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FORMULAS IN MODAL LOGIC **S4**

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Abstract. Here, we provide a detailed description of the mutual relation of formulas with finite propositional variables $p_1, \ldots, p_m$ in modal logic **S4**. Our description contains more information on **S4** than those given in Shehtman (1978) and Moss (2007); however, Shehtman (1978) also treated Grzegorczyk logic and Moss (2007) treated many other normal modal logics. Specifically, we construct normal forms, which behave like the principal conjunctive normal forms in the classical propositional logic. The results include finite and effective methods to find a normal form equivalent to a given formula A by clarifying the behavior of connectives and giving a finite method to list all exact models.

§1. Introduction. In the present section, we introduce formulas, the modal logic **S4**, and some types of Kripke models, and also describe the purpose of the present paper.

1.1. Formulas. Formulas are constructed from $\bot$ (contradiction) and the propositional variables $p_1, p_2, \ldots$ by using logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication), and $\Box$ (necessitation). We use upper case Latin letters, $A, B, C, \ldots$, with or without subscripts, for formulas. Also, we use Greek letters, $\Gamma, \Delta, \ldots$, with or without subscripts, for finite sets of formulas. The expressions $\Box\Gamma$ and $\Gamma^{\Box}$ denote the sets $\{\Box A \mid A \in \Gamma\}$ and $\{\Box A \mid \Box A \in \Gamma\}$, respectively. The *depth* $d(A)$ of a formula A is defined as

$d(p_i) = d(\bot) = 0$,
$d(B \wedge C) = d(B \vee C) = d(B \supset C) = \max\{d(B), d(C)\}$,
$d(\Box B) = d(B) + 1$.

Let **ENU** be an enumeration of the formulas. For a nonempty finite set Γ of formulas, the expressions $\bigwedge \Gamma$ and $\bigvee \Gamma$ denote the formulas

$$(\cdots((A_1 \wedge A_2) \wedge A_3) \cdots \wedge A_n) \quad \text{and} \quad (\cdots((A_1 \vee A_2) \vee A_3) \cdots \vee A_n),$$

respectively, where $\{A_1, \ldots, A_n\} = \Gamma$ and A_i occurs earlier than A_{i+1} in **ENU**. Also, the expressions $\bigwedge \emptyset$ and $\bigvee \emptyset$ denote the formulas $\bot \supset \bot$ and $\bot$, respectively.

The set of propositional variables $p_1, \ldots, p_m$ $(m \geq 1)$ is denoted by $\mathbf{V}$ and the set of formulas constructed from $\mathbf{V}$ and $\bot$ is denoted by $\mathbf{F}$. Also, for any $n = 0, 1, \ldots$, we define $\mathbf{F}^n$ as $\mathbf{F}^n = \{A \in \mathbf{F} \mid d(A) \leq n\}$. In the present paper, we mainly treat the sets $\mathbf{F}$ and $\mathbf{F}^n$.

***1.2.* S4.** By **S4**, we mean the sequent system defined by Ohnishi & Matsumoto (1957). Here, we introduce this system.

A *sequent* is the expression $(\Gamma \to \Delta)$. We often refer to $\Gamma \to \Delta$ as $(\Gamma \to \Delta)$ for brevity and refer to

$$A_1, \ldots, A_i, \Gamma_1, \ldots, \Gamma_j \to \Delta_1, \ldots, \Delta_k, B_1, \ldots, B_\ell$$

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as

$$\{A_1, \ldots, A_i\} \cup \Gamma_1 \cup \cdots \cup \Gamma_j \to \Delta_1 \cup \cdots \cup \Delta_k \cup \{B_1, \ldots, B_\ell\}.$$

We use upper case Latin letters $X, Y, Z, \ldots$, with or without subscripts, for sequents. If $X = (\Gamma \to \Delta)$, then we sometimes refer to $\Gamma \overset{X}{\to} \Delta$ as $\Gamma \to \Delta$. The *antecedent* $\mathbf{ant}(\Gamma \to \Delta)$ and the *succedent* $\mathbf{suc}(\Gamma \to \Delta)$ of a sequent $\Gamma \to \Delta$ are defined as

$$\mathbf{ant}(\Gamma \to \Delta) = \Gamma \quad \text{and} \quad \mathbf{suc}(\Gamma \to \Delta) = \Delta,$$

respectively. Also, for a sequent X and a set $\mathcal{S}$ of sequents, we define $\mathbf{for}(X)$ and $\mathbf{for}(\mathcal{S})$ as

$$\mathbf{for}(X) = \begin{cases} \bigwedge \mathbf{ant}(X) \supset \bigvee \mathbf{suc}(X) & \text{if } \mathbf{ant}(X) \neq \emptyset \\ \bigvee \mathbf{suc}(X) & \text{if } \mathbf{ant}(X) = \emptyset \end{cases}$$

and

$$\mathbf{for}(\mathcal{S}) = \{\mathbf{for}(X) \mid X \in \mathcal{S}\}.$$

For a finite set S of formulas or sequents, the expression $\#(S)$ denotes the number of elements in S.

Ohnishi & Matsumoto (1957) defined the system by adding the following two inference rules to the sequent system **LK** given by Gentzen (1934–1935) for the classical propositional logic:

$$\frac{A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}(\Box \to) \qquad \frac{\Box\Gamma \to A}{\Box\Gamma \to \Box A}(\to \Box).$$

Here, we do not use $\neg$ as a primary connective, so we use the additional axiom $\bot \to$ instead of the inference rules $(\neg \to)$ and $(\to \neg)$. We write $X \in \mathbf{S4}$ if X is provable in **S4**. Ohnishi & Matsumoto (1957) proved that this system enjoys a cut-elimination theorem:

LEMMA 1.1. (Ohnishi & Matsumoto, 1957) *If $X \in \mathbf{S4}$, then there exists a cut-free proof figure for X in* **S4**.

We use $A \equiv B$ instead of $\to (A \supset B) \wedge (B \supset A) \in \mathbf{S4}$. Also, for any two equivalence classes $[A]$ and $[B]$ in $\mathbf{F}/\equiv$, we use $[A] \leq [B]$ instead of $A \to B \in \mathbf{S4}$. Thus, structures $\langle \mathbf{F}^n/\equiv, \leq \rangle$ and $\langle \mathbf{F}/\equiv, \leq \rangle$ express the mutual relation of formulas. Therefore, our purpose is to provide a detailed description of these structures.

1.3. Kripke models. Here, we introduce various Kripke models.

A *Kripke model* is a structure $\langle W, R, P \rangle$ where W is a nonempty set, R is a binary relation on W, and P is a mapping from the set of propositional variables to 2^W. We extend, as usual, the domain of P to include all formulas. We call P a *valuation* and a member of W a *world*. For a Kripke model $M = \langle W, R, P \rangle$, and for a world $\alpha \in W$, we often write $(M, \alpha) \models A$ and $M \models A$ instead of $\alpha \in P(A)$ and $P(A) = W$, respectively. The following lemma is described in several articles (e.g., in Chagrov & Zakharyaschev, 1997):

LEMMA 1.2. $\to A \in \mathbf{S4}$ *if and only if* $M \models A$ *for any reflexive and transitive Kripke model.*

Let S be a set of formulas closed under $\supset$ and $\wedge$. We say that a Kripke model $M = \langle W, R, P \rangle$ is *exact for* S if the following two conditions hold:

- for any $A \in S$, $M \models A$ if and only if $\to A \in \mathbf{S4}$,
- $\{P(A) \mid A \in S\} = 2^W$.

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This model was introduced in de Bruijn (1975). Here, we note that 2^W is either finite or uncountable, whereas $\{P(A) \mid A \in S\}$ is at most countable. Therefore, from the second condition above, we find that W must be finite.

We also consider a Kripke model $M = \langle W, R, P\rangle$ satisfying the following two conditions:

- for any $A \in \mathbf{F}$, $M \models A$ if and only if $\rightarrow A \in \mathbf{S4}$,
- $\{\{\alpha\} \mid \alpha \in W\} \subseteq \{P(A) \mid A \in \mathbf{F}\}$.

If W is finite, then the second condition above is equivalent to the second condition for an exact model. Every exact model is finite, as stated above. Thus, there is no confusion if we refer to the above model as an *exact model for* $\mathbf{F}$.

The following two lemmas are observed easily; therefore, exact models are useful to investigate the structures $\langle \mathbf{F}/\equiv, \leq\rangle$ and $\langle \mathbf{F}^n/\equiv, \leq\rangle$.

LEMMA 1.3. *Let $\langle W, R, P\rangle$ be an exact model for $\mathbf{F}^n$. Then the mapping P^* from $\mathbf{F}^n/\equiv$ to 2^W defined as*

$$P^*([A]) = P(A)$$

is an isomorphism and the structure $\langle \mathbf{F}^n/\equiv, \leq\rangle$ is isomorphic to the structure $\langle 2^W, \subseteq\rangle$.

LEMMA 1.4. *Let $\langle W, R, P\rangle$ be an exact model for $\mathbf{F}$. Then the mapping P^* from $\mathbf{F}/\equiv$ to $\{P(A) \mid A \in \mathbf{F}\}$ defined as*

$$P^*([A]) = P(A)$$

is an isomorphism and the structure $\langle \mathbf{F}/\equiv, \leq\rangle$ is isomorphic to the structure $\langle \{P(A) \mid A \in \mathbf{F}\}, \subseteq\rangle$.

Lemma 1.4 can be achieved without using the second condition of an exact model for $\mathbf{F}$. However, according to the second condition, an exact model for $\mathbf{F}$ has no redundant world, and therefore, an exact model for $\mathbf{F}$ is one of the simplest Kripke models satisfying the condition in Lemma 1.4.

1.4. The purpose. The purpose of the paper is to provide a detailed description of the structures $\langle \mathbf{F}^n/\equiv, \leq\rangle$ and $\langle \mathbf{F}/\equiv, \leq\rangle$, which express the mutual relation of formulas in **S4**. Specifically, we do the following three tasks. In the first two, we treat $\langle \mathbf{F}^n/\equiv, \leq\rangle$, and in the third, we treat $\langle \mathbf{F}/\equiv, \leq\rangle$.

(I) Without using the provability of **S4**, we construct a finite set $\mathbf{ED}^n$ of sequents satisfying

(I-1) $\mathbf{F}^n/\equiv = \{[\bigwedge \mathbf{for}(\mathcal{S})] \mid \mathcal{S} \subseteq \mathbf{ED}^n\}$,

(I-2) for any subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathbf{ED}^n$, $\mathcal{S}_1 \subseteq \mathcal{S}_2$ if and only if $\bigwedge \mathbf{for}(\mathcal{S}_2) \rightarrow \bigwedge \mathbf{for}(\mathcal{S}_1) \in \mathbf{S4}$.

(II) For $\mathbf{ED}^n$ given in (I) and for any $A \in \mathbf{F}^n$, we give finite and effective methods to find $\mathcal{S} \in 2^{\mathbf{ED}^n}$ satisfying $A \equiv \bigwedge \mathbf{for}(\mathcal{S})$.

(III) Similarly to the above two tasks, we treat $\langle \mathbf{F}/\equiv, \leq\rangle$.

Two tasks (I) and (II) prove that a member of $\mathbf{ED}^n$ and a formula $\bigwedge \mathbf{for}(\mathcal{S})$ behave like an elementary disjunction

$$p_1^* \vee \cdots \vee p_m^* \qquad (p_i^* \in \{p_i, p_i \supset \bot\})$$

and a principal conjunctive normal form in the classical propositional logic, respectively.

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There have been many studies on this topic.

Finite structures like $\langle \mathbf{F}^n/\equiv, \leq\rangle$ have also been studied previously. For example, for the set $\mathbf{F}^{[\wedge,\supset]}$ of disjunction free nonmodal formulas in $\mathbf{F}$ and for the equivalence $\equiv_{\mathbf{IPL}}$ of the intuitionistic propositional logic, the structure $\langle \mathbf{F}^{[\wedge,\supset]}/\equiv_{\mathbf{IPL}}, \leq\rangle$ was shown to be finite in Diego (1966). Also, using some types of Kripke models, there are several results concerning with $\langle \mathbf{F}^{[\wedge,\supset]}/\equiv_{\mathbf{IPL}}, \leq\rangle$ (cf. Urquhart, 1974, deBruijn, 1975, Hendriks, 1996, and Sasaki, 2001). Moss (2007) treated $\langle \mathbf{F}^n/\equiv, \leq\rangle$. He constructed the set $\mathcal{C}_{n,m}(\mathbf{S4})$ of formulas and, as a corollary, the set $\{(A \rightarrow) \mid A \in \mathcal{C}_{n,m}(\mathbf{S4})\}$ satisfies (I-1) and (I-2). We will describe the differences between $\mathcal{C}_{n,m}(\mathbf{S4})$ and $\mathbf{ED}^n$ in the detailed description of (I) and (II) below.

Infinite structures like $\langle \mathbf{F}/\equiv, \leq\rangle$ have also been studied in many articles by considering the corresponding algebra. Rieger (1949) and Nishimura (1960) clarified Heyting algebra for the intuitionistic propositional logic in the case that $m = 1$. Urquhart (1973) treated this algebra in general. Esakia & Grigolia (1975) clarified free cyclic algebras for several other modal logics. Esakia & Grigolia (1977) gave a criterion for algebras to be finitely generated.

Also, there have been several works by using an exact model $M = \langle W, R, P\rangle$ for $\mathbf{F}$. Shehtman (1978) and Bellissima (1985) constructed an exact model for $\mathbf{F}$ by an induction on the height of the model. Chagrov & Zakharyaschev (1997) also treated this construction. By these results, it is not hard to see that every exact model for $\mathbf{F}$ is isomorphic to the exact model they constructed. Also, a detailed description of the set W of worlds was given from the results. Therefore, by Lemma 1.4, their description is also a description of the set of generators of the Boolean $\langle \mathbf{F}^n/\equiv, \leq\rangle$. However, in their works, the set $\{P(A) \mid A \in \mathbf{F}\}$ was not treated directly. In other words, a description of generators does not yield the description of the whole Boolean; thus, there still remains a problem to be solved.

Shehtman (1978) also treated the formula corresponding to a world of the model. The formula was constructed by an induction on the height of the corresponding world. However, considering formulas in $\mathbf{F}^n$, there may be redundant worlds among the worlds with height less than k. For example, in the case that $m = 1$, there exist different worlds α and β with the height 2 such that $\{A \in \mathbf{F}^2 \mid (M, \alpha) \models A\} = \{A \in \mathbf{F}^2 \mid (M, \beta) \models A\}$. Our description in task (II) classifies these worlds.

Ghilardi (1995) used a categorical result and treated the algebras for normal modal logics, especially for the smallest normal modal logic $\mathbf{K}$. He constructed the graded algebra corresponding to the structure $\langle \mathbf{F}^n/\equiv_{\mathbf{K}}, \leq\rangle$ by using graded trees as an interpretation of T-objects. Also, he proved that the algebra corresponding to the structure $\langle \mathbf{F}/\equiv_{\mathbf{K}}, \leq\rangle$ can be expressed by using the colimit of the graded algebra. Considering other normal modal logics, we need to remove consistent normal forms as in Fine (1975) and Moss (2007). Ghilardi (1995) explained how to perform this in an algebraic way, but it is nontrivial to restore the necessary details for **S4**.

We describe the details of the above three tasks.

For (I), in order to construct $\mathbf{ED}^n$, we use the construction in Sasaki (2005). As we mentioned above, Moss (2007) also constructed the set $\mathcal{C}_{n,m}(\mathbf{S4})$ of formulas. Both constructions are based on the construction in Fine (1975) and use provability (or consistency) of **S4**. The main difference between $\mathcal{C}_{n,m}(\mathbf{S4})$ and $\mathbf{ED}^n$ is that $\mathbf{for}(\mathbf{ED}^n)$ consists of not only formulas in $\mathbf{F}^n - \mathbf{F}^{n-1}$ but also formulas in $\mathbf{F}^{n-1}$ whereas $\mathcal{C}_{n,m}(\mathbf{S4})$ consists only of formulas in $\mathbf{F}^n - \mathbf{F}^{n-1}$. This difference makes it easier for us on the following three things:

- to give a finite method to list all exact models for $\mathbf{F}^n$ (in Section 4, we will give 14 exact models for $\mathbf{F}^1$),

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- to treat not only $\mathbf{F}^n/\equiv$ but also $\mathbf{F}/\equiv$,
- to avoid the provability of **S4** from the construction (in Section 3, we will avoid it).

The third one, which also occurs in (I) as the first phrase, is also one of the points we emphasize here. Specifically, we provide a detailed description of mutual provability relation of formulas in **S4**. To do so, we construct another expression, which is equivalent to $\langle \mathbf{F}^n/\equiv, \leq\rangle$. Therefore, it is important that our construction of the expression does not depend on the provability of **S4**. In the next section, we will give a construction using the provability of **S4**. In Section 3, such provability will be replaced with another condition, which does not depend on the provability.

