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Real-analytic AbC constructions on the torus

SHILPAK BANERJEE† and PHILIPP KUNDE‡

† *The Pennsylvania State University, Department of Mathematics, University Park, PA 16802, USA*
(e-mail: banerjee.shilpak@gmail.com)
‡ *University of Hamburg, Mathematics, Bundesstrasse 55, Hamburg 20146, Germany*
(e-mail: philipp.kunde@math.uni-hamburg.de)

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Abstract. In this article we demonstrate a way to extend the AbC (approximation by conjugation) method invented by Anosov and Katok from the smooth category to the category of real-analytic diffeomorphisms on the torus. We present a general framework for such constructions and prove several results. In particular, we construct minimal but not uniquely ergodic diffeomorphisms and non-standard real-analytic realizations of toral translations.

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1. *Introduction*

An important question in smooth ergodic theory asks if there are smooth versions to the objects and concepts of abstract ergodic theory. One of the most powerful tools of constructing volume-preserving C^∞-diffeomorphisms with prescribed ergodic or topological properties on any compact connected manifold M of dimension $d \geq 2$ admitting a non-trivial circle action $\mathcal{S} = \{\phi^t\}_{t\in\mathbb{S}^1}$ is the so-called approximation by conjugation (AbC) method developed by Anosov and Katok in their fundamental paper [**AK70**]. These diffeomorphisms are constructed as limits of conjugates $T_n = H_n^{-1} \circ \phi^{\alpha_n} \circ H_n$, where $\alpha_n = p_n/q_n = \alpha_{n-1} + 1/(s_{n-1} \cdot k_{n-1} \cdot l_{n-1} \cdot q_{n-1}^2) \in \mathbb{Q}$, $H_n = h_n \circ H_{n-1}$ and h_n is a measure-preserving diffeomorphism satisfying $\phi^{\alpha_{n-1}} \circ h_n = h_n \circ \phi^{\alpha_{n-1}}$. In each step the conjugation map h_n and the parameters k_{n-1}, l_{n-1} are chosen such that the diffeomorphism T_n imitates the desired property with a certain precision. Then the parameter s_{n-1} is chosen large enough to guarantee closeness of T_n to T_{n-1} in the C^∞-topology and so the convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ to a limit diffeomorphism is provided. This method enables the construction of smooth diffeomorphisms with specific ergodic properties (e.g. weak mixing ones in [**AK70**, §5]) or non-standard smooth realizations of measure-preserving systems (e.g. [**AK70**, §6] and [**FSW07**]). See also the very interesting survey article [**FK04**] for more details and other results of this method.

Unfortunately, there are great challenging differences in the real-analytic category as discussed in [**FK04**, §7.1]: since maps with very large derivatives in the real domain or their inverses are expected to have singularities in a small complex neighbourhood, for a real-analytic family S_t, $0 \leq t \leq t_0$, $S_0 = \mathrm{id}$, the family $h^{-1} \circ S_t \circ h$ is expected to have singularities very close to the real domain for any $t > 0$. So, the domain of analyticity for maps of our form $f_n = H_n^{-1} \circ \phi^{\alpha_n} \circ H_n$ will shrink at any step of the construction and the limit diffeomorphism will not be analytic. Thus, it is necessary to find conjugation maps of a special form which may be inverted more or less explicitly in such a way that one can guarantee analyticity of the map and its inverse in a large complex domain. Using very explicit conjugation maps, Saprykina was able to construct examples of volume-preserving uniquely ergodic real-analytic diffeomorphisms on $\mathbb{T}^2$ (see [**Sa03**]). Fayad and Katok designed such examples on any odd-dimensional sphere in [**FK14**].

The goal of this article is to reproduce some examples of smooth dynamical systems obtained by the AbC scheme in the category of real-analytic diffeomorphisms on the

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torus $\mathbb{T}^d$, $d \geq 2$. For this purpose, we introduce the concept of block-slide type of maps on the torus and demonstrate that these maps in a certain sense can be approximated well enough by measure-preserving real-analytic diffeomorphisms. This allows us to carry out many AbC constructions in the real-analytic category. We briefly summarize certain known results and prove several new ones using real-analytic approximation of block-slide type of maps. Note that all constructions in this article are done on the torus. Real-analytic AbC constructions on arbitrary real-analytic manifolds continue to remain an intractable problem.

Throughout this article $\mathbb{T}^d$ will denote the d-dimensional torus and μ will stand for the usual Lebesgue measure on $\mathbb{T}^d$. We will use $\mathrm{Diff}_\rho^\omega(\mathbb{T}^d\ \mu)$ to denote the set of real-analytic μ-measure-preserving diffeomorphisms of the d-dimensional torus whose lift can be extended holomorphically to a complex neighbourhood of diameter at least ρ.

First we tackle the problem of non-standard realizations (i.e. to find a diffeomorphism which is metrically but not smoothly isomorphic to a given measure-preserving transformation). The first author used the AbC method and the concept of real-analytic approximation of block-slide type of maps to find examples of measure-preserving real-analytic ergodic diffeomorphisms on the torus that are metrically isomorphic to some irrational rotation of the circle. The precise theorem can be stated as follows.

THEOREM 1.1. [**Ba17**] *For any $\rho > 0$ and any integer $d \geq 1$, there exist real-analytic diffeomorphisms $T \in \mathrm{Diff}_\rho^\omega(\mathbb{T}^d, \mu)$ which are metrically isomorphic to some irrational rotations of the circle.*

The diffeomorphisms constructed by Anosov and Katok in [**AK70**, §4] realized circle rotations smoothly with Liouvillean rotation numbers. However, it was not clear from this construction which Liouvillean rotations were realized. Later, Fayad, Saprykina and Windsor extended this result in [**FSW07**] and proved that any Liouvillean rotation of the circle can be realized. In the analytic category, we cannot expect realization of every Liouvillean rotation of the circle but we can give a precise description of a subset of Liouvillean numbers we realize. We introduce the set $\mathcal{L}_*$ of numbers contained in the set of non-Brjuno numbers: $\alpha \in \mathbb{R}$ is in $\mathcal{L}_*$ if for every $k \in \mathbb{N}$ there is $(p, q) \in \mathbb{Z} \times \mathbb{N}$ with p, q relatively prime satisfying

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{e^{e^{kq}}}. \tag{1.2}$$

In a later section, we will examine this set of numbers. In particular, we will show that $\mathcal{L}_*$ is a dense G_δ-subset of $\mathbb{R}$. We will prove the following theorem.

THEOREM A. *For any $\rho > 0$ and every $\alpha \in \mathcal{L}_*$, there exists a real-analytic diffeomorphism $T \in \mathrm{Diff}_\rho^\omega(\mathbb{T}^2, \mu)$ which is metrically isomorphic to the rotation S_α of the circle.*

In the realm of non-standard realizations, there is another set of questions dedicated to the realization of ergodic translations of a torus on another manifold. In the original paper, Anosov and Katok showed that certain ergodic translations on a d-dimensional torus can be realized as measure-preserving smooth diffeomorphisms on any smooth manifold admitting an effective $\mathbb{T}^1$ action (see [**AK70**, §6]). We should note that this

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result was further improved by Benhenda and it was shown that one can realize any ergodic translation on $\mathbb{T}^d$ with one arbitrary Liouvillean coordinate (see [**Be13**]).

It appears that the block-slide type of maps allow enough flexibility for us to realize some of these ergodic translations analytically on another torus of arbitrary dimension. We prove the following theorem.

THEOREM B. *For any $\rho > 0$ and any two integers, $h \geq 1$ and $d \geq 2$, there exists an ergodic real-analytic diffeomorphism $T \in \mathrm{Diff}^{\omega}_{\rho}(\mathbb{T}^d, \mu)$ which is metrically isomorphic to an ergodic translation of $\mathbb{T}^h$.*

And the obvious corollary follows.

THEOREM C. *For any $\rho > 0$ and any two integers, $h \geq 1$ and $d \geq 2$, there exists an ergodic real-analytic diffeomorphism $T \in \mathrm{Diff}^{\omega}_{\rho}(\mathbb{T}^d, \mu)$ such that T has a discrete spectrum generated (over $\mathbb{Z}$) by h linearly independent eigenvalues.*

There is a conjecture of Kolmogorov in [**Ko57**] stating that on a d-dimensional real-analytic manifold an ergodic real-analytic diffeomorphism preserving an analytic measure may have a discrete spectrum with only d distinct eigenvalues. Our result falsifies this conjecture.

Another aspect of the approximation by conjugation scheme deals with the problem of finding diffeomorphisms with a prescribed dynamical property. Originally Anosov and Katok produced examples of measure-preserving smooth diffeomorphisms that are weakly mixing on any manifold admitting an effective $\mathbb{T}^1$ action. Later Fayad and Saprykina constructed weakly mixing diffeomorphisms in the restricted space $\mathcal{A}_{\alpha}(M) = \overline{\{h \circ \phi^{\alpha} \circ h^{-1} : h \in \mathrm{Diff}^{\infty}(M, \mu)\}}^{C^{\infty}}$ for every Liouvillean number α [**FS05**] in case of dimension 2. In case of the disc $\mathbb{D}^2$ and the annulus $\mathbb{A}$, this gives the dichotomy that a number is Diophantine if and only if there is no ergodic C^{∞}-diffeomorphism with that rotation number. In that paper [**FS05**], the authors were even able to construct examples of weakly mixing real-analytic diffeomorphisms of the two-dimensional torus for rotation numbers α that satisfy a condition of similar type as our one above (namely that for some $\delta > 0$ the equation $|\alpha - p/q| < \exp(-q^{1+\delta})$ has an infinite number of relatively prime integer solutions p, q). We should note that the method of reparametrization of linear flows as in [**Fa02**] is more appropriate to get weakly mixing analytic diffeomorphisms on $\mathbb{T}^d$.

Using the AbC method and the concept of real-analytic approximation of block-slide type of maps, the second author showed that on a torus of any dimension greater than one there are examples of weakly mixing real-analytic diffeomorphisms preserving a measurable Riemannian metric.

THEOREM 1.3. [**Ku17**] *For any $\rho > 0$ and any integer $d \geq 2$, there are weakly mixing real-analytic diffeomorphisms $T \in \mathrm{Diff}^{\omega}_{\rho}(\mathbb{T}^d, \mu)$ preserving a measurable Riemannian metric.*

This result solved [**GK00**, Problem 3.9], about the existence of real-analytic volume-preserving IM-diffeomorphisms (i.e. diffeomorphisms preserving an absolutely

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continuous probability measure and a measurable Riemannian metric) in the case of tori. In this before-mentioned paper [**GK00**], Gunesch and Katok constructed volume-preserving weakly mixing C^∞-diffeomorphisms preserving a measurable Riemannian metric (see also [**GKu**] for the same result in the restricted spaces $\mathcal{A}_\alpha(M)$ for arbitrary Liouville number α) and gave a comprehensive consideration of IM-diffeomorphisms and IM-group actions. In particular, the existence of a measurable invariant metric for a diffeomorphism is equivalent to the existence of an invariant measure for the projectivized derivative extension which is absolutely continuous in the fibres. Recently, the second author examined the ergodic behaviour of the derivative extension with respect to such a measure [**Ku1**]. It provides the only known examples of measure-preserving diffeomorphisms whose differential is ergodic with respect to a smooth measure in the projectivization of the tangent bundle. It is an interesting open problem to exhibit such an examination in the real-analytic case.

In another version (called 'topological version' in [**FK04**]) of the AbC method, one tries to exercise control over every orbit of the initial $\mathbb{T}^1$ action, while the original construction by Anosov and Katok was only able to exercise control over almost every orbit of the $\mathbb{T}^1$ action. Such topological constructions deal with minimality and the number of ergodic invariant measures (e.g. unique ergodicity) for instance (see e.g. [**FH77**, **FSW07**, **Wi01**]). We can use such techniques and prove that the limiting diffeomorphisms obtained in Theorem 1.1 are in fact uniquely ergodic with respect to the Lebesgue measure [**Ba17**, Theorem 1.1]: for any $\rho > 0$, there exist uniquely ergodic real-analytic diffeomorphisms $T \in \mathrm{Diff}^\omega_\rho(\mathbb{T}^2, \mu)$ which are metrically isomorphic to some irrational rotations of the circle. We note that minor modifications will extend the above result to the d-dimensional torus but we do not do so in this article. Instead, we can produce more exotic examples. We show that there are minimal but not uniquely ergodic measure-preserving real-analytic diffeomorphisms.

THEOREM D. *For any $\rho > 0$ and any natural number r, there exists a real-analytic diffeomorphism $T \in \mathrm{Diff}^\omega_\rho(\mathbb{T}^2, \mu)$ which is minimal and has exactly r ergodic invariant measures each of which is absolutely continuous with respect to the Lebesgue measure.*

This result parallels a result of Windsor in the smooth category [**Wi01**]. While conversely a uniquely ergodic transformation on a compact metric space preserving a Borel measure is minimal on the support of the measure (e.g. [**HK95**, Proposition 4.1.18]), the first example showing that minimality does not imply unique ergodicity is due to Markov (see [**NS60**, §9.35]). In the analytic category, Furstenberg constructed skew-products admitting uncountably many ergodic measures (see [**Fu61**] or [**HK95**, Corollary 12.6.4]). In fact, these counterexamples bear a great meaning in the history of ergodic theory: they showed that the so-called *quasi-ergodic hypothesis* (i.e. each orbit is dense in each surface of constant energy) does not imply the equality of space means and time means and so helped to find the right notion of ergodicity.

In a forthcoming paper, the authors use the AbC method with the real-analytic approximation of block-slide type of maps to construct $T \in \mathrm{Diff}^\omega_\rho(\mathbb{T}^d, \mu)$ with disjoint convolutions and a homogeneous spectrum of multiplicity 2 for its Cartesian square $T \times T$. In [**Ku16**], C^∞-diffeomorphisms in $\mathcal{A}_\alpha(M)$ with these properties were constructed.

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2. *Preliminaries*

Here we introduce the basic concepts and establish notations that we will use for the rest of this article.

For a natural number d, we will denote the d-dimensional torus by $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$. The standard Lebesgue measure on $\mathbb{T}^d$ will be denoted by μ. We define ϕ, a measure-preserving $\mathbb{T}^1$ action on the torus $\mathbb{T}^d$, as follows:

$$\phi^t(x_1, \ldots, x_d) = (x_1 + t, x_2, \ldots, x_d). \tag{2.1}$$

2.1. *The topology of real-analytic diffeomorphisms on the torus.*

We give a description of the space of diffeomorphisms that are interesting to us. Any real-analytic diffeomorphism on $\mathbb{T}^d$ homotopic to the identity admits a lift to a map from $\mathbb{R}^d$ to $\mathbb{R}^d$ and has the following form:

$$F(x_1, \ldots, x_d) = (x_1 + f_1(x_1, \ldots, x_d), \ldots, x_d + f_d(x_1, \ldots, x_d)), \tag{2.2}$$

where $f_i : \mathbb{R}^d \to \mathbb{R}$ are $\mathbb{Z}^d$-periodic real-analytic functions. Any real-analytic $\mathbb{Z}^d$-periodic function defined on $\mathbb{R}^d$ can be extended to some complex neighbourhood† of $\mathbb{R}^d$ as a holomorphic (complex-analytic) function. For a fixed $\rho > 0$, let

$$\Omega_\rho := \{(z_1, \ldots, z_d) \in \mathbb{C}^d : |\mathrm{Im}(z_1)| < \rho, \ldots, |\mathrm{Im}(z_d)| < \rho\} \tag{2.3}$$

and, for a function f defined on this set, put

$$\|f\|_\rho := \sup_{(z_1,\ldots,z_d)\in\Omega_\rho} |f(z_1, \ldots, z_d)|. \tag{2.4}$$

We define $C^\omega_\rho(\mathbb{T}^d)$ to be the space of all $\mathbb{Z}^d$-periodic real-analytic functions on $\mathbb{R}^d$ that extends to a holomorphic function on Ω_ρ and $\|f\|_\rho < \infty$.

We define $\mathrm{Diff}^\omega_\rho(\mathbb{T}^d, \mu)$ to be the set of all measure-preserving real-analytic diffeomorphisms of $\mathbb{T}^d$ homotopic to the identity, whose lift $F(x) = (x_1 + f_1(x), \ldots, x_d + f_d(x))$ to $\mathbb{R}^d$ satisfies $f_i \in C^\omega_\rho(\mathbb{T}^d)$ and we also require the lift $\tilde{F}(x) = (x_1 + \tilde{f}_1(x), \ldots, x_d + \tilde{f}_d(x))$ of its inverse to $\mathbb{R}^d$ to satisfy $\tilde{f}_i \in C^\omega_\rho(\mathbb{T}^d)$. The metric d in $\mathrm{Diff}^\omega_\rho(\mathbb{T}^d, \mu)$ is defined by

$$d_\rho(f, g) = \max\{\tilde{d}_\rho(f, g), \tilde{d}_\rho(f^{-1}, g^{-1})\},$$
$$\text{where } \tilde{d}_\rho(f, g) = \max_{i=1,\ldots,d} \left\{ \inf_{n\in\mathbb{Z}} \|f_i - g_i + n\|_\rho \right\}.$$

Let $F = (F_1, \ldots, F_d)$ be the lift of a diffeomorphism in $\mathrm{Diff}^\omega_\rho(\mathbb{T}^d, \mu)$; we define the norm of the total derivative

$$\|DF\|_\rho := \max_{\substack{i=1,\ldots,d \\ j=1,\ldots,d}} \left\| \frac{\partial F_i}{\partial x_j} \right\|_\rho.$$

Next, with some abuse of notation, we define the following two spaces:

$$C^\omega_\infty(\mathbb{T}^d) := \bigcap_{n=1}^{\infty} C^\omega_n(\mathbb{T}^d), \tag{2.5}$$

† We identify $\mathbb{R}^d$ inside $\mathbb{C}^d$ via the natural inclusion $(x_1, \ldots, x_d) \mapsto (x_1 + i0, \ldots, x_d + i0)$.

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$$\mathrm{Diff}^{\omega}_{\infty}(\mathbb{T}^d, \mu) := \bigcap_{n=1}^{\infty} \mathrm{Diff}^{\omega}_{n}(\mathbb{T}^d, \mu). \tag{2.6}$$

Note that the functions in (2.5) can be extended to $\mathbb{C}^d$ as entire functions. We also note that $\mathrm{Diff}^{\omega}_{\infty}(\mathbb{T}^d, \mu)$ is closed under composition. To see this, let $f, g \in \mathrm{Diff}^{\omega}_{\infty}(\mathbb{T}^d, \mu)$ and F and G be their corresponding lifts. Then note that $F \circ G$ is the lift of $f \circ g$ (with $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ as the natural projection, $\pi \circ F \circ G = f \circ \pi \circ G = f \circ g \circ \pi$). Now, for the complexification of F and G, note that the composition $F \circ G(z) = (z_1 + g_1(z) + f_1(G(z)), \ldots, z_d + g_d(z) + f_d(G(z)))$. Since $g_i \in C^{\omega}_{\infty}(\mathbb{T}^d)$, we have for any ρ, $\sup_{z\in\Omega_\rho} |\mathrm{Im}(G(z))| \le \max_i(\sup_{z\in\Omega_\rho} |\mathrm{Im}(z_i) + \mathrm{Im}(g_i(z))|) \le \max_i(\sup_{z\in\Omega_\rho} |\mathrm{Im}(z_i)| + \sup_{z\in\Omega_\rho} |\mathrm{Im}(g_i(z))|) \le \rho + \max_i(\sup_{z\in\Omega_\rho} |g_i(z)|) < \rho + \mathrm{const} < \rho' < \infty$ for some ρ'. So, $\sup_{z\in\Omega_\rho} |z_i + g_i(z) + f_i(G(z))| \le |z_i| + |g_i(z)| + |f_i(G(z))| < \infty$ since $z \in \Omega_\rho$, $g_i \in C^{\omega}_{\infty}(\mathbb{T}^d)$ and $G(z) \in \Omega_{\rho'}$. An identical treatment gives the result for the inverse.

All intermediate diffeomorphisms constructed with the AbC method in this paper will belong to this category†.

This completes the description of the analytic topology necessary for our construction. Also, throughout this paper, the word 'diffeomorphism' will refer to a real-analytic diffeomorphism. Also, the phrase 'analytic topology' will refer to the topology of $\mathrm{Diff}^{\omega}_{\rho}(\mathbb{T}^d, \mu)$ described above. See [**Sa03**] for a more extensive treatment of these spaces.

2.2. *Some partitions of the torus.* First we recall a few definitions. A sequence of partitions $\{\mathcal{P}_n\}_n$ of a *Lebesgue space*‡ (M, μ) is called *generating* if there exists a measurable subset M' of full measure such that $\{x\} = \bigcap_{n=1}^{\infty} \mathcal{P}_n(x)$ for every $x \in M'$, where $\mathcal{P}_n(x)$ is the partition element in $\mathcal{P}_n$ containing x. We say that the sequence $\{\mathcal{P}_n\}_n$ is *monotonic* if $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$.

There are some partitions of $\mathbb{T}^d$ that are of special interest to us. They appear repeatedly in this article and we summarize them here.

Assume that we are given three natural numbers l, k, q and a function $a : \{0, 1, \ldots, k-1\} \to \{0, 1, \ldots, q-1\}$. We define the following three partitions of $\mathbb{T}^d$:

$$\mathcal{T}_q := \left\{ \Delta_{i,q} := \left[\frac{i}{q}, \frac{i+1}{q}\right) \times \mathbb{T}^{d-1} : i = 0, 1, \ldots, q-1 \right\}, \tag{2.7}$$

$$\mathcal{G}_{l,q} := \left\{ \left[\frac{i_1}{lq}, \frac{i_1+1}{lq}\right) \times \left[\frac{i_2}{l}, \frac{i_2+1}{l}\right) \times \cdots \times \left[\frac{i_d}{l}, \frac{i_d+1}{l}\right) : i_1 = 0, 1, \ldots, lq-1, (i_2, \ldots, i_d) \in \{0, 1, \ldots, l-1\}^{d-1} \right\}, \tag{2.8}$$

$$\mathcal{G}_{j,l,q} := \left\{ \left[\frac{i_1}{l^{d+1-j}q}, \frac{i_1+1}{l^{d+1-j}q}\right) \times \left[\frac{i_2}{l}, \frac{i_2+1}{l}\right) \times \cdots \times \left[\frac{i_j}{l}, \frac{i_j+1}{l}\right) \times \mathbb{T}^{d-j} : i_1 = 0, 1, \ldots, l^{d+1-j}q - 1, (i_2, \ldots, i_j) \in \{0, 1, \ldots, l-1\}^{j-1} \right\}, \tag{2.9}$$

† We note that the existence of such real-analytic functions whose complexification is entire or, as in this case, the complexification of their lift is entire is central to a real-analytic AbC method. As of now we only know how to construct such functions on the torus, odd-dimensional spheres and certain homogeneous spaces.

‡ Also known as a *standard probability space* or a *Lebesgue–Rokhlin space*. We consider those spaces which are isomorphic mod 0 to the unit interval with the usual Lebesgue measure.

