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Mean diameter, regularity and diam-mean equicontinuity

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Abstract. In the context of (not necessarily minimal) actions, we consider the mean diameter and use it to characterize regular factor maps. Building on this characterization, we prove that an action is diam-mean equicontinuous if and only if it is a regular extension of its maximal equicontinuous factor. Furthermore, we establish the existence of a maximal diam-mean equicontinuous factor and discuss stability properties of regular factor maps. For this, we work in the context of actions of locally compact and σ-compact amenable groups.

Keywords: regularity, diam-mean equicontinuity, mean diameter, amenability
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1. *Introduction*

Equicontinuous actions are fundamental building blocks of the theory of topological dynamics. To study more intricate systems, researchers often turn to weaker derivatives of equicontinuity, such as distality [**Aus88**], mean equicontinuity [**FGL22**, **Fom51**, **LTY15**] and diam-mean equicontinuity [**GR17**, **GRJY21**], among others.

A central technique in the field is to relate a given dynamical system to an equicontinuous one, typically via the maximal equicontinuous factor. A natural and important question is how dynamical properties of the original system are reflected in the structure of the corresponding factor map.

For instance, a landmark result due to Furstenberg states that a minimal action is distal if and only if the factor map to its maximal equicontinuous factor can be expressed as a (possibly transfinite) composition of equicontinuous factor maps [**Aus88**]. Another elegant result concerns mean equicontinuity: an action is mean equicontinuous if and only if the factor map to its maximal equicontinuous factor is topo-isomorphic. See [**DG16**, **FGL22**] for further details.

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In this article, we focus on diam-mean equicontinuous actions and their relation to regular factor maps [GR17, GRJY21, Hau25, HJL25]. For this, we will work in the context of actions of amenable locally compact and σ-compact topological groups, which we abbreviate as *amenable* lc_σ *groups*. This is motivated by the natural appearance of actions of amenable lc_σ groups, such as $\mathbb{R}^n$, in the context of aperiodic order [BG13, Sol20] and the study of regular Toeplitz actions for countable amenable (residually finite) groups [BJL16, CP14, Dow05].

Our central object of study is the mean diameter. To define it, let X be an action of an amenable lc_σ group G on a compact metric space and denote by d the metric of X. We fix a (left) Haar measure on G and denote it by $|\cdot|$. A sequence (F_n) of compact subsets of G with $|F_n| > 0$ is called *left Følner* if for every non-empty compact $K \subseteq G$, we have $|F_n \Delta K F_n|/|F_n| \to 0$. For a left Følner sequence $\mathcal{F} = (F_n)$ and a closed subset $A \subseteq X$, we define the *mean diameter of A with respect to* $\mathcal{F}$ as

$$\mathrm{Diam}_{\mathcal{F}}(A) := \limsup_{n\to\infty} \frac{1}{|F_n|} \int_{F_n} \mathrm{diam}(g.A)\, dg,$$

where we abbreviate $g.A := \{g.x \mid x \in A\}$ and the *diameter* of A is given by $\mathrm{diam}(A) := \max_{x,x'\in A} d(x, x')$. The *mean diameter* of A is defined by

$$\mathrm{Diam}(A) := \sup_{\mathcal{F}} \mathrm{Diam}_{\mathcal{F}}(A),$$

where the supremum is taken over all left Følner sequences $\mathcal{F}$ of G. In §4, we present basic properties of Diam together with several methods for its computation, such as

$$\mathrm{Diam}(A) = \inf_K \sup_{h\in G} \frac{1}{|Kh|} \int_{Kh} \mathrm{diam}(g.A)\, dg,$$

where the infimum is taken over all compact subsets K of G with $|K| > 0$.

We will use the mean diameter to characterize regular factor maps. For this, we call a factor map $\pi : X \to Y$ *diam-mean proximal* if for all $y \in Y$, we have $\mathrm{Diam}(\pi^{-1}(y)) = 0$. Recall that a factor map $\pi : X \to Y$ is called *regular* (also *almost surely one-to-one*) if the set of injectivity points X_0 of π satisfies $\mu(X_0) = 1$ for all invariant Borel probability measures on X. See §5 for further details. In Theorem 5.2, we establish the following characterization of regularity.

THEOREM. *A factor map is regular if and only if it is diam-mean proximal.*

It is also natural to study the following notion. For a left Følner sequence $\mathcal{F}$ in G, we call a factor map $\pi : X \to Y$ *$\mathcal{F}$-diam-mean proximal* if for all $y \in Y$, we have $\mathrm{Diam}_{\mathcal{F}}(\pi^{-1}(y)) = 0$. Clearly, any diam-mean proximal factor map is $\mathcal{F}$-diam-mean proximal. In Theorem 5.2, we will see that also the converse holds and show the following theorem.

THEOREM. *A factor map is diam-mean proximal if and only if it is $\mathcal{F}$-diam-mean proximal for some left Følner sequence $\mathcal{F}$ in G.*

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Another important application of the mean diameter is the concept of diam-mean equicontinuity [**GR17**, **GRJY21**]. An action is called *diam-mean equicontinuous* (also *Banach diam-mean equicontinuous* [**GRJY21**]) if for all $\epsilon > 0$ and every $x \in X$, there exists a compact neighbourhood A of x with $\mathrm{Diam}(A) \leq \epsilon$. We will see in §6 that this notion is independent of the choice of a metric on X and provide some basic characterizations.

Recall that diam-mean proximality and regularity are equivalent concepts for all factor maps, so it is to be expected that the concepts of diam-mean equicontinuity and regularity are strongly related. The first result of this interplay was achieved in [**GR17**, Theorem 54] in the context of minimal almost automorphic subshifts [**GR17**, Definition 50]. Generalizing the result of [**GR17**], it was presented in [**GRJY21**, Theorem 4.12] that whenever X is a minimal action of $\mathbb{Z}$, then X is diam-mean equicontinuous if and only if the factor map $\pi_{\mathrm{eq}} \colon X \to X_{\mathrm{eq}}$ onto the maximal equicontinuous factor is regular. This result was later generalized to minimal actions of more general acting groups, such as Abelian lc_σ groups [**Hau25**, Theorem 4.5]. Recent progress was made in [**HJL25**, Theorem 5.2] for the context of actions of countable amenable groups, where it was shown that a minimal action that satisfies the local Bronstein property [**HJL25**, Definition 2.14] is diam-mean equicontinuous if and only if π_{eq} is regular.

Nevertheless, the question remained whether the characterization holds for all minimal actions of countable amenable groups. Furthermore, it was asked in [**GRJY21**, Question 2] whether this characterization also holds in the non-minimal context. In §7, we will give an affirmative answer to both questions and show the following theorem.

THEOREM 1.1. *An action of an amenable* lc_σ *group on a compact metric space is diam-mean equicontinuous if and only if the factor map onto the maximal equicontinuous factor is regular.*

Note that our proof significantly differs from [**GRJY21**, **Hau25**, **HJL25**] and does not involve the concept of frequent stability [**GRJY21**, §3].

Motivated by Theorem 1.1, we will explore stability properties of regular factor maps in §8. This will allow us to prove that diam-mean equicontinuity is stable under (countable) products and to establish the following theorem in §9.

THEOREM 1.2. (Maximal diam-mean equicontinuous factor) *Let X be an action of an amenable* lc_σ *group G on a compact metric space.*

There exists a maximal diam-mean equicontinuous factor of X, that is, a factor map $\pi_{\mathrm{dm}} \colon X \to X_{\mathrm{dm}}$ onto a diam-mean equicontinuous action X_{dm} of G, such that for any factor map $\pi : X \to Y$ onto a diam-mean equicontinuous action of G, there exists a factor map $\psi \colon X_{\mathrm{dm}} \to Y$ such that $\pi = \psi \circ \pi_{\mathrm{dm}}$. The maximal diam-mean equicontinuous factor of X is unique up to conjugacy.

1.1. *Convention.* Throughout this article, G denotes an amenable lc_σ group, unless explicitly stated otherwise. We fix a (left) Haar measure on G, denoted by $|\cdot|$. For readability, we often omit explicit mention of the group G, and refer simply to an action,

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a factor map or a Følner sequence. Unless stated otherwise, it is understood that all such objects refer to the same fixed group G.

2. *Preliminaries*

We denote $\mathbb{R}^+ := [0, \infty)$. If X is a topological space, we denote by $\mathfrak{F}_1(X)$ and $\mathcal{H}(X)$ the sets of all singletons and non-empty compact subsets, respectively. Whenever we speak of *measurability*, we refer to the Borel σ-algebra. We write $\mathcal{H}'(G)$ for the set of all compact subsets K of G with $|K| > 0$.

2.1. *Hyperspaces.* Let (X, d) be a compact metric space. We write $\mathbf{B}_\epsilon^d(x) := \{x' \in X \mid d(x, x') \leq \epsilon\}$ and $B_\epsilon^d(x) := \{x' \in X \mid d(x, x') < \epsilon\}$ for $\epsilon > 0$ and $x \in X$. For $A \in \mathcal{H}(X)$, we denote $\operatorname{diam}_d(A) = \max_{x,x' \in A} d(x, x')$. If the metric is understood from the context, we simply write $\mathbf{B}_\epsilon(x)$, $B_\epsilon(x)$ and $\operatorname{diam}(A)$. For $A \in \mathcal{H}(X)$ and $\epsilon > 0$, we denote $\mathbf{B}_\epsilon(A) := \bigcup_{x \in A} \mathbf{B}_\epsilon(x)$. The *Hausdorff metric* is given by

$$d_{\mathcal{H}}(A, A') := \min\{\epsilon > 0 \mid A \subseteq \mathbf{B}_\epsilon(A') \text{ and } A' \subseteq \mathbf{B}_\epsilon(A)\}$$

for $A, A' \in \mathcal{H}(X)$. Note that, as a straightforward consequence of the compactness of X, the infimum in the above formula is attained. It is well known that $d_{\mathcal{H}}$ is a metric on $\mathcal{H}(X)$ and that $\mathcal{H}(X)$ equipped with the Hausdorff metric $d_{\mathcal{H}}$ is a compact metric space. We recommend [**Nad92**] for more details on hyperspaces and note that $\operatorname{diam}\colon \mathcal{H}(X) \to \mathbb{R}^+$ is continuous.

2.2. *Amenable groups and Følner sequences.* Let G be a locally compact and σ-compact group (lc_σ *group*) and choose a (left) Haar measure $|\cdot|$ on G. A sequence (F_n) in $\mathcal{H}'(G)$ is called *left Følner* if $|F_n \Delta K F_n|/|F_n| \to 0$ holds for all $K \in \mathcal{H}(G)$, where Δ denotes the symmetric difference. It is called *right Følner* whenever $|F_n \Delta F_n K|/|F_n| \to 0$ holds for all $K \in \mathcal{H}(G)$. An lc_σ group that allows for a left Følner sequence is called *amenable*.

2.3. *Actions.* Let G be a topological group and X be a compact metric space. A *(continuous) action* of G on X is a group homomorphism α from G into the group of homeomorphisms $X \to X$ such that $G \times X \to X\colon (g, x) \mapsto \alpha(g)(x)$ is continuous. For $g \in G$ and $x \in X$, we suppress the symbol for the action by simply writing $g.x := \alpha(g)(x)$. This allows us to simply speak of an *action X of G*. Whenever the group G is given by the context, we simply speak of an action X. See [**Aus88**] for more details on actions.

For $x \in X$, we denote $O(x) := \{g.x \mid g \in G\}$ for the *orbit* of x. A point $x \in X$ is called *transitive* if it has a dense orbit. An action is called *transitive* if it allows for a transitive point. It is called *minimal* if every point is transitive.

A subset $M \subseteq X$ is called *invariant* if $M = g.M := \{g.x \mid x \in M\}$ holds for all $g \in G$. For an invariant $A \in \mathcal{H}(X)$, the action of G on X naturally induces an action of G on A. We speak of a *subaction (of X)* in this case. Whenever $(X_n)_{n \in \mathbb{N}}$ is a countable family of actions, then we consider the action of G on $\prod_n X_n$ given by $g.(x_n)_n := (g.x_n)_n$ for $g \in G$ and $(x_n)_n \in \prod_n X_n$. Furthermore, whenever X is an action, then $(g, A) \mapsto g.A$ induces an

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action of G on the hyperspace $\mathcal{H}(X)$. We simply write $\prod_n X_n$ and $\mathcal{H}(X)$ for the respective actions.

2.4. *Factor maps.* A continuous surjection $\pi\colon X \to Y$ between actions is called a *factor map* (also *factor of* X) if $\pi(g.x) = g.\pi(x)$ holds for all $g \in G$ and $x \in X$. A factor map is called a *conjugacy* if it is a homeomorphism. Note that any factor map is *closed*, that is, that $\pi(A)$ is closed for all closed $A \subseteq X$.

An action X is called *equicontinuous* if there exists a continuous metric d on X such that $d(g.x, g.x') = d(x, x')$ for all $x, x' \in X$. For any action X, there exists a *maximal equicontinuous factor* of X, that is, a factor map $\pi_{\mathrm{eq}}\colon X \to X_{\mathrm{eq}}$ onto an equicontinuous action X_{eq} such that for any factor map $\pi\colon X \to Y$ onto an equicontinuous action, there exists a factor map $\psi\colon X_{\mathrm{eq}} \to Y$ with $\pi = \psi \circ \pi_{\mathrm{eq}}$. For details, see [**Aus88**].

