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Ergodic sums of non-integrable functions under one-dimensional dynamical systems with indifferent fixed points

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Abstract. We consider one-dimensional dynamical systems with indifferent fixed points (fixed points with derivative one). Many such maps have absolutely continuous ergodic infinite invariant measures. We study the limit of the ratio of the ergodic sum of f_A to that of f_B, where the integrals of f_A and f_B are infinite with respect to the absolutely continuous ergodic infinite invariant measure. If f_A and f_B are analytic functions on [0, 1], the result in this paper makes it clear whether the ratio of the ergodic sum of f_A to that of f_B converges in the Lebesgue measure or not.

1. *Introduction*

Let T be a map from [0, 1] onto itself. A fixed point p is called an indifferent fixed point of T, if

$$\lim_{x\to p} T(x) = p \quad \text{and} \quad \lim_{x\to p} |T'(x)| = 1.$$

The dynamical system we study in this paper is composed of the iterates $\{T^n(x)\}_{n\geq 0}$ of a map T with indifferent fixed points. In most cases T has an infinite invariant measure.

For simplicity, we mainly consider the following class of maps. (As described in the final section, we can obtain similar results for more general one-dimensional maps.)

We assume that the map $T : [0, 1] \to [0, 1]$ satisfies the following conditions.

(i) There exists a partition $0 = c_0 = p_1 < c_1 < p_2 < c_2 < \cdots < p_{r-1} < c_{r-1} < c_r = p_r = 1$ of [0, 1] such that the restrictions of T to the intervals (p_i, c_i) and (c_i, p_{i+1}) are C^2 functions which can be extended as C^2 functions to $(p_i, c_i]$ and $[c_i, p_{i+1})$, and $T|_{(p_i,c_i)}$ is convex in a right neighborhood of p_i and $T|_{(c_i,p_{i+1})}$ is concave in a left neighborhood of p_{i+1} for each $i = 1, \ldots, r-1$.

(ii) $T(p_i) = p_i$ and T is continuous at p_i for each $i = 1, \ldots, r$.

(iii) $T(c_{i-1}, c_i) = (0, 1)$ for each $i = 1, \ldots, r$.

(iv) $T'(x) > 1$ if x is not a fixed point of T.

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This kind of map is studied in [**I3, I4, T1, T2, T3**].

Remark 1. The maps satisfying our assumption have Lebesgue equivalent ergodic σ-finite invariant measures μ. We can choose a density function of μ which is bounded away from 0 and continuous except on all the fixed points of T (cf. [**T1**]).

For this map let p and q be fixed points (it is not necessary that $p \neq q$). Let A and B be a small one- or two-sided neighborhood of p and a small one- or two-sided neighborhood of q respectively. (We allow $A = [p, p+\varepsilon)$, $(p-\varepsilon, p]$, $(p-\varepsilon, p+\varepsilon)$ if p is not 0, 1. If $p = q$, we should consider $A = [p, p+\varepsilon)$ and $B = (p-\varepsilon, p]$.)

We assume that

$$|T(x) - x| = \theta_0 |x-p|^{d_0} + o(|x-p|^{d_0}) \quad \text{in } A \tag{1.1}$$

and

$$|T(x) - x| = \theta_1 |x-q|^{d_1} + o(|x-q|^{d_1}) \quad \text{in } B, \tag{1.2}$$

where $\theta_0 > 0$, $\theta_1 > 0$, $d_0 \geq 1$ and $d_1 \geq 1$ are constants.

If both d_0 and d_1 are greater than 1, p and q are indifferent fixed points of T and an intermittent chaos with the two laminar states A and B appears in the trajectory $\{T^k(x)\}_{k=0}^n$.

In previous papers [**I3, I4, I5**], we have studied the limit of the ratio of the sojourn time of the trajectory $\{T^k(x)\}_{k=0}^n$ in A to that in B, that is

$$\lim_{n\to\infty} \frac{\sum_{k=0}^n 1_A(T^k(x))}{\sum_{k=0}^n 1_B(T^k(x))}, \tag{1.3}$$

where 1_A is the indicator function on A.

Remark 2. Let μ be a Lebesgue equivalent T-ergodic σ-finite invariant measure. $\mu(A) = \infty$ if and only if $d_0 \geq 2$, and $\mu(B) = \infty$ if and only if $d_1 \geq 2$ (cf. [**I1, I2, T1**]).

By this remark and the Hopf Ratio Ergodic Theorem [**A, H, K**], if $d_0 < 2$ or $d_1 < 2$, the limit value is almost clear as we have explained in previous papers. In the case when both d_0 and d_1 are ≥ 2, we have already obtained the results related to the limit (1.3). Here we summarize the following.

For $2 \leq d_0 < d_1$ (or $2 \leq d_1 < d_0$) [**I3**], the limit (1.3) is 0 (or ∞) for Leb-a.e. x.

For $d_0 = d_1 > 2$ [**I4**],

$$\limsup_{n\to\infty} \frac{\sum_{k=0}^n 1_A(T^k(x))}{\sum_{k=0}^n 1_B(T^k(x))} = \infty \quad \text{for Leb-a.e. } x,$$

$$\liminf_{n\to\infty} \frac{\sum_{k=0}^n 1_A(T^k(x))}{\sum_{k=0}^n 1_B(T^k(x))} = 0 \quad \text{for Leb-a.e. } x.$$

For $d_0 = d_1 = 2$ [**I5**], the limit (1.3) diverges for Leb-a.e. x and this converges to a positive constant ρ in Lebesgue measure on $[0, 1]$, that is

$$\lim_{n\to\infty} \mathrm{Leb}\left(\left\{x; \left|\frac{\sum_{k=0}^n 1_A(T^k(x))}{\sum_{k=0}^n 1_B(T^k(x))} - \rho\right| > \varepsilon\right\}\right) = 0 \quad \text{for any } \varepsilon > 0.$$

Some related results are found in [**AN, ATZ, T3**].

In this paper we consider the problem as follows.

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PROBLEM. *If 1_A and 1_B are replaced by other functions f_A and f_B with $\int f_A\,d\mu = \infty$ and $\int f_B\,d\mu = \infty$, then does a limit like (1.3) exist in some sense? What is the limit value if it exists?*

If f_A and f_B are continuous on $[0, 1]$ and bounded away from 0 at the fixed points, we have no problem as, in this case, the answer of the problem is easily obtained from the previous summary for 1_A and 1_B.

However, even if f_A and f_B are analytic, the answer of the problem cannot be easily obtained from the previous summary when $f_A(p) = 0$ (and/or $f_B(q) = 0$).

So, we are essentially interested in the following f_A and f_B:

$$f_A(x) = |p - x|^{m_0} \cdot 1_A(x) \quad \text{and} \quad f_B(x) = |q - x|^{m_1} \cdot 1_B(x), \tag{1.4}$$

where m_0 and m_1 are real constants. We allow that m_0 and m_1 are negative.

We study the limit in the case when $d_0, d_1 > 1$ and when $\int f_A\,d\mu$ and $\int f_B\,d\mu$ are infinity. So, the results in this paper include the case when $1 < d_0, d_1 < 2$ ($\mu(A \cup B) < \infty$) and when both $\int f_A\,d\mu$ and $\int f_B\,d\mu$ are infinite.

In the proof of Theorem 2 in [**I3**], the author found an uncorrectable error when he came to write this paper. Here a proof of the corrected version of the theorem is given because Theorem 2 in this paper includes the fact announced in [**I5**].

2. *Results*

In this section, let $T : [0, 1] \to [0, 1]$ be a map satisfying conditions (i)–(iv) in §1. Define $d_0, d_1 > 1, \theta_0, \theta_1 > 0$ as in (1.1) and (1.2) and set f_A and f_B as in (1.4).

First we note the following theorem, which follows from Thaler's estimate (see Theorem 1 in [**T1**]).

THEOREM 1. *Let μ be an absolutely continuous T-invariant measure. $f_A \in L^1(\mu)$ if and only if $d_0 - m_0 < 2$.*

Of course a similar result holds for f_B. So, by the Hopf Ratio Ergodic Theorem, we have nothing to discuss if $d_0 - m_0 < 2$ or $d_1 - m_1 < 2$.

When both $d_0 - m_0$ and $d_1 - m_1$ are greater than or equal to two, we have the following theorems.