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$$\mathcal{R}_{a,k,q} := \left\{ R_{j,q} := \phi^{j/q}\left(\bigcup_{i=0}^{k-1} \Delta_{a(i)k+i,kq}\right),\ j = 0, \ldots, q-1 \right\}, \tag{2.10}$$

$$\mathcal{S}_{kq,l} := \{S_{i,j}^{kq,l} := [i/(kq), (i+1)/(kq)) \times [j/l, (j+1)/l),\ 0 \le i < kq,\ 0 \le j < l\}. \tag{2.11}$$

We note that ϕ^α acts on the partitions (2.7)–(2.10) as a permutation for any choice of p when $\alpha = p/q$.

2.3. *Block-slide type of maps and their analytic approximations.* We recall that a *step function* on the unit interval is a finite linear combination of indicator functions on intervals. We define for $1 \le i,\ j \le d$ and $i \ne j$ the following piecewise continuous map on the d-dimensional torus:

$$\mathfrak{h} : \mathbb{T}^d \to \mathbb{T}^d \quad \text{defined by}$$
$$\mathfrak{h}(x_1, \ldots, x_d) := (x_1, \ldots, x_{i-1}, x_i + s(x_j) \bmod 1, x_{i+1}, \ldots, x_d), \tag{2.12}$$

where s is a step function on the unit interval. We refer to any finite composition of maps of the above kind as a *block-slide type of map* on the torus. The nomenclature is motivated from the fact that a finite composition of maps of the above kind has the effect of sliding solid blocks on the torus much like the game of nine.

Inspired by [**Ka73**], the purpose of the subsection is to demonstrate that a block-slide type of map can be approximated extremely well by measure-preserving real-analytic diffeomorphisms outside a set of arbitrarily small measure. This can be achieved because step functions can be approximated well by real-analytic functions whose complexification is entire.

LEMMA 2.13. *Let l and N be two positive integers and $\beta = (\beta_0, \ldots, \beta_{l-1}) \in [0, 1)^l$. Assume that l is even. Consider a step function of the form*

$$\tilde{s}_{\beta,N} : [0, 1) \to \mathbb{R} \quad \textit{defined by } \tilde{s}_{\beta,N}(x) = \sum_{i=0}^{lN-1} \tilde{\beta}_i \chi_{[i/lN,(i+1)/lN)}(x). \tag{2.14}$$

Here $\tilde{\beta}_i := \beta_j$, where $j := i \bmod l$. Then, given any $\varepsilon > 0$ and $\delta > 0$, there exists a periodic real-analytic function $s_{\beta,N} : \mathbb{R} \to \mathbb{R}$ satisfying the following properties.

(1) *Entirety: the complexification of $s_{\beta,N}$ extends holomorphically to $\mathbb{C}$.*

(2) *Proximity criterion: $s_{\beta,N}$ is L^1-close to $\tilde{s}_{\beta,N}$. We can say more:*

$$\sup_{x \in [0,1)\setminus F} |s_{\beta,N}(x) - \tilde{s}_{\beta,N}(x)| < \varepsilon. \tag{2.15}$$

(3) *Periodicity: $s_{\beta,N}$ is $1/N$-periodic. More precisely, the complexification will satisfy*

$$s_{\beta,N}(z + n/N) = s_{\beta,N}(z) \quad \textit{for all } z \in \mathbb{C} \textit{ and } n \in \mathbb{Z}. \tag{2.16}$$

(4) *Bounded derivative: the derivative is small outside a set of small measure,*

$$\sup_{x \in [0,1)\setminus F} |s'_{\beta,N}(x)| < \varepsilon, \tag{2.17}$$

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where $F = \bigcup_{i=0}^{lN-1} I_i \subset [0, 1)$ *is a union of intervals centred around* i/lN, $i = 1, \ldots, lN-1$, $I_0 = [0, \delta/2lN] \cup [1-\delta/2lN, 1)$ *and* $\lambda(I_i) = \delta/lN$ *for all* i *where* λ *is the length.*

Proof. See [**Ba17**, Lemma 4.7] and [**Ku17**, Lemma 3.6]. □

Note that the condition 2.16 in particular implies that

$$\sup_{z:\mathrm{Im}(z)<\rho} s_{\beta,N}(z) < \infty \quad \text{for all } \rho > 0.$$

Indeed, for any $\rho > 0$, put $\Omega'_\rho = \{z = x + iy : x \in [0, 1], |y| < \rho\}$ and note that entirety of $s_{\beta,N}$ combined with compactness of $\overline{\Omega'_\rho}$ implies that $\sup_{z\in\Omega'_\rho} |s_{\beta,N}(z)| < C$ for some constant C. Periodicity of $s_{\alpha,N}$ in the real variable and the observation that $\Omega_\rho = \bigcup_{n\in\mathbb{Z}}(\Omega'_\rho + n)$ implies that $\sup_{z\in\Omega_\rho} |s_{\beta,N}(z)| < C$. We have essentially concluded that $s_{\beta,N} \in C^\omega_\infty(\mathbb{T}^1)$.

We also make the observation that the condition l is even is not really a necessary one. One can drop the condition after replacing F with a different error set.

In order to prove convergence with a prescribed rotation number, we require the following refinement of the lemma.

LEMMA 2.18. *Let* l, $N \in \mathbb{N}$, l *even and* $\beta = (\beta_0, \ldots, \beta_{l-1}) \in [0, 1]^l$. *We consider a step function of the form as in equation (2.14), where* $\tilde{\beta}_i := \beta_j$ *in case of* $j \equiv i \bmod l$. *Given any* $\varepsilon \in (0, \frac{1}{8})$ *and* $\delta \in (0, 1)$ *let the number* $A > 0$ *fulfil the conditions*

$$A > -\frac{2l}{\pi \cdot \delta} \cdot \ln\left(-\ln\left(1 - \frac{\varepsilon}{8}\right)\right) \tag{A1}$$

and

$$A > \frac{2l}{\pi \cdot \delta} \cdot \ln\left(-\ln\left(\frac{\varepsilon}{2l}\right)\right). \tag{A2}$$

Then the $1/N$*-periodic real entire function* $s_{\beta,N,\varepsilon,\delta}$ *given by*

$$\begin{aligned} &s_{\beta,N,\varepsilon,\delta}(z) \\ &\quad= \left(\sum_{i=0}^{l/2-1} \beta_i \cdot (e^{-e^{-A\cdot\sin(2\pi(Nz-i/l))}} - e^{-e^{-A\cdot\sin(2\pi(Nz-(i+1)/l))}})\right) \cdot e^{-e^{-A\cdot\sin(2\pi Nz)}} \\ &\qquad+ \left(\sum_{i=l/2}^{l-1} \beta_i \cdot (e^{-e^{-A\cdot\sin(2\pi(Nz-i/l))}} - e^{-e^{-A\cdot\sin(2\pi(Nz-(i+1)/l))}})\right) \cdot e^{-e^{A\cdot\sin(2\pi Nz)}} \end{aligned}$$

satisfies

$$\sup_{x\in[0,1)\setminus F} |s_{\beta,N,\varepsilon,\delta}(x) - \tilde{s}_{\beta,N}(x)| < \varepsilon, \tag{2.19}$$

where $F = \bigcup_{i=0}^{lN-1} I_i \subset [0, 1)$ *is a union of intervals centred around* i/lN, $i = 1, \ldots, lN-1$, $I_0 = [0, \delta/2lN] \cup [1-\delta/2lN, 1)$ *and* $\lambda(I_i) = \delta/lN$ *for every* i.

Proof. First of all, we point out that $s_{\beta,N,\varepsilon,\delta}$ is a $1/N$-periodic real entire function. Let $x \in [0, 1) \setminus F$, namely $x \in [j/lN + \delta/2lN, (j+1)/lN - \delta/2lN]$ for some $j \in \mathbb{Z}$,

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$0 \le j \le lN - 1$. We write $x = j/lN + \Delta$. Exploiting the fact that $\sin(x) > x/2$ for $0 < x < \pi/2$, we get

$$e^{-e^{-A\cdot\sin(2\pi N\Delta)}} \ge e^{-e^{-A\cdot\pi N\Delta}}.$$

Using equation (A1), this implies that

$$e^{-e^{-A\cdot\sin(2\pi(Nx-s/l))}} > 1 - \frac{\varepsilon}{8} \tag{2.20}$$

in case of $0 \le j - s < l/2$ or $-l < j - s < -l/2$. On the other hand, we use the fact that $\sin(x) < x/2$ for $-\pi/2 < x < 0$ and get

$$e^{-e^{-A\cdot\sin(-2\pi N\Delta)}} \le e^{-e^{A\cdot\pi N\Delta}}.$$

By applying condition (A2), this yields

$$e^{-e^{-A\cdot\sin(2\pi(Nx-s/l))}} < \frac{\varepsilon}{2l} \tag{2.21}$$

in case of $l/2 \le j - s < l$ or $-l/2 \le j - s < 0$. By the above estimates in equations (2.20) and (2.21), we get

$$s_{\beta,N,\varepsilon,\delta}(x) \ge \beta_j \cdot \left(\left(1 - \frac{\varepsilon}{8}\right) \cdot \left(1 - \frac{\varepsilon}{8}\right) - \frac{\varepsilon}{2l}\right) - (l-1) \cdot \frac{\varepsilon}{2l}$$

and

$$s_{\beta,N,\varepsilon,\delta}(x) \le \beta_j + (l-1) \cdot \frac{\varepsilon}{2l}.$$

Altogether, we conclude that

$$|s_{\beta,N,\varepsilon,\delta}(x) - \beta_j| < \varepsilon. \qquad \square$$

Finally, we piece together everything and demonstrate how a block-slide type of map on the torus can be approximated by a measure-preserving real-analytic diffeomorphism.

PROPOSITION 2.22. *Let $\mathfrak{h} : \mathbb{T}^d \to \mathbb{T}^d$ be a block-slide type of map which commutes with $\phi^{1/q}$ for some natural number q. Then, for any $\varepsilon > 0$ and $\delta > 0$, there exists a real-analytic diffeomorphism $h \in \mathrm{Diff}^{\omega}_{\infty}(\mathbb{T}^d, \mu)$ satisfying the following conditions.*

(1) *Proximity property: there exists a set $E \subset \mathbb{T}^d$ such that $\mu(E) < \delta$ and $\sup_{x\in\mathbb{T}^d\setminus E} \|h(x) - \mathfrak{h}(x)\| < \varepsilon$.*
(2) *Commuting property: $h \circ \phi^{1/q} = \phi^{1/q} \circ h$.*

In this case we say that the diffeomorphism h is (ε, δ)-close to the block-slide type of map $\mathfrak{h}$.

Proof. First we assume that for some step function $\tilde{s}_{\beta,1}$ and any integer i with $1 < i \le d$, the block-slide type of map $\mathfrak{h}$ is of the following type:

$$\mathfrak{h} : \mathbb{T}^d \to \mathbb{T}^d \quad \text{defined by } \mathfrak{h}(x) = (x_1 + \tilde{s}_{\beta,1}(x_i), x_2, \ldots, x_d). \tag{2.23}$$

Then we define the following function using $s_{\beta,1}$ as in Lemma 2.13:

$$h : \mathbb{T}^d \to \mathbb{T}^d \quad \text{defined by } h(x) = (x_1 + s_{\beta,1}(x_i), x_2, \ldots, x_d). \tag{2.24}$$

With F as in Lemma 2.13, we put $E = \mathbb{T}^{i-1} \times F \times \mathbb{T}^{d-i}$ and observe that h satisfies all the conditions of the proposition.

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Similarly, if for some step function $\tilde{s}_{\beta,q}$ and any integer i with $1 < i \le d$, we have a block-slide type of map $\mathfrak{h}$ of the following type:

$$\mathfrak{h} : \mathbb{T}^d \to \mathbb{T}^d \quad \text{defined by } \mathfrak{h}(x) = (x_1, \ldots, x_{i-1}, x_i + \tilde{s}_{\beta,q}(x_1), x_{i+1}, \ldots, x_d), \quad (2.25)$$

then we define the following function using $s_{\beta,q}$ as in Lemma 2.13:

$$h : \mathbb{T}^d \to \mathbb{T}^d \quad \text{defined by } h(x) = (x_1, \ldots, x_{i-1}, x_i + s_{\beta,q}(x_1), x_{i+1}, \ldots, x_d). \quad (2.26)$$

With F as in Lemma 2.13, we put $E = F \times \mathbb{T}^{d-1}$ and observe that h satisfies all the conditions of the proposition.

So, for a general block-slide type of map which is obtained by a composition of several maps of the above type, we just take a composition of the individual approximations and a union of all the component error sets. □

2.4. *Real-analytic AbC method.* Our objective now is to recall the approximation by conjugation scheme developed by Anosov and Katok in [**AK70**]. Though we modify this scheme slightly to be more suitable for our purpose and fit the notations of our article, we insist that the method presented here is almost identical to the original construction. In more modern works, this method is often presented in a less formal way (see [**FK04**]) avoiding most technicalities but for our purpose we find the original scheme to be most suitable and we stick close to it.

The AbC method is an inductive process where a sequence of diffeomorphisms $T_n \in \mathrm{Diff}^{\omega}_{\infty}(\mathbb{T}^d, \mu)$ is constructed inductively. The diffeomorphisms T_n converge to some diffeomorphism $T \in \mathrm{Diff}^{\omega}_{\rho}(\mathbb{T}^d, \mu)$. Additionally, T_ns are chosen carefully so that they satisfy some finite version of the desired property of T.

We now give an explicit description. At the beginning of the construction we fix a constant $\rho > 0$ and note that all parameters chosen will depend on this ρ.

Assume that the construction has been carried out up to the nth stage and we have the following information available to us.

(1) We have sequences of natural numbers $\{p_m\}_{m=1}^n$, $\{q_m\}_{m=1}^n$, $\{k_m\}_{m=1}^{n-1}$, $\{l_m\}_{m=1}^{n-1}$, $\{s_m\}_{m=1}^{n-1}$, a sequence of functions $\{a_m : \{0, \ldots, k_m - 1\} \to \{0, \ldots, q_m - 1\}\}_{m=1}^{n-1}$ and a sequence of numbers $\{\varepsilon_m\}_{m=1}^n$. They satisfy the following condition:

$$\begin{gathered} p_m = s_{m-1}k_{m-1}l_{m-1}q_{m-1}p_{m-1} + 1, \quad q_m = s_{m-1}k_{m-1}l_{m-1}q_{m-1}^2, \\ \alpha_m = \frac{p_m}{q_m}, \quad \varepsilon_m < 2^{-q_m}. \end{gathered} \quad (2.27)$$

(2) The sequence of diffeomorphisms $\{T_m\}_{m=1}^n$ is constructed as conjugates of a periodic translation. More precisely,

$$T_m := H_m^{-1} \circ \phi^{\alpha_m} \circ H_m, \quad H_m := h_m \circ H_{m-1}, \quad h_m \in \mathrm{Diff}^{\omega}_{\infty}(\mathbb{T}^d, \mu). \quad (2.28)$$

The diffeomorphisms $\{h_m\}_{m=1}^n$ satisfy the following commuting condition:

$$h_m \circ \phi^{\alpha_{m-1}} = \phi^{\alpha_{m-1}} \circ h_m. \quad (2.29)$$

(3) For $m = 1, \ldots, n$, the diffeomorphism T_m preserves and permutes two sequences of partitions, namely, $H_m^{-1}\mathcal{R}_{a_m,k_m,q_m}$ and $\mathcal{F}_{q_m} := H_m^{-1}\mathcal{T}_{q_m}$.

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(4) For $m = 1, \ldots, n$,
$$\mu(h_m^{-1} R_{i,q_{m-1}} \triangle \Delta_{i,q_{m-1}}) < \varepsilon_{m-1}$$
for any $R_{i,q_{m-1}} \in \mathcal{R}_{a_{m-1},k_{m-1},q_{m-1}}$ and $\Delta_{i,q_{m-1}} \in \mathcal{T}_{q_{m-1}}$ with the same i.

(5) For $m = 1, \ldots, n$, $\operatorname{diam}(\mathcal{F}_{q_m} \cap E_m) < \varepsilon_m$† for some measurable set E_m satisfying $\mu(E_m) > 1 - \varepsilon_m$. (Note that this means that $\mathcal{F}_{q_m}$ is a generating but not necessarily monotonic sequence of partitions.)

(6) For $m = 1, \ldots, n$, $d_\rho(T_m, T_{m-1}) < \varepsilon_m$ and $d_0(T_m^i, T_{m-1}^i) < 1/2^{m-1}$ for $0 \le i < q_{m-1} - 1$.

Now we show how to do the construction at the $(n+1)$th stage of this induction process. We proceed in the following order.

(1) We choose k_n and our function $a_n : \{0, \ldots, k_n - 1\} \to \{0, \ldots, q_n - 1\}$. This choice will depend on the construction we are doing and the specific properties we are targeting to prove.

(2) We choose l_n to be a large enough integer so that the following condition is satisfied:
$$l_n > 2^n \| D H_n^{-1} \|_0. \tag{2.30}$$

(3) Find a block-slide type of map $\mathfrak{h}_{a_n,k_n,l_n,q_n}$ which commutes with ϕ^{α_n}, maps the partition $\mathcal{G}_{l_n k_n, q_n}$ to the partition $\mathcal{T}_{l_n^d k_n^d q_n}$ and maps the partition $\mathcal{T}_{q_n}$ to the partition $\mathcal{R}_{a_n,k_n,q_n}$.

(4) Use Proposition 2.22 to construct h_{n+1} which is $(\varepsilon_n', \delta_n)$-close to $\mathfrak{h}_{a_n,k_n,l_n,q_n}$ for sufficiently small $\varepsilon_n' > 0$ and $\delta_n > 0$. Put E_n to be the error set in Proposition 2.22.

(5) Ensure that $|\alpha_{n+1} - \alpha_n|$ is small enough to guarantee that $d_\rho(T_{n+1}, T_n) < \varepsilon_m$ and $d_0(T_{n+1}^i, T_n^i) < 1/2^n$ for $0 \le i < q_n - 1$ by choosing s_n sufficiently large.

This completes the construction at the $(n+1)$th stage. Note that in this way convergence of T_n to some $T \in \operatorname{Diff}_\rho^\omega(\mathbb{T}^d, \mu)$ is guaranteed.

Remark 1. By $d_0(T_{n+1}^i, T_n^i) < 1/2^n$ for $0 \le i < q_n - 1$ and every $n \in \mathbb{N}$, we get $d_0(T^i, T_{n+1}^i) < 1/2^n$ for $0 \le i < q_{n+1} - 1$.

We need another important construction, which is very handy for some applications. Note that the partitions $\mathcal{F}_{q_n}$ are not necessarily monotonic. However, the following proposition shows that a generating monotonic partition which is cyclically permuted by T_n can be constructed from $\mathcal{F}_{q_n}$. This is identical to [**AK70**, Lemma 3.1].

PROPOSITION 2.31. *With notations as in the approximation by conjugation scheme, we can find a sequence of partitions $\mathcal{M}_n$ of $\mathbb{T}^d$ satisfying the following three properties.*

(1) *Monotonicity condition: $\mathcal{M}_{n+1} > \mathcal{M}_n$.*

(2) *Cyclic permutation: the diffeomorphisms T_n cyclically permute the atoms of $\mathcal{M}_n$.*

(3) *Generating condition: $\mathcal{M}_n \to \varepsilon$ as $n \to \infty$.*

Proof. For any n, we define a measurable map $\mathfrak{c}_{n+1,n}^{(1)} : \mathbb{T}^d/\mathcal{T}_{q_n} \to \mathbb{T}^d/\mathcal{R}_{a_n,k_n,q_n}$ by $\mathfrak{c}_{n+1,n}^{(1)}(\Delta_{i,q_n}) = R_{i,q_n}$ and another measurable map $\mathfrak{c}_{n+1,n}^{(2)} : \mathbb{T}^d/\mathcal{T}_{q_{n+1}} \to \mathbb{T}^d/\mathcal{R}_{a_n,k_n,q_n}$ by $\mathfrak{c}_{n+1,n}^{(2)}(\Delta_{i,q_{n+1}}) = R_{j,q_n}$, where j is such that $\Delta_{i,q_{n+1}} \subset R_{j,q_n}$. So, at the level of $\mathbb{T}^d$,

† This means that the diameter of the intersection of any atom of $\mathcal{F}_{q_m}$ and E_m is less than ε_m.

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the composition $\mathfrak{c}_{n+1,n} = (\mathfrak{c}^{(2)}_{n+1,n})^{-1} \circ \mathfrak{c}^{(1)}_{n+1,n}$ gives us a correspondence which associates each atom of $\mathcal{T}_{q_n}$ with a union of atoms of $\mathcal{T}_{q_{n+1}}$. So, we can define a new partition $\mathcal{T}_{q_{n+1},q_n} := \{\mathfrak{c}_{n+1,n}(\Delta_{i,q_n}) : 0 \le i < q_n\}$. Continuing this procedure, we obtain for any $m > n$ a partition $\mathcal{T}_{q_m,q_n} := \{\mathfrak{c}_{m,m-1} \circ \cdots \circ \mathfrak{c}_{n+2,n+1} \circ \mathfrak{c}_{n+1,n}(\Delta_{i,q_n}) : 0 \le i < q_n\}$. We note that this partition satisfies the following three conditions for any three integers n, l and m with $m > l > n$.

(1) Monotonicity condition: $\mathcal{T}_{q_m,q_{n+1}} > \mathcal{T}_{q_m,q_n}$.
(2) Cyclic permutation: ϕ^{1/q_n} cyclically permutes the atoms of $\mathcal{T}_{q_m,q_n}$.
(3) $\mathfrak{c}_{m,l} \circ \mathfrak{c}_{l,n} = \mathfrak{c}_{m,n}$.

Now we define two new partitions as follows:

$$\mathcal{F}_{q_m,q_n} := H_m^{-1}\mathcal{T}_{q_m,q_n}, \tag{2.32}$$

$$\mathcal{F}_{q_n} := H_n^{-1}\mathcal{T}_{q_n}. \tag{2.33}$$

We define the correspondence $\mathfrak{p}_{m,n} : \mathcal{F}_{q_n} \to \mathcal{F}_{q_m,q_n}$ by $\mathfrak{p}_{m,n}(H_n^{-1}(\Delta_{i,q_n})) = H_m^{-1}(\mathfrak{c}_{m,n}(\Delta_{i,q_n}))$. Now note that for any three integers $m > l > n$, we have the following three properties.