2.5. *Invariant measures.* We denote by $\mathcal{M}(X)$ the set of all Borel probability measures on X. Equipped with the weak*-topology, $\mathcal{M}(X)$ becomes a compact metric space [**FGH25**, **Gla03**, **Vil03**]. For $\mu \in \mathcal{M}(X)$ and a continuous surjection $\pi\colon X \to Y$, we denote $\pi_*\mu$ for the *pushforward measure*, given by $\pi_*\mu(M) := \mu(\pi^{-1}(M))$ for all measurable $M \subseteq Y$. Clearly, we have $\pi_*\mu \in \mathcal{M}(Y)$.

For $g \in G$ and $\mu \in \mathcal{M}(X)$, we denote $g_*\mu$ for the pushforward under the homeomorphism induced by g. A measure $\mu \in \mathcal{M}(X)$ is called *invariant* if $g_*\mu = \mu$ holds for all $g \in G$. We denote by $\mathcal{M}_G(X)$ the set of all invariant $\mu \in \mathcal{M}(X)$. An action is called *uniquely ergodic* if it allows for a unique invariant probability measure. An invariant measure $\mu \in \mathcal{M}(X)$ is called *ergodic* if for each invariant measurable $M \subseteq X$, we have that $\mu(M) \in \{0, 1\}$. Let $\pi\colon X \to Y$ be a factor map. Note that for $\mu \in \mathcal{M}_G(X)$, we have $\pi_*\mu \in \mathcal{M}_G(Y)$ and that ergodicity is preserved under pushing forward.

2.6. *Actions of amenable* lc_σ *groups.* Let X be an action of an amenable lc_σ group G. For $x \in X$, we denote by δ_x the Dirac measure at x. For $K \in \mathcal{H}'(G)$ and $\mu \in \mathcal{M}(X)$, we denote $K_*\mu := (1/|K|)\int_K (g^*\mu)\, dg$, that is,

$$(K_*\mu)(M) := \frac{1}{|K|}\int_K \mu(g^{-1}.M)\, dg$$

for all $M \subseteq X$ measurable. Note that $K_*\mu \in \mathcal{M}(X)$. For a left (or right) Følner sequence $\mathcal{F} = (F_n)$ in G and $\mu \in \mathcal{M}(X)$, we say that x is *$\mathcal{F}$-generic for μ* if $((F_n)_*\delta_x)_{n\in\mathbb{N}}$ converges to μ in $\mathcal{M}(X)$. We say that x is *$\mathcal{F}$-generic* if it is $\mathcal{F}$-generic for some $\mu \in \mathcal{M}(X)$. Note that for $x \in X$, any left Følner sequence $\mathcal{F}$ allows for a subsequence $\mathcal{F}'$ such that x is $\mathcal{F}'$-generic. Furthermore, from [**FGL22**, Theorem 2.4], we know that for an ergodic measure $\mu \in \mathcal{M}_G(X)$ and any left Følner sequence $\mathcal{F}$, there exists a subsequence $\mathcal{F}'$ such that μ-almost every point is $\mathcal{F}$-generic for μ.

It follows from a standard Krylov–Bogolyubov argument that for a left Følner sequence $\mathcal{F} = (F_n)$ and $x \in X$, any cluster point of $((F_n)_*\delta_x)$ is invariant. Thus, any point in a uniquely ergodic action is $\mathcal{F}$-generic with respect to all left Følner sequences $\mathcal{F}$. Furthermore, from the sequential compactness of $\mathcal{M}(X)$, it is straightforward to observe that $\mathcal{M}_G(X) \neq \emptyset$. Another straightforward argument yields that for a factor map $\pi\colon X \to Y$

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and an ergodic $\nu \in \mathcal{M}_G(Y)$, there exists $\mu \in \mathcal{M}_G(X)$ with $\pi_*\mu = \nu$. These observations rely on the amenability of G.

2.7. *Subadditivity techniques.* A function $H\colon \mathcal{H}'(G) \to \mathbb{R}^+$ is called *right dominated* if for all $K \in \mathcal{H}'(G)$, we have $\sup_{g\in G} H(Kg)/|Kg| < \infty$. It is called *right measurable* if the function $G \to \mathbb{R}^+$ defined by $g \mapsto H(Kg)$ is measurable for all $K \in \mathcal{H}'(G)$. A right measurable function $H\colon \mathcal{H}'(G) \to \mathbb{R}^+$ is said to satisfy (Ś), if for all $K, F \in \mathcal{H}'(G)$, we have

$$H(F) \le \frac{1}{|K|} \int_{K^{-1}F} H(Kg)\, dg. \tag{Ś}$$

Note that the integral in the above formula is well defined, since $g \mapsto H(Kg) \in \mathbb{R}^+$ is measurable.

Remark 2.1. For countable discrete amenable groups G, Shearer's inequality [**DFR16**, Definition 2.1] implies (Ś). This implication follows from the arguments presented in the proof of [**DFR16**, Proposition 3.3]. However, since we are not aware of an appropriate formulation of Shearer's inequality for non-discrete amenable groups, we introduce (Ś), which is better suited to our setting. Note that (Ś) serves as an intermediate condition between Shearer's inequality and the so-called 'infimum rule' [**DFR16**, Definition 3.1], which was implicitly employed in the proof of [**DFR16**, Proposition 3.3].

Example 2.2. For bounded and measurable $f\colon G \to \mathbb{R}^+$, the function

$$H\colon \mathcal{H}'(G) \to \mathbb{R}^+\colon K \mapsto \int_K f(g)\, dg$$

is right dominated, right measurable and satisfies (Ś).

Proof. Since compact subsets of G have finite Haar measure, and f is bounded and measurable, the integral defining H is well defined and finite. Furthermore, the boundedness of f yields

$$\sup_{g\in G} \frac{H(Kg)}{|Kg|} = \sup_{g\in G} \frac{1}{|Kg|} \int_{Kg} f(h)\, dh \le \|f\|_\infty$$

for all $K \in \mathcal{H}'(G)$, that is, that H is right dominated. From the continuity of the multiplication $G \times G \to G$, we know that $(h, g) \mapsto f(hg)$ is measurable. Using the Fubini–Tonelli theorem, we observe that H is right measurable and compute the following for $K, F \in \mathcal{H}'(G)$:

$$\begin{aligned}\int_{K^{-1}F} H(Kg)\, dg &= \int_{K^{-1}F} \int_K f(hg)\, dh\, dg = \int_K \int_{K^{-1}F} f(hg)\, dg\, dh \\ &\ge \int_K \int_{h^{-1}F} f(hg)\, dg\, dh = |K| \int_F f(g)\, dg = |K| H(F).\end{aligned}$$

This verifies that H satisfies (Ś). □

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PROPOSITION 2.3. *Let $H\colon \mathcal{H}'(G) \to \mathbb{R}^+$ be a right dominated and right measurable function that satisfies ($\mathfrak{S}$), and (F_n') a Følner sequence in G. We have*

$$\lim_{n\to\infty} \sup_{g\in G} \frac{H(F_n'g)}{|F_n'g|} = \inf_{K\in\mathcal{H}'(G)} \sup_{g\in G} \frac{H(Kg)}{|Kg|} = \sup_{(F_n)} \limsup_{n\to\infty} \frac{H(F_n)}{|F_n|} < \infty,$$

where the supremum, taken over all Følner sequences (F_n), is attained. Moreover, the same statement holds with lim sup *replaced by* lim inf.

Remark 2.4. G is called *unimodular* if $|\cdot|$ is additionally right invariant. Note that Abelian lc_σ groups, as well as discrete groups, are unimodular. If G is a unimodular amenable lc_σ group, then the above formulae can be simplified by rewriting $H(Kg)/|Kg| = H(Kg)/|K|$ for $K \in \mathcal{H}'(G)$.

The *modular function* $\chi_G\colon G \to (0,\infty)$ is defined by $\chi_G(g) := |Kg|/|K|$. We regard $(0,\infty)$ as a multiplicative group. It is known that χ_G is a continuous group homomorphism and that it is independent of the choice of $K \in \mathcal{H}'(G)$ [**DE14**, Ch. 1]. For non-unimodular groups, $\chi_G \not\equiv 1$ and, hence, $\chi_G(G)$ is a non-trivial subgroup of $(0,\infty)$. It follows that $\chi_G(G)$ is unbounded. For H defined by $H(K) := |K|$, we know from Example 2.2 (with $f \equiv 1$) that H is right dominated, right measurable and satisfies ($\mathfrak{S}$). Nevertheless, for any $K \in \mathcal{H}'(G)$, we have $\sup_{g\in G}(H(Kg)/|Kg|) = 1 < \infty = \sup_{g\in G} \chi_G(g) = \sup_{g\in G}(H(Kg)/|K|)$. Thus, the discussed simplification is only possible in the context of unimodular groups.

Remark 2.5. A function $H\colon \mathcal{H}'(G) \to \mathbb{R}^+$ is called *right invariant* if $H(Kg) = H(K)$ holds for all $g \in G$. If G is unimodular, and H is right invariant and satisfies ($\mathfrak{S}$), then H is right dominated and right measurable, and Proposition 2.3 yields that for any Følner sequence (F_n) in G, we have

$$\frac{H(F_n)}{|F_n|} \to \inf_{K\in\mathcal{H}'(G)} \frac{H(K)}{|K|}.$$

Recall from Remark 2.1 that, for countable discrete groups, Shearer's inequality implies ($\mathfrak{S}$). Thus, Proposition 2.3 extends [**DFR16**, Proposition 3.3], from which the present result was inspired.

Proof of Proposition 2.3. Since H is right dominated, for each $n \in \mathbb{N}$, we have $\sup_{g\in G}(H(F_n'g)/|F_n'g|) < \infty$ and find $g_n \in G$ with

$$\sup_{g\in G} \frac{H(F_n'g)}{|F_n'g|} \le \frac{H(F_n'g_n)}{|F_n'g_n|} + \frac{1}{n}.$$

Since $(F_n'g_n)$ is a left Følner sequence and $F_n' \in \mathcal{H}'(G)$ for every n, we have

$$\begin{aligned}\inf_{K\in\mathcal{H}'(G)} \sup_{g\in G} \frac{H(Kg)}{|Kg|} &\le \liminf_{n\to\infty} \sup_{g\in G} \frac{H(F_n'g)}{|F_n'g|} \le \limsup_{n\to\infty} \sup_{g\in G} \frac{H(F_n'g)}{|F_n'g|} \\ &\le \limsup_{n\to\infty} \frac{H(F_n'g_n)}{|F_n'g_n|} \le \sup_{(F_n)} \limsup_{n\to\infty} \frac{H(F_n)}{|F_n|},\end{aligned}$$

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where the supremum is taken over all Følner sequences (F_n). Similarly, we observe

$$\inf_{K\in\mathcal{H}'(G)} \sup_{g\in G} \frac{H(Kg)}{|Kg|} \leq \sup_{(F_n)} \liminf_{n\to\infty} \frac{H(F_n)}{|F_n|} \leq \sup_{(F_n)} \limsup_{n\to\infty} \frac{H(F_n)}{|F_n|}.$$

Since H is right dominated, we have $\inf_{K\in\mathcal{H}'(G)} \sup_{g\in G}(H(Kg)/|Kg|) < \infty$. Thus, it remains to show that for every Følner sequence (F_n), we have

$$\limsup_{n\to\infty} \frac{H(F_n)}{|F_n|} \leq \inf_{K\in\mathcal{H}'(G)} \sup_{g\in G} \frac{H(Kg)}{|Kg|}. \tag{♠}$$

For this, consider $K \in \mathcal{H}'(G)$ and $\epsilon > 0$. Recall from Remark 2.4 that χ_G denotes the modular function. Since $K^{-1}F_n$ is compact and χ_G is continuous, we find $h_n \in K^{-1}F_n$ with $\chi_G(h_n) = \max_{g\in K^{-1}F_n} \chi_G(g)$. For $g \in (Kh_n)^{-1}F_n = h_n^{-1}K^{-1}F_n$, we have $h_n g \in K^{-1}F_n$ and, hence,

$$\chi_G(g) = \chi_G(h_n^{-1})\chi_G(h_n g) \leq \chi_G(h_n^{-1})\chi_G(h_n) = 1.$$

Thus, from (𝔖), we observe

$$\begin{aligned} H(F_n) &\leq \frac{1}{|Kh_n|} \int_{(Kh_n)^{-1}F_n} H(Kh_n g)\, dg = \int_{(Kh_n)^{-1}F_n} \frac{|Kh_n g|}{|Kh_n|} \frac{H(Kh_n g)}{|Kh_n g|}\, dg \\ &\leq \left(\int_{(Kh_n)^{-1}F_n} \chi_G(g)\, dg \right) \left(\sup_{g\in G} \frac{H(Kg)}{|Kg|} \right) \\ &\leq |(Kh_n)^{-1}F_n| \left(\sup_{g\in G} \frac{H(Kg)}{|Kg|} \right). \end{aligned}$$

For large n, we have $|K^{-1}F_n \Delta F_n| \leq \epsilon |F_n|$ and, hence,

$$|(Kh_n)^{-1}F_n| = |h_n^{-1}K^{-1}F_n| = |K^{-1}F_n| \leq |K^{-1}F_n \Delta F_n| + |F_n| \leq (1+\epsilon)|F_n|.$$

This shows

$$H(F_n) \leq |(Kh_n)^{-1}F_n| \left(\sup_{g\in G} \frac{H(Kg)}{|Kg|} \right) \leq (1+\epsilon)|F_n| \left(\sup_{g\in G} \frac{H(Kg)}{|Kg|} \right)$$

for large n and, hence,

$$\limsup_{n\to\infty} \frac{H(F_n)}{|F_n|} \leq (1+\epsilon) \sup_{g\in G} \frac{H(Kg)}{|Kg|}.$$

The claim (♠) follows from the arbitrariness of $\epsilon > 0$. □

2.8. *Upper densities.* Let G be an amenable lc_σ group. Consider a measurable subset $G' \subseteq G$. For a left (or right) Følner sequence $\mathcal{F} = (F_n)$, we define the *upper asymptotic density* as $\mathrm{ua}\text{-}\mathrm{dens}_{\mathcal{F}}(G') := \limsup_{n\to\infty} |G' \cap F_n|/|F_n|$ and the *lower asymptotic density* as $\mathrm{la}\text{-}\mathrm{dens}_{\mathcal{F}}(G') := \liminf_{n\to\infty} |G' \cap F_n|/|F_n|$. Furthermore, we define the *upper Banach density* as $\mathrm{ub}\text{-}\mathrm{dens}(G') := \sup_{\mathcal{F}} \mathrm{ua}\text{-}\mathrm{dens}_{\mathcal{F}}(G')$, where the supremum is taken over all left Følner sequences $\mathcal{F}$ in G. The following lemma shows that our definition agrees with the definition in the literature. See [**DHZ19**, §2.2] and the references therein for more details in the context of actions of countable amenable groups.