THEOREM 2. *If $2 \le d_0 - m_0 < d_1 - m_1$,*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^n f_A(T^k(x))}{\sum_{k=0}^n f_B(T^k(x))} = 0 \quad \textit{for Leb-a.e. } x.$$

THEOREM 3. *Assume $d_0 - m_0 = d_1 - m_1 = 2$. Then*

$$\limsup_{n\to\infty} \frac{\sum_{k=0}^n f_A(T^k(x))}{\sum_{k=0}^n f_B(T^k(x))} = \infty \quad \textit{for Leb-a.e. } x,$$

$$\liminf_{n\to\infty} \frac{\sum_{k=0}^n f_A(T^k(x))}{\sum_{k=0}^n f_B(T^k(x))} = 0 \quad \textit{for Leb-a.e. } x,$$

$$\lim_{n\to\infty} \mathrm{Leb}\left(\left\{x;\ \left|\frac{\sum_{k=0}^n f_A(T^k(x))}{\sum_{k=0}^n f_B(T^k(x))} - \rho\right| > \varepsilon\right\}\right) = 0 \quad \textit{for any } \varepsilon > 0,$$

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where ρ is a positive constant. In particular, let T_i be a continuous extension to $[c_{i-1}, c_i]$ of the restriction of T to (c_{i-1}, c_i) for each i and let g be a density function of invariant measure which is continuous except on all the fixed points of T. Then the following hold.

(1) *If A and B are small one-sided neighborhoods of $p = p_{i_0}$ and $q = p_{i_1}$ respectively, then*

$$\rho = \frac{\theta_1 \sum_{i \neq i_0} g(T_i^{-1}(p)) \cdot (T_i'(T_i^{-1}(p)))^{-1}}{\theta_0 \sum_{i \neq i_1} g(T_i^{-1}(q)) \cdot (T_i'(T_i^{-1}(q)))^{-1}},$$

where $T_i'(c_{i-1})$ and $T_i'(c_i)$ mean the right derivative of T at c_{i-1} and the left derivative of T at c_i respectively.

(2) *If A and B are a small right neighborhood of p and a small left neighborhood of p respectively, then $\rho = \theta_1/\theta_0$.*

(3) *If $r = 2$, $p = 0$ and $q = 1$, then $\rho = (\theta_1 M_1)/(\theta_0 M_2)$, where $M_1 = \lim_{x \uparrow c_1} T'(x)$ and $M_2 = \lim_{x \downarrow c_1} T'(x)$.*

(4) *If A and B are small two-sided neighborhoods of $p = p_{i_0}$ and $q = p_{i_1}$ respectively (if T is a C^2 mapping on S^1 with endpoints identified, we allow $p = 0 = 1$), then*

$$\rho = \frac{\theta_1 \sum_{i \neq i_0} g(T_i^{-1}(p)) \cdot (T_i'(T_i^{-1}(p)))^{-1}}{\theta_0 \sum_{i \neq i_1} g(T_i^{-1}(q)) \cdot (T_i'(T_i^{-1}(q)))^{-1}}.$$

In the case $d_0 - m_0 = d_1 - m_1 > 2$, the limit superior of the ratio is infinity and the limit inferior of the ratio is 0, which can be proved in essentially the same way as the main result of [**I4**].

Before closing this section, we give an example to show how our results work.

Example. Put

$$T(x) = \begin{cases} x + 4x^3, & [0, \frac{1}{2}) \\ x - 2(1-x)^2, & [\frac{1}{2}, 1] \end{cases}$$

and consider the following ratio:

$$\frac{\sum_{k=0}^{n} T^k(x)}{\sum_{k=0}^{n} (T^k(x))^2}. \tag{2.1}$$

Let μ be an absolutely continuous invariant measure. By Theorem 1, we know $\int_0^{1/2} x \, d\mu = \infty$, $\int_{1/2}^1 x \, d\mu = \infty$ and $\int_{1/2}^1 x^2 \, d\mu = \infty$. It follows from Theorem 3(3) that

$$\frac{\sum_{k=0}^{n} T^k(x) \, 1_{[0,1/2)}(T^k(x))}{\sum_{k=0}^{n} (T^k(x))^2 \, 1_{(1/2,1]}(T^k(x))}$$

converges to $\frac{2}{3}$ in Lebesgue measure as n goes to infinity. By this fact we can easily obtain that (2.1) converges to $\frac{5}{3}$ in the Lebesgue measure as n goes to infinity. Of course, the limit superior of the ratio is infinity and the limit inferior of the ratio is 0 for Leb-a.e. x.

3. *Basic definition and lemmas*

In this section, we first define the ergodicity and give a basic corollary of the Hopf Ratio Ergodic Theorem, and we state a basic property of the first return maps which are important

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tools in this paper. We also give two lemmas of probability theory. Furthermore, we prepare notation which we will often use later.

First, we define the ergodicity for a transformation T on a σ-finite measure space $(X, \mathcal{F}, \mu)$.

Definition. (T, μ) is called *ergodic* if $\mu(D) = 0$ or $\mu(X \setminus D) = 0$ for every $D \in \mathcal{F}$ with $T^{-1}D = D$ μ-a.e.

The following lemma is a basic corollary of the Hopf Ratio Ergodic Theorem.

LEMMA 3.1. *Let T be a measure-preserving ergodic transformation on a σ-finite measure space $(X, \mathcal{F}, \mu)$, let $f \in L^1(\mu)$ and let $\int g\, d\mu = \infty$, $g \geq 0$. Then*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} f(T^k(x))}{\sum_{k=0}^{n} g(T^k(x))} = 0 \quad \text{for } \mu\text{-a.e. } x.$$

The proof of this is easy (see Lemma 2.1 in [**I2**]).

The first return map of T on D is defined as $T^{n(x)}(x)$, where $n(x)$ is $\inf\{n \geq 1;\ T^n(x) \in D\}$. In the following lemma, let T be a transformation on a σ-finite measure space $(X, \mathcal{F}, \mu)$ and let $D \subset X$ be a measurable set with $D \subset \bigcup_{n=1}^{\infty} T^{-n}(D)$. Then the first return map is well defined.

LEMMA 3.2. *Assume that μ is a T-ergodic σ-finite invariant measure. Let $\mu|_D$ be the restriction of μ to D. Then the first return map R of T on D is $\mu|_D$-ergodic invariant. As a consequence, if $\tilde{\mu}$ is a μ-absolutely continuous σ-finite R-invariant measure, then $\tilde{\mu} = \text{constant} \cdot \mu|_D$.*

The proof of this is a minor modification of the proof of Lemma 1 in [**P**].

Next, we give two lemmas of probability theory. The following lemma is important in the next section.

LEMMA 3.3. *Let $v : N \to N$ be a function. Let $\zeta_{t,k}$ be a random variable on a probability space $(X, \mathcal{F}, \mu)$ for $k = 0, \ldots, t$. Assume that*

$$\mathrm{E}(\zeta_{t,0}) = \mathrm{E}(\zeta_{t,k}) \geq 0,$$
$$\mathrm{E}(\zeta_{t,k} \cdot \zeta_{t,i}) - \mathrm{E}(\zeta_{t,0})^2 \leq K\gamma^{-|k-i|} v(t) \mathrm{E}(\zeta_{t,0}),$$

where $\mathrm{E}(\zeta)$ is the expectation of ζ on the probability space $(X, \mathcal{F}, \mu)$, $K > 0$ and $\gamma > 1$ are constants. Then, for $\varepsilon > 0$,

$$\mu\left(\left\{\left|\frac{1}{(t+1)\mathrm{E}(\zeta_{t,0})}\sum_{k=0}^{t}\zeta_{t,k} - 1\right| > \varepsilon\right\}\right) \leq \frac{\text{constant} \cdot v(t)}{\varepsilon^2(t+1)\mathrm{E}(\zeta_{t,0})}.$$

The proof of this is similar to Lemma 4.2 of [**I3**].

The following lemma is a generalization of the well-known Borel–Cantelli lemma.

LEMMA 3.4. (Borel–Cantelli lemma for dependent events, see [**B**].) *Let $\{A_n\}$ be a sequence of events on a probability space $(X, \mathcal{F}, \mu)$. Suppose that there exists M such that $\mu(A_m \cap A_n) \leq M\mu(A_m)\mu(A_n)$ for all $m \neq n$. If $\sum_n \mu(A_n) = \infty$, $\mu(A_n \text{ i.o.}) > 0$.*

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The following notation will be often used in the rest of this paper:

$$v_n \sim w_n \text{ stands for } \lim_{n\to\infty}(v_n/w_n) = 1.$$

4. *Lemmas for measure-preserving transformations on a probability space*

In this section, let S be a measure-preserving transformation on a probability space $(X, \mathcal{F}, \mu)$ and let $\{E_l\}$ be a suitable sequence of measurable sets with $\mu(E_l) \sim \text{constant} \cdot l^{-\chi}$, where χ is a positive constant. Put

$$\hat{\zeta}_{u,k}(x) = \sum_{l=1}^{u} {a_l}^m 1_{E_l}(S^k(x)) \quad \text{for } k \geq 0,\ u \geq 1 \text{ and } x \in X,$$

where $a_l > 0$ and m are constants. And put $v(u) = \sum_{l=1}^{u} {a_l}^m$.