For (II), we give two methods: one is syntactical and the other uses exact models for $\mathbf{F}^n$. The first method depends on the behavior of connectives $\wedge$, $\vee$, $\supset$, $\Box$, and $\bot$ in $\mathbf{ED}^n$. This method will be given in the proofs of (I-1) and (I-2) in the next section.

The second method depends on the exact models. In Section 4, we will give a finite method to list all exact models for $\mathbf{F}^n$, and for each exact model $M = \langle W, R, P\rangle$, we will obtain finite method for (II). We note that the construction of the methods does not depend on the provability of **S4**. Moss (2007) also constructed the Kripke model $\mathbb{C}_{n,m}(\mathbf{S4}) = \langle \mathcal{C}_{n,m}(\mathbf{S4}), R_1, P_1\rangle$, which can be shown to be exact for $\mathbf{F}^n$. He also introduced the Kripke model $\mathbb{M}_{n,m}(\mathbf{S4}) = \langle \mathcal{C}_{n,m}(\mathbf{S4}), R_2, P_2\rangle$ and proved $\mathbb{C}_{n,m}(\mathbf{S4}) = \mathbb{M}_{n,m}(\mathbf{S4})$. However, the relations R_1 and R_2 also depend on the consistency of **S4**. He used the set $\mathcal{C}_{n,m}(\mathbf{S4})$ of his normal forms as the sets of worlds of the models. Fine (1975) also constructed a Kripke model M using normal forms in $\mathbf{F}^n$ as the worlds in M. Here, in order to construct exact models for $\mathbf{F}^n$, we also use some members of $\mathbf{ED}^n$, but not all. We also use members of $\mathbf{ED}^{n+i}$ $(i = 1, 2, \ldots)$ for worlds of exact models for $\mathbf{F}^n$.

Also, by the description to list all exact models for $\mathbf{F}^n$, it is clear which worlds with height k in the exact model for $\mathbf{F}$ are necessary to treat formulas in $\mathbf{F}^n$. For example, two sequents $T2.2.1^*$ and $T2.2.5^*$, which are also worlds of exact model **EM** for **F** with the height 2 (see Figure 3 in Section 4), satisfy $\{A \in \mathbf{F}^2 \mid (\mathbf{EM}, T2.2.1^*) \models A\} = \{A \in \mathbf{F}^2 \mid (\mathbf{EM}, T2.2.5^*) \models A\}$. Therefore, to treat the formulas in $\mathbf{F}^2$, we need only one of the two worlds.

For (III), using a subset of $\bigcup_{i=0}^{\infty} \mathbf{ED}^i$, we construct an exact model for **F**. However, as we mentioned above, we need a detailed description of the set $\{P(A) \mid A \in \mathbf{F}\}$ corresponding to the set $\mathbf{F}/\equiv$. In Section 5, we construct an exact model for **F** with a detailed description of $\{P(A) \mid A \in \mathbf{F}\}$, and using this model, we provide a detailed description of the structure $\langle \mathbf{F}/\equiv, \leq\rangle$.

§2. A construction of $\mathbf{ED}^n$. In the present section, Section 3, and Section 4, we treat the structure $\langle \mathbf{F}^n/\equiv, \leq\rangle$. The purposes of the present section are the following:

- to construct the set $\mathbf{ED}^n$ satisfying the conditions (I-1) and (I-2) in Subsection 1.4,
- to give a syntactical method for (II) in Subsection 1.4 by clarifying the behavior of the connectives in $\mathbf{F}^n/\equiv$.

For the first purpose, we use the provability of **S4**. Therefore, the task (I) in Subsection 1.4 has not yet been complete. However, in the next section, we will avoid the provability and complete (I).

For $\mathbf{ED}^n$, as we mentioned in the previous section, we use the construction in Sasaki (2005) based on the construction of normal forms in Fine (1975). Fine's construction also

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occurred in Ghilardi (1995) and Moss (2007). We introduce our construction comparing it with these works.

We begin with Fine's normal form for **K**. As in Fine (1975), Ghilardi (1995), and Moss (2007), Fine's normal form F can be considered as a world of an exact model M for $\mathbf{F}^n$ such that $(M, F) \models F$. Comparing Fine's normal form with ours, there are three differences.

First, a member of $\mathbf{ED}^n$ corresponds to the negation of their normal form. Our purpose is to construct a conjunctive normal form, whereas Fine constructed a disjunctive one. In other words, for Fine's normal form $A_1 \wedge \cdots \wedge A_k \wedge \neg B_1 \wedge \cdots \wedge \neg B_\ell$ (A_i is either a member of $\mathbf{V}$ or of the form of $\Box C$), we consider the sequent $(A_1, \ldots, A_k \to B_1, \ldots, B_\ell)$ as a member of $\mathbf{ED}^n$. We can also consider a member $X \in \mathbf{ED}^n$ as a world of an exact model M for $\mathbf{F}^n$ such that $(M, X) \not\models \mathbf{for}(X)$; however, in the present paper, we do not construct any exact model in this style. Below, we define the set $\mathbf{Fine}(n)$ of normal forms corresponding to Fine, in our sequent style.

$$\mathbf{Fine}(0) = \{(\mathbf{V} - V_1 \to V_1) \mid V_1 \subseteq \mathbf{V}\},$$
$$\mathbf{Fine}(n+1) = \bigcup_{X \in \mathbf{Fine}(n)} \mathbf{next}^+(X),$$

where

$$\mathbf{next}^+(X) = \{(\Box\Gamma, \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\Delta) \mid \Gamma \cup \Delta = \mathbf{for}(\mathbf{Fine}(n)), \Gamma \cap \Delta = \emptyset\}. \quad (*)$$

A sequent in $\mathbf{Fine}(n+1)$, the next step of $\mathbf{Fine}(n)$, is constructed from a sequent $X \in \mathbf{Fine}(n)$ by adding each member of $\Box\mathbf{for}(\mathbf{Fine}(n))$ to either the antecedent or the succedent of X.

Second, we remove the sequents provable in **S4**. Moss (2007) constructed $\mathcal{C}_{n,m}(\mathbf{S4})$ by removing the normal forms that are consistent in **S4** from Fine's normal form with the depth n. In our case, to construct $\mathbf{ED}^n$, we remove the sequents provable in **S4** from $\mathbf{Fine}(n)$. Therefore, in $(*)$, we can assume that $\mathbf{for}(X) \in \Delta$ because $(\Box\mathbf{for}(X), \mathbf{ant}(X) \to \mathbf{suc}(X)) \in \mathbf{S4}$.

Third, we distinguish the sequents with a maximal antecedent. This is the main difference compared with earlier works. We define two sets $\mathbf{G}(n)$ and $\mathbf{G}^*(n)$ of sequents. $\mathbf{G}^*(n)$ is the set of the sequents with the maximal antecedent. Two sets $\mathbf{G}(n)$ and $\mathbf{G}^*(n)$ can be considered as sets of generators of the Boolean $\langle \mathbf{F}^n/\equiv, \leq\rangle$. A member of $\mathbf{G}(n) (\supseteq \mathbf{G}^*(n))$ can be considered as a generator of $\langle \mathbf{F}^n/\equiv, \leq\rangle$, whereas a member of $\mathbf{G}^*(n)$ can also be considered as a generator of $\langle \mathbf{F}^{n+i}/\equiv, \leq\rangle$ and $\langle \mathbf{F}/\equiv, \leq\rangle$. In Definition 2.2, the relation among the three sets $\mathbf{G}(n)$, $\mathbf{G}^*(n)$, and $\mathbf{ED}^n$ is shown.

We construct $\mathbf{ED}^n$ as follows.

DEFINITION 2.1. *The sets* $\mathbf{G}(n)$ *and* $\mathbf{G}^*(n)$ *of sequents are defined inductively as follows.*

$$\mathbf{G}(0) = \{(\mathbf{V} - V_1 \to V_1) \mid V_1 \subseteq \mathbf{V}\},$$
$$\mathbf{G}^*(0) = \emptyset,$$
$$\mathbf{G}(k+1) = \bigcup_{X \in \mathbf{G}(k) - \mathbf{G}^*(k)} \mathbf{next}(X),$$

$\mathbf{G}^*(k+1) = \{X \in \mathbf{G}(k+1) \mid (\mathbf{ant}(X))^\Box \subseteq (\mathbf{ant}(Y))^\Box$ *implies* $(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box$, *for any* $Y \in \mathbf{G}(k+1)\}$,

where for any $X \in \mathbf{G}(k)$,

$\mathbf{next}^+(X) = \{(\Box\Gamma, \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\Delta) \mid \Gamma \cup \Delta = \mathbf{for}(\mathbf{G}(k)), \Gamma \cap \Delta = \emptyset, \mathbf{for}(X) \in \Delta\}$,

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$\mathbf{prov}(X) = \{Y \in \mathbf{next}^+(X) \mid Y \in \mathbf{S4}\}$,
$\mathbf{next}(X) = \mathbf{next}^+(X) - \mathbf{prov}(X)$.

DEFINITION 2.2. *We define the sets* $\mathbf{ED}^n$ *and* $\mathbf{G}^+(n)$ *as*

$$\mathbf{ED}^n = \mathbf{G}(n) \cup \bigcup_{i=0}^{n-1} \mathbf{G}^*(i) \quad \textit{and} \quad \mathbf{G}^+(n) = \begin{cases} \mathbf{G}(0) & \textit{if } n = 0 \\ \bigcup_{X \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)} \mathbf{next}^+(X) & \textit{if } n > 0. \end{cases}$$

We will use the set $\mathbf{G}^+(n)$ for Lemma 2.9 and the next section.

EXAMPLE 2.3. *We list members of* $\mathbf{G}(n)$ *and* $\mathbf{G}^*(n)$ *in the case that* $m = 1$ *and* $n = 0, 1, 2$. *We use* $(\quad)^*$ *for a sequent in* $\mathbf{G}^*(n)$.

$\mathbf{G}(0) = \{T, F\}$,
$\mathbf{next}^+(T) = \mathbf{next}(T) = \{T1^*, T2\}$, $\mathbf{next}^+(F) = \mathbf{next}(F) = \{F1^*, F2\}$,
$\mathbf{G}(1) = \{T1^*, T2, F1^*, F2\}$, $\mathbf{G}^*(1) = \{T1^*, F1^*\}$,
$\mathbf{next}^+(T2) = \{T2.1^*, T2.2, T2.3, T2.4, T2.5^*, T2.6, T2.7, T2.8\}$,
$\mathbf{next}(T2) = \mathbf{next}^+(T2) - \{T2.7, T2.8\}$,
$\mathbf{G}(2) = \{T2.1^*, T2.2, T2.3, T2.4, T2.5^*, T2.6, F2.1^*, F2.2, F2.3, F2.4, F2.5^*, F2.6\}$,
$\mathbf{G}^*(2) = \{T2.1^*, T2.5^*, F2.1^*, F2.5^*\}$,
$\mathbf{ED}^2 = \mathbf{G}^*(1) \cup \mathbf{G}(2)$,
where

$$\begin{array}{lcrcl}
T & = & (p_1 & \to &), \\
F & = & (& \to & p_1), \\
T1^* & = & (\Box\mathbf{for}(F), \mathbf{ant}(T) & \to & \mathbf{suc}(T), \Box\mathbf{for}(T)), \\
T2 & = & (\mathbf{ant}(T) & \to & \mathbf{suc}(T), \Box\mathbf{for}(\{T, F\})), \\
F1^* & = & (\Box\mathbf{for}(T), \mathbf{ant}(F) & \to & \mathbf{suc}(F), \Box\mathbf{for}(F)), \\
F2 & = & (\mathbf{ant}(F) & \to & \mathbf{suc}(F), \Box\mathbf{for}(\{T, F\})), \\
T2.1^* & = & (\Box\mathbf{for}(\{T1^*, F1^*\}), \mathbf{ant}(T2) & \to & \mathbf{suc}(T2), \Box\mathbf{for}(\{T2, F2\})), \\
T2.2 & = & (\Box\mathbf{for}(T1^*), \mathbf{ant}(T2) & \to & \mathbf{suc}(T2), \Box\mathbf{for}(\{F1^*, T2, F2\})), \\
T2.3 & = & (\Box\mathbf{for}(F1^*), \mathbf{ant}(T2) & \to & \mathbf{suc}(T2), \Box\mathbf{for}(\{T1^*, T2, F2\})), \\
T2.4 & = & (\mathbf{ant}(T2) & \to & \mathbf{suc}(T2), \Box\mathbf{for}(\{T1^*, F1^*, T2, F2\})), \\
T2.5^* & = & (\Box\mathbf{for}(\{T1^*, F2\}), \mathbf{ant}(T2) & \to & \mathbf{suc}(T2), \Box\mathbf{for}(\{F1^*, T2\})), \\
T2.6 & = & (\Box\mathbf{for}(F2), \mathbf{ant}(T2) & \to & \mathbf{suc}(T2), \Box\mathbf{for}(\{T1^*, F1^*, T2\})), \\
T2.7 & = & (\Box\mathbf{for}(\{F1^*, F2\}), \mathbf{ant}(T2) & \to & \mathbf{suc}(T2), \Box\mathbf{for}(\{T1^*, T2\})), \\
T2.8 & = & (\Box\mathbf{for}(\{T1^*, F1^*, F2\}), \mathbf{ant}(T2) & \to & \mathbf{suc}(T2), \Box\mathbf{for}(T2)), \\
F2.1^* & = & (\Box\mathbf{for}(\{T1^*, F1^*\}), \mathbf{ant}(F2) & \to & \mathbf{suc}(F2), \Box\mathbf{for}(\{T2, F2\})), \\
F2.2 & = & (\Box\mathbf{for}(T1^*), \mathbf{ant}(F2) & \to & \mathbf{suc}(F2), \Box\mathbf{for}(\{F1^*, T2, F2\})), \\
F2.3 & = & (\Box\mathbf{for}(F1^*), \mathbf{ant}(F2) & \to & \mathbf{suc}(F2), \Box\mathbf{for}(\{T1^*, T2, F2\})), \\
F2.4 & = & (\mathbf{ant}(F2) & \to & \mathbf{suc}(F2), \Box\mathbf{for}(\{T1^*, F1^*, T2, F2\})), \\
F2.5^* & = & (\Box\mathbf{for}(\{F1^*, T2\}), \mathbf{ant}(F2) & \to & \mathbf{suc}(F2), \Box\mathbf{for}(\{T1^*, F2\})), \\
F2.6 & = & (\Box\mathbf{for}(T2), \mathbf{ant}(F2) & \to & \mathbf{suc}(F2), \Box\mathbf{for}(\{T1^*, F1^*, F2\})).
\end{array}$$

In Figure 1, we also show the relation between X and a member of $\mathbf{next}(X)$. We note that the lines in Figure 1 do not express the relation of any exact model.

By an induction on n, we can show the following lemma.

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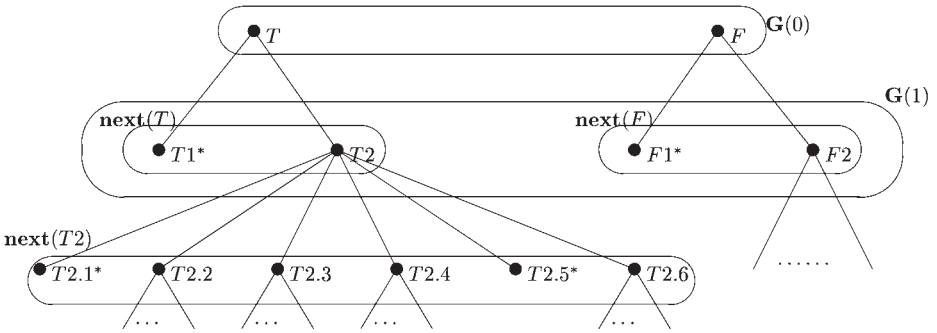


Fig. 1. $\mathbf{G}(0) \cup \mathbf{G}(1) \cup \cdots$

LEMMA 2.4.

(1) *None of the members in* $\mathbf{G}(n)$ *is provable in* **S4**.

(2) *For any* $X, Y \in \mathbf{ED}^n$, $X \neq Y$ *implies* $\mathbf{for}(X) \vee \mathbf{for}(Y) \in \mathbf{S4}$.

The main purpose in the present section is to prove the following two theorems.

THEOREM 2.5.

(1) $\bot \equiv \bigwedge \mathbf{for}(\mathbf{ED}^n)$.

(2) $p_i \equiv \bigwedge \mathbf{for}(\{X \in \mathbf{ED}^n \mid p_i \in \mathbf{suc}(X)\})$.