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LEMMA 2.6. *For any measurable $G' \subseteq G$, we have*

$$\text{ub-dens}(G') = \sup_{\mathcal{F}} \text{ua-dens}_{\mathcal{F}}(G') = \sup_{\mathcal{F}} \text{la-dens}_{\mathcal{F}}(G') = \inf_{K \in \mathcal{H}'(G)} \sup_{g \in G} \frac{|G' \cap Kg|}{|Kg|},$$

where the suprema, taken over all left Følner sequences $\mathcal{F}$ in G, are attained. Furthermore, for every left Følner sequence (F_n), we have

$$\sup_{g \in G} \frac{|G' \cap F_n g|}{|F_n g|} \to \text{ub-dens}(G').$$

Proof. For $K \in \mathcal{H}'(G)$, denote $H(K) := |G' \cap K|$. Since the characteristic function $\mathbf{1}_{G'} \colon G \to \mathbb{R}^+$ is bounded, measurable and satisfies $H(K) = \int_K \mathbf{1}_{G'}(g)\, dg$, we observe from Example 2.2 that $H \colon \mathcal{H}'(G) \to \mathbb{R}^+$ is right dominated, right measurable and satisfies ($\mathfrak{S}$). Thus, the statement follows from Proposition 2.3. □

3. *Averages of functions*

Throughout this section, let X be an action of an amenable lc_σ group G. We denote by $\mathcal{B}(X)$ the space of all bounded functions $X \to \mathbb{R}$ and by $C(X)$ the subspace of continuous functions. We equip $\mathcal{B}(X)$ and $C(X)$ with the supremum norm and the pointwise ordering. For $g \in G$ and $f \in \mathcal{B}(X)$, we denote $g^* f := f \circ g$, where g is identified with the homeomorphism $x \mapsto g.x$. A function $f \in \mathcal{B}(X)$ is called *invariant* if $g^* f = f$ for all $g \in G$. It is called *positive* if $f(x) \geq 0$ holds for all $x \in X$. We denote by $\mathcal{B}_G(X)$ and $C_G(X)$ the sets of invariant functions in $\mathcal{B}(X)$ and $C(X)$, respectively. Similarly, we write $C^+(X)$ and $\mathcal{B}^+(X)$ for the respective sets of positive functions, and abbreviate $C_G^+(X) := C^+(X) \cap C_G(X)$ and $\mathcal{B}_G^+(X) := \mathcal{B}^+(X) \cap \mathcal{B}_G(X)$. For $K \in \mathcal{H}'(G)$, $f \in C(X)$ and $x \in X$, we denote

$$(K^* f)(x) := \frac{1}{|K|} \int_K f(g.x)\, dg$$

and observe that this establishes a linear contraction $K^* \colon C(X) \to \mathcal{B}(X)$. For a left (or right) Følner sequence $\mathcal{F} = (F_n)$ in G, we denote $\mathbf{A}_{\mathcal{F}} f := \limsup_{n \to \infty} F_n^* f$. Furthermore, we denote $\mathbf{A} f := \sup_{\mathcal{F}} \mathbf{A}_{\mathcal{F}} f$, where the supremum is taken over all left Følner sequences. We observe that $\mathbf{A}$ and $\mathbf{A}_{\mathcal{F}}$ are mappings $C(X) \to \mathcal{B}(X)$. We next summarize some of their properties, omitting the straightforward proofs.

PROPOSITION 3.1. *The mapping $\mathbf{A} \colon C(X) \to \mathcal{B}(X)$ is:*

- *a* contraction, *that is, $\|\mathbf{A} f - \mathbf{A} h\|_\infty \leq \|f - h\|_\infty$ for all $f, h \in C(X)$;*
- monotone, *that is, $\mathbf{A} f \leq \mathbf{A} h$ for all $f, h \in C(X)$ with $f \leq h$;*
- subadditive, *that is, $\mathbf{A}(f + h) \leq \mathbf{A} f + \mathbf{A} h$ for all $f, h \in C(X)$;*
- positively homogeneous, *that is, $\mathbf{A}(\lambda f) = \lambda \mathbf{A} f$ for $f \in C(X)$ and $\lambda \in \mathbb{R}^+$;*
- $C_G^+(X)$-multiplicative, *that is, $\mathbf{A}(hf) = h \mathbf{A} f$ for $f \in C(X)$, $h \in C_G^+(X)$;*
- translation invariant, *that is, $\mathbf{A}(f + \lambda) = (\mathbf{A} f) + \lambda$ for $f \in C(X)$ and $\lambda \in \mathbb{R}$;*
- $C_G(X)$-additive, *that is, $\mathbf{A}(h + f) = h + \mathbf{A} f$ for $f \in C(X)$ and $h \in C_G(X)$.*

The same properties are satisfied for $\mathbf{A}_{\mathcal{F}}$ for any left (or right) Følner sequence $\mathcal{F}$. Furthermore, the mapping $\mathbf{A} \colon C(X) \to \mathcal{B}(X)$:

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- *is* invariant, *that is,* $\mathbf{A}(g^* f) = \mathbf{A} f$ *for all* $f \in C(X)$ *and* $g \in G$*; the same property is satisfied for* $\mathbf{A}_{\mathcal{F}}$ *for any left Følner sequence* $\mathcal{F}$*;*
- *satisfies* $\mathbf{A} f \in \mathcal{B}_G(X)$ *for all* $f \in C(X)$*; the same property is satisfied for* $\mathbf{A}_{\mathcal{F}}$ *for any right Følner sequence* $\mathcal{F}$*.*

We next summarize some equivalent approaches for computing $\mathbf{A}$.

PROPOSITION 3.2. *For* $f \in C(X)$*, we have*

$$\mathbf{A} f = \sup_{\mathcal{F}} \limsup_{n\to\infty} F_n^* f = \sup_{\mathcal{F}} \liminf_{n\to\infty} F_n^* f = \inf_{K\in\mathcal{H}'(G)} \sup_{g\in G} (Kg)^* f,$$

where the suprema, taken over all left Følner sequences $\mathcal{F} = (F_n)$*, are attained pointwise. Furthermore, for every left Følner sequence* (F_n) *in* G*, we have that* $\sup_{g\in G}(F_n g)^* f$ *converges pointwise to* $\mathbf{A} f$*.*

Proof. For $f \in C(X)$, there exists $\lambda \in \mathbb{R}$ such that $f + \lambda \geq 0$. Since all terms above are easily seen to be translation invariant, we assume without loss of generality that f is positive. Fix $x \in X$ and define $f_x \colon G \to \mathbb{R}^+$ by $f_x(g) := f(g.x)$. Since f_x is bounded and measurable, Example 2.2 implies that $H \colon \mathcal{H}'(G) \to \mathbb{R}^+$ defined by $H(K) := \int_K f_x(g)\, dg$ is right dominated, right measurable and satisfies (ℌ). We have $H(K)/|K| = (K^* f)(x)$ for all $K \in \mathcal{H}'(G)$ and conclude the statement from Proposition 2.3. □

PROPOSITION 3.3. *For* $f \in C(X)$ *and a countable family* $N \subseteq X$*, there exists a left Følner sequence* $\mathcal{F}$ *in* G *with* $\mathbf{A} f(x) = \mathbf{A}_{\mathcal{F}} f(x)$ *for all* $x \in N$*.*

Proof. Denote $I := N \times \mathbb{N}$ and note that I is countable. For $i = (x, k) \in I$, we consider a left Følner sequence $\mathcal{F}^{(i)} = (F_n^{(i)})$ that satisfies

$$\mathbf{A} f(x) \leq \lim_{n\to\infty} (F_n^{(i)})^* f(x) + \frac{1}{k}.$$

Since $(\mathcal{F}^{(i)})_{i\in I}$ is a countable family of left Følner sequences, there exists a left Følner sequence $\mathcal{F} = (F_n)$ that has a common subsequence with each $\mathcal{F}^{(i)}$ [**FGH25**, Proposition 2.6]. For $x \in N$ and $k \in \mathbb{N}$, we have $i := (x, k) \in I$ and observe

$$\mathbf{A} f(x) \leq \lim_{n\to\infty} (F_n^{(i)})^* f(x) + \frac{1}{k} \leq \limsup_{n\to\infty} F_n^* f(x) + \frac{1}{k} = \mathbf{A}_{\mathcal{F}} f(x) + \frac{1}{k}.$$

We thus have $\mathbf{A} f(x) = \mathbf{A}_{\mathcal{F}} f(x)$ for all $x \in N$. □

For $f, h \in \mathcal{B}^+(X)$ and $\epsilon, \delta > 0$, we say that f is *ϵ-δ-majorizing for h* if for all $x \in X$ with $f(x) \leq \delta$, we have $h(x) \leq \epsilon$.

LEMMA 3.4. *Consider* $\epsilon, \delta' > 0$ *and* $f, h \in C^+(X)$*.*
Whenever f *is* $(\epsilon/2)$*-*δ'*-majorizing for* h*, then there exists* $\delta > 0$ *such that:*

(i) $\mathbf{A} f$ *is* ϵ*-*δ*-majorizing for* $\mathbf{A} h$*; and*
(ii) $\mathbf{A}_{\mathcal{F}} f$ *is* ϵ*-*δ*-majorizing for* $\mathbf{A}_{\mathcal{F}} h$ *for all left (or right) Følner sequences* $\mathcal{F}$*.*

Proof. By the positive homogeneity of $\mathbf{A}_{\mathcal{F}}$ and $\mathbf{A}$, we assume without loss of generality that $h \leq 1$. Observe that statement (i) is a straightforward consequence of statement (ii).

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Denote $\delta := \epsilon\delta'/4$. For $x \in X$, we define positive functions $f_x, h_x \colon G \to \mathbb{R}$ by $f_x(g) := f(g.x)$ and $h_x(g) := h(g.x)$.

For a left (or right) Følner sequence $\mathcal{F} = (F_n)$, we consider $x \in X$ with $\mathbf{A}_{\mathcal{F}}(f)(x) \le \delta$. For all sufficiently large n, we observe that $F_n^* f(x) \le 2\delta$. Note that $\mu := |(\cdot) \cap F_n|/|F_n|$ defines a probability measure on G. We have $2\delta \ge F_n^* f(x) = \mu(f_x) \ge \int_{f_x > \delta'} \delta' d\mu$ and, hence, $\int_{f_x > \delta'} 1 \, d\mu \le \epsilon/2$. Since $h_x \le 1$ holds and since $f_x(g) \le \delta'$ implies $h_x(g) \le \epsilon/2$ for all $g \in G$, we have

$$F_n^* h(x) = \mu(h_x) \le \int_{f_x \le \delta'} \frac{\epsilon}{2} \, d\mu + \int_{f_x > \delta'} 1 \, d\mu \le \epsilon.$$

We observe $\mathbf{A}_{\mathcal{F}}(h)(x) = \limsup_{n\to\infty} F_n^* h(x) \le \epsilon$. □

The idea of the following lemma can be found in [**GRLZ19**, Remark 2.1]. We include the short proof for the convenience of the reader.

