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A property of vector fields without singularity in $\mathcal{G}^1(M)$

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Abstract. We prove the following property: if X in $\mathcal{G}^1(M)$ has no singularity and $x \in \Sigma(X)$, then $\overline{\text{orbit}(x)} \cap \overline{\text{per}(X)} \neq \emptyset$. In addition, if we assume $\overline{\text{per}_i(X)} \cap \overline{\text{per}_j(X)} = \emptyset$ for $i \neq j$, then $\overline{\text{per}(X)} = \bigcup_{i=0}^{n-1} \overline{\text{per}_i(X)}$ is a hyperbolic set. Moreover, we shall give a proof of the Ω-stability conjecture for flows.

1. *Introduction*

Let M be an n-dimensional compact smooth manifold without boundary and let $\mathcal{X}^1(M)$ be the set of C^1 vector fields on M with the C^1 topology. We denote by X_t, $t \in \mathbb{R}$, the C^1 flow on M generated by $X \in \mathcal{X}^1(M)$. We denote by $\overline{\text{per}(X)}$ the closure of the set of periodic orbits of X and by $\overline{\text{per}_j(X)}$ the closure of the set of hyperbolic periodic orbits with index j (index is the dimension of the stable subspace). We denote by $\Omega(X)$ the non-wandering set of $X \in \mathcal{X}^1(M)$. A set $\Lambda \subset M$ is said to be a hyperbolic set of $X \in \mathcal{X}^1(M)$, if it is compact, X_t-invariant (i.e. $X_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$) and there is a continuous splitting $TM|\Lambda = E^0 \oplus E^s \oplus E^u$ ($E^0(x) = \mathbb{R} \cdot X(x)$, $x \in \Lambda$) invariant under $D_x X_t$ such that there exist $K > 0$, $0 < \lambda < 1$, satisfying

$$\|(D_x X_t)|E^s_x\| \leq K\lambda^t \quad \text{and} \quad \|(D_x X_{-t})|E^u_x\| \leq K\lambda^t$$

for all $t \geq 0$, $x \in \Lambda$.

When $\Omega(X)$ is hyperbolic and the periodic points are dense in $\Omega(X)$, we say that X satisfies Axiom A. Let $\mathcal{G}^1(M)$ denote the set of $X \in \mathcal{X}^1(M)$ which has a neighborhood $\mathcal{U}$ such that if $Y \in \mathcal{U}$ then all the periodic orbits and singularities of Y are hyperbolic. For $X \in \mathcal{X}^1(M)$, let $B_\varepsilon(X, x) = \{y \in M : d(X_t(x), y) \leq \varepsilon \text{ for some } t \in \mathbb{R}\}$. Define $\Sigma(X)$ as the set of points $x \in M$ such that for every neighborhood $\mathcal{U}$ of X and every $\varepsilon > 0$, there exist $Y \in \mathcal{U}$, $y \in \text{per}(Y)$, $T_0 > 0$ and $t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$ such that $Y_{T_0}(y) = y$, $X = Y$ on $M - B_\varepsilon(X, x)$, $d(Y_t(y), X_t(x)) \leq \varepsilon$ for all $0 \leq t \leq T_0$, $\{X_t(x) : t_0 \leq t \leq t_1\} \subset \{Y_t(y) : t \geq 0\}$ and $(t_1 - t_0)/T_0 > 1 - \varepsilon$. Note that $\Sigma(X)$ is X_t-invariant.

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We shall prove Theorem A in §2, Theorem B in §3, and in §4, using the argument in Theorem A, we shall prove the Ω-stability conjecture for flows. In 1997, Hayashi [**5**] announced a proof of this conjecture: however, it seems that his proof has flaws. The proof of Lemma VII.4 in [**5**] is not clear.

THEOREM A. *If X is in $\mathcal{G}^1(M)$ and has no singularities, then for any $x \in \Sigma(X)$, $\overline{\mathrm{orbit}(x)} \cap \overline{\mathrm{per}(X)} \neq \emptyset$.*

THEOREM B. *Let X be in $\mathcal{G}^1(M)$ and have no singularities. If $\overline{\mathrm{per}_i(X)} \cap \overline{\mathrm{per}_j(X)} = \emptyset$ (for all $0 \leq i < j \leq n-1$), then $\overline{\mathrm{per}(X)}$ is hyperbolic.*

THEOREM C. (Ω-stability conjecture for flows) *If X is Ω-stable, then X satisfies Axiom A.*

2. *Proof of Theorem A*

For the proof of Theorem A, we need some properties of $\Sigma(X)$ and $\mathcal{G}^1(M)$. Let $\mathcal{U}_0$ be a neighborhood of X in $\mathcal{G}^1(M)$ such that for any Y in $\mathcal{U}_0$, Y has no singularities. Let N^* be the normal bundle to X over M. For each $x \in M$, the fiber N_x is a subspace of T_xM with codimension 1 that is perpendicular to $X(x)$. For any $u \in N_x$, let $P_t^X(u)$ be the orthogonal projection of $D_xX_t(u)$ onto $N_{X_t(x)}$. Then

$$P_t^X : N^* \to N^*$$

is a C^0 flow which is linear on fibers. The next theorem is useful in this section.

