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Some asymptotic formulas related to the Möbius function of higher order

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Abstract. In 1970, Apostol introduced the Möbius function of order k for an integer $k \geq 1$. In 2001, it was generalized by Bege to the Möbius function $\mu_{k,m}$ of two parameters for any integers $m \geq k \geq 1$. In this article, we will establish three kinds of asymptotic formulas related to $\mu_{k,m}$ for $m \geq k \geq 2$ in a unified elementary method. These formulas are related to the shifted convolution sums of Fourier coefficients of holomorphic cusp forms, Bergelson and Richter's dynamical generalization of the prime number theorem, and the Titchmarsh divisor problem.

1 Introduction and statement of results

Let $\mu(n)$ be the Möbius function defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } p^2 \mid n \text{ for some prime } p, \\ (-1)^r & \text{if } n = p_1 \dots p_r, \text{ where the } p_i \text{ are distinct primes.} \end{cases}$$

Let $k \geq 1$ be an integer. In 1970, Apostol [2] introduced the Möbius function $\mu_k(n)$ of order k, which vanishes if $p^{k+1} \mid n$ for some prime p and equals to $(-1)^r$ if n contains the kth powers of exactly r distinct primes. That is,

$$\mu_k(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } p^{k+1} \mid n \text{ for some prime } p, \\ (-1)^r & \text{if } n = p_1^k \dots p_r^k \prod_{i>r} p_i^{a_i}, \text{ with } 0 \leq a_i < k, \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, $\mu_1(n) = \mu(n)$. Let $m \geq k \geq 1$ be an integer. In [4], Bege generalized Apostol's $\mu_k(n)$ to $\mu_{k,m}(n)$ as follows:

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$$\mu_{k,m}(n)=\begin{cases}1 & \text{if } n=1,\\ 1 & \text{if } p^k\nmid n \text{ for each prime } p,\\ (-1)^r & \text{if } n=p_1^m\dots p_r^m\prod_{i>r}p_i^{\alpha_i},\ \text{with } 0\le\alpha_i<k,\\ 0 & \text{otherwise.}\end{cases}$$

Clearly, $\mu_{k,k}(n)=\mu_k(n)$ for all $k\ge 1$. Moreover, by [4, Lemma 2.1], Bege's generalization $\mu_{k,m}$ of the Möbius function satisfies the following relation to μ:

$$\mu_{k,m}(n)=\sum_{\substack{d^m\delta=n\\(d,\delta)=1}}\mu(d)q_k(\delta) \tag{1.1}$$

for any $m\ge k\ge 1$, where $q_k(n)$ is the characteristic function of the k-free integers:

$$q_k(n)=\begin{cases}0 & \text{if } p^k\mid n \text{ for some prime } p;\\ 1 & \text{if } p^\alpha\mid n \text{ implies } \alpha<k.\end{cases}$$

Using (1.1), Bege obtained the asymptotic formula for the partial sum of $\mu_{k,m}$

$$\sum_{n\le x}\mu_{k,m}(n)=A_{k,m}x+O\Big(x^{\frac{1}{k}}\exp\Big\{-c\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\Big\}\Big) \tag{1.2}$$

for some absolute constant $c>0$, where

$$A_{k,m}=\frac{1}{\zeta(k)}\prod_p\Big(1-\frac{1}{p^{m-k+1}+p^{m-k+2}+\cdots+p^m}\Big), \tag{1.3}$$

the product runs over all primes p, and $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ is the Riemann zeta function for $\mathrm{Re}\,s>1$. In this article, based on (1.1), we will establish three kinds of asymptotic formulas related to $\mu_{k,m}$ for $m\ge k\ge 2$, using a unified preliminary estimate in Proposition 3.1.

First, we consider the shifted convolution sums of Fourier coefficients of holomorphic cusp forms. Let H_κ denote the set of normalized primitive holomorphic cusp forms of even integral weight κ for the full modular group $\Gamma=SL(2,\mathbb{Z})$, and suppose $f\in H_\kappa$. Let $\lambda_f(n)$ be the nth normalized Fourier coefficient of $f(z)$. Let $a(n)$ be a function with squarefull kernel introduced by Ivić and Tenenbaum [13], which means that $a(n)=a(s)$ for $n=qs,(q,s)=1$, where q is squarefree and s is squarefull, and $a(n)\ll n^{\varepsilon}$. Let $\tau(n)=\sum_{d\mid n}1$ be the divisor function, and let $\omega(n)=\sum_{p\mid n}1$ be the number of distinct prime factors of n. In [9, equations (1.5) and (1.6)], Erdős and Ivić established the asymptotic formulas for the following shifted convolution sums on $\tau(n)$ and $\omega(n)$:

$$\sum_{n\le x}a(n)\tau(n+1)=C_1x\log x+C_2x+O(x^{\frac{8}{9}+\varepsilon}), \tag{1.4}$$

$$\sum_{n\le x}a(n)\omega(n+1)=D_1x\log\log x+D_2x+O\left(\frac{x}{\log x}\right), \tag{1.5}$$

where $C_1,D_1>0,C_2,D_2$ are constants that can be explicitly evaluated. Inspired by their work, in 2020, Lü and Wang [16, Theorem 1] obtained an asymptotic estimate for the shifted convolution sum of $a(n)$ with $\lambda_f^2(n+1)$: for any $\varepsilon>0$, we have

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$$\sum_{n\le x} a(n)\lambda_f^2(n+1) = Cx + O(x^{13/14+\varepsilon}), \tag{1.6}$$

where C is a constant that can be explicitly evaluated. Based on the work of Lü and Wang, we will establish an asymptotic formula for the shifted convolution sum of $\mu_{k,m}(n)$ with $\lambda_f^2(n+1)$ in our first main theorem. Moreover, for any integer $j \ge 2$, based on the results in [12, 22, 27], using the method in Theorem 1.1, one can establish asymptotic formulas for the shifted convolution sum of $\mu_{k,m}(n)$ with higher power moments $\lambda_f^{2j}(n+1)$ of Fourier coefficients as well.

Theorem 1.1 *Suppose $f \in H_\kappa$ and $m \ge k \ge 2$. Then for any $\varepsilon > 0$, we have*

$$\sum_{n\le x} \mu_{k,m}(n)\lambda_f^2(n+1) = A_{k,m,f}\cdot x + O\Big(x^{1-\frac{k-1}{4(m+k+1)}+\varepsilon}\Big), \tag{1.7}$$

where $A_{k,m,f}$ is a constant defined by (4.5).

Second, we consider a dynamical generalization of the prime number theorem (PNT) with respect to $\mu_{k,m}(n)$. For any $n \in \mathbb{N}$, let $\Omega(n)$ be the number of prime divisors of n counted with multiplicity. Let $\lambda(n) = (-1)^{\Omega(n)}$ be the Liouville function. In 2022, Bergelson and Richter [5, Theorem A] established a new dynamical generalization of the PNT, showing that given any uniquely ergodic topological dynamical system (X, ν, T),

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} g(T^{\Omega(n)}x_0) = \int_X g\,d\nu \tag{1.8}$$

holds for any $g \in C(X)$ and $x_0 \in X$, where $C(X)$ denotes the set of continuous functions on X. They also proved that (1.8) holds over squarefree numbers:

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu^2(n) g(T^{\Omega(n)}x_0) = \frac{6}{\pi^2}\cdot\int_X g\,d\nu \tag{1.9}$$

for any $g \in C(X)$ and $x_0 \in X$. This is another dynamical generalization of the PNT. In [15, Theorem 1.3], Loyd discovered that the sequences capturing the behaviors of the Erdős–Kac theorem and Bergelson–Richter's theorem are asymptotically uncorrelated, that is,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} F\Big(\frac{\Omega(n)-\log\log N}{\sqrt{\log\log N}}\Big) g(T^{\Omega(n)}x_0) = \Big(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(t)e^{-t^2/2}\,dt\Big)\Big(\int_X g\,d\nu\Big)$$

holds for any compactly supported continuous function F on $\mathbb{R}$, $g \in C(X)$ and $x_0 \in X$. In [24–26], the authors also established some analogs of (1.9) related to $\mu^2(n^2+n)$, $\mu^2(n^2+1)$, and $\Omega(n)-\omega(n)$, respectively.

Inspired by these works, we will establish an analog of (1.9) with respect to $\mu_{k,m}$ in our second main result. Similar to [24], we consider an arithmetic function $b : \mathbb{N} \to \mathbb{C}$ of invariant average under multiplications, which means that the average of $b(n)$ exists and satisfies

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$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}b(mn)=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}b(n)$$

for all $m\in\mathbb{N}$. For any $g\in C(X)$ and $x_0\in X$ in (1.8), the orbit $\{g(T^{\Omega(n)}x_0)\}_{n\in\mathbb{N}}$ is a typical example of a sequence of invariant average $\int_X g\,dv$ under multiplications. We refer readers to [24] for more interesting examples.