LEMMA 3.5. *Let $f \in C^+(X)$ and $\mathcal{F}$ be a left (or right) Følner sequence in G. Let $x \in X$ and $\epsilon > 0$, and denote $G_{x,\epsilon} := \{g \in G \mid f(g.x) > \epsilon\}$.*

(i) *Whenever $\mathbf{A}_{\mathcal{F}} f(x) \le \epsilon^2$, then we have* ua-dens$_{\mathcal{F}}(G_{x,\epsilon}) \le \epsilon$.
(ii) *Whenever* ua-dens$_{\mathcal{F}}(G_{x,\epsilon}) \le \epsilon$, *then we have $\mathbf{A}_{\mathcal{F}} f(x) \le (\|f\|_\infty + 1)\epsilon$.*
(iii) *Whenever $\mathbf{A} f(x) \le \epsilon^2$, then we have* ub-dens$(G_{x,\epsilon}) \le \epsilon$.
(iv) *Whenever* ub-dens$(G_{x,\epsilon}) \le \epsilon$, *then we have $\mathbf{A} f(x) \le (\|f\|_\infty + 1)\epsilon$.*

Proof. Observe that $G_{x,\epsilon}$ is open and, hence, measurable. (i) We compute

$$\text{ua-dens}_{\mathcal{F}}(G_{x,\epsilon}) \le \frac{1}{\epsilon} \limsup_{n\to\infty} \frac{1}{|F_n|} \int_{F_n \cap G_{x,\epsilon}} f(g.x) \, dg \le \frac{1}{\epsilon} \mathbf{A}_{\mathcal{F}} f(x) \le \epsilon.$$

(ii) We observe

$$\begin{aligned} \mathbf{A}_{\mathcal{F}} f(x) &\le \limsup_{n\to\infty} \frac{1}{|F_n|} \int_{F_n \cap G_{x,\epsilon}} f(g.x) \, dg + \limsup_{n\to\infty} \frac{1}{|F_n|} \int_{F_n \setminus G_{x,\epsilon}} f(g.x) \, dg \\ &\le \text{ua-dens}_{\mathcal{F}}(G_{x,\epsilon}) \|f\|_\infty + \epsilon \, \text{ua-dens}_{\mathcal{F}}(G \setminus G_{x,\epsilon}) \le (\|f\|_\infty + 1)\epsilon. \end{aligned}$$

Statements (iii) and (iv) are direct consequences of statements (i) and (ii). □

4. *The mean diameter*

Let G be an amenable lc_σ group. Consider an action X of G on a compact metric space. Let d be a continuous metric on X and note that $\mathrm{diam}_d \in C^+(\mathcal{H}(X))$. Considering the induced action of G on $\mathcal{H}(X)$, the *mean diameter (of d)* is given by $\mathrm{Diam}_d := \mathbf{A}(\mathrm{diam}_d)$. Similarly, for a left (or right) Følner sequence $\mathcal{F}$, we consider the *mean diameter (of d) with respect to* $\mathcal{F}$, which is defined by $\mathrm{Diam}_{\mathcal{F},d} := \mathbf{A}_{\mathcal{F}}(\mathrm{diam}_d)$. If the metric is clear from the context, we simply write Diam and Diam$_{\mathcal{F}}$.

Remark 4.1. Proposition 3.1 implies that for any $A \in \mathcal{H}(X)$ and $g \in G$, we have $\mathrm{Diam}(g.A) = \mathrm{Diam}(A)$, as well as $\mathrm{Diam}_{\mathcal{F}}(g.A) = \mathrm{Diam}_{\mathcal{F}}(A)$ for any right Følner sequence $\mathcal{F}$.

The following result is a direct consequence of Proposition 3.2.

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PROPOSITION 4.2. *For $A \in \mathcal{H}(X)$, we have*

$$\begin{aligned}\operatorname{Diam}(A) &= \sup_{\mathcal{F}} \operatorname{Diam}_{\mathcal{F}}(A) = \sup_{\mathcal{F}} \liminf_{n\to\infty} \frac{1}{|F_n|} \int_{F_n} \operatorname{diam}(g.A)\, dg \\ &= \inf_{K\in\mathcal{H}'(G)} \sup_{h\in G} \frac{1}{|Kh|} \int_{Kh} \operatorname{diam}(g.A)\, dg,\end{aligned}$$

where the suprema, taken over all left Følner sequences $\mathcal{F} = (F_n)$, are attained. Furthermore, for every left Følner sequence (F_n), we have

$$\sup_{h\in G} \frac{1}{|F_n h|} \int_{F_n h} \operatorname{diam}(g.A)\, dg \to \operatorname{Diam}(A).$$

Remark 4.3. The Følner sequences attaining the suprema in Proposition 4.2 depend on A.

5. *Regularity and diam-mean proximality*

For a factor map $\pi\colon X \to Y$, we call $x \in X$ an *injectivity point* of π whenever $\pi^{-1}(\pi(x)) = \{x\}$. Note that the set of injectivity points is a G_δ set, i.e. a countable intersection of open sets, [**AG01**, Lemma 1.1], and hence measurable. A factor map $\pi\colon X \to Y$ is called *regular* if for all $\mu \in \mathcal{M}_G(X)$, the set of injectivity points X_0 satisfies $\mu(X_0) = 1$. From the ergodic representation, we observe that π is regular if and only if $\mu(X_0) = 1$ for all ergodic $\mu \in \mathcal{M}_G(X)$. See [**Far62**, Lemma 6] for details.

Remark 5.1. Denote $Y_0 := \pi(X_0)$ and note that $\pi(X \setminus X_0) = Y \setminus \pi(X_0)$. Since closed maps map F_σ sets to F_σ sets, it follows that Y_0 is G_δ and, hence, measurable. (A subset of a topological space is called F_σ if it is the complement of a G_δ set, that is, the countable union of closed sets.)

Since we assume G to be amenable, for ergodic $\nu \in \mathcal{M}_G(Y)$, there exists $\mu \in \mathcal{M}_G(X)$ with $\pi_*\mu = \nu$. We thus observe that π is regular if and only if $\nu(Y_0) = 1$ for all ergodic $\nu \in \mathcal{M}_G(Y)$. Another application of the ergodic representation yields that π is regular if and only if $\nu(Y_0) = 1$ for all $\nu \in \mathcal{M}_G(Y)$.

Let $\pi\colon X \to Y$ be a factor map. We denote $\mathcal{H}_\pi(X)$ for the set of all $A \in \mathcal{H}(X)$ for which there exists $y \in Y$ with $A \subseteq \pi^{-1}(y)$. Observe that $\mathcal{H}_\pi(X)$ and the set of singletons $\mathfrak{F}_1(X)$ are closed invariant (non-empty) subsets of $\mathcal{H}(X)$, and that $\mathfrak{F}_1(X) \subseteq \mathcal{H}_\pi(X)$. In the following, we consider the induced action on $\mathcal{H}_\pi(X)$ and $\mathfrak{F}_1(X)$. Note that X is conjugated to $\mathfrak{F}_1(X)$ via $x \mapsto \{x\}$.

For $A \in \mathcal{H}_\pi(X)$, there exists a unique $y_A \in Y$ with $\pi(A) = \{y_A\}$. The mapping $\mathcal{H}_\pi(X) \to Y$ that assigns $A \mapsto y_A$ will be denoted by $\pi_{\mathcal{H}}$. It is straightforward to verify that $\pi_{\mathcal{H}}\colon \mathcal{H}_\pi(X) \to Y$ is a factor map.

THEOREM 5.2. *For a factor map $\pi\colon X \to Y$, the following statements are equivalent:*

(i) *π is regular;*

(ii) *π is diam-mean proximal, that is, for all $y \in Y$, we have* $\operatorname{Diam}(\pi^{-1}(y)) = 0$*;*

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(ii′) *For all $y \in Y$ and all $\epsilon > 0$, we have*

$$\text{ub-dens}(\{g \in G \mid \text{diam}(\pi^{-1}(g.y)) > \epsilon\}) \leq \epsilon;$$

(iii) *There exists a left Følner sequence $\mathcal{F}$ such that π is $\mathcal{F}$-diam-mean proximal, that is, such that for all $y \in Y$, we have* $\text{Diam}_{\mathcal{F}}(\pi^{-1}(y)) = 0$*;*

(iii′) *There exists a left Følner sequence $\mathcal{F}$ such that for all $y \in Y$ and all $\epsilon > 0$, we have*

$$\text{ua-dens}_{\mathcal{F}}(\{g \in G \mid \text{diam}(\pi^{-1}(g.y)) > \epsilon\}) \leq \epsilon;$$

(iv) *$\pi_{\mathcal{H}} \colon \mathcal{H}_\pi(X) \to Y$ is regular;*

(v) $\mathcal{M}_G(\mathcal{H}_\pi(X)) = \mathcal{M}_G(\mathfrak{F}_1(X))$.

Remark 5.3. Since G induces an action on $\mathcal{H}(X)$ and $\text{diam} \in C^+(\mathcal{H}(X))$, we observe that for $A \in \mathcal{H}(X)$, the set $\{g \in G \mid \text{diam}(g.A) > \epsilon\}$ is open in G and, hence, measurable. From $g.\pi^{-1}(y) = \pi^{-1}(g.y)$, it follows that the subsets of G considered in statements (ii′) and (iii′) are measurable.

Proof of Theorem 5.2. Denote X_0 for the set of injectivity points of π and $Y_0 := \pi(X_0)$. It is straightforward to verify that the injectivity points of $\pi_{\mathcal{H}}$ are given by $\mathfrak{F}_1(X_0) = \pi_{\mathcal{H}}^{-1}(Y_0)$. With this observation, it follows from Remark 5.1 that statements (i) and (iv) are equivalent.

(iv) $\Rightarrow$ (v) for $\mu \in \mathcal{M}_G(\mathcal{H}_\pi(X))$, we have that $\mu(\mathfrak{F}_1(X_0)) = 1$ and, hence, $\mu(\mathfrak{F}_1(X)) = 1$. This shows $\mu \in \mathcal{M}_G(\mathfrak{F}_1(X))$.

(v) $\Rightarrow$ (ii) let $A \in \mathcal{H}_\pi(X)$ and consider a left Følner sequence $\mathcal{F} = (F_n)$. Choose a subsequence $\mathcal{F}' = (F_n')$ such that $\text{Diam}_{\mathcal{F}}(A) = \lim_{n\to\infty}(F_n')^* \text{diam}(A)$. By a standard Krylov–Bogolyubov argument and by possibly considering a further subsequence, we assume without loss of generality that A is $\mathcal{F}'$-generic for some $\mu \in \mathcal{M}_G(\mathcal{H}_\pi(X)) = \mathcal{M}_G(\mathfrak{F}_1(X))$. Since $\text{diam} \equiv 0$ on $\mathfrak{F}_1(X)$, we thus have

$$0 = \mu(\text{diam}) = \lim_{n\to\infty}((F_n')_*\delta_A)(\text{diam}) = \lim_{n\to\infty}(F_n')^*\text{diam}(A) = \text{Diam}_{\mathcal{F}}(A).$$

This shows $\text{Diam}_{\mathcal{F}}(A) = 0$ for any left Følner sequence $\mathcal{F}$. Taking the supremum over all left Følner sequences, we observe $\text{Diam}(A) = \sup_{\mathcal{F}} \text{Diam}_{\mathcal{F}}(A) = 0$.

(ii) $\Rightarrow$ (iii) follows from $\text{Diam}_{\mathcal{F}} \leq \text{Diam}$ for any left Følner sequence $\mathcal{F}$.

(iii) $\Rightarrow$ (i) any left Følner sequence $\mathcal{F}$ allows for a tempered subsequence $\mathcal{F}'$ [**Lin01**, Proposition 1.4]. (A left Følner sequence $\mathcal{F} = (F_n)$ is called *tempered* if $\sup_{n\in\mathbb{N}}(|\bigcup_{k<n} F_k^{-1}F_n|/|F_n|) < \infty$. We will not need this definition and only use that the pointwise ergodic theorem holds along $\mathcal{F}$. See [**Lin01**] for more details.) Since $\text{Diam}_{\mathcal{F}'} \leq \text{Diam}_{\mathcal{F}}$ holds, we assume without loss of generality that $\mathcal{F}$ is tempered.

If π is not regular, then there exist an ergodic $\nu \in \mathcal{M}_G(Y)$ with $\nu(Y_0) < 1$. Since Y_0 is invariant, we observe $\nu(Y_0) = 0$, that is, $\nu(Y \setminus Y_0) = 1$. Denote $A_\epsilon := \{y \in Y \mid \text{diam}(\pi^{-1}(y)) \geq \epsilon\}$ for $\epsilon > 0$. From $Y \setminus Y_0 = \bigcap_{n\in\mathbb{N}} A_{1/n}$, we observe that there exists $\epsilon > 0$ with $\nu(A_\epsilon) > 0$.

By the Lindenstrauss ergodic theorem [**Lin01**, Theorem 1.2], there exists $y \in Y$ such that $|F_n|^{-1}\int_{F_n} \mathbf{1}_{A_\epsilon}(g.y)\, dg \to \nu(A_\epsilon)$. Denoting $F_n' := \{g \in F_n \mid g.y \in A_\epsilon\}$, we observe

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$$\begin{aligned}\mathrm{Diam}_{\mathcal{F}}(\pi^{-1}(y)) &= \limsup_{n\to\infty} \frac{1}{|F_n|}\int_{F_n} \mathrm{diam}(g.\pi^{-1}(y))\,dg \\ &\geq \limsup_{n\to\infty} \frac{1}{|F_n|}\int_{F_n'} \mathrm{diam}(\pi^{-1}(g.y))\,dg \\ &\geq \lim_{n\to\infty} \frac{1}{|F_n|}\int_{F_n} \epsilon \mathbf{1}_{A_\epsilon}(g.y)\,dg = \epsilon\nu(A_\epsilon) > 0.\end{aligned}$$

(ii) $\Rightarrow$ (ii′) for $y \in Y$ and $\epsilon > 0$, consider $f := \mathrm{diam} \in C(\mathcal{H}(X))$ and $A := \pi^{-1}(y) \in \mathcal{H}(X)$. We have $\mathbf{A}f(A) = \mathrm{Diam}(\pi^{-1}(y)) = 0 \leq \epsilon^2$ and Lemma 3.5(iii) yields

$$\text{ub-dens}(\{g \in G \mid \mathrm{diam}(\pi^{-1}(g.x)) > \epsilon\}) = \text{ub-dens}(\{g \in G \mid f(g.A) > \epsilon\}) \leq \epsilon.$$

(ii′) $\Rightarrow$ (ii) consider $y \in Y$ and $\epsilon > 0$. With $f := \mathrm{diam}$ and $A := \pi^{-1}(y)$, we have ub-dens$(\{g \in G \mid f(g.A) > \epsilon\}) \leq \epsilon$ and Lemma 3.5(iv) yields

$$\mathrm{Diam}(\pi^{-1}(y)) = \mathbf{A}f(A) \leq (\|f\|_\infty + 1)\epsilon = (\mathrm{diam}(X) + 1)\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we observe $\mathrm{Diam}(\pi^{-1}(y)) = 0$.