THEOREM 2.1. (Liao [**8**, Theorem 2.1]) *Let X be in $\mathcal{G}^1(M)$. Then there exist a C^1 neighborhood $\tilde{\mathcal{U}}$ of X in $\mathcal{U}_0$ and two numbers $\lambda = \lambda(\tilde{\mathcal{U}}) > 0$ and $T = T(\tilde{\mathcal{U}}) > 0$ such that for any $Y \in \tilde{\mathcal{U}}$ and any periodic point p of Y, the following two estimates hold:*

(a) $\|P_t^Y|E^s(p)\| \cdot \|P_{-t}^Y|E^u(Y_t(p))\| \leq e^{-2\lambda t}$ *for any $t \geq T$;*

(b) *if τ is the period of p, m is any positive integer, and if $0 = t_0 < t_1 < \cdots < t_k = m\tau$ is any partition of the time interval $[0, m\tau]$ with $t_{i+1} - t_i \geq T$, then*

$$\frac{1}{m\tau}\sum_{i=0}^{k-1} \log\|P^Y_{t_{i+1}-t_i}|E^s(Y_{t_i}(p))\| < -\lambda,$$

$$\frac{1}{m\tau}\sum_{i=0}^{k-1} \log\|P^Y_{-(t_{i+1}-t_i)}|E^u(Y_{t_{i+1}}(p))\| < -\lambda.$$

Now suppose that x is in $\Sigma(X)$ and $\overline{\mathrm{orbit}(x)} \cap \overline{\mathrm{per}(X)} = \emptyset$. Since x belongs to $\Sigma(X)$, we may find a sequence $\{Y^n\}$ of vector fields, a sequence $\{P_n\}$ of periodic orbits, a sequence $\{a_n\}$ ($a_n \in P_n$) of points, a sequence $\{T_n\}$ such that $T_n \to \infty$ (as $n \to \infty$), and a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ (as $n \to \infty$) satisfying the following:

(i) $\lim_{n\to\infty} Y^n = X$ and P_n is a periodic orbit of Y^n with period T_n;

(ii) $\lim_{n\to\infty} a_n = x$;

(iii) $d(X_t(x), Y_t^n(a_n)) \leq \varepsilon_n$ (for all $0 \leq t \leq T_n$).

We define l as the integer such that for an infinitely many number of n, the index of P_n (written by $\mathrm{Ind}(P_n)$) is l.

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CLAIM. $l \neq 0$.

Proof. If $l = 0$ on the contrary, we may assume that

$$\text{index of } P_n = 0 \quad \text{for all } n.$$

Since each P_n is compact, a subsequence of P_n, also denoted by P_n, converges to some X_t-invariant closed subset F in $\overline{\text{orbit}(x)}$ by (iii) with respect to Hausdorff metric. As in [**8**], we take an individual measure μ_n corresponding to a point a_n, and we may assume that the sequence μ_n converges with weak-star topology to a probability measure μ on M. Each μ_n is invariant under Y_t^n, and μ is a measure supported on F. Let $\{\varphi_n(a)\}$ be a sequence of real-valued continuous functions on M such that

$$\lim_{n\to\infty} \varphi_n(a) = \varphi(a) \quad \text{uniformly on } M.$$

Then we have

$$\lim_{n\to\infty} \int_M \varphi_n(a)\, d\mu_n = \int_M \varphi(a)\, d\mu. \tag{1}$$

Because

$$\begin{aligned} &\left| \int_M \varphi_n(a)\, d\mu_n - \int_M \varphi(a)\, d\mu \right| \\ &\qquad \leq \left| \int_M \varphi_n(a)\, d\mu_n - \int_M \varphi(a)\, d\mu_n \right| + \left| \int_M \varphi(a)\, d\mu_n - \int_M \varphi(a)\, d\mu \right| \end{aligned}$$

and $\varphi_n(a) \to \varphi(a)$ implies

$$\left| \int_M \varphi_n(a)\, d\mu_n - \int_M \varphi(a)\, d\mu_n \right| \leq \int_M |\varphi_n(a) - \varphi(a)|\, d\mu_n \to 0,$$

while $\mu_n \to \mu$ implies

$$\left| \int_M \varphi(a)\, d\mu_n - \int_M \varphi(a)\, d\mu \right| \to 0.$$

Moreover, since each μ_n is Y_t^n-invariant, μ is X_t-invariant. In fact, by (1) above, for any $t \in \mathbb{R}$,

$$\begin{aligned} \int_M \varphi(X_t(a))\, d\mu &= \lim_{n\to\infty} \int_M \varphi_n(X_t(a))\, d\mu_n - \lim_{n\to\infty} \int_M \varphi_n(Y_t^n(a))\, d\mu_n \\ &\quad + \lim_{n\to\infty} \int_M \varphi_n(Y_t^n(a))\, d\mu_n \\ &= \lim_{n\to\infty} \left(\int_M \varphi_n(X_t(a))\, d\mu_n - \int_M \varphi_n(Y_t^n(a))\, d\mu_n \right) \\ &\quad + \lim_{n\to\infty} \int_M \varphi_n(Y_t^n(a))\, d\mu_n \\ &= \lim_{n\to\infty} \int_M \varphi_n(Y_t^n(a))\, d\mu_n = \lim_{n\to\infty} \int_M \varphi_n(a)\, d\mu_n = \int_M \varphi(a)\, d\mu. \end{aligned}$$

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Remark.

$$\lim_{n\to\infty}\left(\int_M \varphi_n(X_t(a))\,d\mu_n - \int_M \varphi_n(Y_t^n(a))\,d\mu_n\right) = 0.$$

In fact, since $\lim_{n\to\infty} Y^n = X$, $d(X_t(a), Y_t^n(a))$ is very small for sufficiently large n (for any $a \in M$ and a fixed t). On the other hand, as φ_n is uniformly continuous on M,

$$|\varphi_n(X_t(a)) - \varphi_n(Y_t^n(a))| < \varepsilon$$

for any $a \in M$ and sufficiently large n.