(1) Monotonicity: $\mathcal{F}_{q_m,q_{n+1}} > \mathcal{F}_{q_m,q_n}$.
(2) Cyclic permutation: T_n cyclically permutes the atoms of $\mathcal{F}_{q_m,q_n}$ (since ϕ^{1/q_n} commutes with h_j for $j > n$).
(3) $\mathfrak{p}_{m,l} \circ \mathfrak{p}_{l,n} = \mathfrak{p}_{m,n}$. Indeed, for any Δ_{i,q_n}, we observe that

$$\begin{aligned}\mathfrak{p}_{m,l} \circ \mathfrak{p}_{l,n}(H_n^{-1}(\Delta_{i,q_n})) &= \mathfrak{p}_{m,l}(H_l^{-1}(\mathfrak{c}_{l,n}(\Delta_{i,q_n}))) \\ &= H_m^{-1}(\mathfrak{c}_{m,l}(\mathfrak{c}_{l,n}(\Delta_{i,q_n}))) \\ &= H_m^{-1}(\mathfrak{c}_{m,n}(\Delta_{i,q_n})) \\ &= \mathfrak{p}_{m,n}(H_n^{-1}(\Delta_{i,q_n})).\end{aligned}$$

Our next goal is to prove that the limit

$$\mathfrak{p}_{\infty,n}(H_n^{-1}(\Delta_{i,q_n}))) := \lim_{m\to\infty}(\mathfrak{p}_{m,n}(H_n^{-1}(\Delta_{i,q_n}))) \tag{2.34}$$

exists for any n and i. In order to see this, we note at first that $\mathfrak{c}_{m,n}(\Delta_{i,q_n}) = \bigcup_{l\in\sigma} \Delta_{l,q_m}$, where σ is some indexing set of size q_m/q_n. This implies that $\mathfrak{c}_{m+1,n}(\Delta_{i,q_n}) = \bigcup_{l\in\sigma} R_{l,q_m}$. And, hence, $h_{m+1}^{-1}(\mathfrak{c}_{m+1,n}(\Delta_{i,q_n})) = \bigcup_{l\in\sigma} h_{m+1}^{-1}(R_{l,q_m})$. Now note the following estimates:

$$\begin{aligned}&\mu(h_{m+1}^{-1}(R_{l,q_m})\triangle\Delta_{l,q_m}) < \varepsilon_m \\ &\quad\Rightarrow \mu\left(\bigcup_{l\in\sigma} h_{m+1}^{-1}(R_{l,q_m})\triangle\Delta_{l,q_m}\right) < \frac{q_m}{q_n}\varepsilon_m \\ &\quad\Rightarrow \mu\left(\bigcup_{l\in\sigma} h_{m+1}^{-1}(R_{l,q_m})\triangle\bigcup_{l\in\sigma}\Delta_{l,q_m}\right) < \frac{q_m}{q_n}\varepsilon_m \\ &\quad\Rightarrow \mu(h_{m+1}^{-1}(\mathfrak{c}_{m+1,n}(\Delta_{i,q_n}))\triangle\mathfrak{c}_{m,n}(\Delta_{i,q_n}) < \frac{q_m}{q_n}\varepsilon_m \\ &\quad\Rightarrow \mu(H_{m+1}^{-1}(\mathfrak{c}_{m+1,n}(\Delta_{i,q_n}))\triangle H_m^{-1}(\mathfrak{c}_{m,n}(\Delta_{i,q_n})) < \frac{q_m}{q_n}\varepsilon_m\end{aligned}$$

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$$\Rightarrow \mu(\mathfrak{p}_{m+1,n}(H_n^{-1}(\Delta_{i,q_n}))\triangle\mathfrak{p}_{m,n}(H_n^{-1}(\Delta_{i,q_n})) < \frac{q_m}{q_n}\varepsilon_m$$

$$\Rightarrow \sum_{i=0}^{q_n-1} \mu(\mathfrak{p}_{m+1,n}(H_n^{-1}(\Delta_{i,q_n}))\triangle\mathfrak{p}_{m,n}(H_n^{-1}(\Delta_{i,q_n})) < q_m\varepsilon_m$$

$$\Rightarrow \sum_{i=0}^{q_n-1} \mu(\mathfrak{p}_{m_1,n}(H_n^{-1}(\Delta_{i,q_n}))\triangle\mathfrak{p}_{m_2,n}(H_n^{-1}(\Delta_{i,q_n})) < \sum_{m=m_1}^{m_2} q_m\varepsilon_m.$$

This shows that the sequence $\mathcal{F}_{q_m,q_n}$ converges as m goes to infinity and shows the existence of (2.34). So, we define the partition

$$\mathcal{M}_n := \{\mathfrak{p}_{\infty,n}(H_n^{-1}(\Delta_{i,q_n}))) : 0 \le i < q_n\}. \tag{2.35}$$

And we note that this partition has the required properties. □

3. *Non-standard real-analytic realization of some ergodic rotations of the circle*

Non-standard realization problems demonstrate how a dynamical system can live on a non-native manifold. In particular, we are interested in exploring when an ergodic rotation of the circle can be measure-theoretically isomorphic to a measure-preserving real-analytic ergodic diffeomorphism on a torus.

3.1. *Some measure theory.*

In this subsection, we state an abstract lemma from measure theory, which is a slight generalization of [**AK70**, Lemma 4.1]. Essentially we formulate a finite version of the conjugacy we will eventually prove.

Lemma 4.1 from [**AK70**] gave us an easily checkable finite version of the conjugacy that one can use to prove the existence of a metric isomorphism of the limiting diffeomorphisms. Since the generating partitions used in the C^∞ non-standard realization problem can easily made to be monotonic, this lemma was sufficient. But, in the real-analytic case, our construction is not flexible enough to guarantee monotonicity, so we need the following modified version from [**Ba17**].

LEMMA 3.1. *Let $\{M^{(i)}, \mu^{(i)}\}$, $i = 1, 2$, be two Lebesgue spaces. Let $\mathcal{P}_n^{(i)}$ be a sequence of generating finite partitions of $M^{(i)}$. Let $\{\varepsilon_n\}$ be a sequence of positive numbers satisfying $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. In addition, assume that there exists a sequence of sets $\{E_n^{(i)}\}$ in $M^{(i)}$ satisfying*

$$\mu^{(1)}(E_n^{(1)}) = \mu^{(2)}(E_n^{(2)}) < \varepsilon_n, \tag{3.2}$$

$$P_{n+1}^{(1)}(x) \setminus E_{n+1}^{(1)} \subset P_n^{(1)}(x) \quad \textit{for all } x \in M^{(1)} \setminus E_{n+1}^{(1)}, \tag{3.3}$$

$$P_{n+1}^{(2)}(y) \subset P_n^{(2)}(y) \quad \textit{for all } y \in M^{(2)}. \tag{3.4}$$

Let $T_n^{(1)}$ and $T_n^{(2)}$ be two sequences of automorphisms of the spaces $M^{(1)}$ and $M^{(2)}$ satisfying

$$T_n^{(i)}\mathcal{P}_n^{(i)} = \mathcal{P}_n^{(i)}, \quad i = 1, 2, \tag{3.5}$$

$$\lim_{n\to\infty} T_n^{(i)} = T^{(i)}, \quad i = 1, 2, \tag{3.6}$$

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$$T_n^{(i)}\Big(\bigcup_{m=n}^{\infty} E_m^{(i)}\Big) = \bigcup_{m=n}^{\infty} E_m^{(i)}, \quad i = 1, 2. \tag{3.7}$$

Note that the limit in (3.6) is taken in the weak topology. Suppose additionally that there exists a sequence of metric isomorphisms $K_n : M^{(1)}/\mathcal{P}_n^{(1)} \to M^{(2)}/\mathcal{P}_n^{(2)}$ satisfying

$$K_n^{-1} T_n^{(2)}|_{\mathcal{P}_n^{(2)}} K_n = T_n^{(1)}|_{\mathcal{P}_n^{(1)}}, \tag{3.8}$$

$$K_{n+1}(\mathcal{P}_{n+1}^{(1)}(x)) \subset K_n(\mathcal{P}_n^{(1)}(x)) \quad \textit{for all } x \in M^{(1)} \setminus E_{n+1}^{(1)}. \tag{3.9}$$

Then the automorphisms $T^{(1)}$ and $T^{(2)}$ are metrically isomorphic.

Proof. See [**Ba17**, Lemma 3.5]. □

3.2. *Construction of the conjugation map.* Let $\alpha \in \mathcal{L}_*$ be arbitrary. We construct successively a sequence of measure-preserving diffeomorphisms $T_n = H_n^{-1} \circ \phi^{\alpha_n} \circ H_n$, where the conjugation maps $H_n = h_n \circ H_{n-1}$ and rational numbers $\alpha_n = p_n/q_n \in \mathbb{Q}$ are chosen† in such a way that the functions T_n converge to a diffeomorphism in $\mathrm{Diff}_\rho^\omega(\mathbb{T}^2, \mu)$ with the desired properties. We present step $n+1$ of the construction, i.e. we assume that we have already defined H_n and the numbers $\alpha_1, \ldots, \alpha_{n-1}$. Additionally, we choose an even integer $l_n \in \mathbb{N}$ satisfying the condition

$$l_n > 2^{n+1} \cdot \|DH_n^{-1}\|_{\rho_n+1}, \tag{3.10}$$

where $\rho_n = \|H_n\|_\rho$. This condition on l_n will be used to ensure that the sequence of partitions we construct later is generating (see the proof of [**Ba17**, Proposition 6.3]). In this step of the construction, we have to define the conjugation map h_{n+1} and to choose α_n (see equation (3.18) in the proof of Lemma 3.16 for the concrete choice of α_n).

We start by describing the main combinatorial idea behind the proofs of Theorems 1.1 and A. Given any two integers l and q, there exists a block-slide type of map which allows us to break down the partition $\mathcal{T}_{l^2q}$ and reform it into a partition $\mathcal{G}_{l,q}$ whose atoms have diameters less than $2/l$.

First we consider the following three step functions:

$$\psi_{l,q}^{(1)} : [0, 1) \to \mathbb{R} \quad \text{defined by } \psi_{l,q}^{(1)}(x) = \sum_{j=1}^{l-1} \frac{l-j}{l^2 q} \chi_{[j/l,(j+1)/l]}(x),$$

$$\psi_{l,q}^{(2)} : [0, 1) \to \mathbb{R} \quad \text{defined by } \psi_{l,q}^{(2)}(x) = \sum_{j=0}^{l^2q-1} \left(\frac{j}{l} - \left\lfloor \frac{j}{l} \right\rfloor\right) \chi_{[j/l^2q,(j+1)/l^2q]}(x),$$

$$\psi_{l,q}^{(3)} : [0, 1) \to \mathbb{R} \quad \text{defined by } \psi_{l,q}^{(3)}(x) = \sum_{j=0}^{l-1} \frac{j}{l^2 q} \chi_{[j/l,(j+1)/l]}(x). \tag{3.11}$$

Then we define the following three types of block-slide map (see Figure 1):

$$\mathfrak{g}_{l,q}^{(1)} : \mathbb{T}^2 \to \mathbb{T}^2 \quad \text{defined by } \mathfrak{g}_{l,q}^{(1)}(x_1, x_2) = (x_1 + \psi_{l,q}^{(1)}(x_2), x_2),$$

† We point out that this time the numbers α_n are not built according to the form in equation (2.27) but are chosen in the proof of Lemma 3.16.

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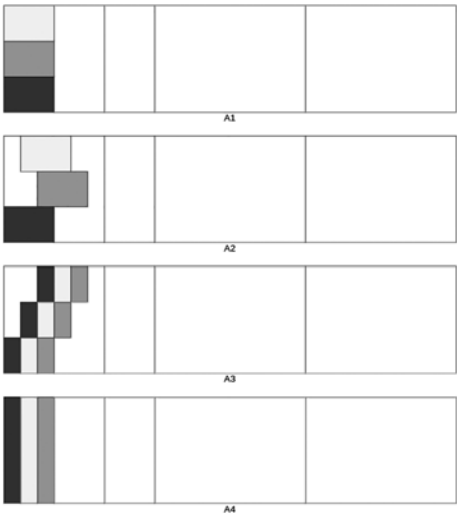


FIGURE 1. Illustration of the action of $\mathfrak{g}_{i,l,q}$ on $\mathbb{T}^2$ with $i = 2$, $l = 3$ and $q = 3$. We note how the blocks are moved by the intermediate maps: $A1 \xrightarrow{\mathfrak{g}^{(1)}_{2,3,3}} A2 \xrightarrow{\mathfrak{g}^{(2)}_{2,3,3}} A3 \xrightarrow{\mathfrak{g}^{(3)}_{2,3,3}} A4$.

$$\mathfrak{g}^{(2)}_{l,q} : \mathbb{T}^2 \to \mathbb{T}^2 \quad \text{defined by } \mathfrak{g}^{(2)}_{l,q}(x_1, x_2) = (x_1, x_2 + \psi^{(2)}_{l,q}(x_1)),$$
$$\mathfrak{g}^{(3)}_{l,q} : \mathbb{T}^2 \to \mathbb{T}^2 \quad \text{defined by } \mathfrak{g}^{(3)}_{l,q}(x_1, x_2) = (x_1 - \psi^{(3)}_{l,q}(x_2), x_2).$$

Note that the composition

$$\mathfrak{g}_{l,q} : \mathbb{T}^2 \to \mathbb{T}^2 \quad \text{defined by } \mathfrak{g}_{l,q} = \mathfrak{g}^{(3)}_{l,q} \circ \mathfrak{g}^{(2)}_{l,q} \circ \mathfrak{g}^{(1)}_{l,q} \tag{3.12}$$

maps the partition $\mathcal{G}_{l,q}$ to $\mathcal{T}_{l^2 q}$.

With the aid of Lemma 2.18, we construct the following entire functions:

$$\psi_{1,n+1} = s_{\beta^{(1)}, N^{(1)}, 1/(3\cdot 2^{n+1}\cdot l_n^2 \cdot q_n), 1/2^{n+1}},$$
$$\text{where } \beta^{(1)}_0 = 0,\ \beta^{(1)}_i = \frac{l_n - i}{l_n^2 \cdot q_n} \text{ for } i = 1, \ldots, l_n - 1,\ N^{(1)} = 1,$$
$$\psi_{2,n+1} = s_{\beta^{(2)}, N^{(2)}, 1/(3\cdot 2^{n+1}\cdot l_n), 1/2^{n+1}},$$
$$\text{where } \beta^{(2)}_i = \frac{i}{l_n} \text{ for } i = 0, \ldots, l_n - 1,\ N^{(2)} = l_n q_n,$$
$$\psi_{3,n+1} = s_{\beta^{(3)}, N^{(3)}, 1/(3\cdot 2^{n+1}\cdot l_n^2 \cdot q_n), 1/2^{n+1}},$$
$$\text{where } \beta^{(3)}_i = \frac{i}{l_n^2 \cdot q_n} \text{ for } i = 0, \ldots, l_n - 1,\ N^{(3)} = 1.$$

Let $A_{i,n+1}$ denote the corresponding number in the construction of $\psi_{i,n+1}$ from Lemma 2.18. Using these functions, we define the conjugation maps

$$h_{1,n+1}(x_1, x_2) = (x_1 + \psi_{1,n+1}(x_2) \bmod 1, x_2),$$
$$h_{2,n+1}(x_1, x_2) = (x_1, x_2 + \psi_{2,n+1}(x_1) \bmod 1),$$
$$h_{3,n+1}(x_1, x_2) = (x_1 - \psi_{3,n+1}(x_2) \bmod 1, x_2)$$

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approximating the maps $\mathfrak{g}^{(\mathrm{i})}_{l_n,q_n}$ from above. Finally, we put

$$h_{n+1} = h_{3,n+1} \circ h_{2,n+1} \circ h_{1,n+1}.$$

We point out that $\phi^{\alpha_n} \circ h_{n+1} = h_{n+1} \circ \phi^{\alpha_n}$.

LEMMA 3.13. *Let $l_n \geq 4$. In the concrete situation of our constructions, we can choose*

$$A_{i,n+1} = 2^{2n+5} \cdot l_n^4 \cdot q_n \quad \textit{for } i = 1 \textit{ and } i = 3,$$
$$A_{2,n+1} = 2^{2n+5} \cdot l_n^3.$$

Proof. Using Taylor expansion and the notation $x = \varepsilon_n/2l_n$, we calculate

$$-\ln(-\ln(1-x)) = -\ln\left(x + \frac{x^2}{2} + \frac{x^3}{3} + O(x^4)\right) \leq -\ln(x) = \ln(x^{-1}).$$

By our explicit definition of the number $\varepsilon_{i,n}$, we get

$$\ln(2l_n \cdot \varepsilon_{i,n}^{-1}) = \ln(2l_n \cdot 3 \cdot 2^{n+1} \cdot l_n^2 \cdot q_n) < \ln(2^{n+4} \cdot l_n^3 \cdot q_n) \quad \text{for } i = 1, 3,$$
$$\ln(2l_n \cdot \varepsilon_{1,n}^{-1}) = \ln(2l_n \cdot 3 \cdot 2^{n+1} \cdot l_n) < \ln(2^{n+4} \cdot l_n^2).$$

Then condition (A1) yields the requirement

$$A_{i,n+1} \geq 2^{n+1} \cdot l_n \cdot \ln(2^{n+4} \cdot l_n^3 \cdot q_n) < 2^{2n+5} \cdot l_n^4 \cdot q_n \quad \text{for } i = 1, 3,$$
$$A_{2,n+1} \geq 2^{n+1} \cdot l_n \cdot \ln(2^{n+4} \cdot l_n^2) < 2^{2n+5} \cdot l_n^3.$$

We note that with this choice condition (A2) is satisfied automatically. □

3.3. *Proof of convergence.* Let $\rho > 0$ be arbitrary. We want to prove convergence of $(T_n)_{n\in\mathbb{N}}$ in $\mathrm{Diff}^{\omega}_{\rho}(\mathbb{T}^m)$. For this purpose, we introduce the numbers

$$\rho_k = \|H_k\|_{\rho} \quad \text{for any } k \in \mathbb{N}.$$

Using the definitions of the conjugation maps, we compute

$$\begin{aligned} h_{n+1}^{-1} \circ \phi^{\alpha_{n+1}} \circ h_{n+1}(x_1, x_2) = (&x_1 + \alpha_{n+1} + \psi_{1,n+1}(x_2) \\ &- \psi_{1,n+1}(x_2 + \psi_{2,n+1}(x_1 + \psi_{1,n}(x_2)) - \psi_{2,n+1}(x_1 + \alpha_{n+1} + \psi_{1,n+1}(x_2))), \\ &x_2 + \psi_{2,n+1}(x_1 + \psi_{1,n+1}(x_2)) - \psi_{2,n+1}(x_1 + \alpha_{n+1} + \psi_{1,n+1}(x_2))). \end{aligned}$$

We recall that $\psi_{2,n+1}$ is $1/q_n$-periodic and get

$$\begin{aligned} h_{n+1}^{-1} \circ \phi^{\alpha_{n+1}} \circ h_{n+1}(x_1, x_2) = (&x_1 + \alpha_{n+1} + \psi_{1,n+1}(x_2) \\ &- \psi_{1,n+1}(x_2 + \psi_{2,n+1}(x_1 + \alpha_n + \psi_{1,n+1}(x_2)) \\ &- \psi_{2,n+1}(x_1 + \alpha_{n+1} + \psi_{1,n+1}(x_2))), \\ &x_2 + \psi_{2,n+1}(x_1 + \alpha_n + \psi_{1,n+1}(x_2)) - \psi_{2,n+1}(x_1 + \alpha_{n+1} + \psi_{1,n+1}(x_2))). \end{aligned}$$

Hereby, we conclude that

$$\begin{aligned} (h_{n+1}^{-1} \circ \phi^{\alpha_{n+1}} \circ h_{n+1} - \phi^{\alpha_n})(x_1, x_2) = (&\alpha_{n+1} - \alpha_n + \psi_{1,n+1}(x_2) \\ &- \psi_{1,n+1}(x_2 + \psi_{2,n+1}(x_1 + \alpha_n + \psi_{1,n+1}(x_2)) \\ &- \psi_{2,n+1}(x_1 + \alpha_{n+1} + \psi_{1,n+1}(x_2))), \\ &\psi_{2,n+1}(x_1 + \alpha_n + \psi_{1,n+1}(x_2)) - \psi_{2,n+1}(x_1 + \alpha_{n+1} + \psi_{1,n+1}(x_2))). \end{aligned}$$

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In the next step, we exploit the closeness of $h_{n+1}^{-1} \circ \phi^{\alpha_{n+1}} \circ h_{n+1}(x_1, x_2)$ to ϕ^{α_n} and find that

$$\begin{aligned} d_\rho(T_{n+1}, T_n) &= d_\rho(H_n^{-1} \circ h_{n+1}^{-1} \circ \phi^{\alpha_{n+1}} \circ h_{n+1} \circ H_n, H_n^{-1} \circ \phi^{\alpha_n} \circ H_n) \\ &\leq \|DH_n^{-1}\|_{\rho_n+1} \cdot \|h_{n+1}^{-1} \circ \phi^{\alpha_{n+1}} \circ h_{n+1} - \phi^{\alpha_n}\|_{\rho_n}. \end{aligned}$$

In order to estimate $\|h_{n+1}^{-1} \circ \phi^{\alpha_{n+1}} \circ h_{n+1} - \phi^{\alpha_n}\|_{\rho_n}$, we will use the subsequent result.

LEMMA 3.14. *Let $\rho > 0$ and $B^\rho = \{z \in \mathbb{C} : |im(z)| < \rho\}$. Then we have*

$$\sup_{z \in B^\rho} |s_{\beta,N,\varepsilon,\delta}(z)| \leq 2\pi \cdot N \cdot A \cdot e^{2e^{A \cdot e^{2\pi N\rho}} + A \cdot e^{2\pi N\rho} + 2\pi N\rho}$$

and

$$\sup_{z_1, z_2 \in B^\rho} |s_{\beta,N,\varepsilon,\delta}(z_1) - s_{\beta,N,\varepsilon,\delta}(z_2)| \leq C \cdot A \cdot l \cdot N \cdot e^{4 \cdot e^{A \cdot e^{2\pi N\rho}}} \cdot |z_1 - z_2|,$$

where C is a constant independent of n, l and N.

Proof. First of all, we observe that

$$\begin{aligned} &\frac{d}{dz} e^{-e^{-A \cdot \sin(2\pi(Nz - i/l))}} \\ &\quad = e^{-e^{-A \cdot \sin(2\pi(Nz - i/l))} - A \cdot \sin(2\pi(Nz - i/l))} \cdot 2\pi \cdot A \cdot N \cdot \cos\left(2\pi\left(Nx - \frac{i}{l}\right)\right). \end{aligned}$$

Using the mean value theorem, this yields

$$\begin{aligned} \sup_{z \in B^\rho} |s_{\beta,N,\varepsilon,\delta}(z)| &\leq l \cdot e^{e^{A \cdot e^{2\pi N\rho}}} \cdot 2\pi \cdot N \cdot A \cdot e^{A \cdot e^{2\pi N\rho}} \cdot e^{2\pi N\rho} \cdot \frac{1}{l} \cdot e^{e^{A \cdot e^{2\pi N\rho}}} \\ &\leq 2\pi \cdot N \cdot A \cdot e^{2e^{A \cdot e^{2\pi N\rho}} + A \cdot e^{2\pi N\rho} + 2\pi N\rho}. \end{aligned}$$

Additionally, we get

$$\begin{aligned} \|Ds_{\beta,N,\varepsilon,\delta}\|_\rho &\leq l \cdot e^{2e^{A \cdot e^{2\pi N\rho}}} \cdot 4\pi \cdot A \cdot N \cdot e^{A \cdot e^{2\pi N\rho}} \cdot e^{2\pi\rho N} \\ &\quad + 2\pi \cdot A \cdot N \cdot e^{2e^{A \cdot e^{2\pi N\rho}}} \cdot e^{2A \cdot e^{2\pi N\rho}} \cdot e^{4\pi\rho N} \\ &\leq 6\pi \cdot A \cdot l \cdot N \cdot e^{4 \cdot e^{A \cdot e^{2\pi N\rho}}}. \end{aligned}$$

By applying the mean value theorem, we obtain the second statement of the lemma. □

In addition to the before-mentioned conditions, we require the number l_n to satisfy

$$l_n > e^{2\pi \cdot (\rho_n + 1)}. \tag{3.15}$$

Finally, we are able to prove convergence of the sequence $(T_n)_{n \in \mathbb{N}}$.

