Since we are only concerned with expressions for elements as iterated Whitehead products it is legitimate to write $|\theta| = n$ if θ is an n-fold product and call $|\theta|$ the *weight* of θ. Then (5) implies

$$|\xi.\theta| = |\theta|, \quad \xi \in \pi_r(X). \tag{13}$$

We argue by induction on weight, the assertion being trivial if $|\theta| \leqslant 2$. We suppose the theorem true for Whitehead products of weight $< n$ and consider a product θ of weight n, where $n \geqslant 3$. Then $\theta = [\rho, \delta]$, say, with $|\rho| + |\delta| = n$. If $|\delta| = 1$ we may apply the inductive hypothesis to ρ and conclude that ρ is expressible as a linear combination of normed products of weight $(n-1)$. Then (11) implies the desired conclusion for θ.

We now assume the theorem true for all products θ' with $|\theta'| = n$ and $\theta' = [\rho', \delta']$ with $|\delta'| < m$, where $m > 1$, and suppose that $\theta = [\rho, \delta]$ with $|\delta| = m$. Then $\delta = [\sigma, \tau]$ with $|\sigma| < m$, $|\tau| < m$, and $\theta = [\rho, [\sigma, \tau]]$. From (12) we deduce that

$$\pm [[\sigma, \tau], \rho] = \pm \sigma.\tau.[[-\tau, \rho], -\sigma] \pm \sigma.\rho.[[-\rho, -\sigma], \tau];$$

since $\theta = \pm [[\sigma, \tau], \rho]$, we have

$$\theta = \pm (\pm \sigma.\tau.[[-\tau, \rho], -\sigma] \pm \sigma.\rho.[[-\rho, -\sigma], \tau]). \tag{14}$$

Now, by (11),

$$-[\alpha, \beta] = [-\alpha, \alpha.\beta], \tag{15}$$

whence (by an evident induction)

$$|-\alpha| = |\alpha|. \tag{16}$$

Then (5), (13), (14), (15) and (16) enable us to apply the inductive hypothesis to deduce that θ is itself expressible as a linear combination of normed products and the theorem is proved.

Since elements of dimension 1 are not excluded in the statement of Theorem 1 we are able to improve Corollary 4·7 of (2); namely, we have

COROLLARY 2. *If K is a connected c.s.s. complex and class $K < n$ then all n-fold Whitehead products in K vanish.*

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RATIONAL EULER CHARACTERISTICS

BY C. T. C. WALL

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Let $\mathscr{L}$ be the class of discrete groups G which admit a finite complex X as classifying space. On $\mathscr{L}$ we may define the Euler characteristic $\chi(G) = \chi(X)$; that this is independent of X follows from the homological interpretation of characteristics.

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For example, if G is the free cyclic group Z, we may take X as a circle, and find $\chi(Z) = 0$. $\mathscr{L}$ is closed under formation of finite direct products, finite free products, and subgroups of finite index; for we may take products, bouquets, and finite coverings for the corresponding classifying spaces. We have the formulae

$$\chi(G \times H) = \chi(G)\chi(H), \tag{1}$$

$$\chi(G * H) = \chi(H) + \chi(H) - 1, \tag{2}$$

$$\chi(K) = r\chi(G), \tag{3}$$

where G, H are $\mathscr{L}$-groups, and K is a subgroup of index r in G. The purpose of this note is to provide a slight generalization of these properties.

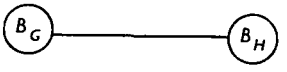
Let $\mathscr{M}$ be the class of groups G having a subgroup K of finite index r which belongs to $\mathscr{L}$. We define $\chi(G) = (1/r)\chi(K)$. It is at once seen that this is independent of the choice of K, for given two such subgroups K, K', their intersection is of the same form, and we apply formula (3) above.

It is clear that condition (3) is still satisfied for $\mathscr{M}$. So is condition (1), for if the $\mathscr{L}$-groups K, L have finite indices r, s in G, H, then $K \times L$ has index rs in $G \times H$.