Theorem 1.2 *Let $b(n)$ be a bounded arithmetic function of invariant average β under multiplications. Then we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu_{k,m}(n)b(n)=A_{k,m}\cdot\beta \tag{1.10}$$

for $m\geq k\geq 2$, where $A_{k,m}$ is defined by (1.3).

Taking $b(n)=g(T^{\Omega(n)}x_0)$ in Theorem 1.2, we obtain the following analog of (1.9) with respect to $\mu_{k,m}$, immediately.

Corollary 1.3 *Let (X,v,T) be a uniquely ergodic topological dynamical system. Then we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu_{k,m}(n)g(T^{\Omega(n)}x_0)=A_{k,m}\cdot\int_X g\,dv \tag{1.11}$$

for any $m\geq k\geq 2$, $g\in C(X)$, and $x_0\in X$, where $A_{k,m}$ is defined by (1.3).

Third, we consider an analog of the Titchmarsh divisor problem with respect to $\mu_{k,m}$. In 1930, under the Generalized Riemann Hypothesis, Titchmarsh [21, Theorem 7] proved that

$$\sum_{p\leq x}\tau(p-a)=\frac{\zeta(2)\zeta(3)}{\zeta(6)}\prod_{p\mid a}\left(1-\frac{p}{p^2-p+1}\right)\cdot x+O\left(\frac{x\log\log x}{\log x}\right) \tag{1.12}$$

for any fixed nonzero integer a. In 1961, Linnik [14, Chapter VIII] proved (1.12) unconditionally by applying his dispersion method. Later, Halberstam [10] gave a very short proof using the Bombieri–Vinogradov theorem. A number of generalizations and analogs of the Titchmarsh divisor problem have been established in the literature (see [1, 3, 6, 8, 19, 20, 23]). In our third main theorem, we will use the Siegel–Walfisz theorem to show an analog of (1.12) for $\mu_{k,m}$.

Theorem 1.4 *Let $m\geq k\geq 2$. For any $A>0$, we have*

$$\sum_{p\leq x}\mu_{k,m}(p-1)=C_{k,m}\,\mathrm{li}(x)+O\left(\frac{x}{(\log x)^A}\right), \tag{1.13}$$

where $\mathrm{li}(x)=\int_2^x\frac{1}{\log t}dt$ and

$$C_{k,m}=\prod_{p}\left(1-\frac{1}{p^{k-1}(p-1)}-\frac{1}{p^m}\right).$$

The implied constant depends on k, m, and A only.

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In Section 2, we will introduce some basic knowledge on holomorphic cusp forms and uniquely ergodic topological dynamical systems. Then in Section 3, we will use (1.1) to build a relationship between the partial sum with weight $\mu_{k,m}$ and the standard partial sum. In the end, in Sections 4–6, we will use this relationship to show Theorems 1.1, 1.2, and 1.4.

Notations: We always use letter p to denote primes. The notations $f(n) = O_{k,m,\varepsilon,\dots}(g(n))$ or $f(n) \ll_{k,m,\varepsilon,\dots} g(n)$ mean that there is a positive constant $C = C(k, m, \varepsilon, \dots) > 0$ such that $|f(n)| \le C|g(n)|$ for all $n \ge 1$. Here, C is called the implied constant of this O-term. Sometimes we write $f = O(g)$ or $f \ll g$ for simplicity, if the implied constant depends at most on some fixed positives.

2 Preliminaries

2.1 Holomorphic cusp forms

Let $f \in H_\kappa$. The Fourier expansion of $f(z)$ at cusp ∞ is given by

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{\kappa-1}{2}} e(nz),$$

where $e(z) = e^{2\pi i z}$. The Fourier coefficients $\lambda_f(n)$ are real and satisfy the following multiplicative property:

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\Big(\frac{mn}{d^2}\Big)$$

for any integers $m, n \ge 1$. By the celebrated work of Deligne [7, Theorem (8.2)], the Ramanujan–Petersson conjecture

$$|\lambda_f(n)| \le \tau(n) \tag{2.1}$$

holds for all $n \ge 1$. Moreover, the L-function associated with $f(z)$ is given by

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \Big(1 - \frac{\alpha_p}{p^s}\Big)^{-1} \Big(1 - \frac{\beta_p}{p^s}\Big)^{-1},$$

where the local parameters α_p and β_p satisfy $\alpha_p + \beta_p = \lambda_f(p)$ and $\alpha_p\beta_p = 1$, and $L(s, f)$ is absolutely convergent for $\mathrm{Re}\, s > 1$.