Assume that $(S, \{E_l\})$ satisfies the following mixing condition:

$$\mathrm{E}(\hat{\zeta}_{u,k} \cdot \hat{\zeta}_{u,i}) - \mathrm{E}(\hat{\zeta}_{u,0})^2 \leq C_1 \gamma^{-|k-i|} v(u) \mathrm{E}(\hat{\zeta}_{u,0}), \tag{M1}$$

where $C_1 > 0$ and $\gamma > 1$ are constants independent of i, k and u.

For each positive integer t, let t^* be the integer satisfying

$$(\log_2 t^*)^{1/2} = [(\log_2 t)^{1/2}],$$

where $[t]$ is the greatest integer not greater than t. Put

$$z_n = 2^{n^2} \quad \text{for } n = 1, 2, \ldots.$$

The following lemma is used to prove Theorem 2.

LEMMA 4.1. *Assume that*

$$\mu(E_l) \sim \text{constant} \cdot a_l \text{ and } {a_l}^m \mu(E_l) \sim \text{constant} \cdot l^{-s}\ (0 < s < 1).$$

For large t, let $u(t)$ be the maximal integer satisfying

$$\log_2 t^* \leq t^* \cdot \mu(E_{u(t)}). \tag{4.1}$$

Then

$$\lim_{t\to\infty} \frac{1}{(t+1)\sum_{l=1}^{u(t)} {a_l}^m \mu(E_l)} \sum_{k=0}^{t} \sum_{l=1}^{u(t)} {a_l}^m 1_{E_l}(S^k(x)) = 1 \quad \textit{for } \mu\text{-a.e. } x.$$

Proof. Under the condition of the lemma, we have

$$v(u(t)) = \sum_{l=1}^{u(t)} {a_l}^m \sim \text{constant} \cdot u(t)^{1-sm/(m+1)}.$$

Putting $\zeta_{t,k}(x) = \hat{\zeta}_{u(t),k}(x)$, we have

$$\mathrm{E}(\zeta_{t,0}) = \sum_{l=1}^{u(t)} {a_l}^m \mu(E_l) \sim \text{constant} \cdot u(t)^{1-s}.$$

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Thus, we have

$$\frac{v(u(t))}{t\mathrm{E}(\zeta_{t,0})} \sim \text{constant} \cdot \frac{u(t)^{s/(m+1)}}{t}.$$

Since it is easy to see that

$$u(t) \sim \text{constant} \cdot \left(\frac{t^*}{\log_2 t^*}\right)^{(m+1)/s},$$

we have

$$\frac{v(u(t))}{t\mathrm{E}(\zeta_{t,0})} \sim \text{constant} \cdot \frac{t^*}{t \log_2 t^*}.$$

Let $\varepsilon > 0$ be arbitrary. By Lemma 3.3 and condition (M1), this relation means that

$$\mu\left(\left\{\left|\frac{1}{(t+1)\sum_{l=1}^{u(t)} {a_l}^m \mu(E_l)} \sum_{k=0}^{t} \sum_{l=1}^{u(t)} {a_l}^m 1_{E_l}(S^k(x)) - 1\right| > \varepsilon\right\}\right) \le \frac{\text{constant} \cdot t^*}{\varepsilon^2 t \log_2 t^*}$$

for large t. Let n be a large positive integer and choose a constant $0 < \delta < 1/2$. For $\kappa = 0, 1, \ldots, 2n$, we similarly have

$$\mu\left(\left\{\left|\frac{1}{(2^\kappa z_n/n^\delta)\sum_{l=1}^{u(t)} {a_l}^m \mu(E_l)} \sum_{k=0}^{2^\kappa z_n/n^\delta} \sum_{l=1}^{u(t)} {a_l}^m 1_{E_l}(S^k(x)) - 1\right| > \varepsilon\right\}\right) \le \frac{\text{constant} \cdot n^\delta}{\varepsilon^2 2^\kappa \log_2 z_n}$$

for $2^\kappa z_n \le t < 2^{\kappa+1} z_n$. This yields

$$\mu\left(\bigcup_{t=2^\kappa z_n}^{2^{\kappa+1} z_n - 1} \left\{\left|\frac{1}{(t+1)\sum_{l=1}^{u(t)} {a_l}^m \mu(E_l)} \sum_{k=0}^{t} \sum_{l=1}^{u(t)} {a_l}^m 1_{E_l}(S^k(x)) - 1\right| > \varepsilon\right\}\right) \le \frac{\text{constant} \cdot n^{2\delta}}{\varepsilon^2 2^\kappa \log_2 z_n}.$$

Therefore, we easily obtain

$$\mu\left(\bigcup_{t=z_n}^{z_{n+1}-1} \left\{\left|\frac{1}{(t+1)\sum_{l=1}^{u(t)} {a_l}^m \mu(E_l)} \sum_{k=0}^{t} \sum_{l=1}^{u(t)} {a_l}^m 1_{E_l}(S^k(x)) - 1\right| > \varepsilon\right\}\right) \le \frac{\text{constant} \cdot n^{2\delta}}{\varepsilon^2 \log_2 z_n} = \frac{\text{constant} \cdot n^{2\delta}}{\varepsilon^2 n^2}.$$

Hence, the lemma follows from the Borel–Cantelli lemma. □

The rest of this section is only used to prove Theorem 3. It is not used to prove Theorem 2. Note that the definition of $u(t)$ in the rest of this section differs from that in (4.1).

LEMMA 4.2. *Assume that*

$$\mu(E_l) \sim \text{constant} \cdot a_l \quad \textit{and} \quad {a_l}^m \mu(E_l) \sim \text{constant} \cdot l^{-1}. \tag{4.2}$$

For large t, let $u(t)$ be the maximal integer satisfying

$$1 \le t^* (\log_2 t^*)^{1/4} \cdot \mu(E_{u(t)}). \tag{4.3}$$

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Then, for any $\varepsilon > 0$ and large integer n,

$$\mu\Bigg(\bigcup_{t=z_n}^{z_{n+1}-1}\Bigg\{\Bigg|\frac{1}{(t+1)\sum_{l=1}^{u(t)} {a_l}^m \mu(E_l)}\sum_{k=0}^{t}\sum_{l=1}^{u(t)} {a_l}^m 1_{E_l}(S^k(x)) - 1\Bigg| > \varepsilon\Bigg\}\Bigg) \le \frac{\text{constant}}{\varepsilon^2 n^{3/2}}.$$

Proof. Applying Lemma 3.3, we easily obtain this in essentially the same way as we proved Lemma 4.1. □

Let $\bar{u} > u$ be a positive integer. Put

$$\xi_k(x) := \sum_{l=u}^{\bar{u}} {a_l}^m 1_{E_l}(S^k(x)) \quad \text{for } k \ge 0 \text{ and } x \in X,$$

and put $\bar{v}(u) := \sum_{l=u}^{\bar{u}} {a_l}^m$. In the next lemma we assume the following mixing condition:

$$\mathrm{E}(\xi_i \cdot \xi_j \cdot \xi_k) - \mathrm{E}(\xi_0)^3 \le C_2 \gamma^{-|i-j|} \bar{v}(u) \mathrm{E}(\xi_0), \tag{M2}$$

where $C_2 > 0$ and $\gamma > 1$ are constants independent of i, j, k, u and $\bar{u}$.