(3) *For any subsets* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *of* $\mathbf{ED}^n$,

(3.1) $\bigwedge \mathbf{for}(\mathcal{S}_1) \wedge \bigwedge \mathbf{for}(\mathcal{S}_2) \equiv \bigwedge \mathbf{for}(\mathcal{S}_1 \cup \mathcal{S}_2)$,

(3.2) $\bigwedge \mathbf{for}(\mathcal{S}_1) \vee \bigwedge \mathbf{for}(\mathcal{S}_2) \equiv \bigwedge \mathbf{for}(\mathcal{S}_1 \cap \mathcal{S}_2)$,

(3.3) $\bigwedge \mathbf{for}(\mathcal{S}_1) \supset \bigwedge \mathbf{for}(\mathcal{S}_2) \equiv \bigwedge \mathbf{for}((\mathbf{ED}^n - \mathcal{S}_1) \cap \mathcal{S}_2)$.

(4) *For any subset* $\mathcal{S}$ *of* $\mathbf{ED}^k$, $\Box \bigwedge \mathbf{for}(\mathcal{S}) \equiv \bigwedge \mathbf{for}(\mathcal{S}_1 \cup \mathcal{S}_2)$, *where*

$$\mathcal{S}_1 = \bigcup_{X \in \mathcal{S}} \{Y \in \mathbf{ED}^{k+1} \mid \Box\mathbf{for}(X) \in \mathbf{suc}(Y)\},$$

$$\mathcal{S}_2 = \bigcup_{i=1}^{k} \bigcup_{X \in \mathcal{S} \cap \mathbf{G}^*(i)} \{Y \in \mathbf{G}^*(i) \mid (\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Y))^{\Box}\}.$$

We note that $\mathcal{S}_1 \cup \mathcal{S}_2 \subseteq \mathbf{ED}^{k+1}$ for $\mathcal{S}_1$ and $\mathcal{S}_2$ in the above (4).

THEOREM 2.6.

(1) $\mathbf{F}^n / \equiv = \{[\bigwedge \mathbf{for}(\mathcal{S}))] \mid \mathcal{S} \subseteq \mathbf{ED}^n\}$.

(2) *For subsets* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *of* $\mathbf{ED}^n$,

$$\mathcal{S}_1 \subseteq \mathcal{S}_2 \textit{ if and only if } \bigwedge \mathbf{for}(\mathcal{S}_2) \rightarrow \bigwedge \mathbf{for}(\mathcal{S}_1) \in \mathbf{S4}.$$

The conditions (I-1) and (I-2) in Subsection 1.4 can be shown by Theorem 2.6. Theorem 2.5 provides a clarification of the behavior of connectives and a syntactical method for (II) in Subsection 1.4.

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Theorem 2.6(1) and Theorem 2.6(2) can be shown by Theorem 2.5 and Lemma 2.4, respectively. In order to prove Theorem 2.5, especially (4), we need some lemmas.

LEMMA 2.7. *Let* Σ, Γ, *and* Δ *be finite sets of formulas. Then*

$$\Lambda, \Gamma \to \Delta \in \mathbf{S4},$$

where $\Lambda = \{\mathbf{for}(\Box\Phi, \Gamma \to \Delta, \Box\Psi) \mid \Phi \cup \Psi = \Sigma, \Phi \cap \Psi = \emptyset\}$.

Proof. Suppose that the sequent is not provable in **S4**. Then by Lemma 1.2, there exist a Kripke model $M = \langle W, R, P\rangle$ and a world $\alpha \in W$ such that $(M, \alpha) \not\models \mathbf{for}(\Lambda, \Gamma \to \Delta)$. Therefore, every member in Λ is true at α. On the other hand, the formula

$$\mathbf{for}(\Box\{B \in \Sigma \mid (M, \alpha) \models B\}, \Gamma \to \Delta, \Box\{B \in \Sigma \mid (M, \alpha) \not\models B\})$$

belongs to Λ, but it is not true at α. □

COROLLARY 2.8. *For any* $X, Y \in \mathbf{G}(n)$,

(1) $\mathbf{for}(\mathbf{next}(X)) \to \mathbf{for}(X) \in \mathbf{S4}$,

(2) $\bigwedge \mathbf{for}(\mathbf{next}(X)) \equiv \mathbf{for}(X)$,

(3) $\{\mathbf{for}(Z) \mid Z \in \mathbf{next}(X), \Box\mathbf{for}(Y) \in \mathbf{suc}(Z)\}, \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\mathbf{for}(Y) \in \mathbf{S4}$.

Proof. We can show (1) and (3) by considering the case that $(\Sigma, \Gamma, \Delta) = (\mathbf{for}(\mathbf{G}(n)), \mathbf{ant}(X), \mathbf{suc}(X) \cup \{\Box\mathbf{for}(X)\})$ and the case that $(\Sigma, \Gamma, \Delta) = (\mathbf{for}(\mathbf{G}(n)), \mathbf{ant}(X), \mathbf{suc}(X) \cup \{\Box\mathbf{for}(X), \Box\mathbf{for}(Y)\})$ in Lemma 2.7, respectively. (2) is clear from (1). □

LEMMA 2.9. *For any* $X \in \mathbf{G}^+(n)$, $\mathbf{ant}(X) \cup \mathbf{suc}(X) = \mathbf{V} \cup \bigcup_{i=0}^{n-1} \Box\mathbf{for}(\mathbf{G}(i))$ *and* $\mathbf{ant}(X) \cap \mathbf{suc}(X) = \emptyset$.

Proof. By an induction on n. □

LEMMA 2.10. *Let* X *and* Y *be sequents in* $\mathbf{G}(n)$ *satisfying* $(\mathbf{ant}(X))^\Box \not\subseteq (\mathbf{ant}(Y))^\Box$. *Then*

(1) $(\to \mathbf{for}(X), \Box\mathbf{for}(Y)) \in \mathbf{S4}$,

(2) *for any* $X_\oplus \in \mathbf{next}^+(X)$, $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_\oplus)$ *implies* $X_\oplus \in \mathbf{S4}$.

Proof. By $(\mathbf{ant}(X))^\Box \not\subseteq (\mathbf{ant}(Y))^\Box$ and Lemma 2.9, there exists a formula $\Box A \in (\mathbf{ant}(X))^\Box - (\mathbf{ant}(Y))^\Box = (\mathbf{ant}(X))^\Box \cap (\mathbf{suc}(Y))^\Box$. Therefore, the sequent $Z = (\mathbf{ant}(X) \to \mathbf{suc}(X), \Box\mathbf{for}(Y))$ is of the form of

$$\ldots, \Box A \xrightarrow{Z} \Box\mathbf{for}(\cdots \xrightarrow{Y} \Box A, \ldots), \ldots, \tag{3}$$

and provable in **S4**. Hence, we obtain (1). The sequent $X_\oplus$ in (2) is also of the form of (3), and hence, we obtain (2). □

LEMMA 2.11. *Let* X *and* Y *be sequents in* $\mathbf{G}(n)$ *satisfying* $(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box$. *Then*

(1) $X \in \mathbf{G}^*(n)$ *if and only if* $Y \in \mathbf{G}^*(n)$,

(2) $X \in \mathbf{G}^*(n)$ *implies* $\Box\mathbf{for}(Y) \to \mathbf{for}(X) \in \mathbf{S4}$.

Proof. (1) is clear from the definition of $\mathbf{G}^*(n)$. We show (2). If $n = 0$, then the lemma is clear from $\mathbf{G}^*(0) = \emptyset$. We assume that $n > 0$. By $X \in \mathbf{G}^*(n)$ and $Y \in \mathbf{G}(n)$, there exist

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sequents $X_\ominus$ and $Y_\ominus$ in $\mathbf{G}(n-1) - \mathbf{G}^*(n-1)$ such that $X \in \mathbf{next}(X_\ominus)$ and $Y \in \mathbf{next}(Y_\ominus)$. By Corollary 2.8(1), $(\Box \rightarrow)$, and $(\rightarrow \Box)$,

$$\Box\mathbf{for}(\mathbf{next}(Y_\ominus)) \rightarrow \Box\mathbf{for}(Y_\ominus) \in \mathbf{S4}.$$

By $(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box$ and Lemma 2.9, we have $\Box\mathbf{for}(Y_\ominus) \in (\mathbf{suc}(Y))^\Box = (\mathbf{suc}(X))^\Box$. Therefore,

$$\Box\mathbf{for}(\mathbf{next}(Y_\ominus)) \rightarrow \mathbf{for}(X) \in \mathbf{S4},$$

that is,

$$\begin{aligned}\Box\mathbf{for}(\{Z \in \mathbf{next}(Y_\ominus) \mid (\mathbf{ant}(X))^\Box \quad &\subseteq \quad (\mathbf{ant}(Z))^\Box \text{ or } (\mathbf{ant}(X))^\Box \not\subseteq (\mathbf{ant}(Z))^\Box\}) \\ &\rightarrow \quad \mathbf{for}(X) \in \mathbf{S4}.\end{aligned}$$

By $X \in \mathbf{G}^*(n)$, $(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box$ and Lemma 2.9, we have

$$\begin{aligned}\{Z \in \mathbf{next}(Y_\ominus) \mid (\mathbf{ant}(X))^\Box \quad &\subseteq \quad (\mathbf{ant}(Z))^\Box\} = \{Z \in \mathbf{next}(Y_\ominus) \mid (\mathbf{ant}(X))^\Box \\ &= \quad (\mathbf{ant}(Z))^\Box\} = \{Y\},\end{aligned}$$

and therefore,

$$\Box\mathbf{for}(Y), \Box\mathbf{for}(\{Z \in \mathbf{next}(Y_\ominus) \mid (\mathbf{ant}(X))^\Box \not\subseteq (\mathbf{ant}(Z))^\Box\}) \rightarrow \mathbf{for}(X) \in \mathbf{S4}.$$

Hence,

$$\Box\mathbf{for}(Y), \{\mathbf{for}(X) \vee \Box\mathbf{for}(Z) \mid Z \in \mathbf{next}(Y_\ominus), (\mathbf{ant}(X))^\Box \not\subseteq (\mathbf{ant}(Z))^\Box\} \rightarrow \mathbf{for}(X) \in \mathbf{S4}.$$

Using Lemma 2.10, we obtain the lemma. □

We explain the meaning of the above lemma using a Kripke model $\mathbf{EM} = \left\langle \bigcup_{i=0}^{\infty} \mathbf{G}^*(i), R_\mathbf{E}, P_\mathbf{E} \right\rangle$, which will be introduced and shown to be the exact model for $\mathbf{F}$ in Section 4. By Lemma 4.10, $\mathbf{EM}$ has the following property:

- for any $X, Y \in \bigcup_{i=0}^{\infty} \mathbf{G}^*(i)$, $(\mathbf{EM}, X) \not\models \mathbf{for}(Y)$ if and only if $X = Y$.

From the above property, it is not hard to see

- for any $X, Y \in \mathbf{G}^*(n)$, XRY if and only if $\Box\mathbf{for}(Y) \rightarrow \mathbf{for}(X) \in \mathbf{S4}$.

Therefore, by Lemma 2.11(2), for any $X, Y \in \mathbf{G}^*(n)$ satisfying $(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box$, we have XRY and YRX, that is, X and Y belong to the same cluster (the equivalence class with respect to the relation $\{(X', Y') \mid X'RY' \text{ and } Y'RX'\}$).

DEFINITION 2.12. *For any $X \in \mathbf{G}^*(n)$, we define the set* $\mathbf{clus}(X)$ *as*

$$\mathbf{clus}(X) = \{Y \in \mathbf{G}(n) \mid (\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box\}.$$

EXAMPLE 2.13.

(1) $\mathbf{clus}(T2.1^*) = \mathbf{clus}(F2.1^*) = \{T2.1^*, F2.1^*\}$,
(2) $\mathbf{clus}(T2.5^*) = \{T2.5^*\}$.

LEMMA 2.14. *For any $X \in \mathbf{G}(n)$,*

(1) $\Phi(X) \rightarrow \Box\mathbf{for}(X) \in \mathbf{S4}$,

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(2) $X \notin \mathbf{G}^*(n)$ *implies* $(\Phi^\circ(X) \to \Box\mathbf{for}(X)) \in \mathbf{S4}$,

(3) $X \in \mathbf{G}^*(n)$ *implies* $(\mathbf{for}(\mathbf{clus}(X)), \Phi^\circ(X) \to \Box\mathbf{for}(X)) \in \mathbf{S4}$,

where $\Phi(X) = \{\mathbf{for}(Y) \mid Y \in \mathbf{G}(n), (\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(X))^\Box\}$ *and* $\Phi^\circ(X) = \Phi(X) - \mathbf{G}^*(n)$.

Proof. For (1). We use an induction on n.

Basis ($n = 0$) is clear from $\Phi(X) = \mathbf{for}(\mathbf{G}(0))$ and $\mathbf{for}(\mathbf{G}(0)) \to \in \mathbf{S4}$.

Induction step ($n > 0$). By $n > 0$, there exists a sequent $X_\ominus \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$ such that $X \in \mathbf{next}(X_\ominus)$. By the induction hypothesis,

$$\Phi(X_\ominus) \to \Box\mathbf{for}(X_\ominus) \in \mathbf{S4}. \tag{1.1}$$

If $\mathbf{for}(Y_\ominus) \in \Phi(X_\ominus)$, then by $X_\ominus \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$, we have $Y_\ominus \notin \mathbf{G}^*(n-1)$. Hence,

$$\Phi^\circ(X_\ominus) \to \Box\mathbf{for}(X_\ominus) \in \mathbf{S4}. \tag{1.2}$$

Therefore,

$$\Phi^\circ(X_\ominus) \to \Box\mathbf{for}(X) \in \mathbf{S4}. \tag{1.3}$$

In order to prove (1), we show

$$\Sigma, \Phi(X) \to \Box\mathbf{for}(X) \in \mathbf{S4} \tag{1.4}$$

for any subset Σ of $\mathbf{ant}(X)_{n-1}$, where $\mathbf{ant}(X)_{n-1} = \mathbf{ant}(X) \cap \Box\mathbf{for}(\mathbf{G}(n-1))$. In order to show (1.4), we use an induction on $\#(\mathbf{ant}(X)_{n-1} - \Sigma)$. We show Basis ($\Sigma = \mathbf{ant}(X)_{n-1}$) and Induction step ($\Sigma \subsetneq \mathbf{ant}(X)_{n-1}$) simultaneously. We define the set Ψ as

$$\Psi = \mathbf{for}(\{Y \in \mathbf{next}^+(Y_\ominus) \mid Y_\ominus \in \Phi^\circ(X_\ominus), \mathbf{ant}(Y)_{n-1} = \Sigma\}).$$

We note that $\Psi - \{Z \mid Z \in \mathbf{S4}\} \subseteq \Phi(X)$ and for any $\mathbf{for}(Y) \in \Psi$,

$$\mathbf{suc}(Y) \cap \Box\mathbf{for}(\mathbf{G}(n-1)) = (\mathbf{suc}(X) \cap \Box\mathbf{for}(\mathbf{G}(n-1))) \cup (\mathbf{ant}(X)_{n-1} - \Sigma).$$

It is easily seen that

$$A \to \mathbf{for}(X) \in \mathbf{S4} \quad \text{for any } A \in \mathbf{suc}(X) \cap \Box\mathbf{for}(\mathbf{G}(n-1)). \tag{1.5}$$

Also, by the induction hypothesis, we have

$$A, \Sigma, \Phi(X) \to \Box\mathbf{for}(X) \in \mathbf{S4} \quad \text{for any } A \in \mathbf{ant}(X)_{n-1} - \Sigma. \tag{1.6}$$

By (1.3), (1.5), and (1.6), we have

$$\Sigma, \Phi(X), \Psi \to \Box\mathbf{for}(X) \in \mathbf{S4}.$$

Using $\Psi - \{Z \mid Z \in \mathbf{S4}\} \subseteq \Phi(X)$, we have (1.4). Considering the case that $\Sigma = \emptyset$, we obtain (1).

For (2). We can obtain (2) from (1) as in the proof of (1.2) from (1.1).