(iii) $\Leftrightarrow$ (iii′) follows with a similar argument from Lemma 3.5. □

Remark 5.4. Theorem 5.2 also shows that the notion of diam-mean proximality of a factor map $\pi : X \to Y$ is independent of the choice of a metric on X. A similar statement holds about $\mathcal{F}$-diam-mean proximality for any left Følner sequence $\mathcal{F}$. Note that this can also be seen directly from Lemma 3.4.

Remark 5.5. A closely related (but not equivalent) property of factor maps is *Banach proximality*. It was introduced in [**LTY15**] and shown to be equivalent to topo-isomorphy in [**FGL22**, **Hau26**, **QZ20**]. See §7.1 for more details on these notions.

6. *Diam-mean equicontinuity*

Let X be an action (of G) and d be a continuous metric on X. We say that $x \in X$ is *diam-mean equicontinuous (with respect to d)* if for each $\epsilon > 0$, there exists a closed neighbourhood A of x such that $\mathrm{Diam}_d(A) < \epsilon$. The following proposition shows that diam-mean equicontinuity is independent of the choice of d. We will thus simply speak of diam-mean equicontinuous points.

PROPOSITION 6.1. *Let X be an action and d be a continuous metric on X. A point $x \in X$ is diam-mean equicontinuous with respect to d if and only if it is diam-mean equicontinuous with respect to any continuous metric on X.*

Proof. Assume that x is diam-mean equicontinuous with respect to d and let d' be another continuous metric on X. For $\epsilon > 0$, there exists $\delta' > 0$ such that d is $(\epsilon/2)$- δ'-majorizing for d'. We thus observe that diam_d is $(\epsilon/2)$- δ'-majorizing for $\mathrm{diam}_{d'}$. From Lemma 3.4, it follows that there exists $\delta > 0$ such that Diam_d is ϵ- δ-majorizing for $\mathrm{Diam}_{d'}$. Since x is a diam-mean equicontinuous point with respect to d, there exists a compact neighbourhood A of x such that $\mathrm{Diam}_d(A) \leq \delta$. For such A, we observe $\mathrm{Diam}_{d'}(A) \leq \epsilon$. □

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An action X is called *diam-mean equicontinuous* if all points $x \in X$ are diam-mean equicontinuous. Similarly, we define the notion of $\mathcal{F}$-diam-mean equicontinuity (for points and actions) for any left (or right) Følner sequence $\mathcal{F}$, by replacing Diam_d with $\mathrm{Diam}_{\mathcal{F},d}$. Note that a similar argument shows the independence from the choice of a metric also in this context and that any diam-mean equicontinuous point is $\mathcal{F}$-diam-mean equicontinuous.

Remark 6.2. It is straightforward to show that an action X is diam-mean equicontinuous if and only if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in X$, we have $\mathrm{Diam}(\mathbf{B}_\delta(x)) \leq \epsilon$. A similar statement holds for $\mathcal{F}$-diam-mean equicontinuity for any left (or right) Følner sequence $\mathcal{F}$.

Remark 6.3. Any equicontinuous action is diam-mean equicontinuous.

Remark 6.4. From the invariance of Diam (Remark 4.1), it is straightforward to observe that any $x \in X$ for which there exists a diam-mean equicontinuous point (with respect to X) in $\overline{O(x)}$ is diam-mean equicontinuous (with respect to X). In particular, we observe the following.

(i) The set of diam-mean equicontinuous points is invariant.
(ii) Whenever X contains a diam-mean equicontinuous point, then any transitive point in X is diam-mean equicontinuous.
(iii) A minimal action is diam-mean equicontinuous if and only if it allows for a diam-mean equicontinuous point.

A similar statement holds for $\mathcal{F}$-diam-mean equicontinuity whenever $\mathcal{F}$ is a right Følner sequence.

Remark 6.5. It was shown in [**GRJY21**, **Hau25**] that a minimal action of an Abelian group is diam-mean equicontinuous if and only if it allows for an $\mathcal{F}$-diam-mean equicontinuous point for some Følner sequence $\mathcal{F}$. For details, see [**GRJY21**, Theorem 4.12] and [**Hau25**, Theorem 5.6] (with $m = 2$).

Remark 6.6. It remains open whether the $\mathcal{F}$-diam-mean equicontinuity for some Følner sequence implies diam-mean equicontinuity in the context of actions of amenable lc_σ groups. See [**Hau25**, **HJL25**] for some advances in this direction.

Remark 6.7. Diam-mean equicontinuity is also called *Banach diam-mean equicontinuity* [**GRJY21**]. Furthermore, our terminology differs from that of [**GRJY21**], where minimal actions of $\mathbb{Z}$ (actually actions of the semigroup $\mathbb{N}$) were studied. For the standard Følner sequence $\mathcal{F} = (F_n)$ with $F_n = \{0, \ldots, n-1\}$, our concept of $\mathcal{F}$-diam-mean equicontinuity is simply called 'diam-mean equicontinuity' in [**GRJY21**].

In the context of actions of $\mathbb{Z}$, the following characterization was presented in [**GRJY21**, Lemma 4.4] and [**GRJY21**, Lemma 4.7]. It is a straightforward consequence of Lemma 3.5 and we will need it for the proof of Theorem 1.1.

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PROPOSITION 6.8. *Let X be an action.*

(i) *A point $x \in X$ is diam-mean equicontinuous if and only if for any $\epsilon > 0$, there exists a closed neighbourhood A of x such that*

$$\text{ub-dens}(\{g \in G \mid \operatorname{diam}(g.A) > \epsilon\}) \leq \epsilon.$$

(ii) *Let $\mathcal{F}$ be a left (or right) Følner sequence in G. A point $x \in X$ is $\mathcal{F}$-diam-mean equicontinuous if and only if for any $\epsilon > 0$, there exists a closed neighbourhood A of x such that*

$$\text{ua-dens}_{\mathcal{F}}(\{g \in G \mid \operatorname{diam}(g.A) > \epsilon\}) \leq \epsilon.$$

To show Theorem 1.1, we will also need the following characterization.

PROPOSITION 6.9. *Let X be an action.*

(i) *A point $x \in X$ is diam-mean equicontinuous if and only if x is $\mathcal{F}$-diam-mean equicontinuous with respect to all left Følner sequences $\mathcal{F}$.*

(ii) *The action X is diam-mean equicontinuous if and only if X is $\mathcal{F}$-diam-mean equicontinuous with respect to all left Følner sequences $\mathcal{F}$.*

Proof. Observe that statement (ii) is a direct consequence of statement (i). To show statement (i), let $\epsilon > 0$ and note that $\{\mathbf{B}_{1/k}(x) \mid k \in \mathbb{N}\}$ is a countable family in $\mathcal{H}(X)$. Thus, Proposition 3.3 yields the existence of a left Følner sequence $\mathcal{F}$ with

$$\operatorname{Diam}(\mathbf{B}_{1/k}(x)) = \mathbf{A}(\operatorname{diam})(\mathbf{B}_{1/k}(x)) = \mathbf{A}_{\mathcal{F}}(\operatorname{diam})(\mathbf{B}_{1/k}(x)) = \operatorname{Diam}_{\mathcal{F}}(\mathbf{B}_{1/k}(x))$$

for all $k \in \mathbb{N}$. Since x is assumed to be $\mathcal{F}$-diam-mean equicontinuous, there exists $k \in \mathbb{N}$ such that $\operatorname{Diam}(\mathbf{B}_{1/k}(x)) = \operatorname{Diam}_{\mathcal{F}}(\mathbf{B}_{1/k}(x)) \leq \epsilon$, showing x to be diam-mean equicontinuous. □

The following example illustrates that $\mathcal{F}$-diam-mean equicontinuity of points depends on the choice of a left Følner sequence.

Example 6.10. Consider $G := \mathbb{Z}$. Let $X = \mathbb{Z} \cup \{-\infty, \infty\}$ be the two-point compactification. We act on X with G by $g.x := g + x$ for $g \in G$ and $x \in \mathbb{Z}$, and by fixing $+\infty$ and $-\infty$. Consider the Følner sequences $\mathcal{F}$ and $\mathcal{F}'$ given by $F_n := \{0, \ldots, n\}$ and $F'_n := \{-n, \ldots, 0\}$. It is straightforward to observe that ∞ is $\mathcal{F}$-diam-mean equicontinuous, but not $\mathcal{F}'$-diam-mean equicontinuous. We also observe that ∞ is not diam-mean equicontinuous, while it is $\mathcal{F}$-diam-mean equicontinuous.

7. *Regularity and diam-mean equicontinuity*

In this section, we will prove Theorem 1.1. To do so, we begin by recalling some details from the theory of mean equicontinuous actions. See [**FGL22**] and the references therein for more details on mean equicontinuity.

7.1. *Mean equicontinuity.* Let X be an action and let d be a continuous metric on X. We consider the induced action of G on X^2. The *Weyl pseudo-metric* D is defined by $D := \mathbf{A}(d)$. An action is called *mean equicontinuous* whenever for all $x \in X$ and

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$\epsilon > 0$, there exists $\delta > 0$ such that for all $x' \in \mathbf{B}_\delta(x)$, we have $D(x, x') \le \epsilon$. Similarly, we define the *$\mathcal{F}$-Besicovitch pseudo-metric* $D_{\mathcal{F}} := \mathbf{A}_{\mathcal{F}}(d)$ and the concept of *$\mathcal{F}$-mean equicontinuity* for any left Følner sequence. Clearly, any diam-mean equicontinuous action is mean equicontinuous. Furthermore, any $\mathcal{F}$-diam-mean equicontinuous action is $\mathcal{F}$-mean equicontinuous. With an argument similar to that in the proof of Proposition 4.2, we observe the following. See [**ŁS18**, §3.3] for the statement in the context of countable amenable groups.

Remark 7.1. For $(x, x') \in X^2$, we have

$$\begin{aligned} D(x, x') &= \sup_{\mathcal{F}} D_{\mathcal{F}}(x, x') = \sup_{\mathcal{F}} \limsup_{n\to\infty} \frac{1}{|F_n|} \int_{F_n} d(g.x, g.x')\, dg \\ &= \inf_{K \in \mathcal{H}'(G)} \sup_{h \in G} \frac{1}{|Kh|} \int_{Kh} d(g.x, g.x')\, dg, \end{aligned}$$

where the suprema, taken over all left Følner sequences $\mathcal{F} = (F_n)$, are attained. Furthermore, for every left Følner sequence (F_n), we have

$$\sup_{h\in G} \frac{1}{|F_n h|} \int_{F_n h} d(g.x, g.x')\, dg \to D(x, x').$$

A factor map $\pi : X \to Y$ is called *Banach proximal* if for all $y \in Y$ and $x, x' \in \pi^{-1}(y)$, we have $D(x, x') = 0$ [**Hau26**, **LTY15**]. It follows from Theorem 5.2 that any regular factor map is Banach proximal.

Remark 7.2. A factor map $\pi : X \to Y$ is called *topo-isomorphic* if for each $\mu \in \mathcal{M}_G(X)$, the map π induces a measure-theoretic isomorphism between (X, μ) and $(Y, \pi_*\mu)$. It was shown in [**Hau26**, Theorem 8.1] that a factor map $\pi : X \to Y$ is Banach proximal if and only if it is topo-isomorphic. (Although [**Hau26**] presents the argument only for countable amenable groups, it extends straightforwardly to the context of amenable lc_σ groups.) See [**DG16**, **FGL22**, **Hau26**] for further details on topo-isomorphic factor maps.

We will need the following results of [**FGL22**].

PROPOSITION 7.3. [**FGL22**, Theorem 3.16]

(i) *If $\pi : X \to Y$ is a Banach proximal factor map onto an equicontinuous action Y, then π is the factor map onto the maximal equicontinuous factor of X.*

(ii) *An action X is mean equicontinuous if and only if the factor map onto the maximal equicontinuous factor is Banach proximal.*

If X is an action, then there exists a smallest closed subset $A \subseteq X$ such that $\mu(A) = 1$ for all $\mu \in \mathcal{M}_G(X)$. We call A the *maximal support of X*. For reference regarding the following, see [**FGL22**, Proposition 5.7] and [**FGL22**, Theorem 5.8].

PROPOSITION 7.4. *Let X be an action and consider a left Følner sequence $\mathcal{F}$.*

(i) *If X is $\mathcal{F}$-mean equicontinuous, then any point x in the maximal support of X is $\mathcal{F}$-generic.*

(ii) *If X is mean equicontinuous, then any $x \in X$ is $\mathcal{F}$-generic.*

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7.2. *The maximal equicontinuous factor of a diam-mean equicontinuous action.* To relate diam-mean equicontinuity to regularity, we need to relate the mean diameter of fibres to that of balls. For Banach proximal factor maps, this will be achieved in the following lemmas.