As μ_n is an individual measure corresponding to a_n

$$\int_M \varphi_n(a)\,d\mu_n = \lim_{t\to\infty}\frac{1}{t}\int_0^t \varphi_n(Y_s^n(a_n))\,ds. \tag{2}$$

Now, let

$$\xi_{-T}^n(a) = \frac{1}{T}\log\|P_{-T}^{Y^n}(a)\|,$$
$$\xi_{-T}(a) = \frac{1}{T}\log\|P_{-T}^{X}(a)\|.$$

Then $\lim_{n\to\infty}\xi_{-T}^n(a) = \xi_{-T}(a)$ uniformly on M for the number $T = T(\tilde{\mathcal{U}}) > 0$. From Theorem 2.1(b), for sufficiently large n,

$$\frac{T}{T_n}\left\{\sum_{k=1}^{m_n}\xi_{-T}^n(Y_{kT}^n(a_n)) + \frac{1}{T}\log\|P_{-(T_n-m_nT)}^{Y^n}(Y_{T_n}^n(a_n))\|\right\} < -\lambda,$$

where m_n is the greatest integer with $T_n - m_nT \geq T$. Since

$$\frac{1}{T_n}\log\|P_{-(T_n-m_nT)}^{Y^n}(Y_{T_n}^n(a_n))\| \to 0 \quad \text{as } n\to\infty$$

for sufficiently large n, we have

$$\frac{T}{m_nT}\sum_{k=1}^{m_n}\xi_{-T}^n(Y_{kT}^n(a_n)) < -\frac{\lambda}{2}, \quad a_n \in P_n.$$

Thus

$$\frac{1}{pm_n}\left(\sum_{k=1}^{pm_n}\xi_{-T}^n(Y_{kT}^n(a_n))\right) < -\frac{\lambda}{2}, \quad p = 1, 2\ldots.$$

Therefore

$$\begin{aligned}
&\frac{1}{pm_nT}\int_0^{pm_nT}\xi_{-T}^n(Y_s^n(a_n))\,ds \\
&\quad= \frac{1}{pm_nT}\sum_{k=0}^{pm_n-1}\int_{kT}^{(k+1)T}\xi_{-T}^n(Y_s^n(a_n))\,ds \\
&\quad= \frac{1}{T}\int_0^T\frac{1}{pm_n}\sum_{k=0}^{pm_n-1}\xi_{-T}^n(Y_{kT}^n(Y_s^n(a_n)))\,ds < -\frac{\lambda}{2}, \quad p = 1, 2\ldots.
\end{aligned}$$

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Thus, from (2),

$$\int_{P_n} \xi^n_{-T}(a)\, d\mu_n < -\frac{\lambda}{2}.$$

Thus, from (1) we have

$$\int_F \xi_{-T}(a)\, d\mu = \lim_{n\to\infty} \int_{P_n} \xi^n_{-T}(a)\, d\mu_n \le -\frac{\lambda}{2} < 0.$$

Now we need a lemma from Liao [**8**] to proceed.

LEMMA 2.2. (Liao [**8**, Lemma 3.2]) *Let F be a closed subset of M, invariant under X_t. Assume that for a certain $\tilde{T} \in (0, \infty)$, there is a probability measure μ on F, invariant under X_t such that*

$$\int_F \xi_{\tilde{T}}(a)\, d\mu < 0 \quad \text{or} \quad \int_F \xi_{-\tilde{T}}(a)\, d\mu < 0. \tag{$*$}$$

Then, F contains a periodic orbit of X attracting or repelling corresponding to the first or the second inequality of $(*)$.

Applying this lemma to our situation we obtain a repelling periodic orbit in $F \subset \overline{\text{orbit}(x)}$. But this contradicts the assumption $x \in \Sigma(X)$. Because x is a recurrent point of X, $\overline{\text{orbit}(x)}$ could not contain a repelling periodic orbit. □

Now we may assume that $\text{Ind}(P_n) = l \neq 0$ and Y^n is in $\tilde{\mathcal{U}}$ (in Theorem 2.1) for all n. We have

$$\|P_t^{Y^n}(a_n)|E^s(a_n)\| \cdot \|P_{-t}^{Y^n}(Y_t^n(a_n))|E^u(Y_t^n(a_n))\| \le e^{-2\lambda t} \quad (t \ge T).$$

For any $t \in \mathbb{R}$, the subspaces $E^s(Y_t^n(a_n))$ and $E^u(Y_t^n(a_n))$ converge to subspaces of $N_{X_t(x)}$ which we define as $G^s(X_t(x))$ and $G^u(X_t(x))$ respectively. Now we attach for every $y \in \text{orbit}(x)$ two subspaces $G^s(y)$ and $G^u(y)$ with the following properties:

$$\dim G^s_y + \dim G^u_y = \dim M - 1; \tag{3}$$

$$(P_t^X(y))G^\sigma_y = G^\sigma_{X_t(y)}, \quad \sigma = s, u \text{ for any } t \in R; \tag{4}$$

$$\|P_t^X|G^s_y\| \cdot \|P_{-t}^X|G^u_{X_t(y)}\| \le e^{-2\lambda t}, \quad t \ge T; \tag{5}$$

$$G^s(y) \oplus G^u(y) = N_y \quad \text{for any } y \in \text{orbit}(x). \tag{6}$$

By Proposition I.3 of Mañé [**13**], we can extend this splitting to $\overline{\text{orbit}(x)}$ and from properties (3)–(6) it follows that the subspaces $G^s(y)$ and $G^u(y)$ depend continuously on $y \in \overline{\text{orbit}(x)}$. Now we obtain a dominated splitting $G^s \oplus G^u$ over $\overline{\text{orbit}(x)}$ and this dominated splitting is uniquely determined by the sequence of periodic orbits P_n.