For (3). By (1),

$$\{\mathbf{for}(Y) \mid Y \in \mathbf{G}^*(n), (\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(X))^\Box\}, \Phi^\circ(X) \to \Box\mathbf{for}(X) \in \mathbf{S4}.$$

Using the definition of $\mathbf{G}^*(n)$,

$$\{\mathbf{for}(Y) \mid Y \in \mathbf{G}^*(n), (\mathbf{ant}(Y))^\Box = (\mathbf{ant}(X))^\Box\}, \Phi^\circ(X) \to \Box\mathbf{for}(X) \in \mathbf{S4}.$$

Using $X \in \mathbf{G}^*(n)$ and Lemma 2.11(1), we obtain (3). □

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LEMMA 2.15. *For any* $X_n \in \mathbf{G}(n)$ *and for any* $k \in \{1, 2, \ldots\}$,

$$\Box\mathbf{for}(X_n) \equiv \begin{cases} \bigwedge \mathbf{for}(\mathbf{clus}(X_n)) \wedge \bigwedge\{\mathbf{for}(Y) \mid Y \in \mathbf{ED}^{n+k}, \\ \quad \Box\mathbf{for}(X_n) \in \mathbf{suc}(Y)\} & \text{if } X_n \in \mathbf{G}^*(n) \\ \bigwedge\{\mathbf{for}(Y) \mid Y \in \mathbf{ED}^{n+k}, \Box\mathbf{for}(X_n) \in \mathbf{suc}(Y)\} & \text{if } X_n \notin \mathbf{G}^*(n). \end{cases}$$

Proof. We only show the case that $k = 1$. The other cases can be shown by an induction on k. By Lemma 2.9, it is sufficient to show the following four conditions:

(1) $\Box\mathbf{for}(X_n) \to \bigwedge\{\mathbf{for}(Y) \mid Y \in \mathbf{G}(n+1), \Box\mathbf{for}(X_n) \in \mathbf{suc}(Y)\} \in \mathbf{S4}$,

(2) $\Box\mathbf{for}(X_n) \to \bigwedge \mathbf{for}(\mathbf{clus}(X_n)) \in \mathbf{S4}$ if $X_n \in \mathbf{G}^*(n)$,

(3) $\{\mathbf{for}(Y) \mid Y \in \mathbf{G}(n+1), \Box\mathbf{for}(X_n) \in \mathbf{suc}(Y)\} \to \Box\mathbf{for}(X_n) \in \mathbf{S4}$ if $X_n \notin \mathbf{G}^*(n)$,

(4) $\mathbf{for}(\mathbf{clus}(X_n)), \{\mathbf{for}(Y) \mid Y \in \mathbf{G}(n+1), \Box\mathbf{for}(X_n) \in \mathbf{suc}(Y)\} \to \Box\mathbf{for}(X_n) \in \mathbf{S4}$ if $X_n \in \mathbf{G}^*(n)$.

(1) is clear. (2) is also clear from Lemma 2.11(2). We show (3) and (4). By Corollary 2.8(3), for any $Z_n \in \mathbf{G}(n) - \mathbf{G}^*(n)$,

$$\{\mathbf{for}(Z_{n+1}) \mid Z_{n+1} \in \mathbf{next}(Z_n), \Box\mathbf{for}(X_n) \in \mathbf{suc}(Z_{n+1})\} \to \mathbf{for}(Z_n), \Box\mathbf{for}(X_n) \in \mathbf{S4}.$$

Therefore,

$$\{\mathbf{for}(Z_{n+1}) \mid Z_{n+1} \in \mathbf{G}(n+1), \Box\mathbf{for}(X_n) \in \mathbf{suc}(Z_{n+1})\} \to \bigwedge \mathbf{for}(\mathbf{G}(n) - \mathbf{G}^*(n)), \Box\mathbf{for}(X_n) \in \mathbf{S4}.$$

Using Lemma 2.14(2) and Lemma 2.14(3), we obtain (3) and (4), respectively. □

By Corollary 2.8(2), we obtain Theorem 2.5(1). By Lemma 2.15 and Lemma 2.11(1), we obtain Theorem 2.5(4). Theorem 2.5(2) and Theorem 2.5(3) can be shown easily.

§3. A construction without the provability of S4. In Definition 2.1, we use the provability of **S4** to define $\mathbf{prov}(X)$ for $X \in \mathbf{G}(n)$. In the present section, we give the set without using the provability of **S4**. First, we define three sets $\mathbf{prov}_1(X)$, $\mathbf{prov}_2(X)$, and $\mathbf{prov}_3(X)$ for $X \in \mathbf{G}(n)$, and prove that $\mathbf{prov}(X) = \mathbf{prov}_1(X) \cup \mathbf{prov}_2(X) \cup \mathbf{prov}_3(X)$. As a result, we can calculate members of $\mathbf{G}(n)$ and $\mathbf{G}^*(n)$ by checking their succedents. At the end of the section, we list some members in $\mathbf{G}(3)$ and $\mathbf{G}^*(3)$ for the case that $m = 1$.

DEFINITION 3.1. *For any* $X \in \mathbf{G}(n)$, *we define* $\mathbf{prov}_1(X)$, $\mathbf{prov}_2(X)$, *and* $\mathbf{prov}_3(X)$ *as follows:*

$$\mathbf{prov}_1(X) = \{(\Gamma \to \Delta, \Box\mathbf{for}(Y)) \in \mathbf{next}^+(X) \mid Y \in \mathbf{G}(n), (\mathbf{ant}(X))^\Box \not\subseteq (\mathbf{ant}(Y))^\Box\},$$

$$\mathbf{prov}_2(X) = \{(\Gamma \to \Delta, \Box\mathbf{for}(Y)) \in \mathbf{next}^+(X) \mid Y \in \mathbf{G}(n), \Box\mathbf{for}(Z_\ominus) \in \mathbf{suc}(Y), \Box\mathbf{for}(\{Z \in \mathbf{next}(Z_\ominus) \mid (\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(Z))^\Box\}) \subseteq \Gamma \text{ for some } Z_\ominus \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)\},$$

$$\mathbf{prov}_3(X) = \{(\Box\mathbf{for}(Y), \Gamma \to \Delta, \Box\mathbf{for}(Z)) \in \mathbf{next}^+(X) \mid Y, Z \in \mathbf{G}^*(n), (\mathbf{ant}(Y))^\Box = (\mathbf{ant}(Z))^\Box\}.$$

We explain the sets $\mathbf{prov}_1(X)$ and $\mathbf{prov}_2(X)$ for $X \in \mathbf{G}(n)$.

If $X_\oplus \in \mathbf{prov}_1(X)$, then as in the proof of Lemma 2.10, $X_\oplus$ is of the form of (3) in the proof of Lemma 2.10, which is provable in **S4**.

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If $X_{\oplus} \in \mathbf{prov}_2(X)$, then $X_{\oplus}$ is of the form of

$$\ldots, \Box\mathbf{for}(\{Z \in \mathbf{next}(Z_{\ominus}) \mid (\mathbf{ant}(Y))^{\Box} \subseteq (\mathbf{ant}(Z))^{\Box}\})$$
$$\stackrel{X_{\oplus}}{\rightarrow} \Box\mathbf{for}(\ldots \stackrel{Y}{\rightarrow} \Box\mathbf{for}(Z_{\ominus}), \ldots), \ldots. \qquad (**)$$

By Lemma 2.10, for any $Z \in \mathbf{next}(Z_{\ominus})$, $(\mathbf{ant}(Y))^{\Box} \not\subseteq (\mathbf{ant}(Z))^{\Box}$ implies $\mathbf{for}(Y) \vee \Box\mathbf{for}(Z) \in \mathbf{S4}$. Therefore, $(**)$ is equivalent to the sequent obtained from $X_{\oplus}$ by replacing $\{Z \in \mathbf{next}(Z_{\ominus}) \mid (\mathbf{ant}(Y))^{\Box} \subseteq (\mathbf{ant}(Z))^{\Box}\}$ with $\mathbf{next}(Z_{\ominus})$, which is of the form of

$$\ldots, \Box\mathbf{for}(\mathbf{next}(Z_{\ominus})) \rightarrow \Box\mathbf{for}(\cdots \stackrel{Y}{\rightarrow} \Box\mathbf{for}(Z_{\ominus}), \ldots), \ldots.$$

By Corollary 2.8(1), the above sequent is provable in **S4**, and hence, $X_{\oplus} \in \mathbf{S4}$. For example, the sequent $T2.7$ in Example 2.3 is of the form of

$$\ldots, \Box\mathbf{for}(\mathbf{next}(F)) \stackrel{T2.7}{\rightarrow} \Box\mathbf{for}(\cdots \stackrel{T2}{\rightarrow} \Box\mathbf{for}(F), \ldots), \ldots,$$

which belongs to $\mathbf{prov}_2(T2)$ and is provable in **S4**.

The main purpose in the present section is to prove the following theorem. By the theorem and Theorem 2.6, (I) in Subsection 1.4 has been done.

THEOREM 3.2. *For any* $X \in \mathbf{G}(n) - \mathbf{G}^*(n)$,

$$\mathbf{prov}(X) = \mathbf{prov}_1(X) \cup \mathbf{prov}_2(X) \cup \mathbf{prov}_3(X).$$

To prove Theorem 3.2, we provide some preparations.

LEMMA 3.3. *For any* $X \in \mathbf{G}(n) - \mathbf{G}^*(n)$,

(1) $\mathbf{prov}_1(X) \subseteq \mathbf{prov}(X)$,
(2) $\mathbf{prov}_2(X) \subseteq \mathbf{prov}(X)$,
(3) $\mathbf{prov}_3(X) \subseteq \mathbf{prov}(X)$.

Proof. As we explained above, we obtain (1) by Lemma 2.10 and we obtain (2) by Corollary 2.8(1) and Lemma 2.10. By Lemma 2.11(2), we obtain (3). □

LEMMA 3.4. *Let* X_n *and* Y_n *be sequents in* $\mathbf{G}(n) - \mathbf{G}^*(n)$ *and let* X_{n+1} *be a sequent in* $\mathbf{next}^+(X_n) - (\mathbf{prov}_1(X_n) \cup \mathbf{prov}_2(X_n) \cup \mathbf{prov}_3(X_n))$. *If* $\Box\mathbf{for}(Y_n) \in \mathbf{suc}(X_{n+1})$, *then*

$$Y_{n+1} \in \mathbf{next}^+(Y_n) - (\mathbf{prov}_1(Y_n) \cup \mathbf{prov}_2(Y_n) \cup \mathbf{prov}_3(Y_n)),$$

where

$Y_{n+1} = (\Gamma_Y, \mathbf{ant}(X_{n+1}) \cap \Box\mathbf{for}(\mathbf{G}(n)), \mathbf{ant}(Y_n) \rightarrow \mathbf{suc}(Y_n), \Delta_Y)$,
$\Delta_Y = \{\Box\mathbf{for}(Z_n) \in \mathbf{suc}(X_{n+1}) \cap \Box\mathbf{for}(\mathbf{G}(n)) \mid (\mathbf{ant}(Y_n))^{\Box} \subseteq (\mathbf{ant}(Z_n))^{\Box}\}$,
$\Gamma_Y = \{\Box\mathbf{for}(Z_n) \in \mathbf{suc}(X_{n+1}) \cap \Box\mathbf{for}(\mathbf{G}(n)) \mid (\mathbf{ant}(Y_n))^{\Box} \not\subseteq (\mathbf{ant}(Z_n))^{\Box}\}$.

Proof. It is not hard to see that $Y_{n+1} \in \mathbf{next}^+(Y_n) - (\mathbf{prov}_1(Y_n) \cup \mathbf{prov}_3(Y_n))$. We show $Y_{n+1} \notin \mathbf{prov}_2(Y_n)$. Suppose that $Y_{n+1} \in \mathbf{prov}_2(Y_n)$. Then there exist sequents $Y'_n \in \mathbf{G}(n)$ and $Z_{n-1} \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$ such that Y_{n+1} is of the form of

$$\ldots, \Box\mathbf{for}(\{Z_n \in \mathbf{next}(Z_{n-1}) \mid (\mathbf{ant}(Y'_n))^{\Box} \subseteq (\mathbf{ant}(Z_n))^{\Box}\})$$
$$\stackrel{Y_{n+1}}{\rightarrow} \Box\mathbf{for}(\cdots \stackrel{Y'_n}{\rightarrow} \Box\mathbf{for}(Z_{n-1}), \ldots), \ldots. \qquad (1.1)$$

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Therefore, $\Box\mathbf{for}(Y'_n) \in \Delta_Y \subseteq \mathbf{suc}(X_{n+1})$. Also, by $\Box\mathbf{for}(Y'_n) \in \Delta_Y$, we have $(\mathbf{ant}(Y_n))^{\Box} \subseteq (\mathbf{ant}(Y'_n))^{\Box}$; and thus,

$$\Box\mathbf{for}(\{Z_n \in \mathbf{next}(Z_{n-1}) \mid (\mathbf{ant}(Y'_n))^{\Box} \subseteq (\mathbf{ant}(Z_n))^{\Box}\}) \cap \Gamma_Y = \emptyset.$$

Therefore, X_{n+1} is also of the form of (1.1); in other words, $X_{n+1} \in \mathbf{prov}_2(X_n)$, which is in contradiction with Lemma 2.4 and Lemma 3.3(2). □

By Example 2.3, we have $T2 \in \mathbf{next}(T)$ and $T2.3 \in \mathbf{next}(T2)$ (see also Figure 1). Therefore, there is a relation between T and $T2.3$. In order to express this relation, we introduce some notions. By Lemma 2.9, every sequent in $\mathbf{G}^+(n)$ consists of the members in the set $\mathbf{V} \cup \bigcup_{i=0}^{n-1} \Box\mathbf{for}(\mathbf{G}(i))$. We call this set the *base of the sequents in* $\mathbf{G}^+(n)$ and use it to define one of the notions.

DEFINITION 3.5.

(1) *We define the set* $\mathbf{BG}(n)$ *as*

$$\mathbf{BG}(n) = \mathbf{V} \cup \bigcup_{i=0}^{n-1} \Box\mathbf{for}(\mathbf{G}(i)).$$

(2) *For any* $X \in \mathbf{G}^+(n)$ *and for any* k*, we define* $X(k)$ *as*

$$X(k) = (\mathbf{ant}(X) \cap \mathbf{BG}(k) \to \mathbf{suc}(X) \cap \mathbf{BG}(k)).$$

Here, we note that $X(k) = X$ if $k \geq n$.

DEFINITION 3.6. *For any* $X \in \mathbf{G}(n)$*, we define the sets* $X\Downarrow$ *inductively as follows:*

(1) $X \in X\Downarrow$,

(2) *if* $Y \in \mathbf{next}(Z)$ *for some* $Z \in X\Downarrow - \bigcup_{i=1}^{\infty} \mathbf{G}^*(i)$*, then* $Y \in X\Downarrow$.

EXAMPLE 3.7.

(1) $T2.3(3) = T2.3, \quad T2.3(2) = T2, \quad T2.3(1) = T$,
(2) $T\Downarrow = \{T1^*\} \cup T2\Downarrow$,
(3) $T2\Downarrow = \{T2.1^*, T2.5^*\} \cup T2.2\Downarrow \cup T2.3\Downarrow \cup T2.4\Downarrow \cup T2.6\Downarrow$,
(4) $T2.3 \in T2.3\Downarrow \subseteq T2\Downarrow \subseteq T\Downarrow$.

In Figure 2, we can also see the above example.

LEMMA 3.8. *Let* X *and* Y *be sequents in* $\mathbf{G}^+(n)$ *and* $\mathbf{G}(k)$*, respectively. Then*

(1) $n \neq 0$ *implies* $X(n-1) \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$ *and* $X \in \mathbf{next}^+(X(n-1))$,
(2) $n > k$ *implies* $X(k) \in \mathbf{G}(k) - \mathbf{G}^*(k)$, $X \in X(k)\Downarrow \cup \mathbf{prov}(X(n-1))$ *and* $\Box\mathbf{for}(X(k)) \in \mathbf{suc}(X)$,
(3) *if* $X \in \mathbf{G}(n)$*, then the following three conditions are equivalent:*
   (3.1) $\mathbf{ant}(Y) \subseteq \mathbf{ant}(X)$ *and* $\mathbf{suc}(Y) \subseteq \mathbf{suc}(X)$,
   (3.2) $n \geq k$ *and* $Y = X(k)$,
   (3.3) $X \in Y\Downarrow$.

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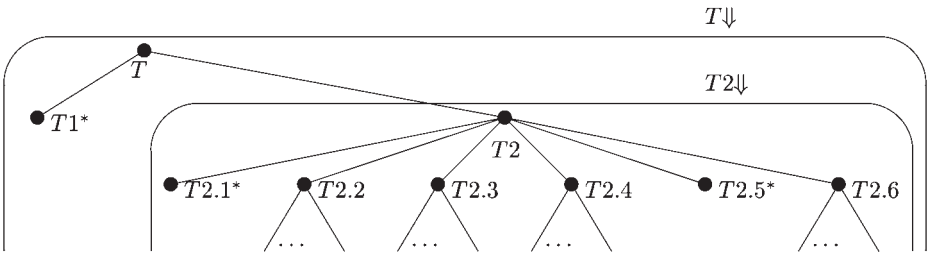


Fig. 2. $T\Downarrow$ and $T2\Downarrow$.