LEMMA 4.3. *Assume (4.2). For large t, let $u(t)$ be the maximal integer satisfying (4.3) and let $\bar{u}(t)$ be the maximal integer satisfying*

$$1 \le t(\log_2 t)^{1/4} \cdot \mu(E_{\bar{u}(t)}). \tag{4.4}$$

Then, for any $\varepsilon > 0$ and large integer n,

$$\mu\Bigg(\bigcup_{t=z_n}^{z_{n+1}-1}\Bigg\{\frac{1}{(t+1)\sum_{l=1}^{u(t)} {a_l}^m \mu(E_l)}\sum_{k=0}^{t}\sum_{l=u(t)}^{\bar{u}(t)} {a_l}^m 1_{E_l}(S^k(x)) > \varepsilon\Bigg\}\Bigg) \le \frac{\text{constant}}{\varepsilon^3 n^2}.$$

Proof. For $\kappa = 0, 1, \ldots, 2n$, we have

$$\begin{aligned}
&\mu\Bigg(\bigcup_{t=2^\kappa z_n}^{2^{\kappa+1}z_n-1}\Bigg\{\frac{1}{(t+1)\sum_{l=1}^{u(t)} {a_l}^m \mu(E_l)}\sum_{k=0}^{t}\sum_{l=u(t)}^{\bar{u}(t)} {a_l}^m 1_{E_l}(S^k(x)) > \varepsilon\Bigg\}\Bigg) \\
&\quad\le \mu\Bigg(\Bigg\{\frac{1}{(2^\kappa z_n+1)\sum_{l=1}^{u(z_n)} {a_l}^m \mu(E_l)}\sum_{k=0}^{2^{\kappa+1}z_n-1}\sum_{l=u(2^\kappa z_n)}^{\bar{u}(2^{\kappa+1}z_n)} {a_l}^m 1_{E_l}(S^k(x)) > \varepsilon\Bigg\}\Bigg) \\
&\quad\le \frac{1}{\big(\varepsilon(2^\kappa z_n+1)\sum_{l=1}^{u(z_n)} {a_l}^m \mu(E_l)\big)^3}\int\Bigg(\sum_{k=0}^{2^{\kappa+1}z_n-1}\sum_{l=u(2^\kappa z_n)}^{\bar{u}(2^{\kappa+1}z_n)} {a_l}^m 1_{E_l}(S^k(x))\Bigg)^3 d\mu.
\end{aligned}$$

By (M2) and (4.2), we obtain

$$\int\Bigg(\sum_{k=0}^{2^{\kappa+1}z_n-1}\sum_{l=u(2^\kappa z_n)}^{\bar{u}(2^{\kappa+1}z_n)} {a_l}^m 1_{E_l}(S^k(x))\Bigg)^3 d\mu \le \text{constant} \cdot (2^{\kappa+1}z_n)^3 n^3.$$

Thus, we have

$$\begin{aligned}
&\mu\Bigg(\bigcup_{t=2^\kappa z_n}^{2^{\kappa+1}z_n-1}\Bigg\{\frac{1}{(t+1)\sum_{l=1}^{u(t)} {a_l}^m \mu(E_l)}\sum_{k=0}^{t}\sum_{l=u(t)}^{\bar{u}(t)} {a_l}^m 1_{E_l}(S^k(x)) > \varepsilon\Bigg\}\Bigg) \\
&\qquad\qquad \le \frac{\text{constant}}{\varepsilon^3 n^3} \quad \text{for } \kappa = 0, 1, \ldots, 2n.
\end{aligned}$$

Therefore, we obtain the lemma. □

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By Lemmas 4.2 and 4.3, we obtain the following proposition.

PROPOSITION 4.4. *Under the same assumption as in Lemma 4.3,*

$$\lim_{t\to\infty} \frac{1}{(t+1)\sum_{l=1}^{u(t)} a_l{}^m \mu(E_l)} \sum_{k=0}^{t} \sum_{l=1}^{\bar{u}(t)} a_l{}^m 1_{E_l}(S^k(x)) = 1 \quad \text{for } \mu\text{-a.e. } x.$$

Here we give some lemmas, which are valid without mixing assumptions (M1) and (M2).

LEMMA 4.5. *If $\bar{u}(t)$ is the maximal integer satisfying (4.4), then*

$$\mu\left(\left\{\sum_{k=0}^{t} 1_{E_{\bar{u}(t)}}(S^k(x)) \geq 1\right\}\right) \leq 2(\log_2 t)^{-1/4} \quad \text{for sufficiently large } t.$$

Proof. By the Chebychev inequality, we easily obtain this. □

LEMMA 4.6. *Let τ be a sufficiently large positive integer and let s be a positive number. Put $\hat{\tau} = [\tau^s \log_2 \tau]$ and $\tilde{\tau} = [\tau^s/\log_2\tau]$. If $\bar{u}(t)$ is the maximal integer satisfying (4.4), then*

$$\mu\left(\bigcup_{t=\tilde{\tau}}^{\hat{\tau}} \{S^t(x) \in E_{\bar{u}(t)}\}\right) \leq \frac{\text{constant} \cdot \log\log \tau}{\log \tau}.$$

Proof. Calculating $\sum_{t=\tilde{\tau}}^{\hat{\tau}} t^{-1}(\log_2 t)^{-1/4}$, we obtain the lemma. □

LEMMA 4.7. *Let τ be a sufficiently large positive integer and let s be a positive number. Put $\tilde{\tau} = [\tau^s/\log_2\tau]$. If $\mu(E_l) \sim \text{constant} \cdot l^{-s}$,*

$$\mu\left(\left\{\sum_{k=0}^{\tilde{\tau}} \sum_{l=1}^{\tau} 1_{E_l}(S^k(x)) \geq \tau\right\}\right) \leq \frac{\text{constant}}{\log \tau}.$$

Proof. By the Chebychev inequality, we easily obtain this. □

Let $\mathcal{A}$ be the smallest σ-algebra including the sets $S^{-n}E_\tau$, $n, \tau \geq 0$. In the next lemma we assume the following condition:

$$\mu(S^{-Mk}A \cap B) \geq C_0 \cdot \mu(A) \cdot \mu(B) \quad \text{for } A, B \in \mathcal{A}, \tag{M0}$$

where M is a positive integer and $C_0 > 0$ is a constant independent of A, B and k.

LEMMA 4.8. *Let τ be a sufficiently large positive integer and let s be a positive number. Put $\hat{\tau} = [\tau^s \log_2 \tau]$. If $\mu(E_l) \sim \text{constant} \cdot l^{-s}$,*

$$\mu\left(\left\{\sum_{k=0}^{\hat{\tau}} 1_{E_\tau}(S^k(x)) \geq 1\right\}\right) \geq 1 - \left(\frac{1}{e}\right)^{\text{constant}\cdot\log\tau}.$$

Proof. This is easily obtained by simple calculation. □

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5. *Piecewise monotonic maps with countable partitions*

Piecewise monotonic maps with countable partitions naturally arise from the first return maps. We will show that $(S, \{E_l\})$ of this section satisfies conditions (M0), (M1) and (M2) of the previous section if S satisfies some assumptions.

Let J be the union of disjoint intervals $J_1, \ldots, J_{r_1}$ with $\mathrm{Leb}(J) < \infty$. In this section, we assume that $S : J \to J$ satisfies the following conditions (1)–(4).

(1) There exists a partition $\{I_\lambda\}_{\lambda\in\Lambda}$ of J such that the restriction of S to the interval I_λ is a C^2 function for each λ.

(2) For each λ, there exists an i such that $S(I_\lambda) = J_i$.

(3) There exists a positive constant K_0 such that

$$|S''(x)|/(S'(x))^2 < K_0 < \infty,$$

whenever the first and the second derivatives of S are defined.

(4) There exists a constant $K_1 > 1$ with $S'(x) > K_1$, whenever the first derivative of S is defined.

Then we have the following lemma.

LEMMA 5.1. (See [**R**]) *S has an absolutely continuous invariant probability measure μ and there exists a non-negative function $g \le K_2$ such that $\mu(U) = \int_U g(x)\mathrm{Leb}(dx)$ for all measurable set U, where K_2 is a positive* constant.

Put $K_0^* = K_0/[1 - (1/K_0)]$. Then it is easy to see that

$$|(S^k)''(x)|/((S^k)'(x))^2 < K_0^*,$$

whenever the first and the second derivatives of S^k are defined. Note that $S^k : J \to J$ satisfies conditions (1)–(4) if we replace S with S^k. Let $\{I_\lambda^{(k)}\}_{\lambda\in\Lambda^k}$ be the partition associated with S^k and let $\varphi_\lambda^{(k)}$ be the inverse function of the restriction to $I_\lambda^{(k)}$ of S^k. Let v_λ and w_λ be the left and right endpoints of $S^k(I_\lambda)$, respectively. The following lemma is used to show Lemma 5.3.

LEMMA 5.2. (See Lemma 5.2 of [**I3**]) *$K_3 \mathrm{Leb}(I_\lambda^{(k)}) \le \varphi_\lambda^{(k)\prime}(x) \le K_4 \mathrm{Leb}(I_\lambda^{(k)})$ for $x \in S^k(I_\lambda^{(k)})$, where K_3 and K_4 are constants independent of k and λ.*

The following lemma means that S has a mixing property.

LEMMA 5.3. *Assume that the invariant density g satisfies $0 < K_5 \le g$, where K_5 is a constant. Let A be a union of disjoint intervals, that is $A = \bigcup_j (x_j, y_j)$, and let $B = \bigcup_{\lambda\in\Lambda_B} I_\lambda^{(k)}$, where Λ_B is a subset of Λ^k. Then*

$$\mu(S^{-k}A \cap B) \le K_6 \cdot \mu(A) \cdot \mu(B),$$

where K_6 is a constant independent of A, B and k.