In fact, if subspaces $E'^s(Y_t^n(a_n))$ and $E'^u(Y_t^n(a_n))$ converge to subspaces $G'^s(X_t(x))$ and $G'^u(X_t(x))$ of $N_{X_t(x)}$, respectively, which are different from $G^s(X_t(x))$, $G^u(X_t(x))$, then, using the same argument as above, we may obtain a dominated splitting $G'^s \oplus G'^u$ over $\overline{\text{orbit}(x)}$. Then we have the following lemma.

LEMMA 2.3. (Wen [**19**, Lemma 3.7]) *Let $A \oplus B$ and $A' \oplus B'$ be two dominated splittings at x with $\dim A = i$ and $\dim A' = i'$. If $i \le i'$, then $A \subset A'$ and $B \supset B'$. In particular, if $i = i'$, then $A = A'$ and $B = B'$.*

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Using this lemma, we obtain $G^s(y) = G'^s(y)$ and $G^u(y) = G'^u(y)$ respectively for any $y \in \overline{\mathrm{orbit}(x)}$. Thus, the dominated splitting $G^s \oplus G^u$ over $\overline{\mathrm{orbit}(x)}$ obtained by the sequence of periodic orbits P_n is uniquely determined.

Let $\mathrm{per}_i(X)$ be a set of hyperbolic periodic points of index i.

LEMMA 2.4. *If G^s is contracting, then $\overline{\mathrm{orbit}(x)} \cap \bigcup_{i=0}^{l} \overline{\mathrm{per}_i(X)} \neq \emptyset$.*

Proof. We may assume Y^n is C^1 close to X and $d(Y_t^n(a_n), X_t(x)) \leq \varepsilon_n$ for all $0 \leq t \leq T_n$. As in [**10**, p. 523], we obtain a vector field $\bar{Y}^n$ close to Y^n such that the following condition also holds.

(**) N^* restricted to the $\bar{Y}^n$-orbit P_n of a_n (which is periodic with the period $\bar{T}_n$ close to T_n) has a $P_t^{\bar{Y}^n}$-invariant splitting $\bar{G}^s \oplus \bar{G}^u$ such that

$$\|P_{-m}^{\bar{Y}^n}(\bar{Y}_{mj}^n(a_n))|\bar{G}^u(\bar{Y}_{mj}^n(a_n))\| = \|P_{-m}^X(X_{mj}(x))|G^u(X_{mj}(x))\|,$$

$$\|P_m^{\bar{Y}^n}(\bar{Y}_{m(j-1)}^n(a_n))|\bar{G}^s(\bar{Y}_{m(j-1)}^n(a_n))\| = \|P_m^X(X_{m(j-1)}(x))|G^s(X_{m(j-1)}(x))\|$$

for all $1 \leq j \leq [T_n/m] - 1$,

$$\|P_{-(T_n - m[T_n/m]+m)}^{\bar{Y}^n}(\bar{Y}_{T_n}^n(a_n))|\bar{G}^u(\bar{Y}_{T_n}^n(a_n))\| = \|P_{-(T_n - m[T_n/m]+m)}^X(X_{T_n}(x))|G^u(X_{T_n}(x))\|,$$

and

$$\|P_{T_n - m[T_n/m]+m}^{\bar{Y}^n}(\bar{Y}_{m[T_n/m]-m}^n(a_n))|\bar{G}^s(\bar{Y}_{m[T_n/m]-m}^n(a_n))\| = \|P_{T_n - m[T_n/m]+m}^X(X_{m[T_n/m]-m}(x))|G^s(X_{m[T_n/m]-m}(x))\|,$$

where m is the smallest integer such that $m \geq T$.

Since the index of P_n associated to Y^n is l, that of P_n associated to $\bar{Y}^n$ is also l. In particular, $\dim G^s = l$.

As we assume that G^s is contracting we may conclude that $\bar{G}^s = E^s$ (E^s is the hyperbolic splitting of $\bar{Y}^n$ on P_n). Therefore $\bar{G}^u = E^u$ and for sufficiently large n,

$$\prod_{j=0}^{k_n-1} \|P_{-m}^X(X_{mj}(x))|G^u(X_{mj}(x))\| = \prod_{j=0}^{k_n-1} \|P_{-m}^{\bar{Y}^n}(\bar{Y}_{mj}^n(a_n))|\bar{G}^u(\bar{Y}_{mj}^n(a_n))\| \leq e^{-(\lambda/2)k_n m} \quad (k_n = [T_n/m]).$$

We shall use the following definitions. We say that a pair of points $(x, X_{ml}(x))$, $l > 0$, is a γ-string if

$$\prod_{j=1}^{l} \|P_{-m}^X|G^u(X_{mj}(x))\| \leq \gamma^l$$

and we say that it is a uniformly γ-string if $(X_{mk}(x), X_{ml}(x))$ is a γ-string for all $0 \leq k < l$. Note that $(x, X_{mk_n}(x))$ is the $\gamma(= e^{-(\lambda/2)m})$-string for sufficiently large n. Now we need the following lemma.