LEMMA 3.16. *Fix $\rho > 0$. Then there is a sequence $(\alpha_n)_{n \in \mathbb{N}}$ of rational numbers converging to α monotonically such that the sequence $(T_n)_{n \in \mathbb{N}}$ converges to T in* $\mathrm{Diff}_\rho^\omega(\mathbb{T}^2)$.

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Proof. First of all, we introduce the number

$$\rho_n' = \rho_n + 2\pi \cdot A_{1,n+1} \cdot e^{2e^{A_{1,n+1}\cdot e^{2\pi\rho_n}} + A_{1,n+1}\cdot e^{2\pi\rho_n} + 2\pi\rho_n}.$$

We recall that C as well as the requirements on l_n in equations (3.10) and (3.15) are independent of q_n. Hence, we can state the subsequent condition on q_n:

$$q_n \geq C^2 \cdot 2^{4n+11} \cdot l_n^{10}. \tag{3.17}$$

Under this restriction on the number q_n, we find that $\alpha_n = p_n/q_n$ with p_n, q_n relatively prime such that

$$|\alpha - \alpha_n| < \frac{1}{e^{e^{(2^{2n+7}\cdot l_n^5 \cdot e^{2\pi\cdot\rho_n'})q_n}}}, \tag{3.18}$$

because $\alpha \in \mathcal{L}_*$. By using ρ_n' and by applying Lemma 3.14 twice, we get

$$\begin{aligned}
&\sup_{(x_1,x_2)\in A^{\rho_n}} |\psi_{2,n+1}(x_1 + \alpha_n + \psi_{1,n+1}(x_2)) - \psi_{2,n}(x_1 + \alpha_{n+1} + \psi_{1,n+1}(x_2))| \\
&\quad \leq \sup_{y\in B^{\rho_n'}} |\psi_{2,n+1}(y + \alpha_n) - \psi_{2,n+1}(y + \alpha_{n+1})| \\
&\quad \leq C \cdot e^{\ln(A_{2,n+1}\cdot l_n \cdot q_n) + 4\cdot e^{A_{2,n+1}\cdot e^{2\pi\cdot q_n\cdot\rho_n'}}} \cdot |\alpha_{n+1} - \alpha_n|.
\end{aligned}$$

Under our conditions on the numbers α_k, its value is less than 1. Hereby, we conclude using equation (3.15) that

$$\begin{aligned}
\|h_{n+1}^{-1} \circ \phi^{\alpha_{n+1}} \circ h_{n+1} - \phi^{\alpha_n}\|_{\rho_n} &\leq |\alpha_{n+1} - \alpha_n| + C \cdot A_{1,n+1} \cdot l_n \cdot e^{4\cdot e^{A_{1,n+1}\cdot e^{2\pi\cdot(\rho_n+1)}}} \cdot C \\
&\quad \times e^{\ln(A_{2,n+1}\cdot l_n\cdot q_n) + 4\cdot e^{A_{2,n+1}\cdot e^{2\pi\cdot q_n\cdot\rho_n'}}} \cdot |\alpha_{n+1} - \alpha_n| \\
&\leq 2C^2 \cdot 2^{4n+10} \cdot l_n^9 \cdot q_n^2 \cdot e^{4\cdot e^{2^{2n+5}\cdot l_n^5 \cdot q_n}} \cdot e^{4\cdot e^{2^{2n+5}\cdot l_n^3 \cdot e^{2\pi\cdot q_n\cdot\rho_n'}}} \cdot |\alpha_{n+1} - \alpha_n|.
\end{aligned}$$

With the aid of conditions (3.10) and (3.17), we can continue the former estimates in the following way:

$$\begin{aligned}
&2^{n+1} \cdot \|DH_n^{-1}\|_{\rho_n+1} \cdot \|h_{n+1}^{-1} \circ \phi^{\alpha_{n+1}} \circ h_{n+1} - \phi^{\alpha_n}\|_{\rho_n} \\
&\quad \leq e^{3\cdot\ln(q_n) + 4\cdot e^{2^{2n+5}\cdot l_n^5\cdot q_n} + 4\cdot e^{2^{2n+5}\cdot l_n^3\cdot(e^{2\pi\cdot\rho_n'})q_n}} \cdot |\alpha_{n+1} - \alpha_n| \\
&\quad \leq e^{e^{2^{2n+7}\cdot l_n^5\cdot(e^{2\pi\cdot\rho_n'})q_n}} \cdot |\alpha_{n+1} - \alpha_n| \\
&\quad \leq e^{e^{(2^{2n+7}\cdot l_n^5\cdot e^{2\pi\cdot\rho_n'})q_n}} \cdot |\alpha_{n+1} - \alpha_n|.
\end{aligned}$$

Using the above estimates, we conclude that

$$\begin{aligned}
d_\rho(T_{n+1}, T_n) &\leq \|DH_n^{-1}\|_{\rho_n+1} \cdot \|h_{n+1}^{-1} \circ \phi^{\alpha_{n+1}} \circ h_{n+1} - \phi^{\alpha_n}\|_{\rho_n} \\
&\leq e^{e^{(2^{2n+7}\cdot l_n^5\cdot e^{2\pi\cdot\rho_n'})q_n}} \cdot \frac{1}{2^{n+1}} |\alpha_{n+1} - \alpha_n| \\
&\leq e^{e^{(2^{2n+7}\cdot l_n^5\cdot e^{2\pi\cdot\rho_n'})q_n}} \cdot \frac{1}{2^{n+1}} \cdot 2 \cdot |\alpha - \alpha_n| \\
&< \frac{1}{2^n}.
\end{aligned}$$

Hence, $(T_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{Diff}_\rho^\omega(\mathbb{T}^2)$. Since this is a complete space, we obtain convergence of $(T_n)_{n\in\mathbb{N}}$ to a real-analytic diffeomorphism $T \in \mathrm{Diff}_\rho^\omega(\mathbb{T}^2)$. □

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3.4. *Proof of conjugacy of T to the rotation R_α of the circle.* This subsection is identical to [**Ba17**, §6] and we omit very detailed proofs, which are available in that paper. We have a sequence of real-analytic diffeomorphisms T_n converging to a real-analytic diffeomorphism T, a generating sequence of partitions $\mathcal{F}_{q_n}$ of $\mathbb{T}^2$ (see §2.4) and each $\mathcal{F}_{q_n}$ is cyclically permuted by T_n. We also know the convergence of $(\alpha_n)_{n\in\mathbb{N}}$ to the prescribed number α. Additionally, we define the *error set* $E_n \subset \mathbb{T}^2$ by

$$H_n(E_n) = \bigcup_{i=0}^{l_n^2 q_n - 1} \left[\frac{i}{l_n^2 q_n} - \frac{1}{2^{n+2} l_n^2 q_n}, \frac{i}{l_n^2 q_n} + \frac{1}{2^{n+2} l_n^2 q_n} \right] \times \mathbb{T}^1.$$

Note that $H_n(E_n)$ is the set where we do not have any control under h_{n+1}.

On the other hand, we approximate an irrational rotation of the circle by rational rotations. Let $\alpha_n = p_n/q_n$ be the sequence of rational numbers constructed in Lemma 3.16 above and consider a sequence of partitions of the circle as follows:

$$\mathcal{C}_{q_n} := \left\{ \Gamma_{i,q_n} := \left[\frac{i}{q_n}, \frac{i+1}{q_n} \right) : i = 0, 1, \ldots, q_n - 1 \right\}.$$

Clearly, this is a sequence of partitions which are monotonic and generating. We also define a sequence of maps:

$$R_{\alpha_n} : S^1 \to S^1 \quad \text{defined by } x \mapsto x + \alpha_n.$$

So, we have $R_{\alpha_n} \to R_\alpha$. We also define

$$\tilde{E}_n := \bigcup_{i=0}^{q_n} \left[\frac{i}{q_n} - \frac{\mu(E_n)}{2q_n}, \frac{i}{q_n} + \frac{\mu(E_n)}{2q_n} \right].$$

Following the notation of Lemma 3.1, we let $M^{(1)} := \mathbb{T}^2$, $\mu^{(1)} := \mu$, $\mathcal{P}_n^{(1)} := \mathcal{F}_{q_n}$, $E_n^{(1)} := E_n$, $M^{(2)} := \mathbb{T}^1$, $\mu^{(2)} := \lambda$, $\mathcal{P}_n^{(2)} := \mathcal{C}_{q_n}$ and $E_n^{(2)} := \tilde{E}_n$. Finally, we define the conjugacy K_n by $K_n(H_n^{-1}\Delta_{i,q_n}) = \Gamma_{i,q_n}$. This gives us that $\mathcal{P}_n^{(i)}$ is generating and conditions (3.2), (3.3), (3.5) and (3.6) in Lemma 3.1. Conditions (3.8) and (3.9) follow from the definition. Now note that ϕ^{α_n} preserves $H_n(E_n)$ and hence T_n preserves E_n. This gives us (3.7) and completes the proof of Theorem A.

3.5. *Set of numbers $\mathcal{L}_*$.* As announced in the introduction, we examine the set of obtained rotation numbers. By well-known arguments for spaces like $\mathcal{L}_*$ (e.g. [**Br15**, Appendix A.2]), we prove the following result.

LEMMA 3.19. *$\mathcal{L}_*$ is a dense G_δ-subset of $\mathbb{R}$.*

Proof. For each pair $(p, q) \in \mathbb{Z} \times \mathbb{N}$ with p, q relatively prime and every $k \in \mathbb{N}$, we define the following open set:

$$O_k(p, q) = \left\{ x \in \mathbb{R} \;\middle|\; 0 < \left| x - \frac{p}{q} \right| < \frac{1}{e^{e^{kq}}} \right\}.$$

Then the countable union

$$U_k = \bigcup_{\substack{(p,q)\in\mathbb{Z}\times\mathbb{N} \text{ with} \\ p,q \text{ relatively prime}}} O_k(p, q)$$

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is also open in $\mathbb{R}$ for every $k \in \mathbb{N}$. In the next step, we fix $k \in \mathbb{N}$. Obviously, each rational number $\omega \in \mathbb{Q}$ written in its lowest form $\omega = p/q$ lies in the closure of $O_k(p, q)$. Hence, each rational number lies in the closure of U_k. Since the rational numbers are dense in $\mathbb{R}$, U_k is dense in $\mathbb{R}$. This applies to all $k \in \mathbb{N}$. Moreover, we observe that

$$\mathcal{L}_* = \bigcap_{k\in\mathbb{N}} U_k.$$

By the Baire category theorem, $\mathcal{L}_*$ as a countable intersection of open dense sets is dense in $\mathbb{R}$. □

4. *Non-standard real-analytic realization of some ergodic translations of the torus*

Our goal in this section is to produce a proof of Theorem B. The proof of this theorem in the smooth category was done by Anosov and Katok in [**AK70**]. Later, Benhenda in [**Be13**] produced an estimated version of this result showing that every ergodic toral translation with one arbitrary Liouvillean coordinate can be realized smoothly on any manifold admitting a circle action.

In our article we prove a real-analytic version of this result. Unfortunately, we do not have the techniques to prove results like [**AK70**, Theorems 1.1 and 1.2] for arbitrary real-analytic manifolds. However, the concept of block-slide type of maps and their real-analytic approximations allow us just about enough flexibility on the torus to produce pretty much any combinatorial picture the approximation by conjugation scheme requires. So, we prove that there exist real-analytic diffeomorphisms on $\mathbb{T}^d$ which are metrically conjugated to ergodic translations on $\mathbb{T}^h$ for arbitrary $d \geq 2$ and $h \geq 1$.

As of now we do not know if this concept of block-slide type of maps can be successfully generalized to other types of real-analytic manifolds. It is possible that these maps can be generalized to odd-dimensional spheres using transitive flows like those described in [**FK14**]. The main obstruction to this generalization seems to be the fact that the analytic flows they use commute with the Hopf fibration but not with each other.

4.1. *Description of the required combinatorics.*

Our objective here is to demonstrate that one can reproduce the approximation by conjugation scheme as described in [**AK70**] in its full generality in the real-analytic category on $\mathbb{T}^d$, $d \geq 2$. We show three basic kinds of rearrangement techniques in this section. Together, these three kinds of rearrangements will be sufficient to produce all constructions done in [**AK70**].

4.1.1. *Periodic interchange of two consecutive atoms.*

Fix any two integers k and q. Our objective here is to show that one can interchange two consecutive atoms of $\mathcal{T}_{kq}$ periodically inside each atom of $\mathcal{T}_q$. More precisely, we show that there exists a block-slide type of map $\mathfrak{f}_{k,q}$ that interchanges the atom $\Delta_{ik,kq}$ with the atom $\Delta_{ik+1,kq}$ for $i = 0, \ldots, q-1$ and leaves all other atoms of $\mathcal{T}_{kq}$ unchanged.

We begin by considering the following step functions (or more appropriately piecewise constant functions):

$$\sigma_{kq}^{(1)} : (0, 1] \to \mathbb{R} \quad \text{defined by } \sigma_{kq}^{(1)}(t) = \begin{cases} 0 & \text{if } t \in (0, 1/2], \\ 1/(kq) & \text{if } t \in (1/2, 1], \end{cases} \tag{4.1}$$

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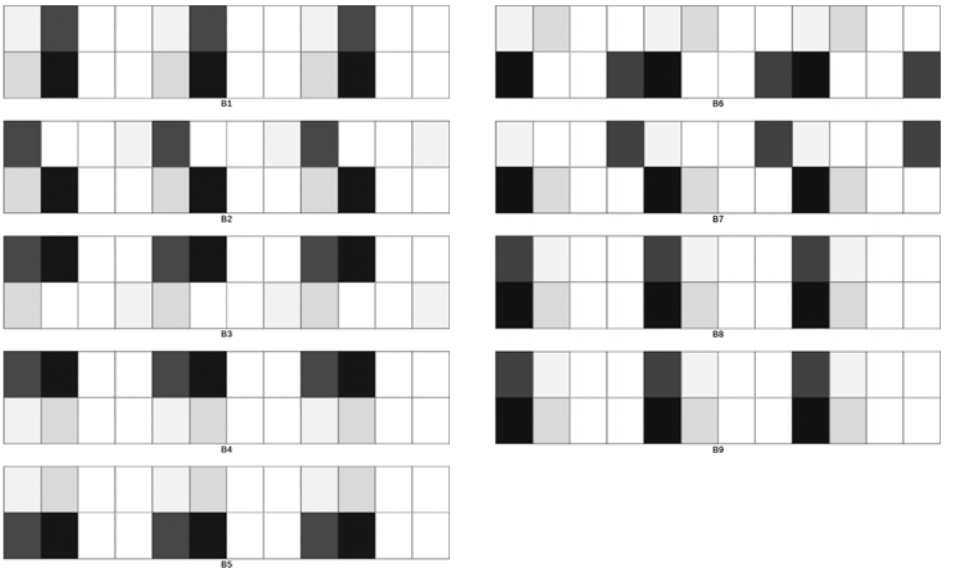


FIGURE 2. Illustration of the action of $\mathfrak{f}_{kq}$ on $\mathbb{T}^2$ with $k = 4$ and $q = 3$. We note how the blocks are moved by the intermediate maps: $B1 \xrightarrow{\mathfrak{f}_{kq}^{(1)}} B2 \xrightarrow{\mathfrak{f}_{kq}^{(2)}} B3 \xrightarrow{\mathfrak{f}_{kq}^{(3)}} B4 \xrightarrow{\mathfrak{f}_{kq}^{(4)}} B5 \xrightarrow{\mathfrak{f}_{kq}^{(5)}} B6 \xrightarrow{\mathfrak{f}_{kq}^{(6)}} B7 \xrightarrow{\mathfrak{f}_{kq}^{(7)}} B8 \xrightarrow{\mathfrak{f}_{kq}^{(8)}} B9$. Note that at the end the first two columns are interchanged. $\mathfrak{f}_{kq}$ acts as an identity on every other column. Also note that $B8$ and $B9$ appear to be identical but the unmarked rectangles are flipped.

$$\sigma_{kq}^{(2)} : (0, 1] \to \mathbb{R} \quad \text{defined by } \sigma_{kq}^{(2)}(t) = \begin{cases} 1/(kq) & \text{if } t \in (0, 1/2], \\ 0 & \text{if } t \in (1/2, 1], \end{cases} \tag{4.2}$$

$$\sigma_{kq}^{(3)} : (0, 1] \to \mathbb{R} \quad \text{defined by } \sigma_{kq}^{(3)}(t) = \begin{cases} 0 & \text{if } qt \bmod 1 \in \left(0, \dfrac{1}{k}\right], \\ 1/2 & \text{if } qt \bmod 1 \in \left(\dfrac{1}{k}, 1\right], \end{cases} \tag{4.3}$$

$$\sigma_{kq}^{(4)} : (0, 1] \to \mathbb{R} \quad \text{defined by } \sigma_{kq}^{(4)}(t) = \begin{cases} 0 & \text{if } qt \bmod 1 \in \left(0, \dfrac{2}{k}\right], \\ 1/2 & \text{if } qt \bmod 1 \in \left(\dfrac{2}{k}, 1\right]. \end{cases} \tag{4.4}$$

Note that $\sigma_{kq}^{(3)}$ is $1/q$-periodic and we can define the following piecewise continuous functions $\mathfrak{f}_{kq}^{(\mathrm{i})} : \mathbb{T}^d \to \mathbb{T}^d$, $\mathrm{i} = 1, \ldots, 8$, by (see Figure 2)

$$\mathfrak{f}_{kq}^{(1)}((x_1, \ldots, x_d)) = (x_1 - \sigma_{kq}^{(1)}(x_2), x_2, \ldots, x_d), \tag{4.5}$$

$$\mathfrak{f}_{kq}^{(2)}((x_1, \ldots, x_d)) = (x_1, x_2 + \sigma_{kq}^{(3)}(x_1), \ldots, x_d), \tag{4.6}$$

$$\mathfrak{f}_{kq}^{(3)}((x_1, \ldots, x_d)) = (x_1 + \sigma_{kq}^{(2)}(x_2), x_2, \ldots, x_d), \tag{4.7}$$

$$\mathfrak{f}_{kq}^{(4)}((x_1, \ldots, x_d)) = (x_1, x_2 + 1/2, \ldots, x_d), \tag{4.8}$$

$$\mathfrak{f}_{kq}^{(5)}((x_1, \ldots, x_d)) = (x_1 - \sigma_{kq}^{(2)}(x_2), x_2, \ldots, x_d), \tag{4.9}$$

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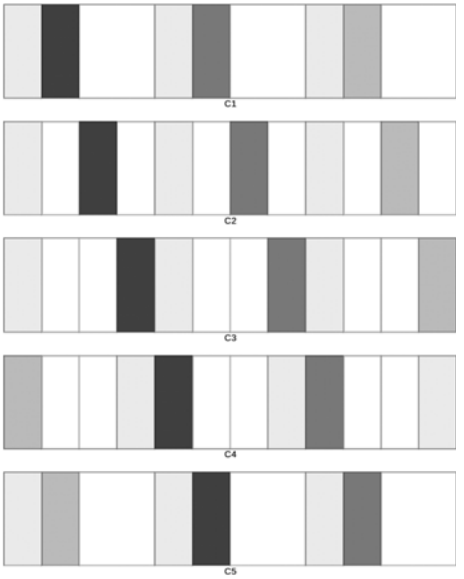


FIGURE 3. Illustration of the action of $\mathfrak{w}_{i,l,k,q}$ on $\mathbb{T}^2$ with $l=1$, $i=1$, $k=4$ and $q=3$. We note how the blocks are moved by the intermediate maps: $C1 \xrightarrow{\mathfrak{w}_{1,4,3}} C2 \xrightarrow{\mathfrak{w}_{2,4,3}} C3 \xrightarrow{\mathfrak{w}_{3,4,3}} C4 \xrightarrow{\phi^{1/(12)}} C5$. Note that all the remaining columns are kept constant at the end.

$$\mathfrak{f}_{kq}^{(6)}((x_1,\dots,x_d)) = (x_1, x_2 + \sigma_{kq}^{(3)}(x_1),\dots,x_d), \tag{4.10}$$

$$\mathfrak{f}_{kq}^{(7)}((x_1,\dots,x_d)) = (x_1 - \sigma_{kq}^{(1)}(x_2), x_2,\dots,x_d), \tag{4.11}$$

$$\mathfrak{f}_{kq}^{(8)}((x_1,\dots,x_d)) = (x_1, x_2 + \sigma_{kq}^{(4)}(x_2),\dots,x_d). \tag{4.12}$$

We compose all the functions above into the following function:

$$\mathfrak{f}_{k,q} : \mathbb{T}^d \to \mathbb{T}^d \quad \text{defined by } \mathfrak{f}_{k,q} := \mathfrak{f}_{kq}^{(8)} \circ \mathfrak{f}_{kq}^{(7)} \circ \mathfrak{f}_{kq}^{(6)} \circ \mathfrak{f}_{kq}^{(5)} \circ \mathfrak{f}_{kq}^{(4)} \circ \mathfrak{f}_{kq}^{(3)} \circ \mathfrak{f}_{kq}^{(2)} \circ \mathfrak{f}_{kq}^{(1)}. \tag{4.13}$$

4.1.2. *Periodic rearrangement of atoms.* Now we show that for any given l and i with $0 \le l < k$ and $0 \le i < q$, there exists a map of the block-slide kind which will allow us to rearrange the atoms of $\mathcal{T}_{k,q}$ so that for any i with $0 \le i < q$, the atom $\Delta_{l+kj,kq}$ is moved to $\Delta_{l+k(j+i),kq}$, while any atom that is not of the form $\Delta_{l+kj',kq}$ is left invariant.

Now we describe this map. Consider the following block-slide type of map:

$$\mathfrak{w}_{l,k,q} : \mathbb{T}^d \to \mathbb{T}^d \quad \text{defined by } \mathfrak{w}_{l,k,q} = \phi^{l/(kq)} \circ \mathfrak{f}_{k,q} \circ \phi^{-l/(kq)} \tag{4.14}$$

using the circle action $\{\phi^t\}_{t\in\mathbb{S}^1}$ and consider the following composition (see Figure 3):

$$\mathfrak{w}_{i,l,k,q} : \mathbb{T}^d \to \mathbb{T}^d \quad \text{defined by } \mathfrak{w}_{i,l,k,q} = \phi^{1/(kq)} \circ \mathfrak{w}_{l+ki-2,k,q} \circ \cdots \mathfrak{w}_{l+1,k,q} \circ \mathfrak{w}_{l,k,q}. \tag{4.15}$$

Note that this map is the one we desired at the beginning.