Furthermore, let $\{B_t\}$ be sequences of sets like B satisfying $\lim_{t\to\infty} \mu(B_t) = 1$. Then, for sufficiently large t,

$$K_7 \cdot \mu(A) \le \mu(S^{-k}A \cap B_t),$$

where K_7 is a constant independent of A, B_t and k.

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Proof. For simplicity, we omit the index (k) in the proof. Without loss of generality for each j, we assume that there exists an i such that $(x_j, y_j) \subset J_i$. By the assumption on g, we have

$$\frac{\mu(S^{-k}A \cap B)}{\mu(B)} \leq \frac{K_2 \sum_\lambda \sum_j (\varphi_\lambda(y_j) - \varphi_\lambda(x_j))}{K_5 \operatorname{Leb}(B)},$$

where $\sum_\lambda$ is the sum over $\lambda \in \Lambda_B$ with $S^k(I_\lambda) \supset (x_j, y_j)$. By the Mean Value Theorem, there exists $z_j \in (x_j, y_j)$ such that

$$\varphi_\lambda(y_j) - \varphi_\lambda(x_j) = \varphi'_\lambda(z_j)(y_j - x_j).$$

This equality and the previous lemma imply

$$\begin{aligned}\frac{\mu(S^{-k}A \cap B)}{\mu(B)} &\leq \frac{K_2 K_4 \sum_{\lambda \in \Lambda_B} \sum_j \operatorname{Leb}(I_\lambda)(y_j - x_j)}{K_5 \operatorname{Leb}(B)} \\ &\leq \frac{K_2 K_4 \operatorname{Leb}(A)}{K_5} \leq \frac{K_2 K_4 \mu(A)}{K_5^2},\end{aligned}$$

which shows the first half of the lemma.

To prove the latter half, choose i_0 such that $\mu(A \cap J_{i_0}) \geq r_1^{-1}\mu(A)$. Let B' be the union of I_λ satisfying $\lambda \in \Lambda_{B_t}$ such that $S^k I_\lambda \supset J_{i_0}$. Then B' is not the empty set for sufficiently large t. In a similar way as we proved the first half of the lemma, we obtain

$$K_8 \cdot \mu(A \cap J_{i_0}) \cdot \mu(B') \leq \mu(S^{-k}(A \cap J_{i_0}) \cap B') \leq \mu(S^{-k}A \cap B_t),$$

where K_8 is a positive constant. Therefore, we obtain the lemma. □

Let $\{\Lambda_l\}$ be a decreasing sequence of subsets of Λ. Put

$$E_l = \bigcup_{\lambda \in \Lambda_l} I_\lambda.$$

Then the following lemma is proved similarly to Lemma 5.3.

LEMMA 5.4. *Assume that S satisfying conditions (1)–(4) is ergodic and has the invariant density g with $0 < K_5 \leq g$. And assume that J is a union of $L_1, \ldots, L_M$ such that $S^{-1}(L_l) = L_{\alpha(l)}$, where α is a cyclic permutation of $\{1, \ldots, M\}$. Then $(S, \{E_l\})$ satisfies condition (M0) in §4.*

Let P be the Frobenius–Perron operator corresponding to (S, Leb). (The definition of the Frobenius–Perron operator is found in [**LM**].)

The following lemma can be obtained similarly to Lemma 5.5 of [**I3**].

LEMMA 5.5. *Assume that S satisfying conditions (1)–(4) is ergodic and that P is asymptotically periodic, that is for any probability density function f,*

$$\lim_{n \to \infty} \left\| P^n f - \sum_{i=1}^{M} \lambda_i(f) g_{\alpha^n(i)} \right\|_{L^1(\text{Leb})} = 0,$$

where $g_1, \ldots, g_M$ are probability density functions with $g_i(x) \cdot g_j(x) = 0$ for all $i \neq j$ and α is a cyclic permutation of $\{1, \ldots, M\}$. Then $(S, \{E_l\})$ satisfies conditions (M1) and (M2) in §4 with $X = J$.

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6. *Proof of theorems*

In this section we are going to prove Theorems 2 and 3. Let T be a map satisfying conditions (i)–(iv) in §1. Let T_i be as in Theorem 3.

To prove our theorems, the first return map of T is an important tool. We consider the first return map of T on $\Omega = \bigcup_{i=1}^{r-1}[T_i^{-1}(c_i), T_{i+1}^{-1}(c_i)]$. Then it is clear that the first return map R of T on Ω is well defined except on the set of the endpoints of the countable partition of Ω. Let μ be a Lebesgue equivalent T-ergodic invariant measure (such a measure exists as we have already noted in §1) and let $\tilde{\mu}$ be an absolutely continuous R-ergodic invariant measure. We note that $\tilde{\mu} = \text{constant} \cdot \mu|_\Omega$ (Lemma 3.2). Since it is easy to see that $S = R$ satisfies conditions (1)–(4) in §5, by Lemma 5.1 we can choose $\tilde{\mu}$ with $\tilde{\mu}(\Omega) = 1$. Thus, for convenience, we choose μ satisfying $\mu(\Omega) = 1$.

Now let $p = p_{i_0}$ and $q = p_{i_1}$. We first consider the case when A and B are a small right neighborhood of p and a small left neighborhood of q, respectively. Put $c = c_{i_0}$ and $d = c_{i_1-1}$.

We prepare some notation.

Notation. For all integer $n \geq 0$ and any large integer t,

$$a_n = T_{i_0}^{-n}(c) \quad \text{and} \quad b_n = T_{i_1}^{-n}(d),$$

$$\alpha_{i,n} = T_i^{-1}(a_n) \text{ for } i \neq i_0 \quad \text{and} \quad \beta_{i,n} = T_i^{-1}(b_n) \text{ for } i \neq i_1,$$

$$F(t,x) := \frac{\sum_{k=0}^{t} 1_{(p,c)}(T^k(x)) \cdot (T^k(x) - p)^{m_0}}{\sum_{k=0}^{t} 1_{(d,q)}(T^k(x)) \cdot (q - T^k(x))^{m_1}},$$

$$E(\alpha_n) := \bigcup_{i \neq i_0} [c, \alpha_{i,n}], \quad E(\beta_n) := \bigcup_{i \neq i_1} [\beta_{i,n}, d],$$

$$\tau_t := \max\left\{\tau; \sum_{k=1}^{\tau} 1_\Omega(T^k(x)) = t\right\},$$

$$t^* : \text{the integer satisfying } (\log_2 t^*)^{1/2} = [(\log_2 t)^{1/2}],$$

where $[t]$ is the greatest integer not greater than t,

$n(0,t)$: the maximal integer satisfying

$$\log_2 t^* \leq t^* \mu(E(\alpha_{n(0,t)})) \quad \text{if } d_0 - m_0 > 2$$
$$1 \leq t(\log_2 t)^{1/4} \mu(E(\alpha_{n(0,t)})) \quad \text{if } d_0 - m_0 = 2,$$

$n(1,t)$: the maximal integer satisfying:

$$\log_2 t^* \leq t^* \mu(E(\beta_{n(1,t)})) \quad \text{if } d_1 - m_1 > 2$$
$$1 \leq t(\log_2 t)^{1/4} \mu(E(\beta_{n(1,t)})) \quad \text{if } d_1 - m_1 = 2,$$

$$H_0(t,x) := \sum_{k=0}^{\tau_t} 1_{[p, a_{n(0,t)})}(T^k(x)) \cdot (T^k(x) - p)^{m_0},$$

$$H_a(t,x) := \sum_{k=0}^{\tau_t} 1_{[a_{n(0,t)}, c)}(T^k(x)) \cdot (T^k(x) - p)^{m_0},$$

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$$H_b(t,x) := \sum_{k=0}^{\tau_t} 1_{(d,b_{n(1,t)}]}(T^k(x)) \cdot (q - T^k(x))^{m_1},$$

$$H_1(t,x) := \sum_{k=0}^{\tau_t} 1_{(b_{n(1,t)},q]}(T^k(x)) \cdot (q - T^k(x))^{m_1},$$

$$\eta(t,x) := \frac{H_a(t,x)}{H_b(t,x)}.$$

The following lemma is important in applying the result in §4.

LEMMA 6.1. *Put $S = R$ and $E_l = E(\alpha_l)$ or $E_l = E(\beta_l)$. Then $(S, \{E_l\})$ satisfies the conditions (M0), (M1) and (M2) in §4.*

Proof. It is easy to see that $(S, \{E_l\})$ satisfies conditions (1)–(4) in §5. Furthermore, as we have already noted, S has an absolutely continuous ergodic invariant measure and we can choose a density function which is bounded away from 0. The asymptotic periodicity of the Frobenius–Perron operator follows from Theorems 1 and 4 in [**R**] and the ergodicity of S. So, all the assumptions of Lemmas 5.4 and 5.5 are satisfied. Therefore, we obtain the lemma. □

As we see in the following lemma, it is essential for our purpose to calculate the limit value of $F(t,x)$.