Proof. (1) is clear. We obtain (2) by an induction on n. By (2), we have that (3.2) implies (3.3). By an induction on $X \in Y\Downarrow$, we can show that (3.3) implies (3.1). By Lemma 2.9, (2), and (3.1), we have $n \geq k$, $X(k) \in \mathbf{G}(k)$, and $\mathbf{ant}(Y) = \mathbf{ant}(X(k))$; and hence, we obtain that (3.1) implies (3.2). □

LEMMA 3.9. *Let X_n and Y_k be sequents in $\mathbf{G}(n)$ and $\mathbf{G}^*(k)$, respectively. If $n \geq k$ and $(\mathbf{ant}(Y_k))^{\Box} \neq (\mathbf{ant}(X_n(k)))^{\Box}$, then $(\Box\mathbf{for}(X_n) \supset \mathbf{for}(Y_k)) \equiv \mathbf{for}(Y_k)$.*

Proof. We show

$$(\mathbf{ant}(Y_k))^{\Box} \neq (\mathbf{ant}(X_n(k)))^{\Box} \text{ implies } (\rightarrow \mathbf{for}(Y_k), \Box\mathbf{for}(X_n)) \in \mathbf{S4}$$

by an induction on n. Basis ($n = k$) can be shown by Lemma 2.10. Induction step ($n > k$) can be shown by Lemma 3.8(1) and $\Box\mathbf{for}(X_n(n-1)) \rightarrow \Box\mathbf{for}(X) \in \mathbf{S4}$. □

LEMMA 3.10. *Let X_n be a sequent in $\mathbf{G}(n) - \mathbf{G}^*(n)$. Let X_{n+1} and Y_k be sequents in $\mathbf{next}^+(X_n) - (\mathbf{prov}_1(X_n) \cup \mathbf{prov}_2(X_n) \cup \mathbf{prov}_3(X_n))$ and $\mathbf{G}(k)$, respectively. If $n \geq k$ and $\Box\mathbf{for}(Y_k) \in \mathbf{suc}(X_{n+1})$, then*

(1) *$Y_k \in \mathbf{G}^*(k)$ implies $(\Box\mathbf{for}(Z) \supset \mathbf{for}(Y_k)) \equiv \mathbf{for}(Y_k)$, for any $Z \in \{Z' \mid \Box\mathbf{for}(Z') \in \mathbf{ant}(X_{n+1})\}$,*

(2) *there exists a sequent $Y \in \mathbf{ED}^n$ such that $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_{n+1})$ and $Y \in Y_k\Downarrow$.*

Proof. For (1). First, we note that $X_n \notin \mathbf{S4}$ by Lemma 2.4(1). Suppose that $\Box\mathbf{for}(Z) \in (\mathbf{ant}(X_{n+1}))^{\Box} \cap \Box\mathbf{for}(\mathbf{G}(i))$ for some $i \leq n$. We divide the cases.

The case that $i \geq k$. By Lemma 3.9, we can assume that $(\mathbf{ant}(Y_k))^{\Box} = (\mathbf{ant}(Z(k)))^{\Box}$. Using $Y_k \in \mathbf{G}^*(k)$ and Lemma 2.11(1), we have $Z(k) \in \mathbf{G}^*(k)$. Using Lemma 3.8(2), we have $i = k$ and $Z(k) = Z$. Therefore, $k = n$ is in contradiction with $X_{n+1} \notin \mathbf{prov}_3(X_n)$; and $k < n$ is in contradiction with Lemma 2.11(2) and $X_n \notin \mathbf{S4}$.

The case that $i < k$. If $\Box\mathbf{for}(Z) \in (\mathbf{ant}(Y_k))^{\Box}$, then clearly, $(\Box\mathbf{for}(Z) \supset \mathbf{for}(Y_k)) \equiv \mathbf{for}(Y_k)$. We assume that $\Box\mathbf{for}(Z) \notin (\mathbf{ant}(Y_k))^{\Box}$. Using $\Box\mathbf{for}(Z) \in (\mathbf{ant}(X_{n+1}))^{\Box} \cap \Box\mathbf{for}(\mathbf{G}(i)) \subseteq (\mathbf{ant}(X_{n+1}(k)))^{\Box}$, we have $(\mathbf{ant}(X_{n+1}(k)))^{\Box} \not\subseteq (\mathbf{ant}(Y_k))^{\Box}$. Therefore, $k = n$ is in contradiction with $X_{n+1} \notin \mathbf{prov}_1(X_n)$; and $k < n$ is in contradiction with Lemma 2.10 and $X_n \notin \mathbf{S4}$.

For (2). We use an induction on n. Basis ($n = k$) is clear since Y_k satisfies the conditions.

Induction step ($n > k$). By $n > k$, Lemma 3.8, and Lemma 3.3, we have $X_n \in \mathbf{next}(X_n(n-1)) \subseteq \mathbf{next}^+(X_n(n-1)) - (\mathbf{prov}_1(X_n(n-1)) \cup \mathbf{prov}_2(X_n(n-1)) \cup \mathbf{prov}_3(X_n(n-1)))$ and $\Box\mathbf{for}(Y_k) \in \mathbf{suc}(X_n)$. Using the induction hypothesis, there exists a sequent $Y' \in \mathbf{ED}^{n-1}$ such that $\Box\mathbf{for}(Y') \in \mathbf{suc}(X_{n+1}(n))$ and $Y' \in Y_k\Downarrow$.

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If $Y' \in \bigcup_{i=0}^{n-1} \mathbf{G}^*(i)$, then Y' satisfies the conditions, and hence, we obtain (2). We assume that $Y' \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$. Then X_{n+1} is of the form of

$$\cdots \overset{X_{n+1}}{\rightarrow} \Box\mathbf{for}(\cdots \overset{X_{n+1}(n)}{\rightarrow} \Box\mathbf{for}(Y'), \ldots), \ldots.$$

By $X_{n+1} \notin \mathbf{prov}_2(X_n)$,

$$\Box\mathbf{for}(\{Z_n \in \mathbf{next}(Y') \mid (\mathbf{ant}(X_{n+1}(n)))^{\Box} \subseteq (\mathbf{ant}(Z_n))^{\Box}\}) \not\subseteq \mathbf{ant}(X_{n+1}).$$

Therefore, there exists a sequent $Y_n \in \mathbf{next}(Y')$ such that $\Box\mathbf{for}(Y_n) \notin \mathbf{ant}(X_{n+1})$. Using Lemma 2.9, Y_n satisfies the conditions, and hence we obtain (2). □

Below, we define the *saturation* $\mathbf{sat}(X)$ of a sequent $X \in \mathbf{G}^+(n)$, which satisfies the relation:

$$X \notin \mathbf{S4} \text{ implies } \mathbf{sat}(X) \notin \mathbf{S4}.$$

If we confirm

- $(A, B, \Gamma \rightarrow \Delta) \notin \mathbf{S4}$ implies $(A \wedge B, A, B, \Gamma \rightarrow \Delta) \notin \mathbf{S4}$,
- $(A, \Gamma \rightarrow \Delta) \notin \mathbf{S4}$ implies $(A \vee C, A, \Gamma \rightarrow \Delta) \notin \mathbf{S4}$,

and so on, then we can observe the above relation.

DEFINITION 3.11. *The saturation* $\mathbf{sat}(X)$ *of a sequent* $X \in \mathbf{G}^+(n)$ *is defined as follows:*

$$\mathbf{sat}(X) = \begin{cases} X & \text{if } n = 0 \\ \Gamma_d, \Gamma_c, \mathbf{ant}(X), \{A \mid \Box A \in \mathbf{ant}(X)\} \rightarrow \mathbf{suc}(X), \Delta_c, \Delta_d, \Delta_f & \text{if } n > 0, \end{cases}$$

where

$\Gamma_c = \{\bigwedge \Sigma \mid \Sigma \subseteq \mathbf{ant}(X), \Sigma \subseteq \mathbf{BG}(n-1), \#(\Sigma) > 1\}$,
$\Gamma_d = \{\bigvee \Sigma \mid \Sigma \cap \mathbf{ant}(X) \neq \emptyset, \Sigma \subseteq \mathbf{BG}(n-1), \#(\Sigma) > 1\}$,
$\Delta_c = \{\bigwedge \Sigma \mid \Sigma \cap \mathbf{suc}(X) \neq \emptyset, \Sigma \subseteq \mathbf{BG}(n-1), \#(\Sigma) > 1\}$,
$\Delta_d = \{\bigvee \Sigma \mid \Sigma \subseteq \mathbf{suc}(X), \Sigma \subseteq \mathbf{BG}(n-1), \#(\Sigma) > 1\}$,
$\Delta_f = \{\mathbf{for}(X(\ell)) \mid \ell \leq n-1, \mathbf{ant}(X(\ell)) \neq \emptyset\}$.

We note that $\mathbf{ant}(X) \subseteq \mathbf{ant}(\mathbf{sat}(X))$ and $\mathbf{suc}(X) \subseteq \mathbf{suc}(\mathbf{sat}(X))$.

LEMMA 3.12. *Let* $\mathcal{P}$ *be a cut-free proof figure in* **S4** *whose end sequent is* $\Phi \rightarrow \Psi$. *Then for any* $X_n \in \mathbf{G}(n) - \mathbf{G}^*(n)$ *and for any* $X_{n+1} \in \mathbf{next}^+(X_n) - (\mathbf{prov}_1(X_n) \cup \mathbf{prov}_2(X_n) \cup \mathbf{prov}_3(X_n))$,

$$(\Phi \rightarrow \Psi) \notin \{(\Phi^* \rightarrow \Psi^*) \mid \Phi^* \subseteq \mathbf{ant}(\mathbf{sat}(X_{n+1})), \Psi^* \subseteq \mathbf{suc}(\mathbf{sat}(X_{n+1}))\}.$$

Proof. We use an induction on $\mathcal{P}$.

Basis ($\mathcal{P}$ consists of an axiom). By Lemma 2.9, we can show $\bot \notin \mathbf{ant}(\mathbf{sat}(X_{n+1}))$ and $\mathbf{ant}(\mathbf{sat}(X_{n+1})) \cap \mathbf{suc}(\mathbf{sat}(X_{n+1})) = \emptyset$, and hence, none of the axioms of **S4** belongs to the set.

Induction step ($\mathcal{P}$ has the inference rule I introducing the end sequent $\Phi \rightarrow \Psi$). Suppose that

$$(\Phi \rightarrow \Psi) \in \{(\Phi^* \rightarrow \Psi^*) \mid \Phi^* \subseteq \mathbf{ant}(\mathbf{sat}(X_{n+1})), \Psi^* \subseteq \mathbf{suc}(\mathbf{sat}(X_{n+1}))\}.$$

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We only show the case that I is $(\to \Box)$. Then there exists a sequent $Y_k \in \mathbf{G}(k)$ $(k \leq n)$ such that

$$I \text{ is } \frac{\Phi \to \mathbf{for}(Y_k)}{\Phi \to \Box\mathbf{for}(Y_k)}, \tag{1}$$

$$\{\Box\mathbf{for}(Y_k)\} = \Psi \subseteq (\mathbf{suc}(X_{n+1}))^{\Box}. \tag{2}$$

Also, by the form of $(\to \Box)$,

$$\Phi \subseteq (\mathbf{ant}(X_{n+1}))^{\Box}. \tag{3}$$

By (2) and Lemma 3.10(2), there exists a sequent $Y \in \mathbf{ED}^n$ such that

$$\Box\mathbf{for}(Y) \in (\mathbf{suc}(X_{n+1}))^{\Box}, \tag{4}$$

$$Y \in Y_k\Downarrow . \tag{5}$$

We divide the cases.

The case that $Y \in \mathbf{G}^*(i)$ for some $i \leq n$. By (1) and (5), we have $\Phi \to \mathbf{for}(Y) \in \mathbf{S4}$. Also, by (3), (4), and Lemma 3.10(1), we have $\mathbf{for}(\Phi \to \mathbf{for}(Y)) \equiv \mathbf{for}(Y)$. Therefore, we have $Y \in \mathbf{S4}$, which is in contradiction with $Y \in \mathbf{ED}^n$ and Lemma 2.4(1).

The case that $Y \in \mathbf{G}(n) - \mathbf{G}^*(n)$. We use Y_{n+1} as in Lemma 3.4. By (4) and Lemma 3.4, we have

$$Y_{n+1} \in \mathbf{next}^+(Y) - (\mathbf{prov}_1(Y) \cup \mathbf{prov}_2(Y) \cup \mathbf{prov}_3(Y)).$$

By (5) and Lemma 3.8, we have

$$\mathbf{for}(Y_k) = \mathbf{for}(Y(k)) = \mathbf{for}(Y_{n+1}(k)) \in \mathbf{suc}(\mathbf{sat}(Y_{n+1})). \tag{6}$$

On the other hand, by (4) and $X_{n+1} \notin \mathbf{prov}_1(X_n)$, we have $(\mathbf{ant}(X_n))^{\Box} \subseteq (\mathbf{ant}(Y))^{\Box}$. Using (3), we have

$$\Phi \subseteq (\mathbf{ant}(X_{n+1}))^{\Box} \subseteq (\mathbf{ant}(X_n))^{\Box} \cup ((\mathbf{ant}(X_{n+1}))^{\Box} \cap \Box\mathbf{for}(\mathbf{G}(n)))$$

$$\subseteq (\mathbf{ant}(Y))^{\Box} \cup ((\mathbf{ant}(X_{n+1}))^{\Box} \cap \Box\mathbf{for}(\mathbf{G}(n))) \subseteq (\mathbf{ant}(Y_{n+1}))^{\Box} \subseteq \mathbf{ant}(\mathbf{sat}(Y_{n+1})).$$

Using (6), the upper sequent of I belongs to

$$\{(\Phi^* \to \Psi^*) \mid \Phi^* \subseteq \mathbf{ant}(\mathbf{sat}(Y_{n+1})), \Psi^* \subseteq \mathbf{suc}(\mathbf{sat}(Y_{n+1}))\}$$

for $Y_{n+1} \in \mathbf{next}^+(Y_n) - (\mathbf{prov}_1(Y_n) \cup \mathbf{prov}_2(Y_n) \cup \mathbf{prov}_3(Y_n))$. This is in contradiction with the induction hypothesis. □

By the above lemma and Lemma 1.1, we obtain

COROLLARY 3.13. *Let X be a sequent in $\mathbf{G}(n) - \mathbf{G}^*(n)$. Then*

$$\mathbf{prov}_1(X) \cup \mathbf{prov}_2(X) \cup \mathbf{prov}_3(X) \supseteq \mathbf{prov}(X).$$

From Lemma 3.3 and Corollary 3.13, we obtain Theorem 3.2.

Theorem 3.2 and Theorem 2.6 make it possible to calculate the members of $\mathbf{G}(n)$ and $\mathbf{G}^*(n)$ by checking their succedent. Below, we list some members in $\mathbf{G}(3)$ and $\mathbf{G}^*(3)$ in the case that $m = 1$. As in Example 2.3, we use $(\quad)^*$ for a sequent in $\mathbf{G}^*(n)$. Also, for brevity's sake, we define $\mathbf{n}(X, \mathcal{S})$ for $\mathcal{S} \in 2^{\mathbf{G}(n)}$ as follows.

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DEFINITION 3.14.

$$\mathbf{n}(X, \mathcal{S}) = (\Box\mathbf{for}(\mathbf{G}(n) - (\{X\} \cup \mathcal{S})), \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\mathbf{for}(\{X\} \cup \mathcal{S})).$$

We note that $\mathbf{n}(X, \mathcal{S}) \in \mathbf{next}^{+}(X)$ if $X \in \mathbf{G}(n)$.

EXAMPLE 3.15. *In Table 1, we list the members of* $\mathbf{G}(3) - (\mathbf{next}(T2.4) \cup \mathbf{next}(F2.4))$ *and eight members of* $(\mathbf{next}(T2.4) \cup \mathbf{next}(F2.4))$ *containing the six members in* $(\mathbf{next}(T2.4) \cup \mathbf{next}(F2.4)) \cap \mathbf{G}^{*}(3)$. *For brevity's sake, we refer to* **cl** *as two members* $T2.1^{*}$ *and* $F2.1^{*}$. *Also,* $\#(\mathbf{next}(T2.4)) = \#(\mathbf{next}(F2.4)) = 708$.

In the next section, we also use the following examples to list exact models for $\mathbf{F}^{1}$.

EXAMPLE 3.16. *Two sequents* $T2.4.4.1^{*}$ *and* $F2.4.4.1^{*}$ *defined as*

$$T2.4.4.1^{*} = \mathbf{n}(T2.4.4, \{F2.6.1^{*}\}) \quad \textit{and} \quad F2.4.4.1^{*} = \mathbf{n}(F2.4.4, \{T2.6.1^{*}\})$$

belong to $\mathbf{G}^{*}(4)$ *in the case that* $m = 1$.

