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The vorticity equation as an angular momentum equation

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In a Newtonian fluid, in which the body forces are conservative, in which the pressure is a function only of density and in which the kinematic viscosity ν is uniform, the vorticity $\boldsymbol{\omega}$ satisfies the equation

$$\frac{D\boldsymbol{\omega}}{Dt} - (\boldsymbol{\omega}.\nabla)\,\mathbf{u} + \boldsymbol{\omega}(\nabla.\mathbf{u}) = \nu\nabla^2\boldsymbol{\omega}, \tag{1}$$

where $\mathbf{u}$ is the velocity field, so that $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. This equation is normally derived by taking the curl of the Navier–Stokes equations. However, the vorticity has many interpretations in terms of the angular velocity of elements of fluid and it is natural to expect that (1) can be derived by equating the rate of change of the angular momentum of a small material volume element about its centre of mass with the moment of the forces acting on the element. Such a derivation is presented here in the hope that it may be of pedagogic interest.

Consider a small material volume element τ. Let O be a fixed origin and let G, with position vector $\bar{\mathbf{r}}$ relative to O, be the centre of mass of the element. Thus

$$\bar{\mathbf{r}}\int_\tau \rho\, d\tau = \int_\tau \rho\mathbf{r}\, d\tau, \tag{2}$$

where ρ is the density of the fluid at the point with position vector $\mathbf{r}$. Now $\rho\, d\tau$ is the mass of a small quantity of fluid contained in τ and so does not change with time. Hence if $\bar{\mathbf{u}}$ is the rate of change of $\bar{\mathbf{r}}$,

$$\bar{\mathbf{u}}\int_\tau \rho\, d\tau = \int_\tau \rho\mathbf{u}\, d\tau. \tag{3}$$

The angular momentum of the element about G, say $\mathbf{H}$, satisfies

$$\mathbf{H} = \int_\tau (\mathbf{r}-\bar{\mathbf{r}}) \times \mathbf{u}(\mathbf{r})\,\rho\, d\tau. \tag{4}$$

Now $\mathbf{u}(\mathbf{r}) = \mathbf{u}|_{\bar{\mathbf{r}}} + (\mathbf{s}.\nabla)\,\mathbf{u}|_{\bar{\mathbf{r}}} + O(|\mathbf{s}|^2)$ where $\mathbf{s} = \mathbf{r}-\bar{\mathbf{r}}$, and from (2),

$$\int_\tau \mathbf{s}\rho\, d\tau = \mathbf{0}.$$

Thus, to highest order in the linear dimensions of τ,

$$H_i = \epsilon_{ijk}\frac{\partial u_k}{\partial x_l}\bigg|_{\bar{\mathbf{r}}}\int_\tau s_j s_l \rho\, d\tau = \epsilon_{ijk}\frac{\partial u_k}{\partial x_l}\bigg|_{\bar{\mathbf{r}}} T_{jl}, \tag{5}$$

say. For a spherical element with ρ having spherical symmetry about G, it is easily shown that $T_{jl} = \frac{1}{2}I\,\delta_{jl}$ where I is the moment of inertia of the element about an axis

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through its centre, and (5) reduces to the well-known results $\mathbf{H} = \frac{1}{2}I\boldsymbol{\omega}$ ((1), p. 82). In general there is no simple relationship between T_{jl} and the inertia tensor. From (5) it follows that

$$\frac{dH_i}{dt} = \epsilon_{ijk}\frac{D}{Dt}\left(\frac{\partial u_k}{\partial x_l}\bigg|_{\bar{\mathbf{r}}}\right)T_{jl} + \epsilon_{ijk}\frac{\partial u_k}{\partial x_l}\bigg|_{\bar{\mathbf{r}}}\frac{dT_{jl}}{dt}. \tag{6}$$

Now
$$\frac{dT_{jl}}{dt} = \int_\tau (u_j s_l + s_j u_l)\,\rho\,d\tau = \frac{\partial u_j}{\partial x_p}\bigg|_{\bar{\mathbf{r}}} T_{pl} + \frac{\partial u_l}{\partial x_p}\bigg|_{\bar{\mathbf{r}}} T_{pj},$$

to the highest order in the linear dimensions of τ. Thus

$$\frac{dH_i}{dt} = \epsilon_{ijk}\left\{\frac{D}{Dt}\left(\frac{\partial u_k}{\partial x_l}\right) + \frac{\partial u_k}{\partial x_m}\frac{\partial u_m}{\partial x_l}\right\}\bigg|_{\bar{\mathbf{r}}} T_{jl}. \tag{7}$$

For the case of a spherical element with ρ having spherical symmetry, the first term in (6) (and (7)) refers to the rate of change of the angular velocity of the element, while the second term refers to the rate of change of the inertia tensor of the element due to its deformation into an ellipsoid (the rigid rotation of the element with angular velocity $\frac{1}{2}\boldsymbol{\omega}$ causing no change in the inertia tensor for a spherical element).