LEMMA 7.5. *Let X be an action. For $A \in \mathcal{H}(X)$, $K \in \mathcal{H}'(G)$ and $\epsilon > 0$, there exists a finite non-empty subset $E \subseteq A$ such that*

$$K^*\mathrm{diam}(A) \leq K^*\mathrm{diam}(E) + \epsilon.$$

Proof. For any $g \in G$, we choose $x_g, x'_g \in A$ with $d(g.x_g, g.x'_g) = \mathrm{diam}(g.A)$. Recall that $\mathrm{diam}\colon \mathcal{H}(X) \to \mathbf{R}^+$ is continuous. Furthermore, note that $K \times X \to X$ and $K \times \mathcal{H}(X) \to \mathcal{H}(X)$ are uniformly continuous. Thus, there exists a neighbourhood V of the identity element of G such that for all $x \in X$, $A' \in \mathcal{H}(X)$ and $g, g' \in K$ with $g'g^{-1} \in V$, we have $d(g.x, g'.x) \leq \epsilon/3$ and $|\mathrm{diam}(g.A') - \mathrm{diam}(g'.A')| \leq \epsilon/3$. Choose a finite subset F of K with $K \subseteq VF$ and denote $E := \{x_h, x'_h \mid h \in F\}$. For $g \in K$, there exists $h \in F$ with $gh^{-1} \in V$. Such h satisfies

$$\mathrm{diam}(h.A) = d(h.x_h, h.x'_h) = \mathrm{diam}(h.E) \leq \mathrm{diam}(g.E) + \tfrac{1}{3}\epsilon$$

and allows to compute

$$\begin{aligned}\mathrm{diam}(g.A) &= d(g.x_g, g.x'_g)\\ &\leq d(g.x_g, h.x_g) + d(h.x_g, h.x'_g) + d(h.x'_g, g.x'_g)\\ &\leq \mathrm{diam}(h.A) + \tfrac{2}{3}\epsilon \leq \mathrm{diam}(g.E) + \epsilon.\end{aligned}$$

This shows that for all $g \in K$, we have $\mathrm{diam}(g.A) \leq \mathrm{diam}(g.E) + \epsilon$. We thus conclude that $K^*\mathrm{diam}(A) \leq K^*\mathrm{diam}(E) + \epsilon$. □

LEMMA 7.6. *Let $\pi\colon X \to Y$ be a Banach proximal factor map and $y \in Y$. For all finite non-empty subsets E of $\pi^{-1}(y)$, we have* $\mathrm{Diam}(E) = 0$. *In particular, for $\delta > 0$, there exists $g \in G$ with* $\mathrm{diam}(g.E) \leq \delta$.

Proof. For $g \in G$, we have

$$\mathrm{diam}(g.E) = \max_{x,x' \in E} d(g.x, g.x') \leq \sum_{x,x' \in E} d(g.x, g.x').$$

We thus observe $\mathrm{Diam}(E) \leq \sum_{x,x' \in E} D(x, x') = 0$ from a straightforward subadditivity argument. □

LEMMA 7.7. *Let $\pi\colon X \to Y$ be a Banach proximal factor map, $y \in Y$ and $\delta > 0$. For $K \in \mathcal{H}'(G)$ and $\epsilon > 0$, there exist $g \in G$ and $x \in X$ with*

$$K^*\mathrm{diam}(\pi^{-1}(y)) \leq (Kg)^*\mathrm{diam}(\mathbf{B}_\delta(x)) + \epsilon.$$

Proof. By Lemma 7.5, there exists a finite non-empty subset $E \subseteq \pi^{-1}(y)$ with $K^*\mathrm{diam}(\pi^{-1}(y)) \leq K^*\mathrm{diam}(E) + \epsilon$. From Lemma 7.6, we know that there exists $g \in G$ with $\mathrm{diam}(g^{-1}.E) \leq \delta$. For any $x \in g^{-1}.E$, we observe $g^{-1}.E \subseteq \mathbf{B}_\delta(x)$, that is,

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$E \subseteq g.\mathbf{B}_\delta(x)$ and compute $K^*\mathrm{diam}(\pi^{-1}(y)) - \epsilon \leq K^*\mathrm{diam}(E) \leq (K^*\mathrm{diam})(g.\mathbf{B}_\delta(x))$ $= ((Kg)^*\mathrm{diam})(\mathbf{B}_\delta(x))$. □

LEMMA 7.8. *Let $\pi : X \to Y$ be a Banach proximal factor map, $y \in Y$ and $\delta > 0$. We have* $\mathrm{Diam}(\pi^{-1}(y)) \leq \sup_{x\in X} \mathrm{Diam}(\mathbf{B}_\delta(x))$.

Proof. Let $\hat{\mathcal{F}} = (\hat{F}_n)$ be a left Følner sequence. Consider a subsequence $\mathcal{F} = (F_n)$ such that $\mathrm{Diam}_{\hat{\mathcal{F}}}(\pi^{-1}(y)) = \lim_n F_n^*\mathrm{diam}(\pi^{-1}(y))$ and a sequence (ϵ_n) in $(0, 1)$ satisfying $\epsilon_n \to 0$. From Lemma 7.7, we observe the existence of sequences (g_n) in G and (x_n) in X such that

$$F_n^*\mathrm{diam}(\pi^{-1}(y)) \leq (F_n g_n)^*\mathrm{diam}(\mathbf{B}_{\delta/2}(x_n)) + \epsilon_n.$$

Since X is compact (and by possibly considering a further subsequence of $\mathcal{F}$), we assume without loss of generality that (x_n) converges to some $x \in X$. For large n, we have $\mathbf{B}_{\delta/2}(x_n) \subseteq \mathbf{B}_\delta(x)$ and, hence,

$$F_n^*\mathrm{diam}(\pi^{-1}(y)) - \epsilon_n \leq (F_n g_n)^*\mathrm{diam}(\mathbf{B}_{\delta/2}(x_n)) \leq (F_n g_n)^*\mathrm{diam}(\mathbf{B}_\delta(x)).$$

Since $\mathcal{F}' := (F_n g_n)$ is a left Følner sequence, it follows that

$$\mathrm{Diam}_{\hat{\mathcal{F}}}(\pi^{-1}(y)) = \mathrm{Diam}_{\mathcal{F}}(\pi^{-1}(y)) \leq \mathrm{Diam}_{\mathcal{F}'}(\mathbf{B}_\delta(x)) \leq \sup_{x\in X} \mathrm{Diam}(\mathbf{B}_\delta(x)).$$

Taking the supremum over all left Følner sequences $\hat{\mathcal{F}}$ yields the statement. □

PROPOSITION 7.9. *Let X be a diam-mean equicontinuous action. Then, the factor map π onto the maximal equicontinuous factor of X is regular.*

Proof. Consider a diam-mean equicontinuous action X and denote $\pi : X \to Y$ for the factor map onto the maximal equicontinuous factor. Since X is mean equicontinuous, we know from Proposition 7.3 that π is Banach proximal.

To show that π is regular, we use Theorem 5.2 and show that for $y \in Y$, we have $\mathrm{Diam}(\pi^{-1}(y)) = 0$. Consider $y \in Y$ and $\epsilon > 0$. Let $\delta > 0$ be such that for all $x \in X$, we have $\mathrm{Diam}(\mathbf{B}_\delta(x)) \leq \epsilon$. From Lemma 7.8, we observe that $\mathrm{Diam}(\pi^{-1}(y)) \leq \sup_{x\in X} \mathrm{Diam}(\mathbf{B}_\delta(x)) \leq \epsilon$. Since $\epsilon > 0$ was arbitrary, this shows $\mathrm{Diam}(\pi^{-1}(y)) = 0$. □

7.3. *Lifting of diam-mean equicontinuity along regular factor maps.*

PROPOSITION 7.10. *Let $\pi : X \to Y$ be a regular factor map between actions of G. Let $\mathcal{F} = (F_n)$ be a left (or right) Følner sequence in G. If y is a $\mathcal{F}$-diam-mean equicontinuous point of Y that is $\mathcal{F}$-generic (for some invariant Borel probability measure), then any $x \in \pi^{-1}(y)$ is $\mathcal{F}$-diam-mean equicontinuous in X.*

Proof. Consider $x \in \pi^{-1}(y)$. Furthermore, denote by ν the measure for which y is $\mathcal{F}$-generic. Recall that the notion of a diam-mean equicontinuous point does not depend on the choice of a continuous metric of the respective action. We thus assume without loss of generality that the metrics on X and Y satisfy $d(x_1, x_2) \geq d(\pi(x_1), \pi(x_2))$ for all $x_1, x_2 \in X$. Denote X_0 for the set of injectivity points of π and $Y_0 := \pi(X_0)$. Since π is regular, we observe that $\nu(Y_0) = 1$.

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Let $\epsilon > 0$. For $\delta > 0$, we denote

$$G_x^\delta := \{g \in G \mid \operatorname{diam}(g.\mathbf{B}_\delta(x)) > \epsilon\}.$$

Considering the characterization presented in Proposition 6.8, it suffices to show that there exists $\delta > 0$ with $\text{ua-dens}_{\mathcal{F}}(G_x^\delta) \leq \epsilon$.

Since any factor map is closed, we know that for $y' \in Y$, there exists $\delta_{y'} \in (0, \epsilon/2)$ such that $\pi^{-1}(\mathbf{B}_{2\delta_{y'}}(y')) \subseteq \mathbf{B}_{\epsilon/2}(\pi^{-1}(y'))$. For details, see [**AB06**, Theorem 17.7(1)]. Exploring the regularity of the Borel measure ν, we choose a compact subset $A \subseteq Y_0$ with $\nu(A) \geq 1 - \epsilon/2$. Furthermore, we choose a finite subset $E \subseteq A$ satisfying $A \subseteq \bigcup_{y' \in E} B_{\delta_{y'}}(y') =: O$ and denote $\delta' := \min_{y' \in E} \delta_{y'}$. Since y is assumed to be $\mathcal{F}$-diam-mean equicontinuous, the characterization presented in Proposition 6.8 allows to observe that there exists $\delta > 0$ such that

$$G_y^\delta := \{g \in G \mid \operatorname{diam}(g.\mathbf{B}_\delta(y)) > \delta'\}$$

satisfies $\text{ua-dens}_{\mathcal{F}}(G_y^\delta) \leq \delta'$.

We will next show that, with this choice of δ, we also have $\text{ua-dens}_{\mathcal{F}}(G_x^\delta) \leq \epsilon$. This will be achieved by considering

$$G' := \{g \in G \mid g.y \notin O\}$$

and showing: (i) $\text{ua-dens}_{\mathcal{F}}(G' \cup G_y^\delta) \leq \epsilon$ and (ii) $G_x^\delta \subseteq G' \cup G_y^\delta$.

(i) Clearly, $O = \bigcup_{y' \in E} B_{\delta_{y'}}(y')$ is open and satisfies $\nu(O) \geq \nu(A) \geq 1 - \epsilon/2$. Thus, $A' := Y \setminus O$ is closed and satisfies $\nu(A') \leq \epsilon/2$. We thus observe from $(F_n)_* \delta_y \to \nu$ that

$$\text{ua-dens}_{\mathcal{F}}(G') = \limsup_{n \to \infty} \frac{|F_n \cap G'|}{|F_n|} = \limsup_{n \to \infty} (F_n)_* \delta_y(A') \leq \nu(A') \leq \frac{\epsilon}{2}.$$

Since $\delta' = \min_{y' \in E} \delta_{y'} \leq \epsilon/2$, we conclude

$$\text{ua-dens}_{\mathcal{F}}(G' \cup G_y^\delta) \leq \text{ua-dens}_{\mathcal{F}}(G') + \text{ua-dens}_{\mathcal{F}}(G_y^\delta) \leq \frac{\epsilon}{2} + \delta' \leq \epsilon.$$

(ii) Consider $g \in G_x^\delta$ and assume that $g \notin G' \cup G_y^\delta$. From $g \notin G_y^\delta$, we observe $\operatorname{diam}(g.\mathbf{B}_\delta(y)) \leq \delta'$ and, hence, $g.\mathbf{B}_\delta(\pi(x)) = g.\mathbf{B}_\delta(y) \subseteq \mathbf{B}_{\delta'}(g.y)$. Furthermore, since $g \notin G'$, we have $g.y \in O = \bigcup_{y' \in E} B_{\delta_{y'}}(y')$. Thus, there exists $y' \in E$ with $d(g.y, y') < \delta_{y'}$. For such y', we observe $\mathbf{B}_{\delta'}(g.y) \subseteq \mathbf{B}_{\delta_{y'}}(g.y) \subseteq \mathbf{B}_{2\delta_{y'}}(y')$. Using our particular choice of the metrics on X and Y, we compute

$$\pi(g.\mathbf{B}_\delta(x)) = g.\pi(\mathbf{B}_\delta(x)) \subseteq g.\mathbf{B}_\delta(\pi(x)) \subseteq \mathbf{B}_{\delta'}(g.y) \subseteq \mathbf{B}_{2\delta_{y'}}(y')$$

and, hence, the choice of $\delta_{y'}$ yields

$$g.\mathbf{B}_\delta(x) \subseteq \pi^{-1}(\pi(g.\mathbf{B}_\delta(x))) \subseteq \pi^{-1}(\mathbf{B}_{2\delta_{y'}}(y')) \subseteq \mathbf{B}_{\epsilon/2}(\pi^{-1}(y')).$$

Since $y' \in E \subseteq A \subseteq Y_0$, we observe that $\pi^{-1}(y')$ is a singleton and, hence, that $\operatorname{diam}(g.\mathbf{B}_\delta(x)) \leq \epsilon$. This shows $g \notin G_x^\delta$, which is a contradiction. □

We observe Theorem 1.1 from Proposition 7.9 and the following theorem.

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THEOREM 7.11. *Let $\pi : X \to Y$ be a regular factor map.*

(i) *If Y is diam-mean equicontinuous, then X is diam-mean equicontinuous.*

(ii) *If Y is equicontinuous, then X is diam-mean equicontinuous and π is the factor map onto the maximal equicontinuous factor of X.*

Proof. Since any equicontinuous action is diam-mean equicontinuous, we obtain condition (ii) from combining condition (i) with Proposition 7.3.