For any integer $m \ge 1$, the mth symmetric power L-function is defined by the following degree $m + 1$ Euler product:

$$L(s, \mathrm{sym}^m f) = \prod_p \prod_{0 \le j \le m} \Big(1 - \frac{\alpha_p^{m-j}\beta_p^j}{p^s}\Big)^{-1}.$$

By the result of Newton and Thorne [18, Theorem A], it can be continued analytically over all of $\mathbb{C}$ for $m \ge 1$. Let χ be a primitive Dirichlet character modulo q.

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By the Rankin–Selberg theory, the twisted mth symmetric power L-function, which is defined by

$$L(s, \mathrm{sym}^m f \otimes \chi) = \prod_p \prod_{0 \le j \le m} \left(1 - \frac{\alpha_p^{m-2j} \chi(p)}{p^s}\right)^{-1}$$

for $\mathrm{Re}\, s > 1$, can be analytically continued over all of $\mathbb{C}$ for $m \ge 1$ as well.

2.2 Uniquely ergodic topological dynamical systems

A *topological dynamical system* is a pair (X, T) for a compact metric space X and a continuous map $T: X \to X$. A Borel probability measure ν on X is said to be *T-invariant* if $\nu(T^{-1}A) = \mu(A)$ for all Borel measurable subsets $A \subset X$. By the Bogolyubov–Krylov Theorem, every topological dynamical system (X, T) possesses at least one T-invariant Borel probability measure. If (X, T) admits only one such measure, then it is called *uniquely ergodic*. It is well-known that (X, ν, T) is uniquely ergodic if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x) = \int_X f \, d\nu \tag{2.2}$$

holds for all $x \in X$ and $f \in C(X)$. For example, let $X = \{0, 1\}$ be a rotation on two points with $\nu(\{0\}) = \nu(\{1\}) = 1/2$ and $T : x \mapsto x + 1 \,(\mathrm{mod}\, 2)$. Then (X, ν, T) is a uniquely ergodic topological dynamical system, and it is the simplest one.

In Bergelson and Richter's theorem (1.8), taking (X, T) as a rotation on two points, we get the following equivalent form of the PNT:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda(n) = 0. \tag{2.3}$$

If we take (X, T) as a rotation on two points in (1.9), then we obtain another well-known equivalent form of the PNT:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n) = 0. \tag{2.4}$$

Thus, either (1.8) or (1.9) provides a dynamical generalization of the PNT. Bergelson and Richter's theorem also unifies some other number-theoretic results from a dynamical point of view. We refer readers to [5] for more interesting applications of (1.8) and (1.9).

3 Partial sum with weight $\mu_{k,m}$

In the following sections, we always let $m \ge k \ge 2$ be integers. To prove Theorems 1.1, 1.2, and 1.4, we build a relationship between the partial sum with weight $\mu_{k,m}$ and the standard partial sum.

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Proposition 3.1 *Let $g : \mathbb{N} \to \mathbb{C}$ be an arithmetic function such that $g(n) \ll_\varepsilon n^\varepsilon$ for any $\varepsilon > 0$. Then for any $\varepsilon > 0$ and $H \le x^{\frac{1}{m+k}}$, we have*

(3.1)
$$\sum_{n\le x} \mu_{k,m}(n)g(n) = \sum_{d\le H} \mu(d) \sum_{\substack{l\le H\\(l,d)=1}} \mu(l) \sum_{r\mid d} \mu(r) \sum_{t\le \frac{x}{d^m l^k r}} g(d^m l^k r t) + O\left(\frac{x^{1+\varepsilon}}{H^{k-1}}\right),$$

where the implied constant depends on ε only.

Moreover, if $g : \mathbb{N} \to \mathbb{C}$ is bounded, then

(3.2)
$$\sum_{n\le x} \mu_{k,m}(n)g(n) = \sum_{d\le H} \mu(d) \sum_{\substack{l\le H\\(l,d)=1}} \mu(l) \sum_{r\mid d} \mu(r) \sum_{t\le \frac{x}{d^m l^k r}} g(d^m l^k r t) + O\left(\frac{x}{H^{k-1}}\right),$$

where the implied constant depends on k, m, and $\|g\|_\infty$ only.