LEMMA 2.5. (Mañé [**12**, Lemma II.3]) *For all $0 < \gamma_0 < \gamma_1 < 1$, there exist $N(\gamma_0, \gamma_1) > 0$ and $0 < c(\gamma_0, \gamma_1) < 1$ such that if $(x, X_{mk_n}(x))$ is a γ_0-string and $k_n \geq N(\gamma_0, \gamma_1)$, then there exist $0 < n_1 < \cdots < n_\theta \leq k_n$, $\theta > 1$ such that $\theta \geq k_n \cdot c(\gamma_0, \gamma_1)$ and $(x, X_{mn_i}(x))$ is a uniformly γ_1-string for all $1 \leq i \leq \theta$.*

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From the above lemma, we may assume that for some $0 < \gamma < \gamma_1 < 1$, for any large n, there exist $0 < n_1 < \cdots < n_\theta \leq k_n$ such that $(x, X_{mn_i}(x))$ is a uniformly γ_1-string. Note that, since $k_n \to \infty$ $(n \to \infty)$, $\theta \to \infty$ and by the compactness of $\overline{\text{orbit}(x)}$, the sequence $\{X_{mn_i}(x)\}$ accumulates to some point z in $\overline{\text{orbit}(x)}$. Thus we obtain a pair of points $\{X_{mn_\alpha}(x), X_{mn_\beta}(x)\}$, $\alpha < \beta$, in $\overline{\text{orbit}(x)}$ which are arbitrarily close to each other. Here we need another lemma.

LEMMA 2.6. (Mañé [**12**, Lemma II.2]) *Given $0 < \hat{\gamma} < 1$ and $\sigma > 0$, there exists $\varepsilon = \varepsilon(\hat{\gamma}, \sigma) > 0$ such that if $(x_i, g^{n_i}(x_i))$, $i = 1, 2, \ldots, k$, are uniformly $\hat{\gamma}$-strings satisfying $d(g^{n_i}(x_i), x_{i+1}) < \varepsilon$ for all $1 \leq i \leq k-1$ and $d(g^{n_k}(x_k), x_1) < \varepsilon$, then there exists a periodic point x of g with period $N = n_1 + n_2 + \cdots + n_k$ such that $d(g^n(x), g^n(x_1)) < \sigma$ for all $0 \leq n \leq n_1$ and $d(g^{N_i+n}(x), g^n(x_{i+1})) < \sigma$, for all $0 \leq n \leq n_{i+1}$, where $N_i = n_1 + \cdots + n_i$.*

Applying this lemma to the pair $\{X_{mn_\alpha}(x), X_{mn_\beta}(x)\}$ we find a periodic point p of X near the $\text{arc}(X_{mn_\alpha}(x), X_{mn_\beta}(x))$. Since $\dim G^s$ is constant over $\overline{\text{orbit}(x)}$, there exists an admissible neighborhood U [**12**, p. 173]. We may assume that this periodic orbit P of p for X is in U. Since G^s is contracting, the fact that periodic orbit P is near the $\text{arc}(X_{mn_\alpha}(x), X_{mn_\beta}(x))$ and P is in U implies that the index of $P \leq l$, completing the proof of Lemma 2.4. □

Now suppose that G^s is not contracting. Then, we can find $y \in \overline{\text{orbit}(x)}$ and a sequence $j_n \to \infty$ such that

$$\lim_{n\to\infty} \frac{1}{j_n} \log \| P^X_{mj_n}(y) | G^s(y) \| \geq 0, \tag{***}$$

where m is the same as above. Without loss of generality we may suppose that the sequence $\{j_n\}$ is such that there exists an X_m-invariant probability measure μ on $\overline{\text{orbit}(x)}$ such that

$$\int_{\overline{\text{orbit}(x)}} \phi \, d\mu = \lim_{n\to\infty} \frac{1}{j_n} \sum_{i=0}^{j_n-1} \phi(X_{mi}(y))$$

for every continuous $\phi : \overline{\text{orbit}(x)} \to \mathbb{R}$. Applying this equality to ϕ defined by $\phi(a) = \log \| P^X_m(a) | G^s(a) \|$ we obtain

$$\begin{aligned} \int_{\overline{\text{orbit}(x)}} \phi \, d\mu &= \lim_{n\to\infty} \frac{1}{j_n} \sum_{i=0}^{j_n-1} \log \| P^X_m(X_{mi}(y)) | G^s(X_{mi}(y)) \| \\ &\geq \lim_{n\to\infty} \frac{1}{j_n} \log \| P^X_{mj_n}(y) | G^s(y) \| \geq 0. \end{aligned} \tag{7}$$

On the other hand, by Birkhoff's theorem,

$$\int_{\overline{\text{orbit}(x)}} \phi \, d\mu = \int_{\overline{\text{orbit}(x)}} \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \| P^X_m(X_{mi}(a)) | G^s(X_{mi}(a)) \| \, d\mu. \tag{8}$$

LEMMA 2.7. (Ergodic closing lemma, Hayashi [**5**, Lemma VII.6])

$$\mu(\Sigma(X) \cup \text{Sing}(X)) = 1$$

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for every X_1-invariant probability measure μ on the borel sets of M, where $\mathrm{Sing}(X)$ *denotes the set of singularities of X.*