4.1.3. *Recovering $\mathcal{T}_{l^d q}$ from a generating partition.* We show that given any two integers l and q, there exists a block-slide type of map which allows us to break down

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the partition $\mathcal{T}_{l^d q}$ and reform it into a partition $\mathcal{G}_{l,q}$ whose atoms have diameters less than d/l. This generalizes the construction from §3.2 to the higher dimensional case.

We consider the following three types of block-slide map:

$\mathfrak{g}_{i,l,q}^{(1)} : \mathbb{T}^d \to \mathbb{T}^d$ defined by

$$\mathfrak{g}_{i,l,q}^{(1)}(x_1, \ldots, x_d) = (x_1 + \psi_{l,q}^{(1)}(x_i), x_2, \ldots, x_d),$$

$\mathfrak{g}_{i,l,q}^{(2)} : \mathbb{T}^d \to \mathbb{T}^d$ defined by

$$\mathfrak{g}_{i,l,q}^{(2)}(x_1, \ldots, x_d) = (x_1, \ldots, x_{i-1}, x_i + \psi_{l,q}^{(2)}(x_1), x_{i+1}, \ldots, x_d),$$

$\mathfrak{g}_{i,l,q}^{(3)} : \mathbb{T}^d \to \mathbb{T}^d$ defined by

$$\mathfrak{g}_{i,l,q}^{(3)}(x_1, \ldots, x_d) = (x_1 - \psi_{l,q}^{(3)}(x_i), x_2, \ldots, x_d)$$

with the maps $\psi_{l,q}^{(\mathrm{i})}$ defined as follows:

$$\psi_{i,l,q}^{(1)} : [0, 1) \to \mathbb{R}, \quad \psi_{i,l,q}^{(1)}(x) = \sum_{j=1}^{l-1} \frac{l-j}{l^{d+2-i}q} \chi_{[j/l,(j+1)/l]}(x),$$

$$\psi_{i,l,q}^{(2)} : [0, 1) \to \mathbb{R}, \quad \psi_{i,l,q}^{(2)}(x) = \sum_{j=0}^{l^{d+2-i}q-1} \left(\frac{j}{l} - \left\lfloor \frac{j}{l} \right\rfloor\right) \chi_{[j/l^{d+2-i}q,(j+1)/l^{d+2-i}q]}(x),$$

$$\psi_{i,l,q}^{(3)} : [0, 1) \to \mathbb{R}, \quad \psi_{i,l,q}^{(3)}(x) = \sum_{j=0}^{l-1} \frac{j}{l^{d+2-i}q} \chi_{[j/l,(j+1)/l]}(x).$$

Then we define the following three types of block-slide map:

$\mathfrak{g}_{i,l,q}^{(1)} : \mathbb{T}^d \to \mathbb{T}^d$ defined by

$$\mathfrak{g}_{i,l,q}^{(1)}(x_1, \ldots, x_d) = (x_1 + \psi_{i,l,q}^{(1)}(x_i), x_2, \ldots, x_d),$$

$\mathfrak{g}_{i,l,q}^{(2)} : \mathbb{T}^d \to \mathbb{T}^d$ defined by

$$\mathfrak{g}_{i,l,q}^{(2)}(x_1, \ldots, x_d) = (x_1, \ldots, x_{i-1}, x_i + \psi_{i,l,q}^{(2)}(x_1), x_i, \ldots, x_d),$$

$\mathfrak{g}_{i,l,q}^{(3)} : \mathbb{T}^d \to \mathbb{T}^d$ defined by

$$\mathfrak{g}_{i,l,q}^{(3)}(x_1, \ldots, x_d) = (x_1 - \psi_{i,l,q}^{(3)}(x_i), x_2, \ldots, x_d).$$

Note that the composition

$$\mathfrak{g}_{i,l,q} : \mathbb{T}^d \to \mathbb{T}^d \quad \text{defined by } \mathfrak{g}_{i,l,q} = \mathfrak{g}_{i,l,q}^{(3)} \circ \mathfrak{g}_{i,l,q}^{(2)} \circ \mathfrak{g}_{i,l,q}^{(1)} \tag{4.16}$$

maps the partition $\mathcal{G}_{i,l,q}$ to $\mathcal{G}_{i-1,l,q}$, where

$$\mathcal{G}_{j,l,q} := \left\{ \left[\frac{i_1}{l^{d+1-j}q}, \frac{i_1+1}{l^{d+1-j}q} \right) \times \left[\frac{i_2}{l}, \frac{i_2+1}{l} \right) \times \cdots \times \left[\frac{i_j}{l}, \frac{i_j+1}{l} \right) \times \mathbb{T}^{d-j} : i_1 = 0, 1, \ldots, l^{d+1-j}q - 1, \ (i_2, \ldots, i_j) \in \{0, 1, \ldots, l-1\}^{j-1} \right\}. \tag{4.17}$$

So, the composition

$$\mathfrak{g}_{l,q} : \mathbb{T}^d \to \mathbb{T}^d \quad \text{defined by } \mathfrak{g}_{l,q} = \mathfrak{g}_{2,l,q} \circ \cdots \circ \mathfrak{g}_{d,l,q} \tag{4.18}$$

maps the partition $\mathcal{G}_{l,q}$ to $\mathcal{T}_{l^d q}$.

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4.1.4. *Piecing everything together.* Our objective now is to demonstrate that there is a $1/q$-periodic block-slide type of map which maps the partition $\mathcal{G}_{l,q}$ to $\mathcal{R}_{a,k,q}$. Such a map is obtained after taking a composition of some of the maps defined above.

Consider the following composition:

$$\mathfrak{h}^{(1)}_{a,k,q} : \mathbb{T}^d \to \mathbb{T}^d \quad \text{defined by } \mathfrak{h}^{(1)}_{a,k,q} = \mathfrak{w}_{a(k-1),k-1,k,q} \circ \cdots \circ \mathfrak{w}_{a(1),1,k,q} \circ \mathfrak{w}_{a(0),0,k,q} \tag{4.19}$$

and note that the above map maps the partition $\mathcal{R}_{a,k,q}$ to the decomposition $\mathcal{T}_q$. Next we define

$$\mathfrak{h}^{(2)}_{a,k,q} : \mathbb{T}^d \to \mathbb{T}^d \quad \text{by } \mathfrak{h}^{(2)}_{l,q} = \mathfrak{g}_{l,q} \tag{4.20}$$

and recall that the above map maps the partition $\mathcal{G}_{l,q}$ to $\mathcal{T}_{l^d q}$. So, the composition

$$\mathfrak{h}_{a,k,l,q} : \mathbb{T}^d \to \mathbb{T}^d \quad \text{defined by } \mathfrak{h}_{a,k,l,q} = (\mathfrak{h}^{(1)}_{a,k,q})^{-1} \circ \mathfrak{h}^{(2)}_{kl,q} \tag{4.21}$$

satisfies the following properties:

(1) $\mathfrak{h}^{-1}_{a,k,l,q}(\mathcal{R}_{a,k,q}) = \mathcal{T}_q$;
(2) $\mathfrak{h}^{-1}_{a,k,l,q}(\mathcal{T}_{l^d k^d q}) = \mathcal{G}_{lk,q}$;
(3) $\mathfrak{h}_{a,k,l,q} \circ \phi^\alpha = \phi^\alpha \circ \mathfrak{h}_{a,k,l,q}$ for any p and $\alpha = p/q$.

4.2. *Periodic approximation of ergodic translations of the torus.* This entire subsection is identical to [**AK70**, §6]. So, we only recall the portions that we need. For exact proofs, one may refer to the original article.

LEMMA 4.22. *There exist sequences* $\alpha_n = (\alpha_n^{(1)}, \ldots, \alpha_n^{(h)})$ *and* $\gamma_n = (\gamma_n^{(1)}, \ldots, \gamma_n^{(h)})$ *satisfying the following properties:*

(1) $\gcd(\gamma_n^{(1)}, \ldots, \gamma_n^{(h)}) = 1$;
(2) *there exist integers* p_n, q_n *such that* $\gcd(p_n, q_n) = 1$ *and* $\alpha_n = (p_n/q_n)\gamma_n$;
(3) *there exist integers* r_n *such that* $q_n = r_n \gamma_{n-1}^{(h)}$;
(4) *there exist integers* s_n *such that* $\gamma_{n+1}^{(h)} = s_n \gamma_n^{(h)}$;
(5) $\gamma_{n+1}^{(i)} \equiv \gamma_n^{(i)} \bmod q_n$ *for* $i = 1, \ldots, h$;
(6) *there exist integers* m_n *such that*

$$\frac{p_{n+1}}{q_{n+1}} = \frac{p_n}{q_n} + \frac{1}{m_n s_n q_n^2}; \tag{4.23}$$

(7) *let* $\Gamma'_n \subset \mathbb{T}^{h-1} \times \{0\} \subset \mathbb{T}^h$ *be a fundamental domain of the flow* $T^{t\gamma_n}$. *Let* $d_n := \mathrm{diam}(\Gamma'_n)$, $\sigma_n = \mu_{h-1}(\partial(\Gamma'_n))$. *Then* $d_{n+1} < 1/(2^n \gamma_n^{(h)} \sigma_n)$;
(8)

$$\left| \frac{\gamma_{n+1}}{\gamma_{n+1}^{(h)}} - \frac{\gamma_n}{\gamma_n^{(h)}} \right| < \frac{1}{2^n \sigma_n q_n}. \tag{4.24}$$

With α_n, p_n, q_n, γ_n, r_n, s_n, m_n and Γ'_n as in Lemma 4.22, we construct the following two sequences of partitions of $\mathbb{T}^{h-1} \times \{0\} \subset \mathbb{T}^h$:

$$\tilde{\mathcal{F}}'_{q_n} := \{\Gamma'_{i,q_n} : \Gamma'_{i,q_n} := T^{i\gamma_n/\gamma_n^{(h)}} \Gamma'_n,\ i = 0, \ldots, q_n - 1\}, \tag{4.25}$$

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$$\tilde{\mathcal{F}}'_{q_n,q_{n+1}} := \Big\{ \Gamma'_{i,q_n,q_{n+1}} : \Gamma'_{i,q_n,q_{n+1}} := T^{i\gamma_n/\gamma_n^{(h)}} \Big(\bigcup \{ \Gamma'_{j,q_{n+1}} : T^{j\gamma_{n+1}/\gamma_{n+1}^{(h)}}(0) \in \Gamma'_n,\ j = 0, \ldots, q_{n+1} - 1,\ i = 0, \ldots, q_n - 1 \} \Big) \Big\}. \tag{4.26}$$

Note that $\mathcal{F}'_{q_{n+1}} > \mathcal{F}'_{q_n,q_{n+1}}$ and they are both generating sequences of partitions. We construct the following two sequences of partitions of $\mathbb{T}^h$ from the above two partitions:

$$\tilde{\mathcal{F}}_{q_n} := \Big\{ \Gamma_{i,q_n} : \Gamma_{i,q_n} := T^{i\gamma_n/q_n} \Big(\bigcup \{ T^{t\gamma_n} \Gamma'_n : 0 \le t < 1/q_n \} \Big),\ i = 0, \ldots, q_n - 1 \Big\}, \tag{4.27}$$

$$\tilde{\mathcal{F}}_{q_n,q_{n+1}} := \Big\{ \Gamma_{i,q_n,q_{n+1}} : \Gamma_{i,q_n,q_{n+1}} := T^{i\gamma_n/q_n} \Big(\bigcup \{ T^{t\gamma_{n+1}} \Gamma'_{0,q_n,q_{n+1}} : 0 \le t < 1/(r_n \gamma_{n+1}^{(h)}) \} \Big), i = 0, \ldots, q_n - 1 \Big\}. \tag{4.28}$$

The following proposition summarizes certain properties of the above partitions. For a proof, one can refer to [**AK70**, pp. 28–29].

PROPOSITION 4.29. *With α_n, p_n, q_n, γ_n, r_n, s_n, m_n as in Lemma 4.22, we can conclude the following.*

(1) *The sequences of partitions $\tilde{\mathcal{F}}_{q_n,q_{n+1}}$ and $\tilde{\mathcal{F}}_{q_n}$ are respectively preserved and permuted by T^{α_n}.*
(2) *$\mu_h(\Gamma_{i,q_n,q_{n+1}} \triangle \Gamma_{i,q_n}) < 1/(2^{n-3} q_n)$† for any $\Gamma_{i,q_n,q_{n+1}} \in \tilde{\mathcal{F}}_{q_n,q_{n+1}}$ and $\Gamma_{i,q_n} \in \tilde{\mathcal{F}}_{q_n}$ with the same i.*
(3) *The sequence of periodic translations $T^{\alpha_n} : \mathbb{T}^h \to \mathbb{T}^h$ converges to an ergodic translation $T^{\alpha} : \mathbb{T}^h \to \mathbb{T}^h$.*

The next proposition is identical to [**AK70**, Lemma 6.2].

PROPOSITION 4.30. *Under the same hypothesis as Proposition 4.29, we can find a sequence of partitions $\tilde{\mathcal{M}}_n$ of $\mathbb{T}^h$ satisfying the following three properties.*

(1) *Monotonicity condition: $\tilde{\mathcal{M}}_{n+1} > \tilde{\mathcal{M}}_n$.*
(2) *Cyclic permutation: the diffeomorphism T^{α_n} cyclically permutes the atoms of $\tilde{\mathcal{M}}_n$.*
(3) *Generating condition: $\tilde{\mathcal{M}}_n \to \varepsilon$ as $n \to \infty$.*

Proof. We use a method similar to the proof of Proposition 2.31. We define, for any n, the following three maps:

$$\mathfrak{q}_{n+1,n}^{(1)} : \mathbb{T}^h/\tilde{\mathcal{F}}_{q_n} \to \mathbb{T}^h/\tilde{\mathcal{F}}_{q_n,q_{n+1}} \quad \text{defined by } \mathfrak{q}_{n+1,n}^{(1)}(\Gamma_{i,q_n}) := \Gamma_{i,q_n,q_{n+1}}, \tag{4.31}$$

$$\mathfrak{q}_{n+1,n}^{(2)} : \mathbb{T}^h/\tilde{\mathcal{F}}_{q_{n+1}} \to \mathbb{T}^h/\tilde{\mathcal{F}}_{q_n,q_{n+1}} \quad \text{defined by}$$
$$\mathfrak{q}_{n+1,n}^{(2)}(\Gamma_{i,q_{n+1}}) := \Gamma_{j,q_n,q_{n+1}} \text{ if } \Gamma_{i,q_{n+1}} \subset \Gamma_{j,q_n,q_{n+1}}, \tag{4.32}$$

$$\mathfrak{q}_{n+1,n} : \mathbb{T}^h/\tilde{\mathcal{F}}_{q_n} \to \mathbb{T}^h/\tilde{\mathcal{F}}_{q_{n+1}} \quad \text{defined by } \mathfrak{q}_{n+1,n} := (\mathfrak{q}_{n+1,n}^{(2)})^{-1} \circ \mathfrak{q}_{n,n+1}^{(1)} \tag{4.33}$$

† Here μ_h denotes the standard Lebesgue measure on $\mathbb{T}^h$

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and more generally we define the composition map for any m and n with $m > n$ as follows:

$$\mathfrak{q}_{m,n} : \mathbb{T}^h/\tilde{\mathcal{F}}_{q_n} \to \mathbb{T}^h/\tilde{\mathcal{F}}_{q_m} \quad \text{defined by } \mathfrak{q}_{m,n} = \mathfrak{q}_{m,m-1} \circ \cdots \circ \mathfrak{q}_{n+1,n}. \tag{4.34}$$

We now define the partition

$$\tilde{\mathcal{M}} = \left\{ \lim_{m\to\infty} \mathfrak{q}_{m,n}(\Gamma_{i,q_n}) : 0 \le i < q_n \right\}$$

(see Proposition 2.31) and finally the correspondence

$$\mathfrak{q}_{\infty,n} : \mathbb{T}^h/\tilde{\mathcal{F}}_{q_n} \to \mathbb{T}^h/\tilde{\mathcal{M}} \quad \text{defined by } \mathfrak{q}_{\infty,n} = \lim_{m\to\infty} \mathfrak{q}_{m,n}. \tag{4.35}$$

The rest of the proof involves proving that $\tilde{\mathcal{M}}$ is indeed a partition satisfying the required conditions. This part can be completed identically to Proposition 2.31 and we do not repeat it. □

4.3. *Real-analytic diffeomorphisms metrically isomorphic to a shift on a torus.* Our goal in this subsection is to prove Theorem B.

Proof. First we introduce the following two correspondences:

$$K_n : \mathbb{T}^h/\tilde{\mathcal{F}}_{q_n} \to \mathbb{T}^d/\mathcal{F}_{q_n} \quad \text{defined by } K_n(\Gamma_{i,q_n}) = H_n^{-1}(\Delta_{i,q_n}), \tag{4.36}$$

$$\tilde{K}_n : \mathbb{T}^h/\tilde{\mathcal{M}}_n \to \mathbb{T}^d/\mathcal{M}_n \quad \text{defined by } \tilde{K}_n(\mathfrak{q}_{\infty,n}(\Gamma_{i,q_n})) = \mathfrak{p}_{\infty,n}(H_n^{-1}(\Delta_{i,q_n})) \tag{4.37}$$

using the conjugation map $H_n = h_n \circ H_{n-1}$. Clearly, the above two maps satisfy $\tilde{K}_n = \mathfrak{p}_{\infty,n} \circ K_n \circ \mathfrak{q}_{\infty,n}^{-1}$. We claim that we can choose parameters carefully so that the following condition can be satisfied:

$$K_{n+1}|_{\tilde{\mathcal{F}}_{q_n,q_{n+1}}} = \mathfrak{p}_{n+1,n} \circ K_n \circ \mathfrak{q}_{n+1,n}^{-1}. \tag{4.38}$$

Before we do that, we make some observation about K_n and $\tilde{K}_n$. First note that we have the following relationship:

$$\tilde{K}_n \circ T^{\alpha_n} = T_n \circ \tilde{K}_n. \tag{4.39}$$

Indeed, observe that using Proposition 4.30, equation (4.37) and Proposition 2.31, we get

$$\begin{aligned}
\tilde{K}_n(T^{\alpha_n}(\mathfrak{q}_{\infty,n}(\Gamma_{i,q_n}))) &= \tilde{K}_n(\mathfrak{q}_{\infty,n}(\Gamma_{p_n+i \bmod q_n,q_n})) \\
&= \mathfrak{p}_{\infty,n}(H_n^{-1}(\Delta_{p_n+i \bmod q_n,q_n})) \\
&= \mathfrak{p}_{\infty,n}(T_n(H_n^{-1}(\Delta_{i,q_n}))) \\
&= T_n(\mathfrak{p}_{\infty,n}(H_n^{-1}(\Delta_{i,q_n}))) \\
&= T_n(\tilde{K}_n(\mathfrak{q}_{\infty,n}(\Gamma_{i,q_n}))).
\end{aligned}$$

The next observation we need is

$$\tilde{K}_{n+1}|_{\mathbb{T}^h/\tilde{\mathcal{M}}_n} = \tilde{K}_n. \tag{4.40}$$

Indeed, observe that using equation (4.37),

$$\begin{aligned}
\tilde{K}_{n+1}(\mathfrak{q}_{\infty,n}(\Gamma_{i,q_n})) &= \tilde{K}_{n+1}(\mathfrak{q}_{\infty,n+1} \circ \mathfrak{q}_{n+1,n}(\Gamma_{i,q_n})) \\
&= \mathfrak{p}_{\infty,n+1}(K_{n+1}(\mathfrak{q}_{n+1,n}(\Gamma_{i,q_n}))) \\
&= \mathfrak{p}_{\infty,n+1}(\mathfrak{p}_{n+1,n}(K_n(\Gamma_{i,q_n}))) \\
&= \mathfrak{p}_{\infty,n}(K_n(\Gamma_{i,q_n})) \\
&= \tilde{K}_n(\mathfrak{q}_{\infty,n}(\Gamma_{i,q_n})).
\end{aligned}$$

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Our job now is to choose parameters correctly in Proposition 4.29 and the real-analytic approximation by conjugation scheme and simultaneously construct the K_n satisfying condition (4.38).

The construction is by induction and assume that we have selected parameters α_j, $p_j, q_j, \gamma_j, r_j, s_j$ for $j = 1, 2, \ldots, n$ in Proposition 4.29 so that these satisfy all the conditions in Lemma 4.22. We recall that the parameter m_n in Lemma 4.22 can be chosen to be arbitrarily large. We will use this freedom to make the approximation by conjugation scheme work. Assume that the approximation by conjugation scheme has been successfully carried out up to the nth stage.

At the $(n+1)$th stage, we choose parameters and proceed with our construction in the following order.

(1) Choose an integer vector γ_{n+1}^h and integers r_n, s_n as in Lemma 4.22.

(2) Choose the parameter $k_n := s_n \gamma_{n+1}^{(h)}$ for the approximation by conjugation scheme. Next we define the partition

$$\tilde{\mathcal{F}}_{k_n q_n} := \left\{ \Gamma_{i,k_n q_n} : \Gamma_{i,k_n q_n} := T^{i/(k_n q_n)} \left(\bigcup \{ T^{t\gamma_{n+1}} \Gamma'_{n+1} : 0 \le t < 1/(k_n q_n) \} \right), \right.$$
$$\left. i = 0, \ldots, k_n q_n - 1 \right\}. \tag{4.41}$$

Note that with this choice of k_n, we have $\tilde{\mathcal{F}}_{k_n q_n} > \tilde{\mathcal{F}}_{q_n, q_{n+1}}$.

(3) Choose the functions a_n in the approximation by conjugation scheme. In order to do so, we define the following correspondence:

$$\hat{K}_n : \mathbb{T}^h / \tilde{\mathcal{F}}_{k_n q_n} \to \mathbb{T}^d / \mathcal{T}_{k_n q_n} \quad \text{defined by } \hat{K}_n(\Gamma_{i,k_n q_n}) = \Delta_{i,k_n q_n}. \tag{4.42}$$

Now we choose a function $a_n : \{0, \ldots, k_n - 1\} \to \{0, \ldots, q_n - 1\}$ so that the following equality holds:

$$R_{0,q_n} := \bigcup_{i=0}^{k_n - 1} \Delta_{a_n(i)k_n + i, k_n q_n} = \hat{K}_n(\Gamma_{0,q_n,k_n q_n}). \tag{4.43}$$

This allows us to define the partition

$$\mathcal{R}_{a_n,k_n,q_n} := \{ R_{i,q_n} : R_{i,q_n} := \phi^{i/q_n} R_{0,q_n} \}. \tag{4.44}$$

Note that $\hat{K}_n(\Gamma_{i,q_n,q_{n+1}}) = R_{i,q_n}$ and hence we conclude that $\hat{K}_n \circ T^{\gamma_n/q_n} = \phi^{1/q_n} \circ \hat{K}_n$.