Proof Since the proof of (3.2) is similar to that of (3.1), we will only show that (3.1) holds. First, using the relation (1.1), we observe that

$$\begin{aligned}\sum_{n\le x} \mu_{k,m}(n)g(n) &= \sum_{n\le x} \sum_{\substack{d^m\delta=n\\(d,\delta)=1}} \mu(d)q_k(\delta)g(n)\\ &= \sum_{\substack{d^m\delta\le x\\(d,\delta)=1}} \mu(d)q_k(\delta)g(d^m\delta)\\ &= \sum_{d\le x^{\frac{1}{m}}} \mu(d) \sum_{\substack{\delta\le \frac{x}{d^m}\\(\delta,d)=1}} q_k(\delta)g(d^m\delta).\end{aligned}$$

Then for $H \le x^{\frac{1}{m+k}}$, we split up the summation into two parts according to the size of the modulus d:

$$\begin{aligned}\sum_{n\le x} \mu_{k,m}(n)g(n) &= \sum_{d\le H} \mu(d) \sum_{\substack{\delta\le \frac{x}{d^m}\\(\delta,d)=1}} q_k(\delta)g(d^m\delta)\\ &\quad + \sum_{H<d\le x^{\frac{1}{m}}} \mu(d) \sum_{\substack{\delta\le \frac{x}{d^m}\\(\delta,d)=1}} q_k(\delta)g(d^m\delta)\\ &=: S_1 + S_2. \qquad (3.3)\end{aligned}$$

To estimate S_2, by $|\mu(d)| \le 1$, $g(n) \ll n^\varepsilon$, and $0 \le q_k(\delta) \le 1$,

$$\begin{aligned}S_2 &\ll_\varepsilon \sum_{H<d\le x^{\frac{1}{m}}} \sum_{\substack{\delta\le \frac{x}{d^m}\\(\delta,d)=1}} (d^m\delta)^\varepsilon \le x^\varepsilon \sum_{H<d\le x^{\frac{1}{m}}} \sum_{\delta\le \frac{x}{d^m}} 1\\ &\le x^{1+\varepsilon} \sum_{H<d\le x^{\frac{1}{m}}} \frac{1}{d^m} \ll \frac{x^{1+\varepsilon}}{(m-1)H^{m-1}},\end{aligned}$$

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which implies that

$$S_2 = O_\varepsilon\left(\frac{x^{1+\varepsilon}}{H^{m-1}}\right). \tag{3.4}$$

Now, we estimate S_1 as follows. For $q_k(n)$, it is well-known that

$$q_k(n) = \sum_{d^k|n} \mu(d).$$

This implies that

$$S_1 = \sum_{d\le H} \mu(d) \sum_{\substack{\delta\le \frac{x}{d^m} \\ (\delta,d)=1}} \Big(\sum_{l^k|\delta} \mu(l)\Big) g(d^m\delta).$$

Writing $\delta = l^k t$, then by $(\delta, d) = 1$, we have $(l, d) = 1$ and $(t, d) = 1$. So

$$S_1 = \sum_{d\le H} \mu(d) \sum_{\substack{l\le \frac{x^{1/k}}{d^{m/k}} \\ (l,d)=1}} \mu(l) \sum_{\substack{t\le \frac{x}{d^m l^k} \\ (t,d)=1}} g(d^m l^k t).$$

By

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1, \end{cases}$$

the condition $(t, d) = 1$ can be by replaced by $\sum_{r|(t,d)} \mu(r)$. We get that

$$\begin{aligned} S_1 &= \sum_{d\le H} \mu(d) \sum_{\substack{l\le \frac{x^{1/k}}{d^{m/k}} \\ (l,d)=1}} \mu(l) \sum_{t\le \frac{x}{d^m l^k}} g(d^m l^k t) \sum_{r|(t,d)} \mu(r) \\ &= \sum_{d\le H} \mu(d) \sum_{\substack{l\le \frac{x^{1/k}}{d^{m/k}} \\ (l,d)=1}} \mu(l) \sum_{r|d} \mu(r) \sum_{t\le \frac{x}{d^m l^k r}} g(d^m l^k r t). \end{aligned}$$