The couple on the element consists of a contribution from the body force and one from the stresses. If the body force is conservative with potential per unit mass V the couple it exerts about G has i component

$$-\int_\tau \epsilon_{ijk} s_j \frac{\partial V}{\partial x_k}\rho\,d\tau. \tag{8}$$

The stresses exert a couple about G with i component

$$\int_s \epsilon_{ijk} s_j p_{km}\,dS_m = \int_\tau \epsilon_{ijk}\frac{\partial}{\partial x_m}(s_j p_{km})\,d\tau,$$

where S is the surface of τ and p_{ij} are the components of the stress tensor. Using the Newtonian constitutive relation

$$p_{ij} = -p\delta_{ij} + 2\mu(e_{ij} - \tfrac{1}{3}e_{pp}\delta_{ij}),$$

and assuming that μ is uniform the stress contribution becomes

$$\int_\tau \epsilon_{ijk} s_j\left\{-\frac{1}{\rho}\frac{\partial p}{\partial x_k} + \nu\frac{\partial^2 u_k}{\partial x_m \partial x_m} + \frac{\nu}{3}\frac{\partial^2 u_m}{\partial x_k \partial x_m}\right\}\rho\,d\tau. \tag{9}$$

Adding (8) and (9) and expanding about $\mathbf{s} = \mathbf{0}$ gives the i component of the total moment about G to highest order in the linear dimensions of τ as

$$\epsilon_{ijk}\left\{-\frac{\partial^2 V}{\partial x_k \partial x_l} - \frac{\partial}{\partial x_l}\left(\frac{1}{\rho}\frac{\partial p}{\partial x_k}\right) + \nu\frac{\partial^3 u_k}{\partial x_l \partial x_m \partial x_m} + \frac{\nu}{3}\frac{\partial^3 u_m}{\partial x_k \partial x_l \partial x_m}\right\}\bigg|_{\bar{\mathbf{r}}} T_{jl}. \tag{10}$$

By a result of elementary mechanics, the rate of change of angular momentum about G, given by (7), is equal to the total moment of the forces about G, given by (10). This gives a relationship of the form $A_{ijl}T_{jl} = 0$. Since T_{jl} is symmetric but otherwise arbitrary, $A_{ijl} = -A_{ilj}$ for all j and l. In particular $A_{ijj} = 0$. This gives the equation

$$\epsilon_{ijk}\left\{\frac{D}{Dt}\left(\frac{\partial u_k}{\partial x_j}\right) + \frac{\partial u_m}{\partial x_j}\frac{\partial u_k}{\partial x_m}\right\}\bigg|_{\bar{\mathbf{r}}} = \nu\epsilon_{ijk}\left\{\frac{\partial^3 u_k}{\partial x_m \partial x_m \partial x_j}\right\}\bigg|_{\bar{\mathbf{r}}}. \tag{11}$$

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In deriving (11), it has been assumed that the pressure is a function only of density so that

$$\epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\frac{\partial p}{\partial x_k}\right) = \epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\left(\int^p \frac{dp'}{\rho(p')}\right) = 0.$$

It is also easy to show that

$$\epsilon_{ijk}\frac{\partial u_m}{\partial x_j}\frac{\partial u_k}{\partial x_m} \equiv w_i\frac{\partial u_j}{\partial x_j} - w_j\frac{\partial u_i}{\partial x_j}.$$

Thus (11) becomes

$$\left\{\frac{D\boldsymbol{\omega}}{Dt} - (\boldsymbol{\omega}.\nabla)\,\mathbf{u} + \boldsymbol{\omega}(\nabla.\mathbf{u})\right\}\Big|_{\bar{\mathbf{r}}} = \{\nu\nabla^2\boldsymbol{\omega}\}\Big|_{\bar{\mathbf{r}}},$$

which is (1) evaluated at G. Since any point is the centre of mass of some material element, it is established that (1) holds everywhere.

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