(4) Choose the parameter l_n large enough so that the analytic approximation by conjugation scheme works.

(5) We choose the parameter m_n in Proposition 4.29. We require that $m_n = \gamma_{n+1}^{(h)} l_n$.

So, after having chosen all the parameters, we can define the partition $\tilde{\mathcal{F}}_{q_{n+1}}$ of $\mathbb{T}^h$ and note that $\tilde{\mathcal{F}}_{q_{n+1}} > \tilde{\mathcal{F}}_{q_n,k_n q_n}$. Now we define the following correspondence:

$$\bar{K}_n : \mathbb{T}^h / \tilde{\mathcal{F}}_{q_{n+1}} \to \mathbb{T}^d / \mathcal{T}_{q_{n+1}} \quad \text{defined by } \bar{K}_n(\Gamma_{i,q_{n+1}}) = \Delta_{i,q_{n+1}} \tag{4.45}$$

and observe that $\bar{K}_n(\Gamma_{i,k_n q_n}) = \hat{K}_n(\Gamma_{i,k_n q_n})$. All that remains is to verify that K_{n+1} satisfies equation (4.38). So, we calculate using definitions and facts from Proposition 2.31 and its proof that

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$$\begin{aligned}
\mathfrak{p}_{n+1,n} \circ K_n(\Gamma_{i,q_n}) &= \mathfrak{p}_{n+1,n} \circ H_n^{-1}(\Delta_{i,q_n}) \\
&= H_{n+1}^{-1} \circ \mathfrak{c}_{n+1,n}(\Delta_{i,q_n}) \\
&= H_{n+1}^{-1}\Bigg(\bigcup_{j:\Delta_{j,q_{n+1}} \subset R_{i,q_n}} \Delta_{j,q_{n+1}} \Bigg) \\
&= H_{n+1}^{-1}(\hat{K}_n(\Gamma_{i,q_n,q_{n+1}})) \quad \ldots \text{(see item 3 above)} \\
&= H_{n+1}^{-1}\Bigg(\hat{K}_n\Bigg(\bigcup_{j:\Gamma_{j,k_nq_n} \subset \Gamma_{i,q_n,q_{n+1}}} \Gamma_{j,k_nq_n} \Bigg)\Bigg) \\
&= H_{n+1}^{-1}\Bigg(\bigcup_{j:\Gamma_{j,k_nq_n} \subset \Gamma_{i,q_n,q_{n+1}}} \hat{K}_n(\Gamma_{j,k_nq_n}) \Bigg) \\
&= H_{n+1}^{-1}\Bigg(\bigcup_{j:\Gamma_{j,k_nq_n} \subset \Gamma_{i,q_n,q_{n+1}}} \bar{K}_n(\Gamma_{j,k_nq_n}) \Bigg) \\
&= H_{n+1}^{-1} \circ \bar{K}_n(\Gamma_{i,q_n,q_{n+1}}) \\
&= H_{n+1}^{-1} \circ \bar{K}_n \circ \mathfrak{q}_{n+1,n}(\Gamma_{i,q_n}) \\
&= K_{n+1} \circ \mathfrak{q}_{n+1,n}(\Gamma_{i,q_n}).
\end{aligned}$$

This completes the proof. □

5. *Minimal diffeomorphisms with a prescribed number of ergodic invariant measures*

Let $\rho > 0$ and $r \in \mathbb{N}$. In order to prove Theorem D, we aim at constructing a minimal $T \in \mathrm{Diff}_\rho^\omega(\mathbb{T}^2, \mu)$ with exactly r ergodic invariant measures. We fix an arbitrary countable set $\Xi = \{\rho_i\}_{i\in\mathbb{N}}$ of Lipschitz functions that is dense in $C(\mathbb{T}^2, \mathbb{R})$. In addition to our usual assumptions, we require the number l_n to satisfy

$$l_n > n^2 \cdot \|DH_n^{-1}\|_0 \cdot \max_{i=1,\ldots,n} \mathrm{Lip}(\rho_i), \tag{5.1}$$

where $\mathrm{Lip}(\rho)$ is the Lipschitz constant of ρ.

First of all, we show that a permutation Π of the partition $\mathcal{S}_{kq,l}$ which commutes with $\phi^{1/q}$ is a block-slide type of map. This property will be required in the construction of our conjugation map in §5.2: $h_n = h_{1,n} \circ h_{2,n}$. This time there are different parts of the torus $\mathbb{T}^2$ introduced with distinct aims. On the one hand, we will divide it into r sets N_t by requirements on the x_2-coordinate. Each set naturally supports an absolutely continuous probability measure μ_t given by the normalized restriction of the Lebesgue measure μ. These will enable us to build the ergodic invariant measures as the limits ξ_t of the sequence $\xi_t^n := (H_n)^*\mu_t$.

On the other hand, we will use stripes corresponding to small parts of the x_1-axis on which the conjugation map $h_{2,n}^{-1}$ will intermingle the sets N_t to prove minimality of the limit diffeomorphism T. These parts are measure-theoretically insignificant because the measure of these sets will converge to zero as $n \to \infty$.

In order to achieve these aims, we need the so-called trapping map $h_{1,n}^{-1}$ introduced in §5.2. On the 'minimality' part, this map captures parts of every orbit $\{\phi^{\alpha_n} \circ H_n(x)\}_{k=0,\ldots,q_n-1}$ so that the conjugation map $h_{2,n}^{-1}$ can spread it over the almost whole

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manifold. Then we can prove minimality in §5.3 by arguing that every element in a family of sufficiently small cubes covering the whole manifold is met by the orbit $\{h_n^{-1} \circ \phi^{k\alpha_n} \circ H_n(x)\}_{k=0,\dots,q_n-1}$ and the image of any such cube under H_{n-1}^{-1} has a small diameter, which converges to 0 as $n \to \infty$. In addition, the trapping map is used to gain control of almost everything of every orbit $\{H_n^{-1} \circ \phi^{k\alpha_n}(x)\}_{k=0,\dots,q_n-1}$. This allows us to prove a convergence result on Birkhoff sums (see Lemma 5.31), which in turn enables us to exclude the existence of further ergodic invariant measures besides the previously mentioned ξ_t.

5.1. *Approximation of arbitrary permutations.* Suppose that we have three natural numbers l, k and q, and a permutation Π of the partition $\mathcal{S}_{kq,l}$ which commutes with $\phi^{1/q}$. Our objective here is to show that Π is a block-slide type of map. This will be achieved in two steps. In the first step we show that there exists a product of two 2-cycles and then we will prove that all transpositions are block-slide type of maps.

5.1.1. *Product of two 2-cycles.* We now show that for any choice of natural numbers l, k and q there exists a block-slide type of map which has the same effect as the product of two 2-cycles in the symmetric group of lkq elements.

In order to make this precise, we need the following notation:

$$\mathcal{S}_{kq,l} := \{S_{i,j}^{kq,l} := [i/(kq), (i+1)/(kq)) \times [j/l, (j+1)/l), 0 \le i < kq, 0 \le j < l\}. \tag{5.2}$$

First we define the following step function:

$$\sigma_{kq}^{(4)} : (0, 1] \to \mathbb{R} \quad \text{defined by } \sigma_{kq}^{(4)}(t) = \begin{cases} 2/(kq) & \text{if } t \in ((l-1)/l, 1], \\ 0 & \text{if } t \in (0, (l-1)/l] \end{cases} \tag{5.3}$$

and then the following two maps of block-slide type:

$$\mathfrak{g}_{kq,l}^{(1)} : \mathbb{T}^2 \to \mathbb{T}^2 \quad \text{defined by } \mathfrak{g}_{kq,l}^{(1)}((x_1, x_2)) = (x_1 - \sigma^{(4)}(x_2), x_2), \tag{5.4}$$

$$\mathfrak{g}_{kq,l}^{(2)} : \mathbb{T}^2 \to \mathbb{T}^2 \quad \text{defined by } \mathfrak{g}_{kq,l}^{(2)}((x_1, x_2)) = (x_1 + \sigma^{(4)}(x_2), x_2). \tag{5.5}$$

Finally, we piece everything together and define the following block-slide type of map (see Figure 4):

$$\mathfrak{g}_{k,q,l} : \mathbb{T}^2 \to \mathbb{T}^2 \quad \text{defined by } \mathfrak{g}_{k,q,l} = \mathfrak{g}_{k,q,l}^{(2)} \circ \mathfrak{f}_{k,q} \circ \mathfrak{g}_{k,q,l}^{(1)} \circ \mathfrak{f}_{k,q} \tag{5.6}$$

using the map $\mathfrak{f}_{k,q}$ from §4.1.

We end this subsubsection after noting that the above block-slide type of map takes $S_{0,l-1}^{kq,l} \to S_{1,l-1}^{kq,l}$, $S_{1,l-1}^{kq,l} \to S_{0,l-1}^{kq,l}$, $S_{2,l-1}^{kq,l} \to S_{3,l-1}^{kq,l}$, $S_{3,l-1}^{kq,l} \to S_{2,l-1}^{kq,l}$ and acts as an identity everywhere else. This is the same as the product of two 2-cycles in the symmetric group on a set of $k \times l$ elements.

5.1.2. *Transposition.* Finally, we show that there exists a block slide on the torus which switches two blocks and leaves all others invariant. Unfortunately if we work with the partition $\mathcal{S}_{kq,l}$, we do not know if such a map exists. The way we circumnavigate this problem is to go to a finer partition, namely $\mathcal{S}_{kq,2l}$, and show that with some care, a product of two 2-cycles in $\mathcal{S}_{kq,2l}$ is a transposition in $\mathcal{S}_{kq,l}$.

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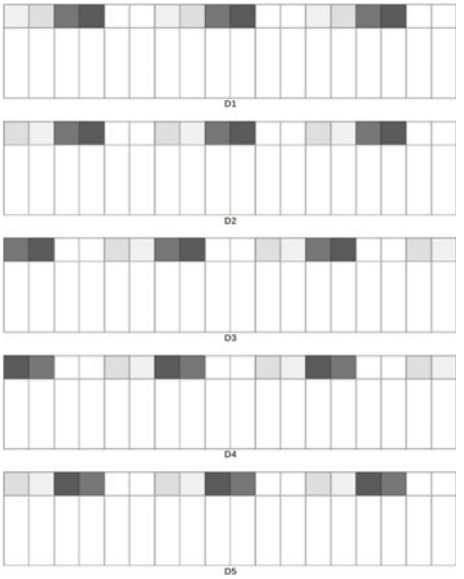


FIGURE 4. Illustration of the action of $\mathfrak{g}_{k,q,l}$ on $\mathbb{T}^2$ with $l = 4$, $k = 6$ and $q = 3$. We note how the blocks are moved by the intermediate maps: $D1 \xrightarrow{\mathfrak{f}_{k,q}} D2 \xrightarrow{\mathfrak{g}^{(1)}_{k,q,l}} D3 \xrightarrow{\mathfrak{f}_{k,q}} D4 \xrightarrow{\mathfrak{g}^{(2)}_{k,q,l}} D5$. Note that all the remaining atoms apart from the four we flipped are kept constant at the end.

First we define the following two step functions:

$$\sigma^{(5)}_{kq,l} : (0,\,1] \to \mathbb{R} \quad \text{defined by } \sigma^{(5)}_{kq,l}(t) = \begin{cases} 0 & \text{if } t \in ((2l-1)/(2l),\,1], \\ 2/(kq) & \text{if } t \in ((2l-2)/(2l), \\ & \qquad (2l-1)/(2l)], \\ 0 & \text{if } t \in (0,\,(2l-2)/(2l)], \end{cases} \tag{5.7}$$

$$\sigma^{(6)}_{kq,l} : (0,\,1] \to \mathbb{R} \quad \text{defined by } \sigma^{(6)}_{kq,l}(t) = \begin{cases} 0 & \text{if } qt \bmod 1 \in (0,\,2/k], \\ 1/(2l) & \text{if } qt \bmod 1 \in (2/k,\,4/k], \\ 0 & \text{if } qt \bmod 1 \in (4/k,\,1]. \end{cases} \tag{5.8}$$

And then the following four block-slide type of maps (see Figure 5):

$$\mathfrak{h}^{(1)}_{kq,l} : \mathbb{T}^2 \to \mathbb{T}^2 \quad \text{defined by } \mathfrak{h}^{(1)}_{kq,l}((x_1,\,x_2)) = (x_1 + \sigma^{(5)}_{kq,l}(x_2),\,x_2), \tag{5.9}$$

$$\mathfrak{h}^{(2)}_{kq,l} : \mathbb{T}^2 \to \mathbb{T}^2 \quad \text{defined by } \mathfrak{h}^{(2)}_{kq,l}((x_1,\,x_2)) = (x_1 - \sigma^{(5)}_{kq,l}(x_2),\,x_2), \tag{5.10}$$

$$\mathfrak{h}^{(3)}_{kq,l} : \mathbb{T}^2 \to \mathbb{T}^2 \quad \text{defined by } \mathfrak{h}^{(3)}_{kq,l}((x_1,\,x_2)) = (x_1,\,x_2 + \sigma^{(6)}_{kq,l}(x_1)), \tag{5.11}$$

$$\mathfrak{h}^{(4)}_{kq,l} : \mathbb{T}^2 \to \mathbb{T}^2 \quad \text{defined by } \mathfrak{h}^{(4)}_{kq,l}((x_1,\,x_2)) = (x_1,\,x_2 - \sigma^{(6)}_{kq,l}(x_1)). \tag{5.12}$$

Finally, we piece everything together and define the following block-slide type of map:

$$\mathfrak{h}_{kq,l} : \mathbb{T}^2 \to \mathbb{T}^2 \quad \text{defined by } \mathfrak{h}_{kq,l} := \mathfrak{h}^{(2)}_{kq,l} \circ \mathfrak{h}^{(4)}_{kq,l} \circ \mathfrak{g}_{k,q,2l} \circ \mathfrak{h}^{(3)}_{kq,l} \circ \mathfrak{h}^{(1)}_{kq,l}. \tag{5.13}$$

As desired, $\mathfrak{h}_{kq,l}$ maps $S^{kq,l}_{0,l-1} \to S^{kq,l}_{1,l-1}$, $S^{kq,l}_{1,l-1} \to S^{kq,l}_{0,l-1}$ and acts as the identity everywhere else. More generally, we can define for any (i, j) $\neq (0, l-1)$ the following

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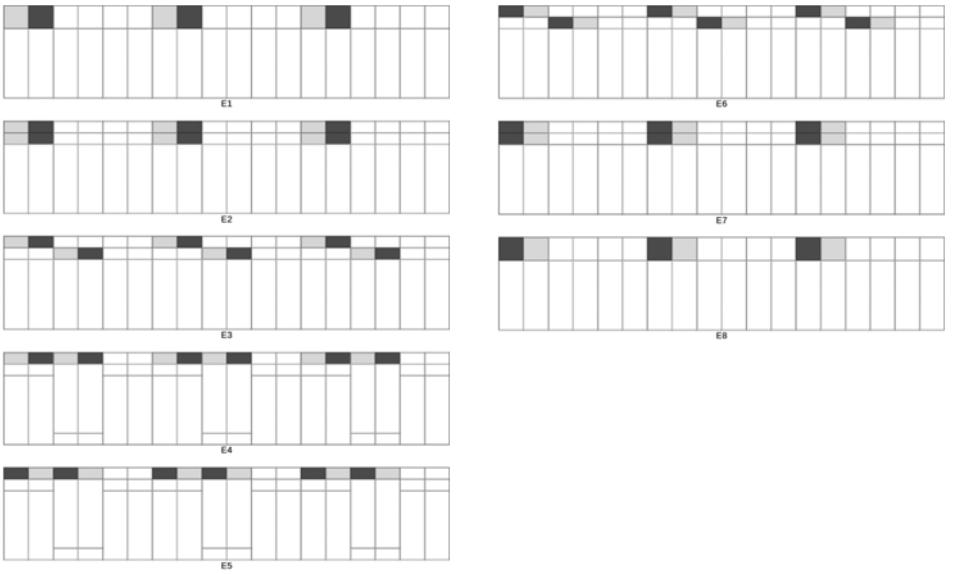


FIGURE 5. Illustration of the action of $\mathfrak{h}_{k,q,l}$ on $\mathbb{T}^2$ with $l = 4$, $k = 6$ and $q = 3$. We note how the blocks are moved by the intermediate maps: $E2 \xrightarrow{\mathfrak{h}^{(1)}_{6,3,4}} E3 \xrightarrow{\mathfrak{h}^{(3)}_{6,3,4}} E4 \xrightarrow{\mathfrak{g}_{6,3,8}} E5 \xrightarrow{\mathfrak{h}^{(4)}_{6,3,4}} E6 \xrightarrow{\mathfrak{h}^{(2)}_{6,3,4}} E7$. Note that every other atom apart from the two we flipped is kept constant at the end.

block-slide type of map:

$$\mathfrak{h}^{(\mathrm{i},\mathrm{j})}_{k,q,l} : \mathbb{T}^2 \to \mathbb{T}^2 \quad \text{defined by}$$
$$\mathfrak{h}^{(\mathrm{i},\mathrm{j})}_{k,q,l} := \phi^{(\mathrm{i}-1)/(kq)} \circ (\mathfrak{h}^{(4)}_{kq,l})^{2(l-1-\mathrm{j})} \circ \mathfrak{h}_{kq,l} \circ (\mathfrak{h}^{(3)}_{k,q,l})^{2(l-1-\mathrm{j})} \circ \phi^{-(\mathrm{i}-1)/(kq)}. \tag{5.14}$$

We end this subsubsection by observing that $\mathfrak{h}^{(\mathrm{i},\mathrm{j})}_{k,q,l}$ maps $S^{kq,l}_{0,l-1} \to S^{kq,l}_{\mathrm{i},\mathrm{j}}$, $S^{kq,l}_{\mathrm{i},\mathrm{j}} \to S^{kq,l}_{0,l-1}$ and acts as the identity everywhere else. So, we have obtained all transpositions of the form $(1, n)$ in the symmetric group on a set of kl elements.

5.1.3. *All permutations are block-slide type of maps.* We now show that any permutation which commutes with $\phi^{1/q}$ can be realized as a block-slide type of map.

THEOREM E. *Let* Π *be any permutation of* kql *elements. We can naturally consider* Π *to be a permutation of the partition* $\mathcal{S}_{kq,l}$ *of the torus* $\mathbb{T}^2$. *Assume that* Π *commutes with* $\phi^{1/q}$. *Then* Π *is a block-slide type of map.*

Proof. This follows from the fact that all permutations are generated by transpositions. □

5.2. *Description of the required combinatorics.* Here we prove Theorem D. We begin by describing the combinatorics we need at the $(n + 1)$th stage of the induction process abstractly.

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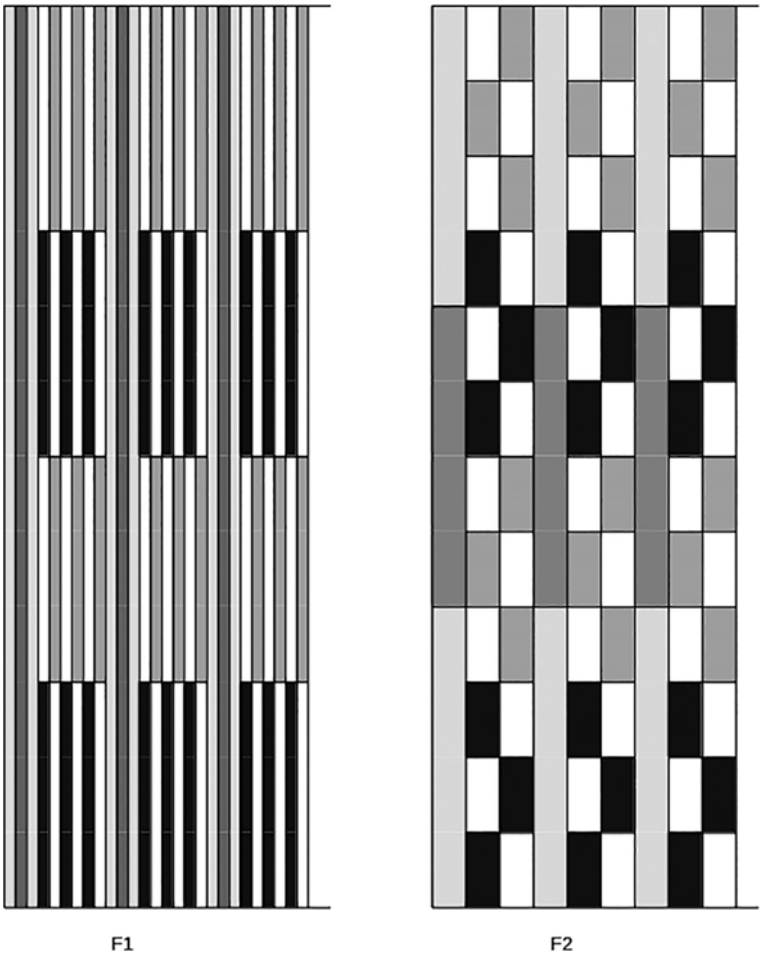


FIGURE 6. Illustration of the action of $(\mathfrak{h}^{(2)})^{-1}$ on the rectangle $[0, 1/q) \times [0, 1)$ inside the torus $\mathbb{T}^2$. The combinatorics depicted here is repeated with periodicity $1/q$. We used $r = 4$ and $l = 3$ for this diagram.

For $t = 0, \ldots, r-1$, we consider the following subsets of $\mathbb{T}^2$:

$$N_t := \mathbb{T}^1 \times \left[\frac{t}{r}, \frac{t+1}{r}\right). \tag{5.15}$$

We denote the restriction of the Lebesgue measure μ to N_t by μ_t.