Similar to (3.3), we divide the summation into two parts according to the size of the modulus l:

$$\begin{aligned} S_1 &= \sum_{d\le H} \mu(d) \sum_{\substack{l\le H \\ (l,d)=1}} \mu(l) \sum_{r|d} \mu(r) \sum_{t\le \frac{x}{d^m l^k r}} g(d^m l^k r t) \\ &\quad + \sum_{d\le H} \mu(d) \sum_{\substack{H<l\le \frac{x^{1/k}}{d^{m/k}} \\ (l,d)=1}} \mu(l) \sum_{r|d} \mu(r) \sum_{t\le \frac{x}{d^m l^k r}} g(d^m l^k r t) \\ &=: S_3 + S_4. \end{aligned}$$

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For S_4, similar to the estimation of S_2, we have

$$\begin{aligned} S_4 &\ll_\varepsilon x^\varepsilon \sum_{d\le H} \sum_{\substack{H<l\le \frac{x^{1/k}}{d^{m/k}}\\(l,d)=1}} \sum_{r\mid d} \sum_{t\le \frac{x}{d^m l^k r}} 1 \\ &\le x^{1+\varepsilon} \sum_{d\le H} \frac{1}{d^m} \sum_{\substack{H<l\le \frac{x^{1/k}}{d^{m/k}}\\(l,d)=1}} \frac{1}{l^k} \sum_{r\mid d} \frac{1}{r} \\ &\le x^{1+\varepsilon} \sum_{d\le H} \frac{\sigma_{-1}(d)}{d^m} \sum_{l>H} \frac{1}{l^k} \\ &\ll \frac{x^{1+\varepsilon}}{(k-1)H^{k-1}}, \end{aligned}$$

where $\sigma_s(n) = \sum_{d\mid n} d^s$. This gives us that

$$S_1 = \sum_{d\le H} \mu(d) \sum_{\substack{l\le H\\(l,d)=1}} \mu(l) \sum_{r\mid d} \mu(r) \sum_{t\le \frac{x}{d^m l^k r}} g(d^m l^k r t) + O_\varepsilon\left(\frac{x^{1+\varepsilon}}{H^{k-1}}\right). \tag{3.5}$$

Thus, (3.1) follows by combining (3.3)–(3.5) together. ∎

As a direct application of Proposition 3.1, taking $g(n) = 1$ for all $n \ge 1$ in (3.2), we obtain an estimate for the partial sum of $\mu_{k,m}$ with a weaker error term than Bege's result (1.2).

Theorem 3.2 *For $m \ge k \ge 2$, we have*

$$\sum_{n\le x} \mu_{k,m}(n) = A_{k,m} x + O\big(x^{\frac{m+1}{m+k}}\big). \tag{3.6}$$

Proof By (3.2) of Proposition 3.1, for $H \le x^{\frac{1}{m+k}}$, we have

$$\begin{aligned} \sum_{n\le x} \mu_{k,m}(n) &= \sum_{d\le H} \mu(d) \sum_{\substack{l\le H\\(l,d)=1}} \mu(l) \sum_{r\mid d} \mu(r) \sum_{t\le \frac{x}{d^m l^k r}} 1 + O\left(\frac{x}{H^{k-1}}\right) \\ &= \sum_{d\le H} \mu(d) \sum_{\substack{l\le H\\(l,d)=1}} \mu(l) \sum_{r\mid d} \mu(r)\left(\frac{x}{d^m l^k r} + O(1)\right) + O\left(\frac{x}{H^{k-1}}\right) \\ &= x \sum_{d\le H} \frac{\mu(d)}{d^m} \sum_{\substack{l\le H\\(l,d)=1}} \frac{\mu(l)}{l^k} \sum_{r\mid d} \frac{\mu(r)}{r} \\ &\quad + O\left(\sum_{d\le H} \tau(d) \sum_{\substack{l\le H\\(l,d)=1}} 1\right) + O\left(\frac{x}{H^{k-1}}\right). \end{aligned}$$

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By the well-known facts that

$$\sum_{d|n}\frac{\mu(d)}{d}=\frac{\varphi(n)}{n},$$

where $\varphi(n)$ is the Euler totient function, and $\sum_{d\le H}\tau(d)\ll H\log H$, we get that

$$\begin{aligned}\sum_{n\le x}\mu_{k,m}(n)&=x\sum_{d\le H}\frac{\mu(d)\varphi(d)}{d^{m+1}}\sum_{\substack{l\le H\\(l,d)=1}}\frac{\mu(l)}{l^k}+O(H^2\log H)+O\left(\frac{x}{H^{k-1}}\right)\\&=x\sum_{d=1}^{\infty}\frac{\mu(d)\varphi(d)}{d^{m+1}}\sum_{\substack{l=1\\(l,d)=1}}^{\infty}\frac{\mu(l)}{l^k}+O(H^2\log H)+O\left(\frac{x}{H^{k-1}}\right).\end{aligned}\tag{3.7}$$