By Lemma 2.7, $\overline{\mathrm{orbit}(x)} \cap \Sigma(X)$ is an X_1-invariant total probability subset of $\overline{\mathrm{orbit}(x)}$. Hence, if we define an X_1-invariant probability measure ν on $\overline{\mathrm{orbit}(x)}$ (see [**10**, p. 522]) by

$$\nu = \frac{1}{m}\sum_{i=0}^{m-1}(X_i)_*(\mu),$$

where μ is an X_m-invariant probability measure on $\overline{\mathrm{orbit}(x)}$, we obtain

$$0 = \nu(\overline{\mathrm{orbit}(x)} - \overline{\mathrm{orbit}(x)} \cap \Sigma(X)) = \frac{1}{m}\sum_{i=0}^{m-1}\mu(X_i(\overline{\mathrm{orbit}(x)} - \overline{\mathrm{orbit}(x)} \cap \Sigma(X)))$$

$$= \frac{1}{m}\sum_{i=0}^{m-1}\mu(\overline{\mathrm{orbit}(x)} - \overline{\mathrm{orbit}(x)} \cap \Sigma(X)) = \mu(\overline{\mathrm{orbit}(x)} - \overline{\mathrm{orbit}(x)} \cap \Sigma(X)).$$

This together with (7) and (8) implies

$$0 \le \int_{\overline{\mathrm{orbit}(x)}\cap\Sigma(X)} \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log\|P_m^X(X_{mi}(a))|G^s(X_{mi}(a))\|\,d\mu(a).$$

Hence there exists $z \in \Sigma(X) \cap \overline{\mathrm{orbit}(x)}$ such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log\|P_m^X(X_{mi}(z))|G^s(X_{mi}(z))\| \ge 0. \tag{9}$$

Now we may take $-\lambda < -\lambda_0 < 0$ such that for n_0 large enough

$$\frac{1}{n_0}\sum_{i=0}^{n_0-1}\log\|P_m^X(X_{mi}(z))|G^s(X_{mi}(z))\| > -\frac{\lambda_0}{2}m. \tag{10}$$

By the hypothesis $\overline{\mathrm{orbit}(x)} \cap \overline{\mathrm{per}(X)} = \emptyset$, we have $z \notin \overline{\mathrm{per}(X)}$. By the same argument used in the proof of Lemma 2.4, we may take $\bar{Y}$ arbitrarily close to X and a periodic orbit P of $\bar{Y}$ with period τ (τ is very large, because $z \notin \overline{\mathrm{per}(X)}$) such that $\bar{Y}$ satisfies $(**)$. That is, N^* restricted to the $\bar{Y}$-orbit P has a $P_t^{\bar{Y}}$-invariant splitting $\bar{G}^s \oplus \bar{G}^u$ such that for $p \in P$,

$$\|P_{-m}^{\bar{Y}}(\bar{Y}_{m(j+1)}(p))|\bar{G}^u(\bar{Y}_{m(j+1)}(p))\| = \|P_{-m}^X(X_{m(j+1)}(z))|G^u(X_{m(j+1)}(z))\|,$$
$$\|P_m^{\bar{Y}}(\bar{Y}_{mj}(p))|\bar{G}^s(\bar{Y}_{mj}(p))\| = \|P_m^X(X_{mj}(z))|G^s(X_{mj}(z))\|$$

for all $0 \le j \le [\tau/m] - 2$,

$$\|P_{-(\tau-m[\tau/m]+m)}^{\bar{Y}}(\bar{Y}_\tau(p))|\bar{G}^u(\bar{Y}_\tau(p))\| = \|P_{-(\tau-m[\tau/m]+m)}^X(X_\tau(z))|G^u(X_\tau(z))\|,$$

and

$$\|P_{\tau-m[\tau/m]+m}^{\bar{Y}}(\bar{Y}_{m[\tau/m]-m}(p))|\bar{G}^s(\bar{Y}_{m[\tau/m]-m}(p))\|$$
$$= \|P_{\tau-m[\tau/m]+m}^X(X_{m[\tau/m]-m}(z))|G^s(X_{m[\tau/m]-m}(z))\|.$$

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Then we have

$$\begin{aligned}
&\|P^{\bar{Y}}_{-\tau}(\bar{Y}_\tau(p))|\bar{G}^u(\bar{Y}_\tau(p))\| \quad (k=[\tau/m]-1)\\
&\quad\le\left(\prod_{i=0}^{k-1}\|P^{\bar{Y}}_{-m}(\bar{Y}_{mk-mi}(p))|\bar{G}^u(\bar{Y}_{mk-mi}(p))\|\right)\times\|P^{\bar{Y}}_{-(\tau-mk)}(\bar{Y}_\tau(p))|\bar{G}^u(\bar{Y}_\tau(p))\|\\
&\quad=\prod_{i=0}^{k-1}(\|P^{\bar{Y}}_{-m}(\bar{Y}_{mk-mi}(p))|\bar{G}^u(\bar{Y}_{mk-mi}(p))\|\\
&\qquad\times\|P^{\bar{Y}}_{m}(\bar{Y}_{m(k-1)-mi}(p))|\bar{G}^s(\bar{Y}_{m(k-1)-mi}(p))\|)\\
&\qquad\times\left(\prod_{i=0}^{k-1}\|P^{X}_{m}(X_{m(k-1)-mi}(z))|G^s(X_{m(k-1)-mi}(z))\|\right)^{-1}\\
&\qquad\times\|P^{\bar{Y}}_{-(\tau-mk)}(\bar{Y}_\tau(p))|\bar{G}^u(\bar{Y}_\tau(p))\|\\
&\quad\le e^{-2\lambda mk}\times e^{(\lambda_0/2)mk}\times\|P^{\bar{Y}}_{-(\tau-mk)}(\bar{Y}_\tau(p))|\bar{G}^u(\bar{Y}_\tau(p))\|.
\end{aligned}$$