LEMMA 6.2. *Assume that*

$$\int f_A \, d\mu = \infty, \quad \int f_B \, d\mu = \infty$$

and that $\lim_{t\to\infty} F(t,x)$ *exists. Then*

$$\lim_{t\to\infty} F(t,x) = \lim_{t\to\infty} \frac{\sum_{k=0}^{t} f_A(T^k(x))}{\sum_{k=0}^{t} f_B(T^k(x))} \quad \text{for } \mu\text{-a.e. } x.$$

Proof. We easily obtain this by Lemma 3.1. □

In the rest of this section, we assume (1.1) and (1.2). Let g be a continuous version of the Radon–Nikodym derivative of the invariant measure μ on Ω.

Put

$$\rho = \lim_{t\to\infty} \frac{\sum_{l=1}^{n(0,t)} (a_l - p)^{m_0} \mu(E(\alpha_l))}{\sum_{l=1}^{n(1,t)} (q - b_l)^{m_1} \mu(E(\beta_l))}. \tag{6.1}$$

The following lemma confirms the existence of ρ and it is important in determining the limit value of $F(t,x)$.

LEMMA 6.3.

(1) *If* $2 \le d_0 - m_0 < d_1 - m_1$, $\rho = 0$.

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(2) *If* $d_0 - m_0 = d_1 - m_1 = 2$,

$$\rho = \frac{\theta_1 \sum_{i \neq i_0} g(T_i^{-1}(p)) \cdot (T_i'(T_i^{-1}(p)))^{-1}}{\theta_0 \sum_{i \neq i_1} g(T_i^{-1}(q)) \cdot (T_i'(T_i^{-1}(q)))^{-1}}.$$

In particular:

(a) $\rho = \theta_1/\theta_0$ *if* $p = q$;

(b) $\rho = (T_1'(c)\theta_1)/(T_2'(c)\theta_0)$ *if* $r = 2$, $p = 0$ *and* $q = 1$.

To prove this, we need the following lemma.

LEMMA 6.4. (See the corollary in §3 of [**T2**])

$$\mathrm{Leb}([p, a_n]) \sim (\theta_0(d_0 - 1)n)^{-1/(d_0-1)}.$$

Proof of Lemma 6.3. Since g is a continuous version of the Radon–Nikodym derivative of the invariant measure μ on Ω, we have

$$(a_n - p)^{m_0} \mu(E(\alpha_n)) \sim (a_n - p)^{m_0} \sum_{i \neq i_0} \int_{T_i^{-1}(p)}^{\alpha_{i,n}} g \, d\mathrm{Leb}.$$

Noting that g is continuous and bounded away from 0, we have

$$(a_n - p)^{m_0} \sum_{i \neq i_0} \int_{T_i^{-1}(p)}^{\alpha_{i,n}} g \, d\mathrm{Leb} \sim (a_n - p)^{m_0} \sum_{i \neq i_0} g(T_i^{-1}(p)) \, \mathrm{Leb}([T_i^{-1}(p), \alpha_{i,n}]).$$

Since T_i has a positive derivative at $T_i^{-1}(p)$ for each $i \neq i_0$, we have

$$\mathrm{Leb}([T_i^{-1}(p), \alpha_{i,n}]) \sim (T_i'(T_i^{-1}(p)))^{-1} \, \mathrm{Leb}([p, a_n]).$$

Thus, it follows from Lemma 6.4 that

$$\mathrm{Leb}([T_i^{-1}(p), \alpha_{i,n}]) \sim (T_i'(T_i^{-1}(p)))^{-1} (\theta_0(d_0 - 1)n)^{-1/(d_0-1)}.$$

Hence, we have

$$(a_n - p)^{m_0} \mu(E(\alpha_n)) \sim \sum_{i \neq i_0} g(T_i^{-1}(p))(T_i'(T_i^{-1}(p)))^{-1} (\theta_0(d_0 - 1)n)^{-(m_0+1)/(d_0-1)}. \tag{6.2}$$

In the rest of the proof, for simplicity we use the following notation:

$$S(p) := \sum_{i \neq i_0} g(T_i^{-1}(p))(T_i'(T_i^{-1}(p)))^{-1}.$$

Now we are going to estimate $n(0, t)$. Similarly to the previous argument, we have

$$\mu(E(\alpha_{n(0,t)})) \sim S(p) \, \mathrm{Leb}([p, a_{n(0,t)}]).$$

This relation and the definition of $n(0, t)$ imply

$$a_{n(0,t)} - p \sim \begin{cases} \dfrac{\log_2 t^*}{S(p)t^*} & \text{if } d_0 - m_0 > 2 \\ \dfrac{1}{S(p)t(\log_2 t)^{1/4}} & \text{if } d_0 - m_0 = 2. \end{cases}$$

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However by Lemma 6.4, we have

$$a_{n(0,t)} - p \sim (\theta_0(d_0 - 1)n(0,t))^{-1/(d_0-1)}.$$

Therefore, we obtain

$$n(0,t) \sim \begin{cases} \dfrac{1}{\theta_0(d_0-1)} \left(\dfrac{S(p)t^*}{\log_2 t^*} \right)^{d_0-1} & \text{if } d_0 - m_0 > 2 \\ \dfrac{1}{\theta_0(d_0-1)} (S(p)t(\log_2 t)^{1/4})^{d_0-1} & \text{if } d_0 - m_0 = 2. \end{cases} \tag{6.3}$$

We are going to estimate

$$\sum_{l=1}^{n(0,t)} (a_l - p)^{m_0} \mu(E(\alpha_l)) \sim \sum_{l=1}^{n(0,t)} S(p)(\theta_0(d_0-1)l)^{-(m_0+1)/(d_0-1)}.$$

In the case $d_0 - m_0 = 2$, we have

$$\sum_{l=1}^{n(0,t)} (a_l - p)^{m_0} \mu(E(\alpha_l)) \sim \frac{S(p)}{\theta_0(d_0-1)} \log n(0,t).$$

This and relation (6.3) imply that

$$\sum_{l=1}^{n(0,t)} (a_l - p)^{m_0} \mu(E(\alpha_l)) \sim \frac{S(p)}{\theta_0} \log t. \tag{6.4}$$

In the case $d_0 - m_0 > 2$, we have

$$\begin{aligned} \sum_{l=1}^{n(0,t)} (a_l - p)^{m_0} \mu(E(\alpha_l)) &\sim \text{constant} \cdot n(0,t)^{1-(m_0+1)/(d_0-1)} \\ &\sim \text{constant} \cdot \left(\frac{t^*}{\log_2 t^*} \right)^{d_0-m_0-2}. \end{aligned} \tag{6.5}$$

In the same way, if $d_1 - m_1 = 2$, we have

$$\sum_{l=1}^{n(1,t)} (q - b_l)^{m_1} \mu(E(\beta_l)) \sim \frac{S(q)}{\theta_1} \log t. \tag{6.6}$$

If $d_1 - m_1 > 2$, we have

$$\sum_{l=1}^{n(1,t)} (q - b_l)^{m_1} \mu(E(\beta_l)) \sim \text{constant} \cdot \left(\frac{t^*}{\log_2 t^*} \right)^{d_1-m_1-2}. \tag{6.7}$$

By relations (6.4)–(6.7) we obtain the lemma. □

LEMMA 6.5. *ρ in (6.1) satisfies the following property:*

$$\lim_{t\to\infty} \frac{\sum_{k=0}^{t} \sum_{l=1}^{n(0,t)} (a_l - p)^{m_0} 1_{E(\alpha_l)}(R^k(x))}{\sum_{k=0}^{t} \sum_{l=1}^{n(1,t)} (q - b_l)^{m_1} 1_{E(\beta_l)}(R^k(x))} = \rho \quad \textit{for } \mu|_\Omega\textit{-a.e. } x. \tag{6.8}$$

As a consequence, $\eta(t,x)$ converges to ρ as t goes to infinity for $\mu|_\Omega$-a.e. x.