Table 1. *List of members in* $\mathbf{G}(3)$

in **next**$(T2.2)$					in **next**$(F2.2)$				
$\mathbf{n}(T2.2, \mathcal{S})$	members in $\mathcal{S}$				$\mathbf{n}(F2.2, \mathcal{S})$	members in $\mathcal{S}$			
$T2.2.1^{*}$	**cl**				$F2.2.1^{*}$	**cl**			
$T2.2.2$	**cl**		$T2.5^{*}$		$F2.2.2$	**cl**		$T2.5^{*}$	
$T2.2.3$	**cl**	$F2.2$			$F2.2.3$	**cl**	$T2.2$		
$T2.2.4$	**cl**	$F2.2$	$T2.5^{*}$		$F2.2.4$	**cl**	$T2.2$	$T2.5^{*}$	
$T2.2.5^{*}$		$F2.2$			$F2.2.5^{*}$		$T2.2$		
$T2.2.6$		$F2.2$	$T2.5^{*}$		$F2.2.6$		$T2.2$	$T2.5^{*}$	
					$F2.2.7^{*}$			$T2.5^{*}$	
in **next**$(T2.3)$					in **next**$(F2.3)$				
$\mathbf{n}(T2.3, \mathcal{S})$	members in $\mathcal{S}$				$\mathbf{n}(F2.3, \mathcal{S})$	members in $\mathcal{S}$			
$T2.3.1^{*}$	**cl**				$F2.3.1^{*}$	**cl**			
$T2.3.2$	**cl**		$F2.5^{*}$		$F2.3.2$	**cl**		$F2.5^{*}$	
$T2.3.3$	**cl**	$F2.3$			$F2.3.3$	**cl**	$T2.3$		
$T2.3.4$	**cl**	$F2.3$	$F2.5^{*}$		$F2.3.4$	**cl**	$T2.3$	$F2.5^{*}$	
$T2.3.5^{*}$		$F2.3$			$F2.3.5^{*}$		$T2.3$		
$T2.3.6$		$F2.3$	$F2.5^{*}$		$F2.3.6$		$T2.3$	$F2.5^{*}$	
$T2.3.7^{*}$			$F2.5^{*}$						
in **next**$(T2.6)$					in **next**$(F2.6)$				
$\mathbf{n}(T2.6, \mathcal{S})$	members in $\mathcal{S}$				$\mathbf{n}(F2.6, \mathcal{S})$	members in $\mathcal{S}$			
$T2.6.1^{*}$					$F2.6.1^{*}$				
$T2.6.2$			$T2.5^{*}$		$F2.6.2$			$F2.5^{*}$	
in **next**$(T2.4)$					in **next**$(F2.4)$				
$\mathbf{n}(T2.4, \mathcal{S})$	members in $\mathcal{S}$				$\mathbf{n}(F2.4, \mathcal{S})$	members in $\mathcal{S}$			
$T2.4.1^{*}$	**cl**				$F2.4.1^{*}$	**cl**			
$T2.4.2^{*}$		$F2.4$			$F2.4.2^{*}$		$T2.4$		
$T2.4.3^{*}$			$F2.5^{*}$		$F2.4.3^{*}$			$T2.5^{*}$	
$T2.4.4$				$T2.6$	$F2.4.4$				$T2.6$
$\vdots$					$\vdots$				

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§4. Exact models for $\mathbf{F}^n$. In the present section, we give a finite method to list all exact models for $\mathbf{F}^n$; as a result, we obtain methods for (II) in Subsection 1.4. For example, in the case that $m = 1$, there are just 4 exact models for $\mathbf{F}^0$ and just 14 exact models for $\mathbf{F}^1$.

First, we introduce the Kripke model **EM**, an exact set $\mathcal{E}$ for $\mathbf{F}^n$, and the Kripke model $\mathbf{EM}_{\mathcal{E}}$. The Kripke model **EM** will be shown to be exact for **F** in the next section. Here, we use **EM** to investigate exact models for $\mathbf{F}^n$. For an exact set $\mathcal{E}$ for $\mathbf{F}^n$, the Kripke model $\mathbf{EM}_{\mathcal{E}}$ is shown to be exact for $\mathbf{F}^n$.

DEFINITION 4.1. *The Kripke model* **EM** *is defined as*

$$\mathbf{EM} = \langle W_{\mathbf{E}}, R_{\mathbf{E}}, P_{\mathbf{E}} \rangle,$$

where $W_{\mathbf{E}} = \bigcup_{n=0}^{\infty} \mathbf{G}^*(n)$, $R_{\mathbf{E}} = \{(X, Y) \mid \Box\mathbf{for}(Y) \in \mathbf{suc}(X)$ *or* $X \in \mathbf{clus}(Y)\}$, *and* $P_{\mathbf{E}}(p_i) = \{X \mid p_i \in \mathbf{ant}(X)\}$.

DEFINITION 4.2.

(1) *A set* $\mathcal{E}$ *is said to be exact for* $\mathbf{F}^n$ *if the following three conditions hold:*

(1.1) $\bigcup_{i=0}^{n} \mathbf{G}^*(i) \subseteq \mathcal{E} \subseteq W_{\mathbf{E}}$,

(1.2) *for any* $X \in \mathbf{ED}^n$, $\#(X\Downarrow \cap \mathcal{E}) = 1$,

(1.3) *for any* $X \in \mathcal{E}$ *and for any* $Y \in W_{\mathbf{E}}$, $X R_{\mathbf{E}} Y$ *implies* $Y \in \mathcal{E}$.

(2) *For an exact set* $\mathcal{E}$ *for* $\mathbf{F}^n$, *the Kripke model* $\mathbf{EM}_{\mathcal{E}}$ *is defined as*

$$\mathbf{EM}_{\mathcal{E}} = \langle \mathcal{E}, R_{\mathcal{E}}, P_{\mathcal{E}} \rangle,$$

where $R_{\mathcal{E}} = R_{\mathbf{E}} \cap \mathcal{E}^2$ *and* $P_{\mathcal{E}}(p_i) = P_{\mathbf{E}}(p_i) \cap \mathcal{E}$.

EXAMPLE 4.3. *We consider the case that* $m = 1$. *We show the relation* $R_{\mathbf{E}}$ *on the set* $\mathbf{G}^*(0) \cup \mathbf{G}^*(1) \cup \mathbf{G}^*(2) \cup \mathbf{G}^*(3) \cup \{T2.4.4.1^*, F2.4.4.1^*\}$ *in Figure 3. In the figure, we write arrows from X to Y if* $X R_{\mathbf{E}} Y$ *and* $X \neq Y$, *but we often omit the arrows from X to Y if* $X R_{\mathbf{E}} Z$ *and* $Z R_{\mathbf{E}} Y$ *for some* $Z \in W_{\mathbf{E}} - \{X, Y\}$.

By Figure 3, we obtain 4 exact sets for $\mathbf{F}^0$ *and 14 exact sets for* $\mathbf{F}^1$. *We list them in Table 2. We can also see that there are only 4 exact sets for* $\mathbf{F}^0$ *and only 14 exact sets for* $\mathbf{F}^1$. *The exact model for* $\mathbf{F}^1$ *from the fourth exact set* $\{T1^*, F1^*, T2.4.2^*, F2.4.2^*\}$ *for* $\mathbf{F}^1$ *in Table 2 is the example given by Moss (2007).*

The main purpose in the present section is to prove the following theorem.

THEOREM 4.4.

(1) *For any exact set* $\mathcal{E}$ *for* $\mathbf{F}^n$, $\mathbf{EM}_{\mathcal{E}}$ *is an exact model for* $\mathbf{F}^n$.

(2) *For any exact model M for* $\mathbf{F}^n$, *there exists an exact set* $\mathcal{E}$ *for* $\mathbf{F}^n$ *such that M is isomorphic to* $\mathbf{EM}_{\mathcal{E}}$.

(3) *Every exact set for* $\mathbf{F}^n$ *is a subset of* $\bigcup_{i=0}^{n+2\#(\mathbf{ED}^n - W_{\mathbf{E}})} \mathbf{G}^*(i)$.

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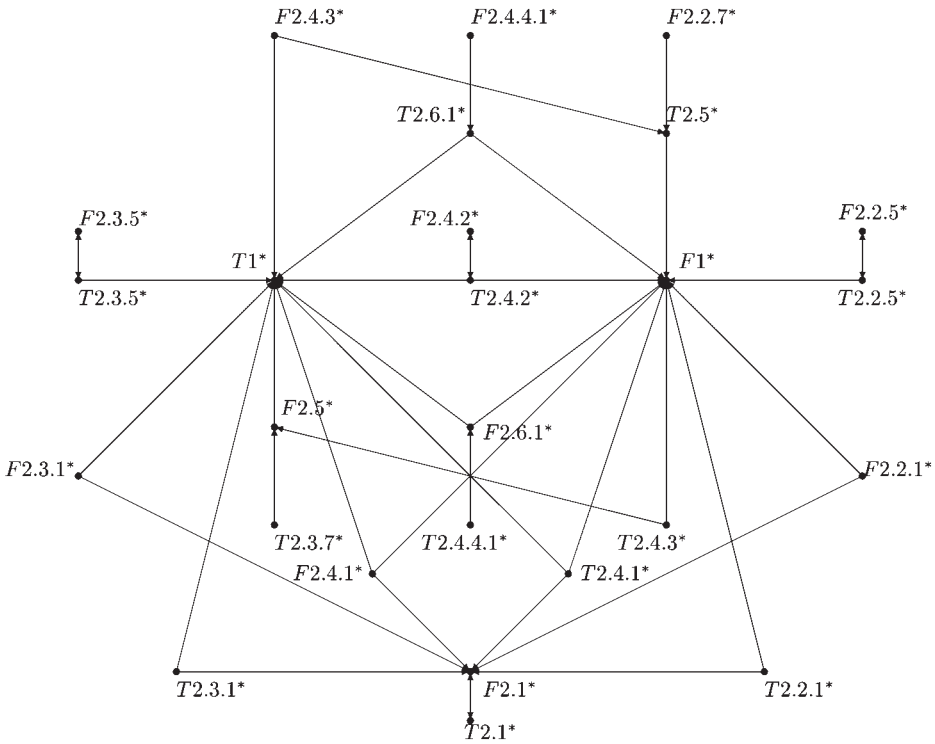


Fig. 3. $R_{\mathbf{E}}$ on $\mathbf{G}^*(1) \cup \mathbf{G}^*(2) \cup \mathbf{G}^*(3) \cup \{T2.4.4.1^*, F2.4.4.1^*\}$.

(4) *Let $\mathcal{E}$ be an exact set for $\mathbf{F}^n$. Then for any $A \in \mathbf{F}^n$,*

$$A \equiv \bigwedge\{\mathbf{for}(X(n)) \mid X \in \mathcal{E}, (\mathbf{EM}_{\mathcal{E}}, X) \not\models A\}.$$

(5) *There exists an exact set for $\mathbf{F}^n$.*

The above theorem has the following meaning. By (1), (2), and (3), we can list all exact models for $\mathbf{F}^n$. By (4), for each exact model for $\mathbf{F}^n$, we obtain a method for (II) in Subsection 1.4. We need (5) to treat the structure $\langle \mathbf{F}/\equiv, \leq \rangle$ in the next section.

To prove the theorem, we need some lemmas. (1) and (4) will be proved by Lemma 4.13. (2) will be proved by Lemma 4.15. (3) can be shown by the proof of Lemma 4.15(2). For (5), we can construct an exact set for $\mathbf{F}^n$ without using the provability of **S4** (see Sasaki (2010)). On the other hand, we can show (5) in another way. As we mentioned in

Table 2. *Exact sets for $\mathbf{F}^0$ and $\mathbf{F}^1$*

Exact sets for $\mathbf{F}^0$	Exact sets for $\mathbf{F}^1$	
$\{T1^*, F1^*\}$	$\{T1^*, F1^*, T2.1^*, F2.1^*\}$	$\{T1^*, F1^*, T2.5^*, F2.6.1^*\}$
$\{T1^*, F2.5^*\}$	$\{T1^*, F1^*, T2.2.5^*, F2.2.5^*\}$	$\{T1^*, F1^*, F2.5^*, T2.3.7^*\}$
$\{F1^*, T2.5^*\}$	$\{T1^*, F1^*, T2.3.5^*, F2.3.5^*\}$	$\{T1^*, F1^*, F2.5^*, T2.4.3^*\}$
$\{T2.1^*, F2.1^*\}$	$\{T1^*, F1^*, T2.4.2^*, F2.4.2^*\}$	$\{T1^*, F1^*, F2.5^*, T2.6.1^*\}$
	$\{T1^*, F1^*, T2.5^*, F2.2.7^*\}$	$\{T1^*, F1^*, T2.6.1^*, F2.4.4.1^*\}$
	$\{T1^*, F1^*, T2.5^*, F2.4.3^*\}$	$\{T1^*, F1^*, T2.6.1^*, F2.6.1^*\}$
	$\{T1^*, F1^*, T2.5^*, F2.5^*\}$	$\{T1^*, F1^*, F2.6.1^*, T2.4.4.1^*\}$

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Subsection 1.4, the Kripke model $\mathbb{C}_{n,m}(\mathbf{S4})$ constructed in Moss (2007) can be shown to be exact for $\mathbf{F}^n$. Using (2), we obtain (5).

LEMMA 4.5. *Let X and Y be sequents in $\mathbf{G}(n)$ and $\mathbf{G}^*(k)$, respectively. If $n \geq k$ and $(\mathbf{ant}(X(k)))^{\Box} = (\mathbf{ant}(Y))^{\Box}$, then $n = k$ and $X \in \mathbf{G}^*(n)$.*

Proof. By Lemma 3.8 and Lemma 2.11(1), we have $X(k) \in \mathbf{G}^*(k)$. Hence, we obtain the lemma. □

LEMMA 4.6. *Let X, Y, and Z be sequent in $\mathbf{G}(n_1)$, $\mathbf{G}(n_2)$, and $\mathbf{G}(n_3)$, respectively. If $\Box\mathbf{for}(X) \in \mathbf{suc}(Y)$ and $\Box\mathbf{for}(Y) \in \mathbf{suc}(Z)$, then $\Box\mathbf{for}(X) \in \mathbf{suc}(Z)$.*

Proof. From $\Box\mathbf{for}(X) \in \mathbf{suc}(Y)$ and $\Box\mathbf{for}(Y) \in \mathbf{suc}(Z)$, we have $(\Box\mathbf{for}(X) \to \mathbf{suc}(Z)) \in \mathbf{S4}$, and using Lemma 2.4(1) and Lemma 2.9, we obtain $\Box\mathbf{for}(X) \in \mathbf{suc}(Z)$. □

LEMMA 4.7.

(1) *$R_{\mathbf{E}}$ is reflexive and transitive.*
(2) *For any exact set $\mathcal{E}$ for $\mathbf{F}^n$, $R_{\mathcal{E}}$ is reflexive and transitive.*

Proof. The reflexivity is from the definition and the transitivity is from Lemma 4.6. □

LEMMA 4.8. *Let X_{n+1} and Y_n be sequents in $\mathbf{G}^*(n+1)$ and $\mathbf{G}(n) - \mathbf{G}^*(n)$, respectively. If $\Box\mathbf{for}(Y_n) \in \mathbf{suc}(X_{n+1})$, then $(\mathbf{ant}(X_{n+1}))^{\Box} \cap \mathbf{BG}(n) = (\mathbf{ant}(Y_n))^{\Box}$.*

Proof. We define Y_{n+1} as in Lemma 3.4. Then by Lemma 3.4 and Theorem 3.2, we have $Y_{n+1} \in \mathbf{next}(Y_n) \subseteq \mathbf{G}(n+1)$. We note that $(\mathbf{ant}(X_{n+1}))^{\Box} \subseteq (\mathbf{ant}(Y_{n+1}))^{\Box}$. Using $X_{n+1} \in \mathbf{G}^*(n+1)$, we have $(\mathbf{ant}(X_{n+1}))^{\Box} = (\mathbf{ant}(Y_{n+1}))^{\Box}$. Hence, $(\mathbf{ant}(X_{n+1}))^{\Box} \cap \mathbf{BG}(n) = (\mathbf{ant}(Y_{n+1}))^{\Box} \cap \mathbf{BG}(n) = (\mathbf{ant}(Y_n))^{\Box}$. □

The following two lemmas are concerned with **EM**, but as a corollary, we have the same results with $\mathbf{EM}_{\mathcal{E}}$. The two lemmas are also useful for the next section.

LEMMA 4.9. *For any $X_n \in \mathbf{G}^*(n)$ and for any $Y_k \in \mathbf{G}(k)$,*

(1) *$(\mathbf{EM}, X_n) \not\models p_i$ if and only if $p_i \in \mathbf{suc}(X_n)$,*
(2) *$(\mathbf{EM}, X_n) \not\models \Box\mathbf{for}(Y_k)$ if and only if either one of the following two conditions holds:*
   (2.1) *$k < n$ and $\Box\mathbf{for}(Y_k) \in \mathbf{suc}(X_n)$,*
   (2.2) *$k = n$ and $(\mathbf{ant}(Y_k))^{\Box} = (\mathbf{ant}(X_n))^{\Box}$.*

Proof. From the definition of $P_{\mathbf{E}}$ and Lemma 2.9, we obtain (1). We show (2) by an induction on $n + k$. A proof of Basis is included in Induction step.

Induction step. We first note that if $k \leq n$, then the following three conditions are equivalent:

- $(\mathbf{EM}, X_n) \not\models \mathbf{for}(Y_k)$,
- $(\mathbf{EM}, X_n) \models A$ for any $A \in \mathbf{ant}(Y_k)$; and $(\mathbf{EM}, X_n) \not\models B$ for any $B \in \mathbf{suc}(Y_k)$,
- $\mathbf{ant}(Y_k) \subseteq \mathbf{ant}(X_n)$ and $\mathbf{suc}(Y_k) \subseteq \mathbf{suc}(X_n)$.