To show condition (i), let $x \in X$ and $\mathcal{F}$ be a left Følner sequence. From Proposition 7.4, we observe that $\pi(x)$ is $\mathcal{F}$-generic. Since $\pi(x)$ is assumed to be diam-mean equicontinuous, Proposition 7.10 yields that x is $\mathcal{F}$-diam-mean equicontinuous. This shows that x is $\mathcal{F}$-diam-mean equicontinuous with respect to any left Følner sequence in G. From Proposition 6.9, we observe that x is diam-mean equicontinuous. □

COROLLARY 7.12. *Let X be an action and π be the factor map onto the maximal equicontinuous factor. Then, X is diam-mean equicontinuous if and only if $\mathcal{H}_\pi(X)$ is diam-mean equicontinuous.*

Proof. Observe that X is conjugated to the subaction $\mathfrak{F}_1(X) \subseteq \mathcal{H}_\pi(X)$ and, hence, the diam-mean equicontinuity of $\mathcal{H}_\pi(X)$ implies the diam-mean equicontinuity of X. To show the converse, assume that X is diam-mean equicontinuous. From Proposition 7.9, we observe that $\pi : X \to X_{\text{eq}}$ is regular and it follows from Theorem 5.2 that $\pi_{\mathcal{H}} : \mathcal{H}_\pi(X) \to X_{\text{eq}}$ is also regular. Since X_{eq} is equicontinuous, we conclude that $\mathcal{H}_\pi(X)$ is diam-mean equicontinuous from Theorem 7.11. □

An action X is called *fully supported* if X equals its maximal support. The maximal support is always a closed invariant and non-empty subset. Thus, any minimal action is fully supported. Whenever X is a diam-mean equicontinuous action that is not equicontinuous, then $\mathfrak{F}_1(X)$ is a proper subset of the diam-mean equicontinuous action $\mathcal{H}_\pi(X)$, where $\pi : X \to X_{\text{eq}}$ denotes the factor map onto the maximal equicontinuous factor of X. With the arguments above, it follows that $\pi_{\mathcal{H}} : \mathcal{H}_\pi(X) \to X_{\text{eq}}$ is regular and Theorem 5.2 yields that the maximal support of $\mathcal{H}_\pi(X)$ is contained in $\mathfrak{F}_1(X)$. We thus observe the following.

Remark 7.13. A diam-mean equicontinuous action does not need to be fully supported.

Proposition 7.10 also allows to say something about regular extensions of $\mathcal{F}$-diam-mean equicontinuous actions. An action X is called *pointwise uniquely ergodic* if $\overline{O(x)}$ is uniquely ergodic for any $x \in X$. In particular, any uniquely ergodic action is pointwise uniquely ergodic.

PROPOSITION 7.14. *Let $\pi : X \to Y$ be a regular factor map and consider a left Følner sequence $\mathcal{F}$. Assume that Y is $\mathcal{F}$-diam-mean equicontinuous.*

(i) *If Y is fully supported, then X is $\mathcal{F}$-diam-mean equicontinuous.*

(ii) *If Y is pointwise uniquely ergodic, then X is $\mathcal{F}$-diam-mean equicontinuous.*

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Proof. (i) Since Y is $\mathcal{F}$-diam-mean equicontinuous, it is $\mathcal{F}$-mean equicontinuous and Proposition 7.4 yields that any point is $\mathcal{F}$-generic. We thus observe the statement from Proposition 7.10.

(ii) Since Y is pointwise uniquely ergodic, any point in Y is $\mathcal{F}$-generic. Thus, the statement follows from Proposition 7.10. □

Remark 7.15. Recall that a factor map $\pi\colon X \to Y$ is called *proximal* if all $x, x' \in X$ with $\pi(x) = \pi(x')$ satisfy $\inf_{g\in G} d(g.x, g.x') = 0$. It is straightforward to observe that any action X for which the factor map π_{eq} onto the maximal equicontinuous factor is proximal satisfies the *local Bronstein property* [**HJL25**, Definition 2.14]. Since any Banach proximal factor map is proximal, we observe from Proposition 7.3 that all mean equicontinuous actions satisfy the local Bronstein property.

Now, recall that any diam-mean equicontinuous action is mean equicontinuous and that any regular factor map is Banach proximal (Theorem 5.2). Thus, this observation allows to apply [**HJL25**, Theorem 5.2] to give an alternative proof for the statement of Theorem 1.1 in the context of minimal actions of countable amenable groups.

8. *Stability properties of regular factor maps*

8.1. *Product stability.* For a sequence $(\pi_n\colon X_n \to Y_n)_{n\in\mathbb{N}}$ of factor maps, we consider the factor map $\prod_{n=1}^{\infty}\pi_n\colon \prod_{n=1}^{\infty} X_n \to \prod_{n=1}^{\infty} Y_n$ given by $(x_n)_n \mapsto (\pi_n(x_n))_n$. See [**CH26**] for further reading.

PROPOSITION 8.1. *Let $(\pi_n\colon X_n \to Y_n)_{n\in\mathbb{N}}$ be a sequence of factor maps. The factor map $\pi := \prod_{n=1}^{\infty}\pi_n$ is regular if and only if $(\pi_n)_n$ is a sequence of regular factor maps.*

Proof. We denote $X := \prod_{n=1}^{\infty} X_n$ and $Y := \prod_{n=1}^{\infty} Y_n$, and write $\pi^{(n)}$ for the projection $X \to X_n$.

'$\Rightarrow$' Let $m \in \mathbb{N}$. We assume without loss of generality that the metrics on X_m and X satisfy $d(x_1, x_2) \geq d(\pi^{(m)}(x_1), \pi^{(m)}(x_2))$ for all $x_1, x_2 \in X$. This implies $\mathrm{Diam}(\pi^{(m)}(A)) \leq \mathrm{Diam}(A)$ for $A \in \mathcal{H}(X)$.

For $y \in Y_m$, we choose a sequence $(y_n)_n \in Y$ with $y_m = y$. Observe that $\pi^{-1}((y_n)_n) = \prod_{n=1}^{\infty}\pi_n^{-1}(y_n)$ and, hence, $\pi^{(m)}(\pi^{-1}((y_n)_n)) = \pi_m^{-1}(y)$. Since π is assumed to be regular, we know from Theorem 5.2 that

$$\mathrm{Diam}(\pi_m^{-1}(y)) = \mathrm{Diam}(\pi^{(m)}(\pi^{-1}((y_n)_n))) \leq \mathrm{Diam}(\pi^{-1}((y_n)_n)) = 0.$$

This shows $\mathrm{Diam}(\pi_m^{-1}(y)) = 0$ for all $y \in Y_m$ and we observe π_m to be regular from another application of Theorem 5.2.

'$\Leftarrow$' For $n \in \mathbb{N}$, we denote $X_{n,0}$ for the set of injectivity points of π_n. We observe that $X_0 := \prod_{n=1}^{\infty} X_{n,0} = \bigcap_{n=1}^{\infty}(\pi^{(n)})^{-1}(X_{n,0})$ is the set of injectivity points of π.

Let $\mu \in \mathcal{M}_G(X)$ and note that $\pi_*^{(n)}\mu \in \mathcal{M}_G(X_n)$ for $n \in \mathbb{N}$. From the regularity of π_n, we obtain

$$\mu((\pi^{(n)})^{-1}(X_{n,0})) = (\pi_*^{(n)}\mu)(X_{n,0}) = 1.$$

Thus, $\mu(X_0) = \mu(\bigcap_{n=1}^{\infty}(\pi^{(n)})^{-1}(X_{n,0})) = 1$. □

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8.2. *(De-)Composition stability.* The following proposition shows that regularity is stable under composition and decomposition.

PROPOSITION 8.2. *Let $\phi\colon X \to Y$ and $\psi\colon Y \to Z$ be factor maps. Then, $\pi := \psi \circ \phi$ is regular if and only if ϕ and ψ are regular.*

Proof. '$\Rightarrow$' We assume without loss of generality that the metrics on X and Y satisfy $d(x_1, x_2) \geq d(\phi(x_1), \phi(x_2))$ for all $x_1, x_2 \in X$. For $y \in Y$ and $z \in Z$, we observe from Theorem 5.2 that we have $\mathrm{Diam}(\phi^{-1}(y)) \leq \mathrm{Diam}(\pi^{-1}(\psi(y))) = 0$ and $\mathrm{Diam}(\psi^{-1}(z)) = \mathrm{Diam}(\phi(\pi^{-1}(z))) \leq \mathrm{Diam}(\pi^{-1}(z)) = 0$. Thus, Theorem 5.2 yields that ϕ and ψ are regular.

'$\Leftarrow$' Denote X_π, X_ϕ and Y_ψ for the injectivity points of π, ϕ and ψ, respectively. It is a straightforward exercise to verify that $\phi^{-1}(Y_\psi) \cap X_\phi \subseteq X_\pi$. For $\mu \in \mathcal{M}_G(X)$, we have $\phi_*\mu \in \mathcal{M}_G(Y)$ and from the regularity of ψ, we observe $\mu(\phi^{-1}(Y_\psi)) = \phi_*\mu(Y_\psi) = 1$. Furthermore, the regularity of ϕ yields $\mu(X_\phi) = 1$ and we thus have $1 = \mu(\phi^{-1}(Y_\psi) \cap X_\phi) \leq \mu(X_\pi) \leq 1$, that is, $\mu(X_\pi) = 1$. □

8.3. *$\mathcal{H}$-(in)stability.* Recall that a factor map $\pi\colon X \to Y$ is called *almost one-to-one* if the set of injectivity points is dense in X. In [**CH26**], it was presented that the properties discussed in the Propositions 8.1 and 8.2 also hold for almost one-to-one factor maps between actions (on compact metric spaces).

For a factor map $\pi\colon X \to Y$, the image mapping $\mathcal{H}(X) \ni A \mapsto \pi(A)$ establishes a factor map $\mathcal{H}(\pi)\colon \mathcal{H}(X) \to \mathcal{H}(Y)$ [**CH26**]. It was also shown in [**CH26**, **DX24**] that a factor map π between actions (on compact metric spaces) is almost one-to-one if and only if $\mathcal{H}(\pi)$ is almost one-to-one. We next present that a similar statement does not hold in the context of regularity.

Example 8.3. Let $Y := \mathbb{Z} \cup \{\infty\}$ be the one-point compactification of $\mathbb{Z}$. We act on Y with $G := \mathbb{Z}$ by fixing ∞, and by defining $g.y := g + y$ for $g \in G$ and $y \in \mathbb{Z}$. Consider two further disjoint copies of $\hat{X} = \{\hat{n} \mid n \in \mathbb{Z}\} \cup \{\hat{\infty}\}$ and $\check{X} = \{\check{n} \mid n \in \mathbb{Z}\} \cup \{\check{\infty}\}$ of Y, and denote $X' := \hat{X} \cup \check{X}$. We obtain X as a quotient of X' after the identification $\infty_{\hat{X}} = \infty_{\check{X}}$ and denote ∞ for the respective element. Note that $X = \{\hat{n} \mid n \in \mathbb{Z}\} \cup \{\check{n} \mid n \in \mathbb{Z}\} \cup \{\infty\}$ is uniquely ergodic and that the invariant Borel probability measure is δ_∞. Thus, the factor map $\pi\colon X \to Y$ given by $\infty \to \infty$ and $\pi(\hat{n}) := \pi(\check{n}) := n$ for $n \in \mathbb{Z}$ is regular.

Consider the factor map $\mathcal{H}(\pi)\colon \mathcal{H}(X) \to \mathcal{H}(Y)$, as well as the elements $A := \{\hat{n} \mid n \in \mathbb{Z}\} \cup \{\infty\}$ and $B := \{\check{n} \mid n \in \mathbb{Z}\} \cup \{\infty\}$ of $\mathcal{H}(X)$. Clearly, we have $\mathcal{H}(\pi)(A) = Y = \mathcal{H}(\pi)(B)$ and, hence, $\{A, B\} \subseteq \mathcal{H}(\pi)^{-1}(Y)$. Nevertheless, the elements A and B are fixed in $\mathcal{H}(X)$. We thus observe $0 < d_{\mathcal{H}}(A, B) \leq \mathrm{Diam}(\mathcal{H}(\pi)^{-1}(Y))$. From Theorem 5.2, we thus conclude that $\mathcal{H}(\pi)$ is not regular.

Let $\pi\colon X \to Y$ be a factor map and denote X_0 and $\mathcal{H}(X)_0$ for the sets of injectivity points of π and $\mathcal{H}(\pi)$, respectively. One readily verifies that $\mathfrak{F}_1(X_0) = \mathfrak{F}_1(X) \cap \mathcal{H}(X)_0$. From this, the following statement follows directly.

PROPOSITION 8.4. *Let $\pi\colon X \to Y$ be a factor map. Whenever $\mathcal{H}(\pi)$ is regular, then π is regular.*

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Remark 8.5. For an action X (of G), the mapping $(g, \mu) \mapsto g_*\mu$ induces an action of G on $\mathcal{M}(X)$. Furthermore, for factor map $\pi\colon X \to Y$, the pushforward $\pi_*\colon \mathcal{M}(X) \to \mathcal{M}(Y)$ is a factor map. It was shown in [**CH26**] that a factor map π (between compact metric spaces) is almost one-to-one if and only if π_* is almost one-to-one.