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Proof. It follows from Lemma 6.4 that

$$\mu(E(\alpha_l)) \sim \text{constant} \cdot (a_l - p)$$

and

$$(a_l - p)^{m_0}\mu(E(\alpha_l)) \sim \begin{cases} \text{constant} \cdot l^{-s}\ (0 < s < 1) & \text{if } d_0 - m_0 > 2 \\ \text{constant} \cdot l^{-1} & \text{if } d_0 - m_0 = 2. \end{cases}$$

We have a similar estimate for $\mu(E(\beta_l))$. Thus, by Lemmas 4.1 and 6.1 and Proposition 4.4, we have

$$\sum_{k=0}^{t}\sum_{l=1}^{n(0,t)}(a_l - p)^{m_0}1_{E(\alpha_l)}(R^k(x)) \sim t\sum_{l=1}^{n(0,t)}(a_l - p)^{m_0}\mu(E(\alpha_l))$$

and

$$\sum_{k=0}^{t}\sum_{l=1}^{n(1,t)}(q - b_l)^{m_1}1_{E(\beta_l)}(R^k(x)) \sim t\sum_{l=1}^{n(1,t)}(q - b_l)^{m_1}\mu(E(\beta_l))$$

for $\mu|_\Omega$-a.e. x. Hence, we can easily obtain equality (6.8).

Obviously, for $\mu|_\Omega$-a.e. x, we have

$$\sum_{k=0}^{\tau_t}\sum_{l=1}^{n(0,t)}(a_l - p)^{m_0}1_{(a_l,a_{l-1}]}(T^k(x)) \le H_a(t,x) \le \sum_{k=0}^{\tau_t}\sum_{l=1}^{n(0,t)}(a_{l-1} - p)^{m_0}1_{(a_l,a_{l-1}]}(T^k(x))$$
$$= \sum_{k=0}^{t}\sum_{l=1}^{n(0,t)}(a_{l-1} - p)^{m_0}1_{E(\alpha_l)}(R^k(x)),$$

and also have a similar estimate for $H_b(t,x)$. From this fact and equality (6.8) it follows that $\eta(t,x)$ converges to ρ for $\mu|_\Omega$-a.e. x. □

We are going to show the following proposition.

PROPOSITION 6.6. *If $2 \le d_0 - m_0 < d_1 - m_1$, $F(t,x)$ converges to 0 as t goes to infinity for μ-a.e. x.*

In order to show this, we prove the following lemma.

LEMMA 6.7. *If $2 \le d_0 - m_0 < d_1 - m_1$, then*

$$\mu|_\Omega\Big(\bigcap_{k=0}^{\infty}\bigcup_{t=k}^{\infty}\{x;\, F(\tau_t,x) > \varepsilon\}\Big) = 0 \quad \text{for any } \varepsilon > 0.$$

To prove Lemma 6.7, we prepare the following lemma.

LEMMA 6.8.

(1) *If $d_0 - m_0 = 2$, $H_a(t,x) \sim \text{constant} \cdot t \log t$ for $\mu|_\Omega$-a.e. x.*

(2) *If $d_1 - m_1 > 2$, $H_b(t,x) \sim \text{constant} \cdot t\ (t^*/\log_2 t^*)^{d_1 - m_1 - 2}$ for $\mu|_\Omega$-a.e. x.*

Proof. We first show (1). Since

$$H_a(t,x) \sim \sum_{k=0}^{t}\sum_{l=1}^{n(0,t)}(a_l - p)^{m_0}1_{E(\alpha_l)}(R^k(x)) \quad \text{for } \mu|_\Omega\text{-a.e. } x,$$

Lemma 6.8(1) follows from Proposition 4.4 and relation (6.4).

(2) is similarly proved. □

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Proof of Lemma 6.7. It is sufficient to prove that

$$H_0(t,x)/H_b(t,x) \to 0 \quad (t \to \infty).$$

We first consider the case $d_0 - m_0 > 2$.

For t choose the integer n satisfying $2^{n^2} \le t < 2^{(n+1)^2}$. Put $z(n) = 2^{n^2}$ as in §4.

It is easy to see that, for almost every x,

$$\lim_{t\to\infty} \frac{1}{(n^2 2^{2n+1})^2} \sum_{k=0}^{z(n+1)} 1_{E(\alpha_{n(0,t)})}(R^k(x)) = 0. \tag{6.9}$$

Let $\varepsilon > 0$ be arbitrary and put

$$C_n = \{x : H_0(z(n+1), x) \ge \varepsilon H_b(z(n), x)\}.$$

Then we have $H_0(z(n+1), x) \ge \text{constant} \cdot \varepsilon z(n)(z(n)/\log_2 z(n))^{d_1 - m_1 - 2}$ for almost every $x \in C_n$ by Lemma 6.8(2). Hence, it follows from (6.9) that for almost every $x \in C_n$, there exists at least one $k \le 2^{(n+1)^2}$ such that $R^k(x)$ enters $E(\alpha_{\tilde{n}})$, where $\tilde{n}$ satisfies the following relation:

$$\sum_{l=n(0,t)}^{\tilde{n}} (a_l - p)^{m_0} \sim \text{constant} \cdot \frac{\varepsilon z(n)}{(n^2 2^{2n+1})^2} \left(\frac{z(n)}{\log_2 z(n)}\right)^{d_1 - m_1 - 2}. \tag{6.10}$$

Since we have

$$\sum_{l=n(0,t)}^{\tilde{n}} (a_l - p)^{m_0} \sim \text{constant} \cdot \tilde{n}^{1 - m_0/(d_0 - 1)} = \text{constant} \cdot \tilde{n}^{(d_0 - m_0 - 1)/(d_0 - 1)},$$

it follows from (6.10) that

$$\tilde{n} \sim \text{constant} \cdot \left(\frac{\varepsilon z(n)}{(n^2 2^{2n+1})^2} \left(\frac{z(n)}{\log_2 z(n)}\right)^{d_1 - m_1 - 2}\right)^{(d_0 - 1)/(d_0 - m_0 - 1)}.$$

Thus, we have the following estimate for $\mu(E(\alpha_{\tilde{n}}))$:

$$\begin{aligned}\mu(E(\alpha_{\tilde{n}})) &\sim \text{constant} \cdot \left(\frac{1}{\tilde{n}}\right)^{1/(d_0 - 1)} \\ &\sim \text{constant} \cdot \left(\frac{2^{2(2n+1)} n^4}{\varepsilon z(n)} \left(\frac{\log_2 z(n)}{z(n)}\right)^{(d_1 - m_1 - 1)}\right)^{1/(d_0 - m_0 - 1)}.\end{aligned}$$

Choose δ_0 such that $(d_1 - m_1 - 1)/(d_0 - m_0 - 1) = 1 + \delta_0$. (Since $2 \le d_0 - m_0 < d_1 - m_1$, we have $\delta_0 > 0$.) Then we have

$$\begin{aligned}&\mu(\{x : R^k(x) \in E(\alpha_{\tilde{n}}) \text{ for some } 0 \le k \le 2^{(n+1)^2}\}) \\ &\quad \le \text{constant} \cdot 2^{(n+1)^2} (2^{2(2n+1)} n^4 / (2^{n^2} \varepsilon))^{1/(d_0 - m_0 - 1)} (n^2 / 2^{n^2})^{1 + \delta_0} \\ &\quad = \text{constant} \cdot 2^{2n+1} (2^{2(2n+1)} n^4 / (2^{n^2} \varepsilon))^{1/(d_0 - m_0 - 1)} n^{2(1+\delta_0)} / 2^{n^2 \delta_0}.\end{aligned}$$

This means that

$$\mu\left(\bigcap_{k=0}^{\infty} \bigcup_{n=k}^{\infty} C_n\right) = 0,$$

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and, hence,

$$\lim_{t\to\infty} \frac{H_0(z(n+1),x)}{H_b(z(n),x)} = 0 \quad \mu|_\Omega\text{-a.e. } x.$$

Therefore, we have

$$\lim_{t\to\infty} \frac{H_0(t,x)}{H_b(t,x)} = 0 \quad \mu|_\Omega\text{-a.e. } x.$$

The case $d_0 - m_0 = 2$ is also proved similarly. □

Proof of Proposition 6.6. Let Q be an arbitrary set with $\mu|_\Omega(Q) = 0$. Since $\mathrm{Leb}\big(\bigcup_{n=0}^\infty T^{-n}\Omega\big) = 1$ and $\mathrm{Leb}\big(\bigcup_{n=0}^\infty T^{-n}Q\big) = 0$, it is sufficient to show that

$$\mu|_\Omega\Big(\bigcap_{k=0}^\infty \bigcup_{t=k}^\infty \{x;\, F(t,x) > \varepsilon\}\Big) = 0 \quad \text{for any } \varepsilon > 0.$$

Obviously, this is equivalent to

$$\mu|_\Omega\Big(\bigcap_{k=0}^\infty \bigcup_{t=k}^\infty \{x;\, F(\tau_t,x) > \varepsilon\}\Big) = 0 \quad \text{for any } \varepsilon > 0.$$

Thus, the proposition follows from Lemma 6.7. □

The lemmas in the rest of this section are essential to prove Theorem 3.