For natural numbers n, l and q, we define the following partition of the torus $\mathbb{T}^2$:

$$\mathcal{G}_{l^3q} := \left\{ G_{i,j,l^3q} : G_{i,j,l^3q} := \left[\frac{i}{l^3q}, \frac{i+1}{l^3q}\right) \times \left[\frac{j}{lr}, \frac{j+1}{lr}\right), 0 \le i < l^3q, 0 \le j < lr \right\}. \tag{5.16}$$

We define the following permutation of the above partition:

$$\mathfrak{h}^{(2)} : \mathbb{T}^2 \to \mathbb{T}^2, \tag{5.17}$$

which acts on the atoms of the partition $\mathcal{G}_{l^3q}$ that are contained in $[0, 1/(lq)) \times \mathbb{T}^1$ in the following way (for $t = 0, \ldots, r-1$) (see Figure 6):

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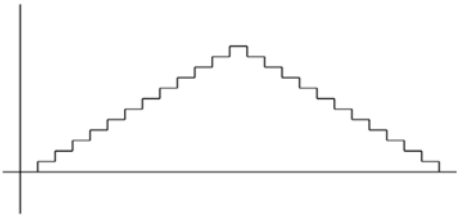


FIGURE 7. Picture of $\kappa^{(1)}$ with $n = 5$ drawn over the interval $(0, 1/(l^3q)]$ on the x-axis.

$$\text{If } 0 \le i < l: \quad (\mathfrak{h}^{(2)})^{-1}(G_{i,j,l^3q}) = G_{i',j',l^3q}, \quad \text{where } i' = \left\lfloor \frac{j}{r} \right\rfloor,\ j' = r \cdot i + j \bmod r, \tag{5.18}$$

$$\text{if } l \le i < l^2: \quad (\mathfrak{h}^{(2)})^{-1}(G_{i,tl+j,l^3q}) = G_{i',tl+j',l^3q}, \quad \text{where } i' = \left\lfloor \frac{i}{l} \right\rfloor \cdot l + j,\ j' = i \bmod l. \tag{5.19}$$

We extend this permutation to the whole of $\mathbb{T}^2$ equivariantly. Since the above description is not very clear, we give a somewhat imprecise but more demonstrative description of the above map. Note that the following rectangles get mapped in the following way:

$$(\mathfrak{h}^{(2)})^{-1}\left(\left[\frac{i}{l^3q}, \frac{i+1}{l^3q}\right) \times \left[0, 1\right)\right) = \left[0, \frac{1}{l^2q}\right) \times \left[\frac{i}{l}, \frac{i+1}{l}\right) \quad \text{if } 0 \le i < l,$$

$$(\mathfrak{h}^{(2)})^{-1}\left(\left[\frac{i}{l^3q}, \frac{i+1}{l^3q}\right) \times \left[\frac{t}{r}, \frac{t+1}{r}\right)\right) = \left[\frac{i'}{l^2q}, \frac{i'+1}{l^2q}\right) \times \left[\frac{tl+j'}{lr}, \frac{(t+1)l+j'+1}{lr}\right) \quad \text{if } l \le i < l^2,$$

where $i' = \lfloor i/l \rfloor$ and $j' = i \bmod l$. Notice that in the first region narrow rectangular stripes of full height get squished and are distributed over the full height of the torus, which will allow us to prove minimality. All other rectangles are mapped to rectangles of small diameter but they remain within the horizontal strip N_t on the torus. These stripes will form the support of a preimage of the invariant measures.

By the previous subsection, we know that this is a block-slide type of map and hence allows good analytic approximations by Proposition 2.22. We denote this $1/lq$-equivariant (ε, δ)-approximation by $h^{(2)}$, the corresponding 'bad set' by E and set $F = \mathbb{T}^2 \setminus E$.

For this number δ and given natural numbers q, l, n, we define the following step function (see Figure 7):

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$\tilde{\kappa}^{(1)} : \left[0, \frac{1}{l^3 q}\right) \to \mathbb{R}$ defined by

$$\tilde{\kappa}^{(1)}(x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{n^2 l^3 q}\right), \\ \frac{\delta}{lr} & \text{if } x \in \left[\frac{1}{n^2 l^3 q}, \frac{2}{n^2 l^3 q}\right), \\ 2\frac{\delta}{lr} & \text{if } x \in \left[\frac{1}{n^2 l^3 q}, \frac{2}{n^2 l^3 q}\right), \\ \dots & \dots \\ \dots & \dots \\ \left(\left\lfloor \frac{n^2}{2} \right\rfloor - 2\right)\frac{\delta}{lr} & \text{if } x \in \left[\frac{\lfloor n^2/2 \rfloor - 1}{n^2 l^3 q}, \frac{\lfloor n^2/2 \rfloor}{n^2 l^3 q}\right), \\ \left(\left\lfloor \frac{n^2}{2} \right\rfloor - 1\right)\frac{\delta}{lr} & \text{if } x \in \left[\frac{\lfloor n^2/2 \rfloor}{n^2 l^3 q}, \frac{\lfloor n^2/2 \rfloor + 1}{n^2 l^3 q}\right), \\ \left(\left\lfloor \frac{n^2}{2} \right\rfloor - 2\right)\frac{\delta}{lr} & \text{if } x \in \left[\frac{\lfloor n^2/2 \rfloor + 1}{n^2 l^3 q}, \frac{\lfloor n^2/2 \rfloor + 2}{n^2 l^3 q}\right), \\ \dots & \dots \\ \dots & \dots \\ \frac{\delta}{lr} & \text{if } x \in \left[\frac{n^2 - 2}{n^2 l^3 q}, \frac{n^2 - 1}{n^2 l^3 q}\right), \\ 0 & \text{if } x \in \left[\frac{n^2 - 1}{n^2 l^3 q}, \frac{1}{l^3 q}\right). \end{cases} \tag{5.20}$$

Let $\kappa^{(1)}$ be the $1/l^3 q$-periodic real-analytic $(\tilde{\varepsilon}, \tilde{\delta})$-approximation of $\tilde{\kappa}^{(1)}$. With the aid of this, we define

$$h^{(1)} : \mathbb{T}^2 \to \mathbb{T}^2 \quad \text{defined by } h^{(1)}(x_1, x_2) := (x_1, x_2 + \kappa^{(1)}(x_1)) \tag{5.21}$$

and we often refer to the above map as a 'trapping map'. The purpose of this map is to capture a large portion of *every* ϕ orbit.

Hereby we introduce the so-called 'trapping zones' (for $t = 0, \dots, r-1$ and $s = 0, \dots, lq - 1$) (see Figure 8):

$$A_{s,i} = h^{(1)}\left(\phi^{s/lq}\left(\bigcup_{j=0}^{lr-1} G_{i,j,l^3 q}\right) \cap F\right) \quad \text{if } 0 \le i < l, \tag{5.22}$$

$$B^t_{s,i} = h^{(1)}\left(\phi^{s/lq}\left(\bigcup_{j=0}^{l-1} G_{i,tl+j,l^3 q}\right) \cap F\right) \quad \text{if } l \le i < l^2. \tag{5.23}$$

In our specific constructions, we define $h_{n+1} = h_{1,n+1} \circ h_{2,n+1}$ using the parameters $q = q_n$, $l = l_n$, $\varepsilon < \varepsilon_n / 2^{l_n q_n}$, $\delta = \delta_n < 1/(n^4 \cdot 2^{l_n q_n})$, $\tilde{\varepsilon} = \tilde{\varepsilon}_n < \delta_n / 2^{l_n q_n}$ and $\tilde{\delta} = \tilde{\delta}_n < \delta_n / 2^{l_n q_n}$.

As announced, we have the following trapping property.

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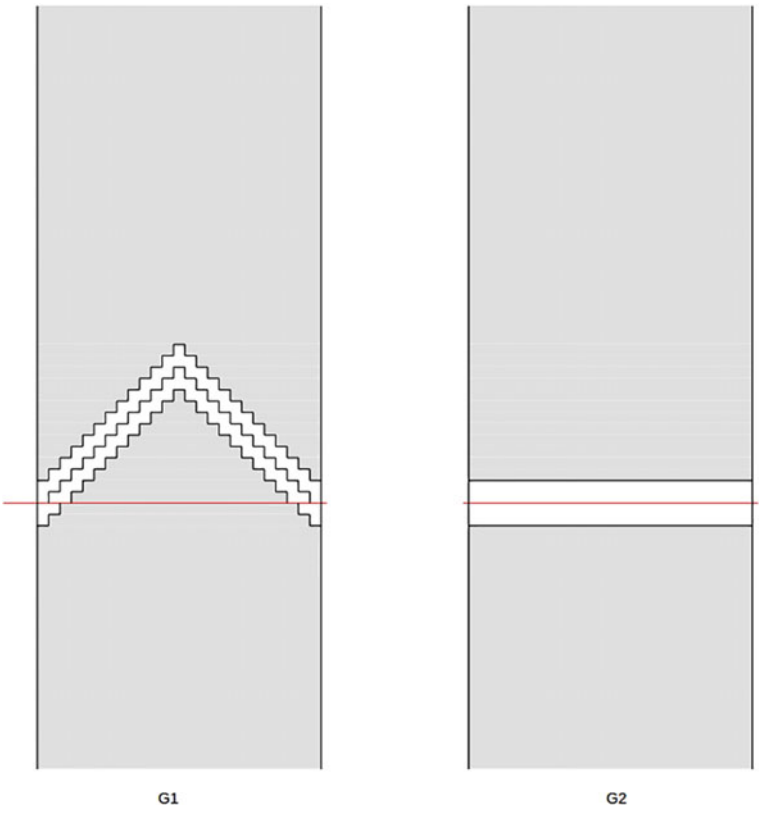


FIGURE 8. Illustration of the action of $(\mathfrak{h}^{(1)})^{-1}$. Note that it maps the partition with A-shaped top and bottom to a rectangle. The combinatorics depicted here is repeated with periodicity $1/(l^3q)$. The A-shaped atoms are able to capture portions of every orbit as depicted in the picture.

LEMMA 5.24. *Let $x \in \mathbb{T}^2$ be arbitrary. Then the orbit $\{\phi^{k\alpha_{n+1}}(x)\}_{k=0,\dots,q_{n+1}}$ meets every set $A_{s,i}$. Moreover, for every $B^t_{s,i}$, at least $\omega^n_t(x) \cdot ((1-8/n^2) \cdot q_{n+1})/l^3_n q_n$ iterates of the orbit $\{\phi^{k\alpha_{n+1}}(x)\}_{k=0,\dots,q_{n+1}}$ lie in $B^t_{s,i}$, where $\omega^n_t(x)$ does not depend on s, i. On the contrary, at most $10/n^2 \cdot q_{n+1}$ iterates are not captured by the collection of sets $B^t_{s,i}$.*

Proof. Let $x = (x_1, x_2) \in \mathbb{T}^2$ and $i \in \{0, \dots, l^3q - 1\}$ be arbitrary. Note that

$$G_{i,j,l^3q} \cap F \supseteq \left[\frac{i+\delta/2}{l^3q}, \frac{i+1-\delta/2}{l^3q}\right] \times \left[\frac{j+\delta/2}{lr}, \frac{j+1-\delta/2}{lr}\right] \tag{5.25}$$

by our approximation (Proposition 2.22). Due to $(n^2 \cdot \delta_n)/l_n r < 1/n^2 l_n r$ and our choice of the approximative step function $\kappa^{(1)}$, there are at most four sections $[i/l^3q + (u+\tilde{\delta}/2)/n^2l^3q, i/l^3q + (u+1-\tilde{\delta}/2)/n^2l^3q]$, where $u \in \{1, \dots, n^2-2\}$, on an arbitrary $[(i+\delta/2)/l^3q, (i+1-\delta/2)/l^3q]$-section such that x_2 does not belong to any of the $h^{(1)}((G_{i,j,l^3q}) \cap F)$-domains for $j = 0, \dots, lr-1$. Since $\{k \cdot \alpha_{n+1}\}_{k=0,\dots,q_{n+1}-1}$ is equidistributed on $\mathbb{S}^1$, the number of iterates k, such that $\{\phi^{k\alpha_{n+1}}(x)\}_{k=0,\dots,q_{n+1}-1}$ is

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captured by one of these domains, is at least

$$\left\lfloor \frac{(1-6/n^2)\cdot(1-\tilde{\delta}_n)\cdot q_{n+1}}{l_n^3 q_n} \right\rfloor \geq \frac{(1-8/n^2)\cdot q_{n+1}}{l_n^3 q_n}. \tag{5.26}$$

Depending on the point $x \in \mathbb{T}^2$, there is a portion $\omega_t^n(x)$ of these iterates spent in trapping regions $B_{s,i}^t$ belonging to N_t. This portion does not depend on the indices s, i. Since there are $l_n q_n \cdot (l_n^2 - l_n)$ such indices, the last claim follows. □

5.3. *Proof of minimality.*

5.3.1. *Criterion for minimality.* We recall the notion of a minimal dynamical system.

Definition 5.27. Let X be a topological space and $f : X \to X$ be a continuous transformation. The map f is called minimal if for every $x \in X$, the orbit $\{f^i(x)\}_{i\in\mathbb{N}}$ is dense in X.

Equivalently, f is minimal if for every $x \in X$ and every non-empty open set $U \subseteq X$, there is $i \in \mathbb{N}$ such that $f^i(x) \in U$. In the case of X being a metric space, every open set contains a γ-ball for γ sufficiently small. Thus, f is minimal if for every $x \in X$, every $\gamma > 0$ and for every γ-ball B_γ there is $i \in \mathbb{N}$ such that $f^i(x) \in B_\gamma$. Hereby, we can deduce the subsequent criterion of minimality in the setting of our constructions.

LEMMA 5.28. *Suppose that the set of iterates* $\{h_{n+1}^{-1} \circ \phi^{i\cdot\alpha_{n+1}} \circ H_{n+1}(x)\}_{i=0,\dots,q_{n+1}-1}$ *meets every set of the form* $[j_1/l_n q_n, (j_1+1)/l_n q_n] \times [j_2/l_n, (j_2+1)/l_n]$ *for every* $x \in \mathbb{T}^2$. *Moreover, we assume that the sequence* $(T_n)_{n\in\mathbb{N}}$ *constructed as in §5.2 converges to a diffeomorphism* T *in the* $\mathrm{Diff}_\rho^\omega$*-topology and satisfies* $d_0(T^i, T_{n+1}^i) < 1/2^n$ *for all* $i = 0, \dots, q_{n+1} - 1$. *Then* $T = \lim_{n\to\infty} T_n$ *is minimal.*

Proof. At first we observe that

$$\operatorname{diam}\left(H_n^{-1}\left(\left[\frac{j_1}{l_n q_n}, \frac{j_1+1}{l_n q_n}\right] \times \left[\frac{j_2}{l_n}, \frac{j_2+1}{l_n}\right]\right)\right) \leq \|DH_n^{-1}\|_0 \cdot \frac{2}{l_n},$$

which converges to 0 as $n \to \infty$ (because of $\|DH_n^{-1}\|_0 < l_n/2^n$ by equation (2.30)), and that the family of sets $[j_1/l_n q_n, (j_1+1)/l_n q_n] \times [j_2/l_n, (j_2+1)/l_n]$ covers the whole space $\mathbb{T}^2$. Hence, for every $\varepsilon > 0$ and $y \in \mathbb{T}^m$, there is $M_1 \in \mathbb{N}$ such that for every $n \geq M_1$ there exists a set $H_n^{-1}([j_1/l_n q_n, (j_1+1)/l_n q_n] \times [j_2/l_n, (j_2+1)/l_n]) \subseteq B_{\varepsilon/2}(y)$.

Let $x \in \mathbb{T}^2$, $\varepsilon > 0$ and an ε-ball $B_\varepsilon(y)$, at which $y \in \mathbb{T}^2$, be arbitrary. Since $d_0(T^i, T_{n+1}^i) < 1/2^n$ for all $i = 0, \dots, q_{n+1} - 1$, there is $M_2 \in \mathbb{N}$ such that $d_0(T^i, T_{n+1}^i) < \varepsilon/2$ for all $i = 0, \dots, q_{n+1} - 1$ and $n \geq M_2$.

We consider $n \geq \tilde{N} := \max\{M_1, M_2\}$. Then there are a set

$$H_n^{-1}\left(\left[\frac{j_1}{l_n q_n}, \frac{j_1+1}{l_n q_n}\right] \times \left[\frac{j_2}{l_n}, \frac{j_2+1}{l_n}\right]\right) \subseteq B_{\varepsilon/2}(y)$$

and, by assumption, an $i < q_{n+1}$ such that $T_{n+1}^i(x) \in H_n^{-1}([j_1/l_n q_n, (j_1+1)/l_n q_n] \times [j_2/l_n, (j_2+1)/l_n]) \subseteq B_{\varepsilon/2}(y)$. By the triangle inequality, we obtain

$$d(T^i(x), y) \leq d(T^i(x), T_{n+1}^i(x)) + d(T_{n+1}^i(x), y) \leq d_0(T^i, T_{n+1}^i) + \frac{\varepsilon}{2} < \varepsilon.$$

Thus, we conclude that $T^i(x) \in B_\varepsilon(y)$. Hence, T is minimal. □

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5.3.2. *Application of the criterion.* The conditions on the convergence of the sequence $(T_n)_{n\in\mathbb{N}}$ and proximity $d_0(T^i, T^i_{n+1}) < 1/2^n$ for all $i = 0, \dots, q_{n+1} - 1$ are fulfilled by Remark 1. Let $x \in \mathbb{T}^2$ and $[j_1/l_nq_n, (j_1+1)/l_nq_n] \times [j_2/l_n, (j_2+1)/l_n]$ be arbitrary. We have to show that the orbit $\{h_{n+1}^{-1} \circ \phi^{i\alpha_{n+1}} \circ H_{n+1}(x)\}_{i=0,\dots,q_{n+1}-1}$ meets $[j_1/l_nq_n, (j_1+1)/l_nq_n] \times [j_2/l_n, (j_2+1)/l_n]$. For this purpose, we note that there is $i \in \{0, \dots, q_{n+1} - 1\}$ with $\phi^{i\alpha_{n+1}} \circ H_{n+1}(x) \in A_{j_1,j_2}$ by Lemma 5.24. Then we compute

$$\begin{aligned} h_{n+1}^{-1}(A_{j_1,j_2}) &= h_{2,n+1}^{-1}\left(\phi^{j_1/l_nq_n}\left(\bigcup_{j=0}^{l_nr-1} G_{j_2,j,l_n^3q_n}\right) \cap F_n\right) \\ &= \phi^{j_1/l_nq_n} \circ h_{2,n+1}^{-1}\left(\bigcup_{j=0}^{l_nr-1} G_{j_2,j,l_n^3q_n} \cap F_n\right) \\ &\subset \phi^{j_1/l_nq_n}\left(\left[0, \frac{1}{l_n^2q_n}\right] \times \left[\frac{j_2}{l_n}, \frac{j_2+1}{l_n}\right]\right) \\ &\subset \left[\frac{j_1}{l_nq_n}, \frac{j_1+1}{l_nq_n}\right] \times \left[\frac{j_2}{l_n}, \frac{j_2+1}{l_n}\right] \end{aligned}$$

and we can apply Lemma 5.28 to prove the minimality of T.

5.4. *The ergodic invariant measures.*

5.4.1. *Construction of the measures.* As announced, we will construct the ergodic invariant measures with the aid of the normalized restrictions μ_t of the Lebesgue measure on the sets N_t, i.e. $\mu_t(A) = \mu(A \cap N_t)/\mu(N_t)$ for any measurable set $A \subseteq \mathbb{T}^2$. Since each set N_t is ϕ^β-invariant for any $\beta \in \mathbb{S}^1$, we have $(\phi^\beta)_*\mu_t = \mu_t$. With these, we define the measures $\xi_t^n := (H_n^{-1})_*\mu_t$ and can prove their T_n-invariance:

$$\begin{aligned} (T_n)_*\xi_t^n &= (T_n)_*((H_n^{-1})_*\mu_t) = (T_n \circ H_n^{-1})_*\mu_t \\ &= (H_n^{-1} \circ \phi^{\alpha_{n+1}})_*\mu_t = (H_n^{-1})_*(\phi^{\alpha_{n+1}})_*\mu_t = \xi_t^n. \end{aligned}$$

Here we used the relation $f_*g_*\mu = (f \circ g)_*\mu$ for maps f, g. This holds because we have for any measurable set A,

$$f_*g_*\mu(A) = g_*\mu(f^{-1}(A)) = \mu(g^{-1}(f^{-1}(A))) = \mu((f \circ g)^{-1}(A)) = (f \circ g)_*\mu(A).$$

In the next step we want to estimate $\mu(H_{n+1}^{-1}(N_t) \triangle H_n^{-1}(N_t))$. For this purpose, we have to examine which parts of the set N_t are not mapped back to N_t under $h_{n+1}^{-1} = h_{2,n+1}^{-1} \circ h_{1,n+1}^{-1}$. The measure difference is composed of our error set $E_{n+1} \cap N_t$, the part where the conjugation map h_{n+1}^{-1} is constructed to prove minimality (i.e. on the l_nq_n sets $[k/l_nq_n, k/l_nq_n + 1/(l_n^2 \cdot q_n)] \times \mathbb{T}$ for $k = 0, \dots, l_nq_n - 1$) and the part that is not mapped back to N_t under $h_{1,n+1}^{-1}$. The last one is caused by the translation about at most $n^2\delta_n/2l_n$ in the x_2-coordinate produced by $h_{1,n+1}^{-1}$. Altogether, we obtain

$$\mu(H_{n+1}^{-1}(N_t) \triangle H_n^{-1}(N_t)) = \mu(h_{n+1}^{-1}(N_t) \triangle N_t) \le \frac{1}{l_n} + \frac{n^2\delta_n}{l_n} + \mu(E_{n+1}) \le \frac{1}{l_n}. \quad (5.29)$$

Now we can use the same approach as in [**Wi01**, Ch. 7].

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By equation (5.29), the sequence $\{H_n^{-1}(N_t)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the metric on the associated measure algebra. Since this space is complete (e.g. [**Pe83**, Proposition 1.4.3]), there exists a limit $B_t := \lim_{n\to\infty} H_n^{-1}(N_t)$ in the measure algebra. For this limit we have $\mu(B_t) = \mu(N_t)$, because H_n^{-1} is measure preserving. The sets B_t and B_s are measurably disjoint due to the disjointness of the sets N_t and N_s. Moreover, we have weak convergence of the measures $(\xi_t^n)_{n\in\mathbb{N}}$ to a measure ξ_t, where $\xi_t(A) = \mu(A \cap B_t)/\mu(B_t)$ for any measurable set $A \subseteq \mathbb{T}^2$. For this absolutely continuous measure ξ_t, we conclude that $\lim_{n\to\infty}(T_n)_*\xi_t^n(A) = T_*\xi_t(A)$ due to the triangle inequality

$$\begin{aligned}\mu(H_n(T_n^{-1}A) \cap N_t) &= \mu(T_n^{-1}A \cap H_n^{-1}(N_t)) \le \mu(T_n^{-1}A \cap B_t) + \mu(H_n^{-1}(N_t)\triangle B_t)\\ &\le \mu(T_n^{-1}A\triangle T^{-1}A) + \mu(T^{-1}A \cap B_t) + \mu(H_n^{-1}(N_t)\triangle B_t)\end{aligned}$$

(where the first summand converges to 0 as $n \to \infty$ because of $T_n \to T$). So, we obtain

$$\xi_t = \lim_{n\to\infty} \xi_t^n = \lim_{n\to\infty} (T_n)_*\xi_t^n = T_*\xi_t$$

using the shown T_n-invariance of the measure ξ_t^n. Thus, the measures ξ_t are T-invariant.

Furthermore, these measures ξ_t are linearly independent because the sets $B_1, \dots, B_d$ are measurably disjoint as noted before. Since any non-ergodic invariant measure can be written as a linear combination of ergodic measures [**Wa75**, Theorem 5.15], there cannot be fewer than d ergodic measures.