Here, we have used the fact that the tail of the first term of (3.7) is bounded by

$$\left|x\sum_{d\le H}\frac{\mu(d)\varphi(d)}{d^{m+1}}\sum_{\substack{l>H\\(l,d)=1}}\frac{\mu(l)}{l^k}\right|\le x\sum_{d=1}^{\infty}\frac{\varphi(d)}{d^{m+1}}\sum_{d>H}\frac{1}{l^k}\ll\frac{x}{H^{k-1}}.$$

For the coefficient of the first term of (3.7),

$$\begin{aligned}\sum_{d=1}^{\infty}\frac{\mu(d)\varphi(d)}{d^{m+1}}\sum_{\substack{l=1\\(l,d)=1}}^{\infty}\frac{\mu(l)}{l^k}&=\sum_{d=1}^{\infty}\frac{\mu(d)\varphi(d)}{d^{m+1}}\cdot\prod_{p\nmid d}\left(1-\frac{1}{p^k}\right)\\&=\frac{1}{\zeta(k)}\sum_{d=1}^{\infty}\frac{\mu(d)\varphi(d)}{d^{m+1}}\cdot\prod_{p|d}\left(1-\frac{1}{p^k}\right)^{-1}\\&=\frac{1}{\zeta(k)}\prod_{p}\left(1-\frac{1}{p^m}\cdot\frac{1-\frac{1}{p}}{1-\frac{1}{p^k}}\right)\\&=\frac{1}{\zeta(k)}\prod_{p}\left(1-\frac{1}{p^m}\cdot\frac{1}{1+\frac{1}{p}+\cdots+\frac{1}{p^{k-1}}}\right),\end{aligned}\tag{3.8}$$

which is equal to $A_{k,m}$ by the definition (1.3). By (3.7), this implies that

$$\sum_{n\le x}\mu_{k,m}(n)=A_{k,m}x+O(H^2\log H)+O\left(\frac{x}{H^{k-1}}\right).\tag{3.9}$$

Thus, (3.6) follows by taking $H=x^{\frac{1}{m+k}}$ in (3.9). ∎

In the following three sections, we will use Proposition 3.1 to prove Theorems 1.1, 1.2, and 1.4.

4 Proof of Theorem 1.1

To prove Theorem 1.1, we cite a proposition in the work [16] of Lü and Wang. This is also the main tool they used to establish the asymptotic formula (1.6) on the shifted convolution sums of Fourier coefficients with squarefull kernel functions.

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Proposition 4.1 [16, Proposition 1] *Let $f \in H_\kappa$. Then for any $\varepsilon > 0$ and $q < x^{1/4}$, we have*

$$\sum_{\substack{n\le x+1\\ n\equiv 1 \,(\mathrm{mod}\, q)}} \lambda_f^2(n) = \frac{x}{q} L(1, \mathrm{sym}^2 f \otimes \chi_0) L(2, \chi_0^2)^{-1} + O\left(\frac{x^{3/5+\varepsilon}}{\varphi(q)}\right) + O(x^{1/2+\varepsilon}),$$

where χ_0 is the principal Dirichlet character modulo q.

Proof of Theorem 1.1. For any $\varepsilon > 0$ and $H \le x^{\frac{1}{m+k}}$, taking $g(n) = \lambda_f^2(n+1)$ in (3.1) of Proposition 3.1,

$$\begin{aligned}
&\sum_{n\le x} \mu_{k,m}(n)\lambda_f^2(n+1)\\
&= \sum_{d\le H} \mu(d) \sum_{\substack{l\le H\\ (l,d)=1}} \mu(l) \sum_{r\mid d} \mu(r) \sum_{t\le \frac{x}{d^m l^k r}} \lambda_f^2(d^m l^k r t + 1) + O\left(\frac{x^{1+\varepsilon}}{H^{k-1}}\right)\\
&= \sum_{d\le H} \mu(d) \sum_{\substack{l\le H\\ (l,d)=1}} \mu(l) \sum_{r\mid d} \mu(r) \sum_{\substack{n\le x+1\\ n\equiv 1 \,(\mathrm{mod}\, d^m l^k r)}} \lambda_f^2(n) + O\left(\frac{x^{1+\varepsilon}}{H^{k-1}}\right). \qquad (4.1)
\end{aligned}$$