LEMMA 6.9. *If $d_0 - m_0 = d_1 - m_1 = 2$,*

$$\limsup_{t\to\infty} \frac{H_0(t,x) + H_a(t,x)}{H_1(t,x) + H_b(t,x)} = \infty \quad \textit{for } \mu|_\Omega\textit{-a.e. } x.$$

Proof. Let ρ_0 be an arbitrary sufficiently large constant. For $t \geq 2$, let $n^*(0,t)$ be the maximal integer satisfying

$$1 \leq 3\rho_0 t(\log t)\mu(E(\alpha_{n^*(0,t)})).$$

Then it follows from Lemma 5.3 that

$$\begin{aligned}&\mu(\{x \in \Omega : R^t(x) \in E(\alpha_{n^*(0,t)}),\, H_1(t,x) = 0\})\\ &\quad \geq \text{constant} \cdot \mu(\{x \in \Omega : R^t(x) \in E(\alpha_{n^*(0,t)})\}) \geq \frac{\text{constant}}{\rho_0 t(\log t)}.\end{aligned}$$

Since $\sum_{t=2}^\infty (t\log t)^{-1} = \infty$, by Lemmas 3.4 and 5.3, we obtain

$$\mu\Big(\bigcap_{k=2}^\infty \bigcup_{t=k}^\infty \{x \in \Omega : R^t(x) \in E(\alpha_{n^*(0,t)}),\, H_1(t,x) = 0\}\Big) > 0.$$

Thus, we have

$$\mu\Big(\bigcap_{k=2}^\infty \bigcup_{t=k}^\infty \{x \in \Omega : H_0(t,x) \geq 2\rho_0 t\log t,\, H_1(t,x) = 0\}\Big) > 0.$$

By the ergodicity, we have

$$\mu\Big(\bigcap_{k=2}^\infty \bigcup_{t=k}^\infty \{x \in \Omega : H_0(t,x) \geq 2\rho_0 t\log t,\, H_1(t,x) = 0\}\Big) = 1.$$

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By Lemma 6.8, for almost every x, there exist infinitely many t such that

$$\frac{H_0(t,x)+H_a(t,x)}{H_1(t,x)+H_b(t,x)} \geq \rho_0.$$

This implies the lemma. □

LEMMA 6.10. *If* $d_0 - m_0 = d_1 - m_1 = 2$,

$$\lim_{t\to\infty} \mathrm{Leb}(\{x : |F(t,x)-\rho| \geq \varepsilon\}) = 0 \quad \textit{for any } \varepsilon > 0.$$

Proof. Let $d_* = \max_p\{d_0 \text{ as in (1.1) for } p\}$ and $s = 1/(d_*-1)$. Put $\tilde{\tau} = \tau^s/\log\tau$ and $\hat{\tau} = \tau^s \log\tau$. Then we have

$$\begin{aligned}
&\mathrm{Leb}(\{x : |F(\tau,x)-\rho| \geq \varepsilon\}) \\
&\quad \leq \mathrm{Leb}(\{x : T^k(x) \notin \Omega \text{ for } k = 0,\dots,\tau/2\}) \\
&\qquad + \mathrm{Leb}(\{x : \sharp\{k \leq \tau : T^k(x) \in \Omega\} \leq \tilde{\tau},\ T^k(x) \in \Omega \text{ for some } k \leq \tau/2\}) \\
&\qquad + \mathrm{Leb}(\{x : \sharp\{k \leq \tau : T^k(x) \in \Omega\} \geq \hat{\tau}\}) \\
&\qquad + \text{constant} \cdot \mathrm{Leb}\Big(\bigcup_{t=\tilde{\tau}}^{\hat{\tau}} \{x \in \Omega : |\eta(t,x)-\rho| > \varepsilon\}\Big) \\
&\qquad + \text{constant} \cdot \mathrm{Leb}\Big(\bigcup_{t=\tilde{\tau}}^{\hat{\tau}} \{x \in \Omega : R^t(x) \in E(\alpha_{n(0,t)}) \cup E(\beta_{n(1,t)})\}\Big) \\
&\qquad + \text{constant} \cdot \mathrm{Leb}\Big(\bigcup_{t=0}^{\tilde{\tau}} \{x \in \Omega : R^t(x) \in E(\alpha_{n(0,\tilde{\tau})}) \cup E(\beta_{n(1,\tilde{\tau})})\}\Big). \qquad (6.11)
\end{aligned}$$

We are going to check that each term in the right-hand side converges to 0 as τ goes to infinity. It is clear that the first term of (6.11) converges to 0. It is easy to see that the second term of (6.11) is less than

$$\text{constant} \cdot \mathrm{Leb}(\{x \in \Omega : \sharp\{k \leq \tau/2 : T^k(x) \in \Omega\} \leq \tilde{\tau}\}).$$

Since we can see that

$$\begin{aligned}
&\mathrm{Leb}(\{x \in \Omega : \sharp\{k \leq \tau/2 : T^k(x) \in \Omega\} \leq \tilde{\tau}\}) \\
&\quad \leq \mathrm{Leb}\Big(\Big\{x \in \Omega : \sum_{k=0}^{\tilde{\tau}} \sum_{l=1}^{\infty} 1_{E_l}(R^k(x)) \geq \tau/2\Big\}\Big) \\
&\quad \leq \mathrm{Leb}\Big(\Big\{x \in \Omega : \sum_{k=0}^{\tilde{\tau}} \sum_{l=1}^{\tau/2} 1_{E_l}(R^k(x)) \geq \tau/2\Big\}\Big) \\
&\qquad + \mathrm{Leb}(\{x \in \Omega : R^k(x) \in E_{\tau/2} \text{ for some } k \leq \tilde{\tau}\}),
\end{aligned}$$

it follows from Lemma 4.7 that the second term of (6.11) converges to 0 as τ goes to infinity.

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It is easy to see that the third term of (6.11) is less than

$$\text{constant} \cdot \text{Leb}(\{x \in \Omega : \sharp\{k \leq \tau : T^k(x) \in \Omega\} \geq \hat{\tau}\}).$$

Since we have

$$\mu(\{x \in \Omega : \sharp\{k \leq \tau : T^k(x) \in \Omega\} \geq \hat{\tau}\}) \leq \mu\left(\left\{x \in \Omega : \sum_{k=0}^{\hat{\tau}} \sum_{l=1}^{\infty} 1_{E_l}(R^k(x)) \leq \tau\right\}\right)$$
$$\leq 1 - \mu\left(\left\{x \in \Omega : \sum_{k=0}^{\hat{\tau}} 1_{E_{\tau+1}}(R^k(x)) \geq 1\right\}\right),$$

by Lemma 4.8 the third term of (6.11) converges to 0.

The fourth, fifth and sixth terms of (6.11) converge to 0 by Lemmas 6.5, 4.6 and 4.5 respectively.

Therefore we obtain the lemma. □

Proof of Theorems 2 and 3. In the case when A and B are a small right neighborhood of p and a small left neighborhood of q respectively, Theorem 2 follows from Lemmas 6.2 and 6.3 and Proposition 6.6, and Theorem 3 follows from Lemmas 6.2, 6.3, 6.9 and 6.10.

The other cases are similarly proved. □

7. *Remark*

Until now we have assumed that T satisfies conditions (i)–(iv) in §1. In this section we consider the map T satisfying more general conditions. We consider non-Markovian maps.

Assume that $T : [0, 1] \to [0, 1]$ with several fixed points satisfies the following conditions:

(a) there exists a partition $0 = c_0 < c_1 < c_2 < \cdots < c_r = 1$ of $[0, 1]$ such that the restrictions of T to the interval (c_{i-1}, c_i) is a C^2 function for each i and can be extended to $[c_{i-1}, c_i]$ as a C^2 function T_i;

(b) $T_i'(x) > 1$ if x is not a fixed point; and

(c) $T([0, 1]) = [0, 1]$ Leb-a.e. and T has a Lebesgue equivalent ergodic invariant measure μ.

(Condition (c) is not very restrictive. Each T satisfying conditions (a) and (b) has an absolutely continuous invariant measure μ and (T, μ) can be decomposed into ergodic components. Furthermore, each ergodic component is supported by a union of a finite number of intervals (see [**Z**]). Thus if T is not ergodic, choose an ergodic component. Then we can obtain the same results as we state later.)

Let p and q be fixed points and let A and B be small one-sided neighborhoods of p and q respectively. We make the same assumptions as in (1.1) and (1.2) for T in A and for T in B respectively.

Then, we can obtain the results which are similar to Theorems 1–3 except the limit value ρ in Theorem 3. In general, the limit value ρ depends on μ but it is a positive constant in the situation corresponding to Theorem 3.

These facts can be proved in essentially the same way as Theorems 1–3.

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