The equivalence between the second one and the third one is from (1) and the induction hypothesis.

We show the "only if" part. Suppose that $(\mathbf{EM}, X_n) \not\models \Box\mathbf{for}(Y_k)$. Then there exist a number ℓ and a sequent $Z_\ell \in \mathbf{G}^*(\ell)$ such that $X_n R_{\mathbf{E}} Z_\ell$ and $(\mathbf{EM}, Z_\ell) \not\models \mathbf{for}(Y_k)$. By

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$X_n R_{\mathbf{E}} Z_\ell$, we have either one of the following two conditions:

$$\ell < n \text{ and } \Box\mathbf{for}(Z_\ell) \in \mathbf{suc}(X_n), \tag{3}$$

$$\ell = n \text{ and } (\mathbf{ant}(X_n))^{\Box} = (\mathbf{ant}(Z_\ell))^{\Box}. \tag{4}$$

We divide the cases.

The case that (3) holds. By $(\mathbf{EM}, Z_\ell) \not\models \mathbf{for}(Y_k)$ and Lemma 4.7, we have $(\mathbf{EM}, Z_\ell) \not\models \Box\mathbf{for}(Y_k)$. Using the induction hypothesis, we have either one of the following two conditions:

$$k < \ell \text{ and } \Box\mathbf{for}(Y_k) \in \mathbf{suc}(Z_\ell), \tag{3.1}$$

$$k = \ell \text{ and } (\mathbf{ant}(Y_k))^{\Box} = (\mathbf{ant}(Z_\ell))^{\Box}. \tag{3.2}$$

If (3.1) holds, then by (3) and Lemma 4.6, we have (2.1). If (3.2) holds, then by (3), Lemma 3.3(3), and Lemma 2.4(1), we also have (2.1).

The case that (4) holds. If $k \leq \ell$, then by $(\mathbf{EM}, Z_\ell) \not\models \mathbf{for}(Y_k)$ and the equivalence we noted first, we have $\mathbf{ant}(Y_k) \subseteq \mathbf{ant}(Z_\ell)$ and $\mathbf{suc}(Y_k) \subseteq \mathbf{suc}(Z_\ell)$. If $k > \ell$, then by $(\mathbf{EM}, Z_\ell) \not\models \mathbf{for}(Y_k)$, we have $(\mathbf{EM}, Z_\ell) \not\models \mathbf{for}(Y_k(\ell))$, and similarly, $\mathbf{ant}(Y_k(\ell)) \subseteq \mathbf{ant}(Z_\ell)$ and $\mathbf{suc}(Y_k(\ell)) \subseteq \mathbf{suc}(Z_\ell)$. Using Lemma 3.8(3), we have either one of the following three conditions:

$$k < \ell \text{ and } \Box\mathbf{for}(Y_k) = \Box\mathbf{for}(Z_\ell(k)) \in \mathbf{suc}(Z_\ell), \tag{4.1}$$

$$k = \ell \text{ and } Y_k = Z_\ell(k) = Z_\ell, \tag{4.2}$$

$$k > \ell \text{ and } Y_k(\ell) = Z_\ell. \tag{4.3}$$

If (4.1) holds, then by (4) and Lemma 2.9, we have (2.1). If (4.2) holds, then by (4), we have (2.2). If (4.3) holds, then by (4), we have $k > \ell = n$ and $(\mathbf{ant}(Y_k(\ell)))^{\Box} = (\mathbf{ant}(Z_\ell))^{\Box} = (\mathbf{ant}(X_n))^{\Box}$, which is in contradiction with Lemma 4.5.

We show the "if" part.

Suppose that (2.1) holds. Then by Lemma 3.10(2) and Theorem 3.2, there exists a sequent $Y \in \mathbf{ED}^{n-1}$ such that $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_n)$ and $Y \in Y_k\Downarrow$. If $Y \in W_{\mathbf{E}} \cap \mathbf{ED}^{n-1}$, then by Lemma 3.8, we have $Y_k = Y(k)$ and either $Y_k = Y$ or $\Box\mathbf{for}(Y_k) \in \mathbf{suc}(Y)$. Using the induction hypothesis, $X_n R_{\mathbf{E}} Y$, and Lemma 4.7, we obtain $(\mathbf{EM}, X_n) \not\models \Box\mathbf{for}(Y_k)$.

We assume that $Y \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$. Then by Lemma 4.8, we have $(\mathbf{ant}(X_n))^{\Box} \cap \mathbf{BG}(n-1) = (\mathbf{ant}(Y))^{\Box}$. We define Z_n as

$$Z_n = \mathbf{n}(Y, \{Z \in \mathbf{G}(n-1) \mid \Box\mathbf{for}(Z) \in \mathbf{suc}X_n\}).$$

Then we have $Z_n \in \mathbf{next}^+(Y)$ and $(\mathbf{ant}(X_n))^{\Box} = (\mathbf{ant}(Z_n))^{\Box}$. Using $X_n \in \mathbf{next}(X_n(n-1))$ and Theorem 3.2, it is not hard to see $Z_n \notin \mathbf{prov}_1(Y) \cup \mathbf{prov}_2(Y) \cup \mathbf{prov}_3(Y)$. Using Theorem 3.2, we have $Z_n \in \mathbf{next}(Y) \subseteq \mathbf{G}(n)$, and using Lemma 2.11(1), $Z_n \in \mathbf{G}^*(n)$. Also, by $Y \in Y_k\Downarrow$ and Lemma 3.8(3), we have $\mathbf{ant}(Y_k) \subseteq \mathbf{ant}(Y) \subseteq \mathbf{ant}(Z_n)$ and $\mathbf{suc}(Y_k) \subseteq \mathbf{suc}(Y) \subseteq \mathbf{suc}(Z_n)$. Using the equivalence we noted first, we have $(\mathbf{EM}, Z_n) \not\models \mathbf{for}(Y_k)$. Using $(\mathbf{ant}(X_n))^{\Box} = (\mathbf{ant}(Z_n))^{\Box}$, we have $X_n R_{\mathbf{E}} Z_n$, and hence, we obtain $(\mathbf{EM}, X_n) \not\models \Box\mathbf{for}(Y_k)$.

Suppose that (2.2) holds. Then by Lemma 2.11(1), we have $Y_k \in \mathbf{G}^*(n)$, and therefore, $Y_k \in W_{\mathbf{E}}$ and $X_n R_{\mathbf{E}} Y_k$. Using the equivalence we noted first, we have $(\mathbf{EM}, Y_k) \not\models \mathbf{for}(Y_k)$. Using $X_n R_{\mathbf{E}} Y_k$, we obtain $(\mathbf{EM}, X_n) \not\models \Box\mathbf{for}(Y_k)$. □

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LEMMA 4.10. *For any* $X_n \in \mathbf{G}^*(n)$ *and for any* $Y_k \in \mathbf{G}(k)$,

$$(\mathbf{EM}, X_n) \not\models \mathbf{for}(Y_k) \text{ if and only if } X_n \in Y_k\Downarrow .$$

Proof. By Lemma 3.8(3) and the equivalence noted first in the proof of Lemma 4.9, we have

$$k \leq n \text{ and } (\mathbf{EM}, X_n) \not\models \mathbf{for}(Y_k) \quad \text{if and only if} \quad X_n \in Y_k\Downarrow .$$

Also, by Lemma 4.7 and Lemma 4.9, we have that $(\mathbf{EM}, X_n) \not\models \mathbf{for}(Y_k)$ implies $k \leq n$. □

COROLLARY 4.11.

(1) *For any* $X \in \mathbf{G}(n)$, $P_{\mathbf{E}}(\mathbf{for}(X)) = W_{\mathbf{E}} - X\Downarrow$ *and* $P_{\mathbf{E}}(\mathbf{for}(X) \supset \bot) = X\Downarrow \cap W_{\mathbf{E}}$.
(2) *For any* $X \in \mathbf{G}^*(n)$, $P_{\mathbf{E}}(\mathbf{for}(X)) = W_{\mathbf{E}} - \{X\}$ *and* $P_{\mathbf{E}}(\mathbf{for}(X) \supset \bot) = \{X\}$.

COROLLARY 4.12. *Let* $\mathcal{E}$ *be an exact set for* $\mathbf{F}^n$. *Then for any* $X_\ell \in \mathbf{G}^*(\ell) \cap \mathcal{E}$ *and for any* $Y_k \in \mathbf{G}(k)$,

$$(\mathbf{EM}_{\mathcal{E}}, X_\ell) \not\models \mathbf{for}(Y_k) \text{ if and only if } X_\ell \in Y_k\Downarrow .$$

LEMMA 4.13. *Let* $\mathcal{E}$ *be an exact set for* $\mathbf{F}^n$. *Then*

(1) *for any* $A \in \mathbf{F}^n$, $\mathbf{EM}_{\mathcal{E}} \models A$ *if and only if* $\rightarrow A \in \mathbf{S4}$,
(2) *for any* $X, Y \in \mathcal{E}$, $(\mathbf{EM}_{\mathcal{E}}, X) \not\models \mathbf{for}(Y(n))$ *if and only if* $X = Y$,
(3) $\{P_{\mathcal{E}}(A) \mid A \in \mathbf{F}^n\} = 2^{\mathcal{E}}$.

Proof. By Lemma 3.8 and Corollary 4.12, we obtain (2). By (2), we have that $\mathcal{S} \in 2^{\mathcal{E}}$ implies $P_{\mathcal{E}}((\bigwedge \mathbf{for}(\{X(n) \mid X \in \mathcal{S}\})) \supset \bot) = \mathcal{S}$, and hence, we obtain (3). The "if" part of (1) is shown by Lemma 1.2 and Lemma 4.7. We show the "only if" part of (1). Suppose that $\rightarrow A \notin \mathbf{S4}$. Then by Theorem 2.6, $A \equiv \bigwedge \mathbf{for}(\mathcal{S})$ for some $\mathcal{S} \in 2^{\mathbf{ED}^n} - \{\emptyset\}$. Therefore, there exists $X \in \mathcal{S}$, and by Definition 4.2(1.2), there exists $Y \in X\Downarrow \cap \mathcal{E} \subseteq W_{\mathbf{E}}$. Using Corollary 4.12, we have $(\mathbf{EM}_{\mathcal{E}}, Y) \not\models \mathbf{for}(X)$. Hence, we obtain $(\mathbf{EM}_{\mathcal{E}}, Y) \not\models A$. □

From the above lemma, we obtain Theorem 4.4(1) and Theorem 4.4(4).

Next, we prove Theorem 4.4(2). As in Definition 2.12, we define $\mathbf{clus}(X)$ for any $X \in \mathbf{G}(n)$. Note that in Definition 2.12, we only defined it for $X \in \mathbf{G}^*(n)$.

LEMMA 4.14. *Let* X *be a sequent in* $\mathbf{G}(n) - \mathbf{G}^*(n)$ $(n > 0)$ *and let* $\mathcal{S}$ *be a subset of* $\mathbf{clus}(X)$. *If* $\mathbf{n}(X, \mathcal{S}) \in \mathbf{next}(X)$, *then* $\mathbf{n}(X, \mathcal{S}) \in \mathbf{G}^*(n+1)$.

Proof. We show an outline of the proof. One can find a detailed proof in Sasaki (2010). Suppose that $\mathbf{n}(X, \mathcal{S}) \notin \mathbf{G}^*(n+1)$. Then there exists a sequent $Y_{n+1} \in \mathbf{G}(n+1)$ such that $(\mathbf{ant}(\mathbf{n}(X, \mathcal{S})))^{\Box} \subsetneq (\mathbf{ant}(Y_{n+1}))^{\Box}$. If $(\mathbf{ant}(X))^{\Box} \subsetneq (\mathbf{ant}(Y_{n+1}(n)))^{\Box}$, then we can show $\Box\mathbf{for}(Y_{n+1}(n)) \in (\mathbf{ant}(Y_{n+1}))^{\Box} \cap (\mathbf{suc}(Y_{n+1}))^{\Box}$, which is in contradiction with Lemma 2.9. We assume that $(\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Y_{n+1}(n)))^{\Box}$. Then by $(\mathbf{ant}(\mathbf{n}(X, \mathcal{S})))^{\Box} \subsetneq (\mathbf{ant}(Y_{n+1}))^{\Box}$ and Lemma 2.9, there exists a sequent $Z \in \{X\} \cup \mathcal{S} \subseteq \mathbf{clus}(X)$ such that $\Box\mathbf{for}(Z) \in (\mathbf{ant}(Y_{n+1}))^{\Box}$. We can show $\Box\mathbf{for}(\mathbf{next}(Z(n-1))) \subseteq (\mathbf{ant}(Y_{n+1}))^{\Box}$ and $\Box\mathbf{for}(Z(n-1)) \in (\mathbf{suc}(Y_{n+1}(n)))^{\Box}$, which is in contradiction with Corollary 2.8(1) and Lemma 2.4(1). □

LEMMA 4.15. *Let* $M = \langle W, R, P \rangle$ *be an exact model for* $\mathbf{F}^n$.

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(1) *There exists one-to-one mapping* f *from* W *onto* $\mathbf{ED}^n$ *such that* $P(\mathbf{for}(f(\alpha))) = W - \{\alpha\}$.

(2) *There exist an exact set* $\mathcal{E}$ *for* $\mathbf{F}^n$ *and an isomorphism from* M *to* $\mathbf{EM}_{\mathcal{E}}$.

Proof. For (1). Let α be a world in W. Then by Lemma 1.3 and Theorem 2.6, there exists only one sequent X in $\mathbf{ED}^n$ such that $(M, \alpha) \not\models \mathbf{for}(X)$. Therefore, we can define a mapping f as $f(\alpha) = X$; and f satisfies the conditions.

For (2). For any $\alpha \in W$, we define the set $\mathbf{clus}(\alpha)$ as $\mathbf{clus}(\alpha) = \{\beta \in W \mid \alpha R\beta, \beta R\alpha\}$. By an induction on $k \in \{n, n+1, n+2, \ldots\}$, we define a mapping g_k from W to the set of sequents as

$$g_k(\alpha) = \begin{cases} f(\alpha) & \text{if } k = n \\ g_{k-1}(\alpha) & \text{if } k > n \text{ and } g_{k-1}(\alpha) \in W_{\mathbf{E}} \\ \mathbf{n}(g_{k-1}(\alpha), \{g_{k-1}(\beta)\mathbf{for}(\in \mathbf{G}(k-1) \mid \alpha R\beta\}) & \text{otherwise} \end{cases}$$

(see Example 4.16). Also, we define the set $\mathcal{E}_k$ as $\mathcal{E}_k = \{g_k(\alpha) \mid \alpha \in W\}$. Then we have the following six conditions:

(2.1) for any $\alpha \in W$, $g_k(\alpha) \in \mathbf{ED}^k$ and $P(\mathbf{for}(g_k(\alpha))) = W - \{\alpha\}$,

(2.2) $\mathbf{ED}^n \cap W_{\mathbf{E}} \subseteq \mathcal{E}_k \subseteq \mathbf{ED}^k$,

(2.3) for any $X \in \mathbf{ED}^n$, $\#(X\Downarrow \cap \mathcal{E}_k) = 1$,

(2.4) for any $\alpha \in W$ and for any $\beta \in \mathbf{clus}(\alpha)$, $(\mathbf{ant}(g_k(\alpha)))^{\Box} = (\mathbf{ant}(g_k(\beta)))^{\Box}$,

(2.5) for any $\alpha \in W$ and for any $X \in W_{\mathbf{E}}$, $g_k(\alpha) \in \mathcal{E}_k \cap W_{\mathbf{E}}$ and $g_k(\alpha) R_{\mathbf{E}} X$ imply $X \in \mathcal{E}_k$,

(2.6) $\mathcal{E}_k \not\subseteq W_{\mathbf{E}}$ implies $\#(\mathcal{E}_k - W_{\mathbf{E}}) > \#(\mathcal{E}_{k+2} - W_{\mathbf{E}})$.

The first five conditions are shown by an induction on k. We use (2.1) and Lemma 4.5 for (2.4).