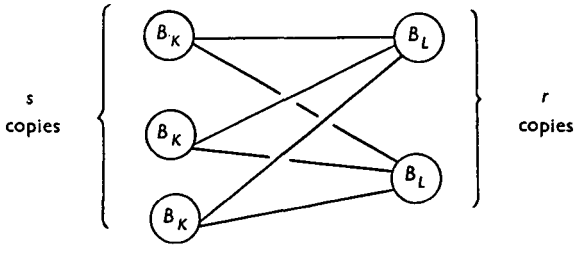
LEMMA. *Condition* (2) *is satisfied for* $\mathscr{M}$.

Proof. Let G, H have $\mathscr{L}$-subgroups K, L of indices r, s. We may suppose K (resp. L) normal, for the intersection of its finite set of conjugates in G has also finite index in G, and is an $\mathscr{L}$-group. $G * H$ has the factor group $(G/K) \times (H/L)$ of finite order, hence the kernel, M, has finite index. We shall show that M is an $\mathscr{L}$-group.

Denote the classifying space of G by B_G; similarly for the others. We may take B_{G*H} as the union of B_G and B_H, joined by an interval.



Now B_G admits an r-fold covering B_K, and B_H an s-fold covering B_L, and we seek the rs-fold covering corresponding to the direct product. Clearly this is given by the diagram



which is thus a B_M. Replacing B_K, B_L by the equivalent finite complexes X, Y we have a finite classifying space for M, and

$$\chi(M) = s\chi(K) + r\chi(L) - rs.$$

Hence

$$\chi(G * H) = \frac{1}{rs}\chi(M) = \frac{1}{r}\chi(K) + \frac{1}{s}\chi(L) - 1$$

$$= \chi(G) + \chi(H) - 1.$$

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Note. In general, given K, L of finite index in G, H, and choosing coset representatives $G = \bigcup_1^r Kg_i$, $H = \bigcup_1^s Lh_j$ with $g_1 = h_1 = 1$, we find that $G * H$ contains a subgroup M isomorphic to the free product of its subgroups $h_j^{-1}Kh_j$, $g_i^{-1}Lg_i$ and of the cyclic subgroups generated by $g_i^{-1}h_j^{-1}g_ih_j$ for $i, j \neq 1$. But the proof of this needs the Kuros subgroup theorem for free products.

The generalized Euler characteristic may be used to discuss the structure of subgroups of finite index, rather as the classical one is used to prove the Nielsen–Schreier theorem for free groups. For example, consider the modular group $Z_2 * Z_3$. We have

$$\chi(Z_2 * Z_3) = \chi(Z_2) + \chi(Z_3) - 1 = \tfrac{1}{2} + \tfrac{1}{3} - 1 = -\tfrac{1}{6}.$$

For a subgroup of finite index, which by Kuros's subgroup theorem is a free product of finite numbers p, q, r say of copies of Z_2, Z_3, Z, its characteristic is equal (by (2)) to

$$\tfrac{1}{2}p + \tfrac{1}{3}q - p - q - r + 1$$

but if it is a subgroup of index g, by (3) its characteristic is $-\frac{1}{6}g$. Hence we get a Diophantine equation

$$3p + 4q + 6r = g + 6. \tag{4}$$

For example, a subgroup of index 2 (if one exists) must be isomorphic to $Z_3 * Z_3$, and one of index 3 to $Z_2 * Z_2 * Z_2$ or to $Z_2 * Z$. (In fact, all these occur.)

Note. Professor Hall has kindly pointed out to me that (4) can be deduced from the Nielsen–Schreier theorem by essentially the same process as above, using free groups for $\mathscr{L}$-groups, and is a known result; nevertheless, it seems the most interesting example to give of the theory.

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A VARIANT ON THE PROOFS OF SOME INEQUALITIES*

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Theorem 1. *If* $a_\nu > 0$ *and* $\sum_{\nu=1}^{\infty} a_\nu$ *is a convergent series, then*

$$\lim_{n\to\infty} n(a_1 a_2 \dots a_n)^{1/n} = 0.$$

* Åkerberg, B. On some inequalities, *Arkiv för matematik, astronomi och fysik* 34B (1947), no. 13.

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