. and Zhu, W. (2008). The algebraic structures of generalized rough set theory. *Information Sciences* **178** (21) 4105–4113.

Mao, X. and Xu, L. (2005). Representation theorems for directed completions of consistent algebraic L-domains. *Algebra Universalis* **54** (4) 435–447.

Pawlak, Z. (1982). Rough sets. *International Journal of Foundations of Computer Science* **11** (5) 341–356.

Rong, Y. and Xu, L. (2017). Join reductions and join saturation reductions of abstract knowledge bases. *International Journal of Contemporary Mathematical Sciences* **12** 109–115.

Scott, D. S. (1972). Continuous lattices. In: *Topos, Algebraic Geometry and Logic*, Lecture Notes in Mathematics, vol. **274**, 97–136.

Spreen, D. (2021). Generalised information systems capture L-domains. *Theoretical Computer Science* **869** 1–28.

Spreen, D., Xu, L. and Mao, X. (2008). Information systems revisited–the general continuous case, *Theoretical Computer Science* **405** 176–187.

Wang, L. and Li, Q. (2020). A representation of proper bc-domains based on conjunctive sequent calculi. *Mathematical Structures in Computer Science* **30** (1) 1–13.

Wang, L. and Li, Q. (2024). L-domains as locally continuous sequent calculi. *Archive For Mathematical Logic* **63** (3-4) 405–425.

Wang, L., Li, Q. and Zhou, X. (2021). Continuous L-domains in logical form. *Annals of Pure and Applied Logic* **172** (9) 1–24.

Wang, L., Zhou, X. and Li, Q. (2022). Information systems for continuous semi-lattices, *Theoretical Computer Science* **913** 138–150.

Wang, L., Zhou, X. and Li, Q. (2023). Topological representations of Lawson compact algebraic L-domains and Scott domains. *Algebra Universalis* **84** (3) 23.

Wang, S. and Li, Q. (2022). A representation of L-domain by formal concept analysis. *Soft Computing* **26** (19) 9751–9760.

Wu, G. and Xu, L. (2023). Representations of domains via CF-approximation spaces. In: *Electronic Notes in Theoretical Informatics and Computer Science, Vol 2-Proceedings of ISDT'9*, 13.

Wu, G. and Xu, L. (2024). Direct approaches for representations of various algebraic domains via closure spaces. *Fuzzy Information and Engineering* **16** (2) 121–143.

Wu, M., Guo, L. and Li, Q. (2016). A representation of L-domains by information systems. *Theoretical Computer Science* **612** 126–136.

Xu, L. (2003) External characterizations of continuous sL-domains. In: Zhang, G., Lawson, J., Liu, Y. and Luo, M. (eds.) *Domain Theory, Logic and Computation*, Semantic Structures in Computation, vol. **3**. Dordrecht, Springer, 137–149.

Xu, L. and Mao, X. (2008a). Formal topological characterizations of various continuous domains. *Computers & Mathematics with Applications* **56** (2) 444–452.

Xu, L. and Mao, X. (2008b). When do abstract bases generate continuous lattices and L-domains. *Algebra Universalis* **58** (1) 95–104.

Yang, L. and Xu, L. (2009). Algebraic aspects of generalized approximation spaces. *International Journal of Approximate Reasoning* **51** (1) 151–161.

Yang, L. and Xu, L. (2011). Topological properties of generalized approximation spaces. *Information Sciences* **181**, 3570–3580.

https://doi.org/10.1017/S0960129525000052 Published online by Cambridge University Press

Zhou, X., Wang, L. and Li, Q. (2024). A direct approach to representing algebraic domains by formal contexts. *International Journal of Approximate Reasoning* **164** 109085.
Zou, J., Zhao, Y., Miao, C. and Wang, L. (2024). A set-theoretic approach to algebraic L-domains. *Mathematical Structures in Computer Science* **34** (3) 244–257.
Zou, Z., Li, Q. and Ho, W. K. (2018). Domains via approximation operators. *Logical Methods In Computer Science* **14**, 1–17.

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