Note that we used (10) to induce the last inequality above. If τ is very large, then k can also be large so that

$$e^{-2\lambda mk}\times e^{(\lambda_0/2)mk}\times\|P^{\bar{Y}}_{-(\tau-mk)}(\bar{Y}_\tau(p))|\bar{G}^u(\bar{Y}_\tau(p))\|<1.$$

Thus, $\bar{G}^u\subset E^u$ (where $E^s\oplus E^u$ is the hyperbolic splitting of P for $\bar{Y}$). Therefore, $\dim E^u\ge\dim\bar{G}^u=(\dim M-1)-l$. That is, $\dim E^s\le l$. If $\dim E^s=l$, then $\bar{G}^u=E^u$, $\bar{G}^s=E^s$. By Theorem 2.1(b) and (10) above, we have,

$$\begin{aligned}
e^{-(\lambda/2)mk}&>\prod_{i=0}^{k-1}\|P^{\bar{Y}}_{m}(\bar{Y}_{mi}(p))|E^s(\bar{Y}_{mi}(p))\|\\
&=\prod_{i=0}^{k-1}\|P^{X}_{m}(X_{mi}(z))|G^s(X_{mi}(z))\|\\
&>e^{-(\lambda_0/2)mk}\quad\text{(from (10))}.
\end{aligned}$$

This is a contradiction. Thus we obtain

$$\mathrm{Ind}(P)<l.$$

Using the same argument as above, we again have a sequence $\{Y^n\}$, a sequence $\{P_n\}$ of periodic orbits, a sequence $\{a_n\}$ with $a_n\in P_n$, a sequence $\{T_n\}$ such that $T_n\to\infty$, a sequence of arcs $\{\mathrm{arc}(z,X_{T_n}(z))\}$, and a sequence $\varepsilon_n\to 0$ such that:

(i) $\lim_{n\to\infty}Y^n=X$ and P_n is a periodic orbit of Y_n with period T_n;

(ii) $\lim_{n\to\infty}a_n=z$;

(iii) $d(X_t(z),Y^n_t(a_n))\le\varepsilon_n$ for all $0\le t\le T_n$.

For all n, we can assume that $\mathrm{Ind}(P_n)=l'$ with $l'<l$. As before, we can obtain the dominated splitting $G^s\oplus G^u$ over $\overline{\mathrm{orbit}(z)}$ $(\subset\overline{\mathrm{orbit}(x)})$ with $\dim G^s=l'$, and if G^s is contracting the situation is as same as Lemma 2.4. Otherwise, we can find $w\in\overline{\mathrm{orbit}(z)}\cap\Sigma(X)$ such that $\overline{\mathrm{orbit}(w)}\subset\overline{\mathrm{orbit}(z)}\subset\overline{\mathrm{orbit}(x)}$ and has a dominated splitting with $\dim G^s<l'<l$.

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In the case when G^s is not contracting, by repeating this argument we eventually obtain a sequence $\{Y^n\}$ of vector fields, and a sequence $\{P_n\}$ of periodic orbits, such that

$$\lim_{n\to\infty} Y^n = X, \quad \mathrm{Ind}(P_n) = 0 \quad \text{for all } n,$$

contradicting the claim. We have finished the proof of Theorem A.

3. *Proof of Theorem B*

If X is in $\mathcal{G}^1(M)$, then the number of attracting or repelling periodic orbits is finite (by Pliss [**17**]). Hence, $\overline{\mathrm{per}_0(X)}$ and $\overline{\mathrm{per}_{n-1}(X)}$ are hyperbolic. It suffices to show that $\overline{\mathrm{per}_j(X)}$ is hyperbolic for each $0 < j < n-1$. Using Theorem 2.1 and the argument of Hayashi [**5**], we obtain the dominated splitting $G^s \oplus G^u$ over $\overline{\mathrm{per}_j(X)}$ such that

$$N^*|\overline{\mathrm{per}_j(X)} = G^s \oplus G^u, \quad \dim G^s = j$$

and satisfying for any $x \in \overline{\mathrm{per}_j(X)}$,

$$\|P_m^X(x)|G^s(x)\| \cdot \|P_{-m}^X(X_m(x))|G^u(X_m(x))\| \le e^{-2\lambda m},$$

where m is the smallest positive integer with $m \ge T$ (T is given in Theorem 2.1). Now we have only to prove that G^s is contracting, as in Hayashi [**5**]. Suppose G^s is not contracting, then by the same argument as in Theorem A, we have $x \in \Sigma(X) \cap \overline{\mathrm{per}_j(X)}$, $x \notin \mathrm{per}(X)$ having property $(***)$. Since $x \in \Sigma(X)$, we can obtain Y, C^1-close to X having a periodic point p with period τ such that, for sufficiently small $\varepsilon > 0$, $d(Y_t(p), X_t(x)) \le \varepsilon$ for all $0 \le t \le \tau$, and we can create $\bar{Y}$ C^1-close to Y such that $\bar{Y}$ has property $(**)$. Thus, as in the proof of Theorem A, the index of the periodic orbit P through p for $\bar{Y}$ is not greater than j. If $\mathrm{Ind}\, P = j$, then as before we can conclude that this is a contradiction. Now assume that $\mathrm{Ind}(P) < j$. As before, we may obtain a sequence $\{Y^n\}$, a sequence $\{P_n\}$ of periodic orbits, a sequence $\{T_n\}$ such that $T_n \to \infty$, a sequence $\{a_n\}$ of points, and a sequence ε_n such that $\varepsilon_n \to 0$ and satisfies:

(i) $\lim_{n\to\infty} Y^n = X$ and P_n is a periodic orbit of Y^n with period T_n;
(ii) $\lim_{n\to\infty} a_n = x$;
(iii) $d(X_t(x), Y_t^n(a_n)) \le \varepsilon_n$ for all $0 \le t \le T_n$;
(iv) $\mathrm{Ind}(P_n) = l$ $(0 \le l < j)$ for all n.