Denote X_0 and $\mathcal{M}(X)_0$ for the respective sets of injectivity points of π and π_*. Abbreviating $\delta(X') := \{\delta_x \mid x \in X'\}$ for $X' \subseteq X$, a straightforward argument yields $\delta(X_0) = \delta(X) \cap \mathcal{M}(X)_0$. With a similar argument as above, it follows that the regularity of π_* implies the regularity of π. It remains open whether the regularity of π is equivalent to the regularity of π_*.

9. *The maximal diam-mean equicontinuous factor*

Next, we establish the existence of a maximal diam-mean equicontinuous factor. We begin with the following observation.

Remark 9.1. Any subaction of a diam-mean equicontinuous action is diam-mean equicontinuous. Furthermore, the canonical action (of G) on a singleton is diam-mean equicontinuous.

Diam-mean equicontinuity is stable under (countable) products.

THEOREM 9.2. *Let $(X_n)_{n\in\mathbb{N}}$ be a countable family of diam-mean equicontinuous actions. Then, also the induced action on $X := \prod_n X_n$ is diam-mean equicontinuous. Furthermore, if Y_n is the maximal equicontinuous factor of X_n, then $Y := \prod_n Y_n$ is the maximal equicontinuous factor of X.*

Proof. For $n \in \mathbb{N}$, denote $\pi_n\colon X_n \to Y_n$ for the respective factor map onto the maximal equicontinuous factor of X_n. Note that Y is equicontinuous [**Aus88**, Ch. 2]. From Proposition 7.9, we observe that all π_n are regular. Thus, Proposition 8.1 yields that the induced factor map $X \to Y$ is also regular and we have established that X is a regular extension of an equicontinuous action. The statement now follows from Theorem 7.11. □

It is well known that a property (P), which holds for the canonical singleton action and which is stable under products and subactions allows for the construction of a 'maximal factor of X that satisfies property (P)' for any action X. For this, [**Aus88**] is often referred to, where this technique is presented in the context of actions on compact Hausdorff spaces and product-stability means stability of property (P) under arbitrary (possibly uncountable) products. However, working with compact metric spaces, one can only consider countable products and the question arises whether the technique of [**Aus88**] can also be used while working only with actions on compact metric spaces. We will present in the following proposition that it can be used and provide a full proof for the convenience of the reader. This allows to establish Theorem 1.2. For a precise terminology, we call two factor maps $\pi\colon X \to Y$, $\pi'\colon X \to Y'$ *conjugated* if there exists a conjugacy $\iota\colon Y \to Y'$ with $\iota \circ \pi = \pi'$.

PROPOSITION 9.3. *Let G be a topological group. Consider a property (P) that is defined for actions of G on compact metric spaces such that:*

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(i) *whenever an action satisfies property (P), then any subaction satisfies property (P);*
(ii) *whenever $(X_n)_{n\in\mathbb{N}}$ is a countable family of actions that satisfy property (P), then $\prod_n X_n$ satisfies property (P); and*
(iii) *the canonical singleton action of G satisfies property (P).*

If X is an action of G (on a compact metric space), then there exists a maximal factor of X that satisfies property (P), that is, a factor map $\pi_P\colon X \to X_P$ that satisfies property (P), such that for all factors $\pi\colon X \to Y$ that satisfy property (P), there exists a factor map $\psi\colon X_P \to Y$ with $\pi = \psi \circ \pi_P$. The maximal factor of X that satisfies property (P) is unique up to conjugacy.

Remark 9.4. Consider properties (E) and (D) of actions such that property (E) is stronger than property (D), that is, any action that satisfies property (E) also satisfies property (D). Whenever properties (E) and (D) satisfy the statements (i), (ii) and (iii), then for an action X, we can consider the maximal factor $\pi_D\colon X \to X_D$ that satisfies property (D). It is easy to verify that the maximal factors $\pi\colon X \to X_E$ and $\phi\colon X_D \to X'_E$ that satisfy property (E) are conjugate, that is, there exists a conjugacy $\iota\colon X'_E \to X_E$ such that $\pi = \iota \circ \phi \circ \pi_D$. In particular, we observe that the maximal equicontinuous factor of an action X is the maximal equicontinuous factor of the maximal diam-mean equicontinuous factor of X.

Remark 9.5. The maximal equicontinuous factor and the maximal diam-mean equicontinuous factor differ in general. For example, any regular Toeplitz subshift of $\{0, 1\}^{\mathbb{Z}}$ is a (non-trivial) regular extension of an odometer and, hence, diam-mean equicontinuous. Here, the odometer is the maximal equicontinuous factor. See [**BJL16**, **CP14**, **Dow05**] for details and further reading.

Before presenting the proof, we briefly discuss some obstacles that arise in this context.

Example 9.6. It is a well-known folklore result that the class of all compact metric spaces, up to homeomorphism, is uncountable. Let X be the Cantor space and G be any group. We act on X by fixing all elements. Since all compact metric spaces are continuous images of X [**Wil04**, Theorem 30.7], we observe that the family of all factors (up to conjugacy) is uncountable.

Proof of Proposition 9.3. Note that the concepts of factor map, conjugacy, invariant subset and product given in §2 can easily be extended to actions of G on a compact Hausdorff space. In particular, this allows to consider arbitrary products of actions. See [**Aus88**] for further reference in this direction.

Let X be an action of G on a compact metric space. For simplicity, a factor of X is called a *P-factor* if it satisfies property (P). For any factor π of X, we denote $R(\pi) := \{x, x' \in X \mid \pi(x) = \pi(x')\}$. Note that two factors π and π' of X are conjugated if and only if $R(\pi) = R(\pi')$. It follows that a family $(\pi_i\colon X \to Y_i)_{i\in I}$ of conjugacy-representatives of P-factors of X can be treated as a set. From statement (iii), we know that I is non-empty. Consider the product action $Y_I := \prod_{i\in I} Y_i$ and the map $\pi_I\colon X \to Y_I$ given by $\pi_I(x) := (\pi_i(x))_i$. Clearly, π_I restricts to a factor map $\pi_P\colon X \to X_P := \pi_I(X)$. It is straightforward to verify that for any P-factor $\pi\colon X \to Y$, there exists a factor $\psi\colon X_P \to Y$ with $\pi = \psi \circ \pi_P$. Furthermore, from $R(\pi_P) = \bigcap_i R(\pi_i)$, it is

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straightforward to observe that X_P is conjugated to any other candidate for a maximal P-factor. This establishes the uniqueness of the maximal P-factor of X up to conjugacy.

It remains to show that X_P satisfies property (P). This does not immediately follow from statement (ii), since I is not necessarily countable. However, since X is metrizable, we observe that X_P is metrizable. Thus, X_P has a countable base $(B^{(n)})_{n\in\mathbb{N}}$ for its topology. Revising the definition of the product topology, we assume without loss of generality that $B^{(n)}$ is of the form $B^{(n)} = X_P \cap (\prod_i U_i^{(n)})$ with $U_i^{(n)} \subseteq Y_i$ open and non-empty, and $U_i^{(n)} = Y_i$ for all but finitely many i.

Let N be the set of all $i \in I$ for which there exists some $n \in \mathbb{N}$ such that $U_i^{(n)} \neq Y_i$ and note that N is countable. Denote $Y_N := \prod_{i\in N} Y_i$ and consider the projection $\pi_{I,N}\colon Y_I \to Y_N$. Let ι be the restriction $\iota\colon X_P \to \iota(X_P)$ of $\pi_{I,N}$ and observe that ι is a factor map. Our choice of N implies that the topology of X_P is given by $\{\pi_{I,N}^{-1}(O) \mid O \in \tau\}$, where τ is the topology of $\iota(X_P)$.

Since $\iota(X_P)$ is Hausdorff, we observe ι to be injective and, hence, a conjugacy. Furthermore, from statement (ii), we know that Y_N satisfies property (P). Thus, $\iota(X_P) \subseteq Y_N$ yields that also $\iota(X_P)$ and, hence, X_P satisfy property (P). □

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References

[AB06] C. D. Aliprantis and K. C. Border. *Infinite Dimensional Analysis. A Hitchhiker's Guide*, 3rd edn. Springer, Berlin, 2006.

[AG01] E. Akin and E. Glasner. Residual properties and almost equicontinuity. *J. Anal. Math.* **84** (2001), 243–286.

[Aus88] J. Auslander. *Minimal Flows and their Extensions (North-Holland Mathematics Studies, 153)*. North-Holland, Amsterdam, 1988.

[BG13] M. Baake and U. Grimm. *Aperiodic Order. Volume 1. A Mathematical Invitation (Encyclopedia of Mathematics and its Applications, 149)*. Cambridge University Press, Cambridge, 2013.

[BJL16] M. Baake, T. Jäger and D. Lenz. Toeplitz flows and model sets. *Bull. Lond. Math. Soc.* **48**(4) (2016), 691–698.

[CH26] M. I. Cortez and T. Hauser. A note on the structural stability of almost one-to-one maps. *Topology Appl.* **377** (2026), 109641.

[CP14] M. I. Cortez and S. Petite. Invariant measures and orbit equivalence for generalized Toeplitz subshifts. *Groups Geom. Dyn.* **8**(4) (2014), 1007–1045.

[DE14] A. Deitmar and S. Echterhoff. *Principles of Harmonic Analysis (Universitext)*, 2nd revised and updated edition. Springer, Cham, 2014.

[DFR16] T. Downarowicz, B. Frej and P.-P. Romagnoli. Shearer's inequality and infimum rule for Shannon entropy and topological entropy. *Dynamics and Numbers (Contemporary Mathematics, 669)*. American Mathematical Society, Providence, RI, 2016, pp. 63–75.

[DG16] T. Downarowicz and E. Glasner. Isomorphic extensions and applications. *Topol. Methods Nonlinear Anal.* **48**(1) (2016), 321–338.

[DHZ19] T. Downarowicz, D. Huczek and G. Zhang. Tilings of amenable groups. *J. Reine Angew. Math.* **747** (2019), 277–298.

[Dow05] T. Downarowicz. Survey of odometers and Toeplitz flows. *Algebraic and Topological Dynamics (Contemporary Mathematics, 385)*. American Mathematical Society, Providence, RI, 2005, pp. 7–37.

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[DX24] X. Dai and Y. Xie. Characterizations of open and semi-open maps of compact Hausdorff spaces by induced maps. *Topology Appl.* **350** (2024), 108921.

[Far62] R. H. Farrell. Representation of invariant measures. *Illinois J. Math.* **6** (1962), 447–467.

[FGH25] G. Fuhrmann, M. Gröger and T. Hauser. On the continuity of Følner averages. *J. Funct. Anal.* **289**(7) (2025), 111039.

[FGL22] G. Fuhrmann, M. Gröger and D. Lenz. The structure of mean equicontinuous group actions. *Israel J. Math.* **247**(1) (2022), 75–123.

[Fom51] S. Fomin. On dynamical systems with a purely point spectrum. *Dokl. Akad. Nauk SSSR* **77** (1951), 22–32.

[Gla03] E. Glasner. *Ergodic Theory via Joinings (Mathematical Surveys and Monographs, 101)*. American Mathematical Society, Providence, RI, 2003.

[GR17] F. García-Ramos. Weak forms of topological and measure-theoretical equicontinuity: relationships with discrete spectrum and sequence entropy. *Ergod. Th. & Dynam. Sys.* **37**(4) (2017), 1211–1237.

[GRJY21] F. García-Ramos, T. Jäger and X. Ye. Mean equicontinuity, almost automorphy and regularity. *Israel J. Math.* **243**(1) (2021), 155–183.

[GRLZ19] F. García-Ramos, J. Li and R. Zhang. When is a dynamical system mean sensitive? *Ergod. Th. & Dynam. Sys.* **39**(6) (2019), 1608–1636.

[Hau25] L. J. F. Haupt. Multivariate frequent stability and diam-mean equicontinuity. *J. Dynam. Differential Equations*. doi:10.1007/s10884-025-10465-z. Published online 8 November 2025.

[Hau26] T. Hauser. Mean equicontinuous factor maps. *Groups Geom. Dyn.* doi:10.4171/GGD/961. Published online 1 April 2026.

[HJL25] L. J. F. Haupt, T. Jäger and C. Liu. A note on multivariate diam mean equicontinuity and frequent stability. *Preprint*, 2025, arXiv:2506.23313 [math.DS].

[Lin01] E. Lindenstrauss. Pointwise theorems for amenable groups. *Invent. Math.* **146**(2) (2001), 259–295.

[ŁS18] M. Łacka and M. Straszak. Quasi-uniform convergence in dynamical systems generated by an amenable group action. *J. Lond. Math. Soc. (2)* **98**(3) (2018), 687–707.

[LTY15] J. Li, T. Siming and X. Ye. Mean equicontinuity and mean sensitivity. *Ergod. Th. & Dynam. Sys.* **35**(8) (2015), 2587–2612.

[Nad92] S. B. J. Nadler. *Continuum Theory. An Introduction (Pure Appl. Math., Marcel Dekker, 158)*. Marcel, New York, 1992.

[QZ20] J. Qiu and J. Zhao. A note on mean equicontinuity. *J. Dynam. Differential Equations* **32**(1) (2020), 101–116.

[Sol20] B. Solomyak. Delone sets and dynamical systems. *Substitution and Tiling Dynamics: Introduction to Self-Inducing Structures. Lecture Notes from the Research School on Tiling Dynamical Systems, CIRM Jean-Morlet Chair, Marseille, France, Fall 2017*. Springer, Cham, 2020, pp. 1–32.

[Vil03] C. Villani. *Topics in Optimal Transportation (Graduate Students in Mathematics, 58)*. American Mathematical Society, Providence, RI, 2003.

[Wil04] S. Willard. *General Topology*, reprint of the 1970 original edition. Dover Publications, Mineola, NY, 2004.

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