We show (2.5). We only treat the case that the three conditions $k > n$, $(\mathbf{ant}(g_k(\alpha)))^{\Box} = (\mathbf{ant}(X))^{\Box}$, and $g_{k-1}(\alpha) \notin W_{\mathbf{E}}$ hold. By $g_{k-1}(\alpha) \notin W_{\mathbf{E}}$ and (2.1), we have $g_k(\alpha) \in \mathbf{G}^*(k)$, and using Lemma 4.5, we have $X \in \mathbf{G}^*(k)$. Using Lemma 2.9, we have $\Box\mathbf{for}(X(k-1)) \in (\mathbf{suc}(X))^{\Box} = (\mathbf{suc}(g_k(\alpha)))^{\Box}$. Considering the form of $g_k(\alpha)$, there exists a world $\beta \in W$ such that $X(k-1) = g_{k-1}(\beta)$ and $\alpha R\beta$. Also, considering the forms of $g_k(\alpha)$ and $g_k(\beta)$, we have $(\mathbf{suc}(g_k(\beta)))^{\Box} \subseteq (\mathbf{suc}(g_k(\alpha)))^{\Box}$. Using Lemma 2.9 and $g_k(\alpha) \in \mathbf{G}^*(k)$, we have $(\mathbf{ant}(g_k(\beta)))^{\Box} = (\mathbf{ant}(g_k(\alpha)))^{\Box} = (\mathbf{ant}(X))^{\Box}$. Using $X(k-1) = g_{k-1}(\beta)$, we obtain $X = g_k(\beta) \in \mathcal{E}_k$.

We show (2.6). By $\mathcal{E}_k \not\subseteq W_{\mathbf{E}}$ and (2.2), there exists a world $\alpha \in W$ such that $g_k(\alpha) \in \mathcal{E}_k \cap (\mathbf{G}(k) - \mathbf{G}^*(k))$ and $g_k(\beta) \in \mathbf{ED}^k \cap W_{\mathbf{E}}$ for any $\beta \in \{\gamma \in W \mid \alpha R\gamma, \gamma \not R\alpha\}$. Therefore,

$$g_{k+1}(\beta) = g_k(\beta) \in \mathbf{ED}^k \cap W_{\mathbf{E}}, \text{ for any } \beta \in \{\gamma \in W \mid \alpha R\gamma, \gamma \not R\alpha\}. \tag{2.6.1}$$

If $g_{k+1}(\alpha) \in W_{\mathbf{E}}$, then $g_{k+2}(\alpha) = g_{k+1}(\alpha) \in W_{\mathbf{E}}$, and hence, we obtain (2.6). We assume that $g_{k+1}(\alpha) \notin W_{\mathbf{E}}$. Then using (2.6.1),

$$\begin{aligned} g_{k+2}(\alpha) &= \mathbf{n}(g_{k+1}(\alpha), \{g_{k+1}(\beta) \in \mathbf{G}(k+1) \mid \alpha R\beta\}) \\ &= \mathbf{n}(g_{k+1}(\alpha), \{g_{k+1}(\beta) \in \mathbf{G}(k+1) \mid \beta \in \mathbf{clus}(\alpha)\}). \end{aligned}$$

Using (2.2), (2.4), and Lemma 4.14, we have $g_{k+2}(\alpha) \in W_{\mathbf{E}}$, and hence, we obtain (2.6).

We consider the number $\kappa = n + 2\#(W - \{\gamma \in W \mid f(\gamma) \in W_{\mathbf{E}}\})$. Then by (2.2) and (2.6), we have $\mathbf{ED}^n \cap W_{\mathbf{E}} \subseteq \mathcal{E}_\kappa \subseteq W_{\mathbf{E}}$. Using (2.3) and (2.5), $\mathcal{E}_\kappa$ is an exact set

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for $\mathbf{F}^n$. Also, by (2.1), Lemma 2.11, and Lemma 4.5, it is not hard to see

(2.7) $\alpha R\beta$ if and only if $g_\kappa(\alpha) R_{\mathcal{E}_\kappa} g_\kappa(\beta)$.

Hence, g_κ is an isomorphism form M to $\mathbf{EM}_{\mathcal{E}_\kappa}$. □

From the above lemma, we obtain Theorem 4.4(2). Also, by the proof of Lemma 4.15(2), we obtain Theorem 4.4(3).

EXAMPLE 4.16. *We consider the following two exact models M_1 and M_2 for $\mathbf{F}^1$:*

$$M_1 = \langle \{\alpha, \beta, \gamma, \delta\}, R_1, P\rangle \quad \text{and} \quad M_2 = \langle \{\alpha, \beta, \gamma, \delta\}, R_2, P\rangle,$$

where $P(p_i) = \{\alpha, \gamma\}$, and R_1 and R_2 are the reflexive and transitive closures of $\{(\gamma, \alpha), (\gamma, \beta), (\delta, \gamma)\}$ and $R_1 \cup \{(\gamma, \delta)\}$, respectively. In Table 3, we list g_1, g_2, g_3, and g_4 defined in the proof of Lemma 4.15(2). Also, we find that g_4 for M_1 and g_3 for M_2 provide exact sets for $\mathbf{F}^1$ (see also Example 4.3).

§5. The structure $\langle \mathbf{F}/\equiv, \leq \rangle$. In the present section, we prove that **EM** is the exact model for **F** and give a detailed description of $\{P_{\mathbf{E}}(A) \mid A \in \mathbf{F}\}$. Also, we give a detailed description of the structure $\langle \mathbf{F}/\equiv, \leq\rangle$.

First, we introduce the set **CNF** of sets of sequents, which behave like the principal conjunctive normal forms in the classical propositional logic.

DEFINITION 5.1. *We define the set* **CNF**, *and for any $\mathcal{S} \in 2^{\mathbf{ED}^n}$, we define* $\mathbf{cnf}(\mathcal{S})$ *and* $\mathcal{S}\Downarrow$ *as follows.*

(1) $$\mathbf{CNF}_k = \begin{cases} 2^{\mathbf{G}(0)} & \text{if } k = 0 \\ \{\mathcal{S} \in 2^{\mathbf{ED}^k} \mid \mathbf{next}(X) \not\subseteq \mathcal{S}, \mathbf{next}(X) \cap \mathcal{S} \neq \emptyset, \\ \quad \text{for some } X \in \mathbf{G}(k-1) - W_{\mathbf{E}}\} & \text{if } k > 0, \end{cases}$$

(2) $$\mathbf{CNF} = \bigcup_{i=0}^{\infty} \mathbf{CNF}_i,$$

(3) $$\mathbf{cnf}(\mathcal{S}) = \begin{cases} \mathcal{S} & \text{if } \mathcal{S} \in \mathbf{CNF} \\ \mathbf{cnf}(\{X \in \mathbf{G}(n-1) \mid \mathbf{next}(X) \subseteq \mathcal{S}\}) & \text{if } \mathcal{S} \notin \mathbf{CNF}, \end{cases}$$

(4) $$\mathcal{S}\Downarrow = \bigcup_{X \in \mathcal{S}} X\Downarrow.$$

We note that each member of **CNF** is a finite set. Also, it is observed easily that

$$\mathbf{CNF}_{k+1} = 2^{\mathbf{ED}^{k+1}} - \{\bigcup_{X \in \mathcal{S}} \mathbf{next}(X) \mid \mathcal{S} \subseteq \mathbf{G}(k) - \mathbf{G}^*(k)\}.$$

Table 3. *g_1, g_2, g_3, and g_4*

	M_1	M_2
$\langle g_1(\alpha), g_1(\beta), g_1(\gamma), g_1(\delta)\rangle$	$\langle T1^*, F1^*, T2, F2\rangle$	$\langle T1^*, F1^*, T2, F2\rangle$
$\langle g_2(\alpha), g_2(\beta), g_2(\gamma), g_2(\delta)\rangle$	$\langle T1^*, F1^*, T2.6, F2.4\rangle$	$\langle T1^*, F1^*, T2.4, F2.4\rangle$
$\langle g_3(\alpha), g_3(\beta), g_3(\gamma), g_3(\delta)\rangle$	$\langle T1^*, F1^*, T2.6.1^*, F2.4.4\rangle$	$\langle T1^*, F1^*, T2.4.2^*, F2.4.2^*\rangle$
$\langle g_4(\alpha), g_4(\beta), g_4(\gamma), g_4(\delta)\rangle$	$\langle T1^*, F1^*, T2.6.1^*, F2.4.4.1^*\rangle$	$\langle T1^*, F1^*, T2.4.2^*, F2.4.2^*\rangle$

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EXAMPLE 5.2. *In the case that* $m = 1$, *the four sets* $\mathbf{CNF}_0$, $\mathbf{CNF}_1$, $\mathbf{cnf}(\mathbf{next}(T) \cup \mathbf{next}(F))$, *and* $\mathbf{cnf}(\mathbf{next}(T2) \cup \{F2.2\})$ *are as follows:*

$\mathbf{CNF}_0 = \{\emptyset, \{T\}, \{F\}, \{T, F\}\}$,

$\mathbf{CNF}_1 = 2^{\mathbf{ED}^1} - \{\emptyset, \mathbf{next}(T2), \mathbf{next}(F), \mathbf{next}(T) \cup \mathbf{next}(F)\}$,

$\mathbf{cnf}(\mathbf{next}(T) \cup \mathbf{next}(F)) = \mathbf{cnf}(\{T, F\}) = \{T, F\}$,

$\mathbf{cnf}(\mathbf{next}(T2) \cup \{F2.2\}) = \mathbf{next}(T2) \cup \{F2.2\}$.

The purpose of the present section is to prove the following three theorems. From these theorems, we obtain the corresponding results to (I) and (II) in Subsection 1.4.

THEOREM 5.3.

(1) $\mathbf{EM}$ *is the exact model for* $\mathbf{F}$.

(2) $\{P_{\mathbf{E}}(A) \mid A \in \mathbf{F}\} = \{W_{\mathbf{E}} - \mathcal{S}\Downarrow \mid \mathcal{S} \in \mathbf{CNF}\}$.

THEOREM 5.4.

(1) $\mathbf{F}/\equiv = \{[\bigwedge \mathbf{for}(\mathcal{S})] \mid \mathcal{S} \in \mathbf{CNF}\}$.

(2) *For any* $\mathcal{S}_1 \in \mathbf{CNF}_n$ *and for any* $\mathcal{S}_2 \in \mathbf{CNF}_k$,

(2.1) $\bigwedge \mathbf{for}(\mathcal{S}_2) \to \bigwedge \mathbf{for}(\mathcal{S}_1) \in \mathbf{S4}$ *if and only if either* $\mathcal{S}_1 \subseteq \mathcal{S}_2\Downarrow$ *or both* $\mathcal{S}_1\Downarrow \cap \mathbf{ED}^k \subseteq \mathcal{S}_2$, *and* $n \leq k$,

(2.2) $\bigwedge \mathbf{for}(\mathcal{S}_1) \equiv \bigwedge \mathbf{for}(\mathcal{S}_2)$ *if and only if* $\mathcal{S}_1 = \mathcal{S}_2$.

(3) *For any* $A \in \mathbf{F}$ *and for any exact set* $\mathcal{E}$ *for* $\mathbf{F}^{d(A)}$, $A \equiv \bigwedge \mathbf{for}(\mathbf{cnf})(\{X(d(A)) \mid (\mathbf{EM}_{\mathcal{E}}, X) \not\models A\})$.

THEOREM 5.5.

(1) $\bot \equiv \bigwedge \mathbf{for}(\mathbf{G}(0))$.

(2) $p_i \equiv \bigwedge \mathbf{for}(\{X \in \mathbf{G}(0) \mid p_i \in \mathbf{suc}(X)\})$.

(3) *For any* $\mathcal{S}_1 \in \mathbf{CNF}_n$ *and for any* $\mathcal{S}_2 \in \mathbf{CNF}_k$ $(n \geq k)$,

(3.1) $\bigwedge \mathbf{for}(\mathcal{S}_1) \wedge \bigwedge \mathbf{for}(\mathcal{S}_2) \equiv \bigwedge \mathbf{for}(\mathbf{cnf}(\mathcal{S}_1 \cup (\mathcal{S}_2\Downarrow \cap \mathbf{ED}^n)))$,

(3.2) $\bigwedge \mathbf{for}(\mathcal{S}_1) \vee \bigwedge \mathbf{for}(\mathcal{S}_2) \equiv \bigwedge \mathbf{for}(\mathbf{cnf}(\mathcal{S}_1 \cap (\mathcal{S}_2\Downarrow \cap \mathbf{ED}^n)))$,

(3.3) $\bigwedge \mathbf{for}(\mathcal{S}_1) \supset \bigwedge \mathbf{for}(\mathcal{S}_2) \equiv \bigwedge \mathbf{for}(\mathbf{cnf}((\mathbf{ED}^n - \mathcal{S}_1) \cap (\mathcal{S}_2\Downarrow \cap \mathbf{ED}^n)))$,

(3.4) $\bigwedge \mathbf{for}(\mathcal{S}_2) \supset \bigwedge \mathbf{for}(\mathcal{S}_1) \equiv \bigwedge \mathbf{for}(\mathbf{cnf}((\mathbf{ED}^n - \mathcal{S}_2\Downarrow) \cap \mathcal{S}_1))$.

(4) *For any* $\mathcal{S} \in \mathbf{CNF}_k$, $\Box \bigwedge \mathbf{for}(\mathcal{S}) \equiv \bigwedge \mathbf{for}(\mathbf{cnf}(\mathcal{S}_1 \cup \mathcal{S}_2))$, *where*

$$\mathcal{S}_1 = \bigcup_{\substack{X \in \mathcal{S} \\ k}} \{Y \in \mathbf{ED}^{k+1} \mid \Box \mathbf{for}(X) \in \mathbf{suc}(Y)\},$$

$$\mathcal{S}_2 = \bigcup_{i=1} \bigcup_{X \in \mathcal{S} \cap \mathbf{G}^*(i)} \mathbf{clus}(X).$$

We describe a reason why we use $\mathbf{CNF}$. By Corollary 4.11, we note

FACT 5.6.

$$\{P_{\mathbf{E}}(A) \mid A \in \mathbf{F}\} = \bigcup_{i=0}^{\infty} \{W_{\mathbf{E}} - \mathcal{S}\Downarrow \mid \mathcal{S} \subseteq \mathbf{ED}^i\}.$$

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Therefore, it seems possible to use this fact instead of Theorem 5.3(2), but the condition corresponding to Theorem 5.4(2.2) does not hold. For example, if $\mathcal{S}_1 = \mathbf{ED}^1$ and $\mathcal{S}_2 = \mathbf{ED}^2$, then we have $\mathcal{S}_1 \neq \mathcal{S}_2$, but by Theorem 2.6, $\bigwedge \mathbf{for}(\mathbf{ED}^1) \equiv \bigwedge \mathbf{for}(\mathbf{ED}^2)$. Hence, we need to define another expression instead of $\bigcup_{i=0}^{\infty} \{W_{\mathbf{E}} - \mathcal{S}\Downarrow \mid \mathcal{S} \subseteq \mathbf{ED}^i\}$. The expression $\{W_{\mathbf{E}} - \mathcal{S}\Downarrow \mid \mathcal{S} \in \mathbf{CNF}\}$ is a solution of this problem.

To prove the above theorems, we need two more lemmas.

LEMMA 5.7. *For any* $A \in \mathbf{F}$,

$$\mathbf{EM} \models A \text{ if and only if } \rightarrow A \in \mathbf{S4}.$$

Proof. By Theorem 4.4(5), for any $X \in \mathbf{G}(n)$, there exists $Y \in W_{\mathbf{E}} \cap X\Downarrow$. Hence, we can prove the lemma similarly to the proof of the Lemma 4.13(1). □

LEMMA 5.8. *For any* $\mathcal{S} \in 2^{\mathbf{ED}^n}$,

(1) $\mathbf{cnf}(\mathcal{S}) \in \mathbf{CNF}$,
(2) $\bigwedge \mathbf{for}(\mathcal{S}) \equiv \bigwedge \mathbf{for}(\mathbf{cnf}(\mathcal{S}))$,
(3) $W_{\mathbf{E}} - \mathcal{S}\Downarrow = W_{\mathbf{E}} - \mathbf{cnf}(\mathcal{S})\Downarrow$.

Proof. We use an induction on $\mathbf{cnf}(\mathcal{S})$. If $\mathcal{S} \in \mathbf{CNF}$, then the lemma is clear from the definition. Suppose that $\mathcal{S} \notin \mathbf{CNF}$. Then from the definition and Corollary 2.8(2), we have

$$\mathbf{cnf}(\mathcal{S}) = \mathbf{cnf}(\mathcal{S}'), \quad \bigwedge \mathbf{for}(\mathcal{S}) \equiv \bigwedge \mathbf{for}(\mathcal{S}'), \quad \text{and} \quad W_{\mathbf{E}} - \mathcal{S}\Downarrow = W_{\mathbf{E}} - \mathcal{S}'\Downarrow,$$

where $\mathcal{S}' = \{X \in \mathbf{G}(n-1) \mid \mathbf{next}(X) \subseteq \mathcal{S}\}$. Using the induction hypothesis, we obtain the lemma. □

From Lemma 5.7 and Corollary 4.11(2), we obtain Theorem 5.3(1). By Fact 5.6 and Lemma 5.8, we obtain Theorem 5.3(2). By Theorem 4.4(4) and Lemma 5.8, we obtain Theorem 5.4(1) and Theorem 5.4(3). By Theorem 2.6 and Lemma 5.8, we obtain Theorem 5.4(2). By Theorem 2.5 and Lemma 5.8, we obtain Theorem 5.5.

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