5.4.2. *Estimates on Birkhoff sums.* In this subsubsection, we show that the measures ξ_t are the only possible ergodic measures for T. For this purpose, we will prove a result on the Birkhoff sums (see Lemma 5.32) and have to gain control over almost everything of every ϕ-orbit. In this connection, the following sets are useful: in case of $0 \le s < l_n q_n$, $0 < j_1 < l_n$ and $0 \le j_2 < l_n$ we introduce

$$\Delta^t_{s,j_1,j_2} = \left[\frac{s}{l_n q_n} + \frac{j_1}{l_n^2 \cdot q_n}, \frac{s}{l_n q_n} + \frac{j_1+1}{l_n^2 \cdot q_n}\right] \times \left[\frac{t}{r} + \frac{j_2}{l_n r}, \frac{t}{r} + \frac{j_2+1}{l_n r}\right].$$

Note that there are $l_n^3 q_n \cdot (1 - 1/l_n)$ such sets Δ^t_{s,j_1,j_2} on N_t. We denote the family of these sets by Ω_n^t as well as the union of these sets by $\tilde{\Omega}_n^t$. Then $\mu(N_t \setminus \tilde{\Omega}_n^t) = 1/l_n r$, i.e. $\mu_t(N_t \setminus \tilde{\Omega}_n^t) = 1/l_n$.

We observe that $\operatorname{diam}(H_n^{-1}(\Delta^t_{s,j_1,j_2})) < \|DH_n^{-1}\|_0 \cdot 1/l_n$. By the requirements on the number l_n in equation (5.1), we obtain

$$|\rho_i(H_n^{-1}(x)) - \rho_i(H_n^{-1}(y))| \le \operatorname{Lip}(\rho_i) \cdot \operatorname{diam}(H_n^{-1}(\Delta^t_{s,j_1,j_2})) < \frac{1}{n^2}$$

for every $x, y \in \Delta^t_{s,j_1,j_2}$ and the function $\rho_i \in \Xi$ in case of $i = 1, \dots, n$. Averaging over all $y \in \Delta^t_{s,j_1,j_2}$, we obtain

$$\left|\rho_i(H_n^{-1}(x)) - \frac{1}{\xi_t^n(H_n^{-1}(\Delta^t_{s,j_1,j_2}))} \int_{H_n^{-1}(\Delta^t_{s,j_1,j_2})} \rho_i \, d\xi_t^n\right| < \frac{1}{n^2}. \tag{5.30}$$

Furthermore, we recall that the image of the trapping region $B^t_{s,i}$ under h_{n+1}^{-1} is contained in $\Delta^t_{s,\lfloor i/l_n\rfloor, i \bmod l_n}$. Vice versa, $B^t_{s,j_1\cdot l_n+j_2}$ is the unique trapping region that is mapped into Δ^t_{s,j_1,j_2}. Hence, we can estimate the number of $i \in \{0, \dots, q_{n+1} - 1\}$ such that $h_{n+1}^{-1} \circ \phi^{i\cdot\alpha_{n+1}}(x)$ is contained in Δ^t_{s,j_1,j_2} by $\varpi_t^n(x) \cdot ((1 - 8/n^2) \cdot q_{n+1})/l_n^3 q_n$ for arbitrary $x \in \mathbb{T}^2$ using Lemma 5.24.

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LEMMA 5.31. *Let $\rho_i \in \Xi$ and $i = 1, \ldots, n$. Then, for every $y \in \mathbb{T}^2$, we have*

$$\inf_{\xi^n \in \Theta_n} \left| \frac{1}{q_{n+1}} \sum_{k=0}^{q_{n+1}-1} \rho_i(T_{n+1}^k y) - \int \rho_i \, d\xi^n \right| < \frac{20}{n^2} \cdot \|\rho_i\|_0 + \frac{1}{n^2},$$

where Θ_n is the simplex generated by $\{\xi_0^n, \ldots, \xi_{r-1}^n\}$.

Proof. Let $x \in \mathbb{T}^2$ be arbitrary. We introduce the measure $\xi_x^n := \sum_{t=0}^{r-1} \varpi_t^n(x) \cdot \xi_t^n \in \Theta_n$.

The set of numbers $k \in \{0, 1, \ldots, q_{n+1} - 1\}$ such that the iterates $\phi^{k \cdot \alpha_{n+1}}(x)$ are not contained in one of the trapping regions of the second kind is denoted by I_a. Referring to Lemma 5.24, there are at most $10/n^2 \cdot q_{n+1}$ numbers in I_a. We obtain $|\sum_{k \in I_a} \rho_i(H_{n+1}^{-1} \circ \phi^{k \cdot \alpha_{n+1}}(x))| \le \|\rho_i\|_0 \cdot 10/n^2 \cdot q_{n+1}$.

Moreover, we denote the set of $k \in \{0, 1, \ldots, q_{n+1} - 1\}$ such that the iterate $h_{n+1}^{-1} \circ \phi^{k\alpha_{n+1}}(x)$ is contained in the corresponding trapping region $\Delta \in \Omega_n^t$ by I_Δ. By the above considerations, there are at least $\varpi_t^n(x) \cdot ((1 - 8/n^2) \cdot q_{n+1})/l_n^3 q_n = \varpi_t^n(x) \cdot q_{n+1} \cdot (1 - 8/n^2) \cdot \mu_t(\Delta)$ and at most $\varpi_t^n(x) \cdot q_{n+1} \cdot \mu_t(\Delta)$ many numbers in I_Δ for an arbitrary $\Delta \in \Omega_n^t$. Thus, we obtain for an arbitrary $\Delta \in \Omega_n^t$ using equation (5.30),

$$\left| \frac{1}{q_{n+1}} \sum_{j \in I_\Delta} \rho_i(H_{n+1}^{-1} \circ \phi^{j\alpha_{n+1}}(x)) - \int_{H_n^{-1}(\Delta)} \rho_i \, d(\varpi_t^n(x)\xi_t^n) \right| \le \frac{(\varpi_t^n(x)\mu_t)(\Delta)}{n^2}$$
$$+ \frac{8}{n^2} \cdot \int_{H_n^{-1}(\Delta)} |\rho_i| \, d(\varpi_t^n(x)\xi_t^n) \le (\varpi_t^n(x)\mu_t)(\Delta) \cdot \left(\frac{1}{n^2} + \frac{8}{n^2} \cdot \|\rho_i\|_0 \right).$$

Altogether, we conclude that

$$\left| \frac{1}{q_{n+1}} \sum_{k=0}^{q_{n+1}-1} \rho_i(H_{n+1}^{-1} \circ \phi^{k\alpha_{n+1}} x) - \int \rho_i \, d\xi_x^n \right| = \left| \frac{1}{q_{n+1}} \sum_{k=0}^{q_{n+1}-1} \rho_i(H_{n+1}^{-1} \circ \phi^{k\alpha_{n+1}}(x)) \right.$$
$$\left. - \sum_{t=0}^{r-1} \left(\sum_{\Delta \in \Omega_n^t} \int_{H_n^{-1}(\Delta)} \rho_i \, d(\varpi_t^n(x)\xi_t^n) + \int_{H_n^{-1}(N_t \setminus \tilde{\Omega}_n^t)} \rho_i \, d(\varpi_t^n(x)\xi_t^n) \right) \right|$$
$$\le \left| \sum_{t=0}^{r-1} \sum_{\Delta \in \Omega_n^t} \left(\frac{1}{q_{n+1}} \sum_{j \in I_\Delta} \rho_i(H_{n+1}^{-1} \circ \phi^{j\alpha_{n+1}}(x)) - \int_{H_n^{-1}(\Delta)} \rho_i \, d(\varpi_t^n(x)\xi_t^n) \right) \right|$$
$$+ \frac{1}{q_{n+1}} \cdot \|\rho_i\|_0 \cdot \frac{10}{n^2} \cdot q_{n+1} + \|\rho_i\|_0 \cdot \sum_{t=0}^{r-1} (\varpi_t^n(x)\mu_t)(N_t \setminus \tilde{\Omega}_n^t)$$
$$\le \frac{1}{n^2} + \frac{8}{n^2} \cdot \|\rho_i\|_0 + \|\rho_i\|_0 \cdot \frac{10}{n^2} + \frac{2 \cdot \|\rho_i\|_0}{l_n} = \frac{20}{n^2} \cdot \|\rho_i\|_0 + \frac{1}{n^2}.$$

With $x = H_{n+1}(y)$, we obtain the claim. □

We point out that the measure ξ_x^n used in the above proof was dependent on the point x, but independent of the function $\rho \in \Xi$.

LEMMA 5.32. *For every $\rho \in \Xi$ and $y \in \mathbb{T}^2$, we have*

$$\inf_{\xi^n \in \Theta_n} \left| \frac{1}{q_{n+1}} \sum_{k=0}^{q_{n+1}-1} \rho(T^k(y)) - \int \rho \, d\xi^n \right| \to 0 \quad \text{as } n \to \infty,$$

where Θ_n is the simplex generated by $\{\xi_0^n, \ldots, \xi_{r-1}^n\}$.

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Proof. By Remark 1, we have

$$d_0^{(q_{n+1})}(T, T_{n+1}) := \max_{i=0,1,\dots,q_{n+1}-1} d_0(T^i, T_{n+1}^i) \overset{n\to\infty}{\to} 0.$$

Then, for every $\rho \in \Xi$, we have $|\rho(T^i(x)) - \rho(T_{n+1}^i(x))| \overset{n\to\infty}{\to} 0$ uniformly for $i = 0, 1, \dots, q_{n+1} - 1$, because every continuous function on the compact space $\mathbb{T}^2$ is uniformly continuous. Thus, we get

$$\left\| \frac{1}{q_{n+1}} \sum_{i=0}^{q_{n+1}-1} \rho(T^i(x)) - \frac{1}{q_{n+1}} \sum_{i=0}^{q_{n+1}-1} \rho(T_n^i(x)) \right\|_0 \overset{n\to\infty}{\to} 0.$$

Applying the previous Lemma 5.31, we obtain the claim. □

Since the family Ξ is dense in $C\left(\mathbb{T}^2, \mathbb{R}\right)$, the convergence holds for every continuous function by an approximation argument.

Now we can prove that the measures $\xi_0, \dots, \xi_{r-1}$ are the only possible ergodic ones: assume that there is another ergodic invariant probability measure ξ. By the Birkhoff ergodic theorem, we have for every $\rho \in C(\mathbb{T}^2, \mathbb{R})$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \rho(T^k(x)) = \int_{\mathbb{T}^2} \rho \, d\xi \quad \text{for } \xi\text{-almost every} x \in \mathbb{T}^2.$$

With the aid of Lemma 5.32, we obtain for every $\rho \in C(\mathbb{T}^2, \mathbb{R})$ and x in a set of ξ-full measure,

$$\int_{\mathbb{T}^2} \rho \, d\xi = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \rho(T^k(x)) = \lim_{n\to\infty} \frac{1}{q_{n+1}} \sum_{k=0}^{q_{n+1}-1} \rho(T^k(x)) = \lim_{n\to\infty} \int_{\mathbb{T}^2} \rho \, d\xi^n,$$

where ξ^n is in the simplex generated by $\{\xi_0^n, \dots, \xi_{r-1}^n\}$. As noted, this measure does not depend on the function ρ. Thus, we have for every $\rho \in C(\mathbb{T}^2, \mathbb{R}) : \lim_{n\to\infty} \int_{\mathbb{T}^2} \rho \, d\xi^n = \int_{\mathbb{T}^2} \rho \, d\xi$. Since the simplex generated by $\{\xi_0, \dots, \xi_{r-1}\}$ is weakly closed, this implies that ξ is in this simplex. We recall that ergodic measures are the extreme points in the set of invariant Borel probability measures (see [**Wa75**, Theorem 5.15]). Then ξ has to be one of the measures $\{\xi_0, \dots, \xi_{r-1}\}$ and we obtain a contradiction. Hence, the measures ξ_t, $t = 0, \dots, r - 1$, are the only possible ergodic ones. Since we have already observed that these are linearly independent and any non-ergodic invariant measure can be written as a linear combination of ergodic ones, we conclude that the ξ_t are exactly the ergodic measures of T.

5.5. *Possible generalizations.*

Remark 2. By putting the combinatorics from [**AK70**, §5] on the rectangles $[i/l^3q, (i+1)/l^3q) \times [t/r, (t+1)/r)$ that are not contained in the minimality region, we can construct the diffeomorphism T to be even weakly mixing with respect to each measure ξ_t. We can realize these combinatorics with the aid of Theorem E.

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Remark 3. With some additional technical and notational effort it is possible to generalize Theorem D and the previous remark to any torus $\mathbb{T}^d$, $d \geq 2$. Indeed, we construct the r ergodic invariant measures on $N_t = \mathbb{S}^1 \times [t/r, (t+1)/r) \times \mathbb{T}^{d-2}$ and consider partition elements

$$\left[\frac{i_1}{l^{d+1}q}, \frac{i_1+1}{l^{d+1}q}\right) \times \left[\frac{i_2}{lr}, \frac{i_2+1}{lr}\right) \times \left[\frac{i_3}{l}, \frac{i_3+1}{l}\right) \times \cdots \times \left[\frac{i_d}{l}, \frac{i_d+1}{l}\right) \tag{5.33}$$

as building blocks in the description of the combinatorics. Then we use the combinatorics from §4.1 to map sets of the form $[i/l^{d+1}q, (i+1)/l^{d+1}q) \times [j/lr, (j+1)/lr) \times [0,1)^{d-2}$ to sets of the form $[i_1/l^3q, (i_1+1)/l^3q) \times [j/lr, (j+1)/lr) \times [i_3/l, (i_3+1)/l) \times \cdots \times [i_d/l, (i_d+1)/l)$.

Remark 4. In this remark, we present modifications in order to prove the existence of a real-analytic diffeomorphism $T \in \mathrm{Diff}^{\omega}_{\rho}(\mathbb{T}^2, \mu)$ which is minimal and has countable many ergodic invariant measures. By weak*-convergence there must be at least one singular ergodic measure. Indeed, we have precisely one singular measure and the other invariant measures are absolutely continuous with respect to Lebesgue measure.

This time we are going to construct the invariant measures on sets N_t, $t \in \mathbb{Z}$. For $t \in \mathbb{N}$, we define $N_t = \mathbb{S}^1 \times [t/(t+1), (t+1)/(t+2)) \subset \mathbb{T}^2$ with x_2-length $1/((t+2)\cdot(t+1))$. For each $n \in \mathbb{N}$, we choose $t_n \in \mathbb{N}$ such that

$$\frac{1}{(t_n+1)\cdot t_n} < \frac{\delta_{n-1}}{2^n \cdot \|DH_n^{-1}\|_0 \cdot \max_{i=1,\ldots,n} \mathrm{Lip}(\rho_i)}. \tag{5.34}$$

Hereby, we define the further sets

$$N_0 = \mathbb{S}^1 \times \left[\frac{1}{(t_1+1)\cdot t_1}, \frac{1}{2}\right),$$
$$N_{-n} = \mathbb{S}^1 \times \left[\frac{1}{(t_{n+1}+1)\cdot t_{n+1}}, \frac{1}{(t_n+1)\cdot t_n}\right) \quad \text{for every } n \in \mathbb{N}.$$

Additionally, for every $n \in \mathbb{N}$, we will use sets

$$\bar{N}_{t_n} = \mathbb{S}^1 \times \left[\frac{t_n}{t_n+1}, 1\right) = \bigcup_{t \geq t_n} N_t,$$
$$N^{(1)}_{-n} = \mathbb{S}^1 \times \left[0, \frac{1}{(t_n+1)\cdot t_n}\right).$$

This time we choose $l_n = (t_n+1)!$. Note that by equation (5.34), the conditions (2.30) and (5.1) are satisfied. Moreover, this choice of l_n allows us to consider building blocks $[i/l_n^3 q_n, (i+1)/l_n^3 q_n) \times [j/l_n, (j+1)/l_n)$ for the $1/l_n q_n$-equivariant combinatorics of $h^{-1}_{2,n+1}$ to be contained in $\bar{N}_{t_n}$, $N^{(1)}_{-n}$ and N_t for $-n < t < t_n$. As before, $h^{-1}_{2,n+1}$ is supposed to map long stripes $[i/l_n^3 q_n, (i+1)/l_n^3 q_n) \times [0,1)$ to $[0, 1/l_n^2 q_n) \times [i/l_n, (i+1)/l_n)$ for $0 \leq i < l_n$. For $l_n \leq i < l_n^2$, $h^{-1}_{2,n+1}$ maps stripes of width $1/l_n^3 q_n$ and full height in the particular set $N^{(1)}_{-n}$ or N_t for $-n < t < t_n$ to sets with height $1/l_n$, while on $\bar{N}_{t_n}$, $h^{-1}_{2,n+1}$ acts approximately as the identity on the building blocks. For the trapping map $h^{(1)}$, the step function is constructed with steps of size δ/l_n in our modification.

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6. *Remarks on future work and difficulties*

We note that Theorem E can be used to upgrade many constructions from the smooth category to the analytic category on the torus. We list some of those results in §§6.1 and 6.2. Full details will appear in forthcoming papers. We also make some comments in §6.3 on the potential of real-analytic constructions on other manifolds and difficulties that come with it.

6.1. *Real-analytic diffeomorphisms with homogeneous spectrum and disjointness of convolutions.*

The second author in [**Ku16**] was able to show that on any smooth compact connected manifold M of dimension $d \geq 2$ admitting a smooth non-trivial circle action $\mathcal{S} = \{\phi^t\}_{t\in\mathbb{S}^1}$, there exists a smooth diffeomorphism $f \in A_\alpha = \{h \circ \phi^\alpha \circ h^{-1} : h \in \mathrm{Diff}^\infty(M, \mu)\}$ for every Liouvillean number α which admits a good approximation of type $(h, h+1)$, a maximal spectral type disjoint with its convolutions and a homogeneous spectrum of multiplicity 2 for the Cartesian square $f \times f$. It is possible to generalize this result to the real-analytic category for some Liouvillean numbers.

THEOREM 6.1. *For any $\rho > 0$, there exist real-analytic diffeomorphisms $T \in \mathrm{Diff}^\omega_\rho(\mathbb{T}^2, \mu)$ that have a maximal spectral type disjoint with their convolutions, a homogeneous spectrum of multiplicity 2 for $T \times T$ and admit a good approximation of type $(h, h+1)$.*

6.2. *Coding untwisted AbC diffeomorphisms and the anti-classification problem.*

This work is motivated from a series of pioneering works done by Belezney, Foreman, Hjorth, Rudolph and Weiss on the interface of ergodic theory and foundations of mathematics. They were able to show that the conjugacy problem in abstract ergodic theory is non-Borel. Later, Foreman and Weiss found a method to code a 'large' class of smooth diffeomorphisms constructed on $\mathbb{T}^2$ or the annulus or the disc by an untwisted version of the AbC method into some symbolic systems known as *uniform circular systems*. This in particular shows that the measure isomorphism relation among pairs (S, T) of measure-preserving diffeomorphisms of M is not a Borel set with respect to the C^∞ topology.

The first author was able to show that the constructions we do in the real-analytic category on $\mathbb{T}^2$ are robust enough to construct a large family of untwisted AbC diffeomorphisms measure-theoretically isomorphic to uniform circular systems. Loosely this can be summarized into the following theorem.

THEOREM 6.2. [**Ba**] *Let T be an ergodic transformation on a standard measure space. Then the following are equivalent.*

(1) *T is measure-theoretically isomorphic to a real-analytic (untwisted) AbC diffeomorphism (satisfying some requirements).*
(2) *T is isomorphic to a uniform circular system (with 'fast'-growing parameters).*

This along with some additional works of Foreman and Weiss would imply an anti-classification result for measure-preserving real-analytic diffeomorphisms. More precisely, we have the following result.

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THEOREM 6.3. *The measure-isomorphism relation among pairs* $(S, T) \in \mathrm{Diff}^{\omega}_{\rho}(\mathbb{T}^2, \mu) \times \mathrm{Diff}^{\omega}_{\rho}(\mathbb{T}^2, \mu)$ *is not a Borel set with respect to the* $\mathrm{Diff}^{\omega}_{\rho}(\mathbb{T}^2, \mu)$ *topology.*

6.3. *Real-analytic constructions on other manifolds.* Constructing real-analytic dynamical systems on arbitrary real-analytic manifolds using the AbC method continues to remain an intractable problem. Some of the difficulties associated with such constructions are well documented in [**FK04**, §7.1]. Here we discuss some of those difficulties along with some new ones.

On arbitrary real-analytic manifolds we are not aware of any method for constructing the conjugating diffeomorphisms h_n in a way so that both h_n and its inverse have 'enough analyticity'. We explain a little on what 'enough analyticity' means. The manifold M can be considered to be an embedded real-analytic submanifold of $\mathbb{R}^n \subset \mathbb{C}^n$ for some large enough n. Then any such diffeomorphism h_n has enough analyticity if h_n and its inverse can be extended to $\mathbb{C}^n$ as entire functions. For, otherwise, when one considers compositions of such maps ($T_n := h_1^{-1} \circ \cdots \circ h_n^{-1} \circ \phi^{\alpha_n} \circ h_n \circ \cdots \circ h_1$), the domain of analyticity will shrink and eventually we cannot guarantee convergence. In [**FK04**], 'enough analyticity' meant that the maps h_n and their inverses can be extended to some fixed complex domain in $\mathbb{C}^n$ containing M, but we emphasize that one needs the diffeomorphisms T_n to belong to some such fixed domain rather than h_n. This is a stronger assumption and we do not know how to do real-analytic constructions without such an assumption.

On certain 'nice' real-analytic manifolds, it seems that such constructions are possible. The key to such constructions are finite collections of flows on the manifold that have the 'transitivity' property. Informally, this means that one can travel from one point to another by riding these flows. The diffeomorphisms h_n are constructed by taking a composition of time-one-maps of time changes of these flows. If these flows had enough analyticity then the diffeomorphisms constructed out of them will have enough analyticity for the AbC construction. Such a construction was successfully carried out on odd-dimensional spheres in [**FK14**] proving existence of real-analytic uniquely ergodic diffeomorphisms.

So, it seems natural to understand what potentially could constitute a class of such 'nice' manifolds supporting existence of such flows. On manifolds where the $\mathbb{T}^1$-action is not free or if it has a boundary, for example in the case of a disc, one can consider the flow of the Hamiltonian $H(x, y) = xy(1 - x^2 - y^2)$ and observe that it has the so-called transitivity property but not enough analyticity. In fact in this case one can check that the h_n constructed do not even belong to any fixed complex domain containing the manifold. As a result we do not know if there exists a zero-entropy real-analytic ergodic diffeomorphism on the disc.

On general real-analytic principal $\mathbb{T}^1$-bundles one can construct flows and diffeomorphisms h_n and their inverses belonging to some fixed complex domain containing the manifold. But then again we cannot guarantee that the composition T_n and its inverse can be extended to some complex domain independent of n. In this case and also the previous one, one can construct vector fields with enough analyticity (in fact polynomial for the disc), but when one integrates such a vector field, one can guarantee that the resulting flows are real analytic but they are not analytic enough in the sense that

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they do not extend to the whole of $\mathbb{C}^n$. The problem here is created by the fact that we are integrating this vector field over a non-compact domain and Picard–Lindelof in general does not guarantee that the flow exists for a fixed time interval over the whole domain. This situation does not arise when the flows or their vector fields are in a certain sense 'affine' as in the case of the tori (translation) and the odd-dimensional spheres (linear). So, one needs some new ideas when working with manifolds where the flows are nonlinear.

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