If $l = 0$, then this is a contradiction as before. So, we assume that $l \ne 0$. Using the same argument as in Theorem A, we can obtain the dominated splitting $G^{s'} \oplus G^{u'}$ over $\overline{\mathrm{orbit}(x)}$ $(\subset \overline{\mathrm{per}_j(X)})$ with $\dim G^{s'} = l < j$. Now if $G^{s'}$ is contracting, then by Lemma 2.4, we obtain

$$\bigcup_{i=0}^{j-1} \overline{\mathrm{per}_i(X)} \cap \overline{\mathrm{per}_j(X)} \supset \bigcup_{i=0}^{l} \overline{\mathrm{per}_i(X)} \cap \overline{\mathrm{orbit}(x)} \ne \emptyset.$$

This contradicts the assumption of Theorem B.

On the other hand, if $G^{s'}$ is not contracting, then we obtain $y \in \overline{\mathrm{orbit}(x)} \cap \Sigma(X)$ with property $(***)$ and y is not a periodic point, because, if y is a periodic point of X, then from Lemma 2.3 we have $G^{s'} \subset E^s$ (where E^s is the hyperbolic splitting of y of X), contradicting $(***)$. As in the proof of Theorem A, we obtain a sequence $\{Y^n\}$ and a

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sequence $\{P_n\}$ of periodic orbits with index smaller than l $(<j)$. Repeating this argument, we obtain a sequence $\{Y^n\}$ and a sequence $\{P_n\}$ of periodic orbits with index 0. But this is a contradiction. Therefore, we may assume that $G^{s'}$ is contracting and therefore $\overline{\mathrm{per}_j(X)}$ is hyperbolic. This proves Theorem B.

4. *Proof of the Ω-stability conjecture for flows*

Using the argument in the proof of Theorem A, we shall prove the Ω-stability conjecture.

As X is Ω-stable, there exists a neighborhood $\mathcal{U}(X)$ of X such that if Y is in $\mathcal{U}(X)$, then there exists a homeomorphism $h_Y : \Omega(Y) \to \Omega(X)$ which satisfies $X_t(h_Y(y)) = h_Y(Y_{t'}(y))$ for any $y \in \Omega(Y)$ and $t, t' \in \mathbb{R}$. If Y is in $\mathcal{U}(X)$, then Y is in $\mathcal{G}^1(M)$ from [**2**] and, by Hayashi's connecting lemma [**5**], $\mathrm{Sing}(Y) \cap \overline{\mathrm{per}(Y)} = \emptyset$. With this property, we may use the same argument as in Ito and Toyoshiba [**7**] (where the connecting lemma is crucial again). We find $Z \in \mathcal{U}(X)$ which is close to X and satisfies Axiom A. Hence, $\overline{\mathrm{per}(Z)}$ is decomposed into a finite union of disjoint basic sets

$$\Omega(Z) = \mathrm{Sing}(Z) \cup \overline{\mathrm{per}(Z)} = \mathrm{Sing}(Z) \cup \Lambda_1 \cup \cdots \cup \Lambda_{s(Z)},$$

where each Λ_i $(1 \le i \le s(Z))$ is a basic set. Since X is Ω-stable, there exists a homeomorphism h_Z such that

$$\begin{aligned}\Omega(X) &= \mathrm{Sing}(X) \cup \overline{\mathrm{per}(X)} \\ &= h_Z(\Omega(Z)) = h_Z(\mathrm{Sing}(Z)) \cup \Lambda_1^X \cup \cdots \cup \Lambda_{s(Z)}^X,\end{aligned}$$

where $\Lambda_i^X = h_Z(\Lambda_i)$ $(1 \le i \le s(Z))$. Note that $\Lambda_i^X \cap \Lambda_j^X = \emptyset$ for $i \ne j$. To complete the proof, we show that each Λ_i^X $(1 \le i \le s(Z))$ is a hyperbolic set. From the argument of Palis [**16**], in each Λ_i^X $(1 \le i \le s(Z))$, periodic orbits have the same index and are dense. So, each Λ_i^X has a dominated splitting. Suppose that Λ_j^X is not hyperbolic and has a dominated splitting $N^*|\Lambda_j^X = G^s \oplus G^u$ $(\dim G^s = l)$. We note that all periodic orbits in Λ_j^X have index l. From the argument by Mañé [**12**] (see [**5**, pp. 131–132] for the flows version of this), we may assume that G^s is not contracting, since otherwise we are done. Then, using the same argument as in the proof of Theorem A, we obtain a point $x \in \Sigma(X) \cap \Lambda_j^X$, $x \notin \mathrm{per}(X)$ such that $\overline{\mathrm{orbit}(x)} \subset \Lambda_j^X$ and

$$\overline{\mathrm{orbit}(x)} \cap \bigcup_{i=0}^{l-1} \overline{\mathrm{per}_i(X)} \ne \emptyset.$$

Therefore, there exists $k \ne j$ such that

$$\Lambda_k^X \cap \Lambda_j^X \ne \emptyset.$$

This is a contradiction. Therefore, we conclude that each Λ_i^X $(1 \le i \le s(Z))$ is hyperbolic, completing the proof.

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