Suppose $H \le x^{\frac{1}{4(m+k+1)}-\varepsilon}$, then $d^m l^k r \le H^{m+k+1} < x^{1/4}$. Set $A_f = L(1, \mathrm{sym}^2 f \otimes \chi_0) L(2, \chi_0^2)^{-1}$. By Proposition 4.1, we have

$$\begin{aligned}
&\sum_{n\le x} \mu_{k,m}(n)\lambda_f^2(n+1)\\
&= \sum_{d\le H} \mu(d) \sum_{\substack{l\le H\\ (l,d)=1}} \mu(l) \sum_{r\mid d} \mu(r) \cdot\\
&\qquad \left(\frac{A_f}{d^m l^k r} x + O\left(\frac{x^{3/5+\varepsilon}}{\varphi(d^m l^k r)}\right) + O(x^{1/2+\varepsilon})\right) + O\left(\frac{x^{1+\varepsilon}}{H^{k-1}}\right)\\
&= A_f A_{k,m} \cdot x + O(x^{3/5+\varepsilon}) + O(x^{1/2+\varepsilon} H^2 \log H) + O\left(\frac{x^{1+\varepsilon}}{H^{k-1}}\right). \qquad (4.2)
\end{aligned}$$

Here, we have used the following estimates for the first three terms in (4.2), respectively:

$$\sum_{d\le H} \frac{\mu(d)}{d^m} \sum_{\substack{l\le H\\ (l,d)=1}} \frac{\mu(l)}{l^k} \sum_{r\mid d} \frac{\mu(r)}{r} = A_{k,m} + O\left(\frac{1}{H^{k-1}}\right),$$

$$\sum_{d\le H} \sum_{\substack{l\le H\\ (l,d)=1}} \sum_{r\mid d} \frac{1}{\varphi(d^m l^k r)} < \sum_{d=1}^{\infty} \sum_{r\mid d} \frac{1}{\varphi(d^m r)} \sum_{l=1}^{\infty} \frac{1}{\varphi(l^k)} < \infty,$$

and

$$\left| \sum_{d\le H} \mu(d) \sum_{\substack{l\le H\\ (l,d)=1}} \mu(l) \sum_{r\mid d} \mu(r) \right| \le \sum_{d\le H} \tau(d) \sum_{l\le H} 1 \ll H^2 \log H. \qquad (4.3)$$

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Thus, taking $H = x^{\frac{1}{4(m+k+1)}-\varepsilon}$ in (4.2), we conclude that

$$\sum_{n\le x}\mu_{k,m}(n)\lambda_f^2(n+1) = A_{k,m,f}\cdot x + O\Big(x^{1-\frac{k-1}{4(m+k+1)}+\varepsilon}\Big), \tag{4.4}$$

where

$$A_{k,m,f} := L(1,\mathrm{sym}^2 f\otimes\chi_0)L(2,\chi_0^2)^{-1}\cdot \zeta(k)^{-1}\prod_p\Big(1-\frac{1}{p^{m-k+1}+p^{m-k+2}+\cdots+p^m}\Big). \tag{4.5}$$

This completes the proof of Theorem 1.1. ∎

5 Proof of Theorem 1.2

In this section, we use (3.2) of Proposition 3.1 and the definition of invariant average under multiplications to prove Theorem 1.2.

Proof of Theorem 1.2. Since $b(n)$ is of invariant average β under multiplications, we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}b(mn) = \beta \tag{5.1}$$

for all $m\in\mathbb{N}$. Then by (3.2) of Proposition 3.1, for $H\le N^{\frac{1}{m+k}}$, we find that

$$\begin{aligned}
&\sum_{n\le N}\mu_{k,m}(n)b(n)\\
&=\sum_{d\le H}\mu(d)\sum_{\substack{l\le H\\(l,d)=1}}\mu(l)\sum_{r|d}\mu(r)\sum_{t\le\frac{N}{d^m l^k r}}b(d^m l^k rt)+O\Big(\frac{N}{H^{k-1}}\Big)\\
&=\sum_{d\le H}\mu(d)\sum_{\substack{l\le H\\(l,d)=1}}\mu(l)\sum_{r|d}\mu(r)\Big(\frac{N}{d^m l^k r}+O(1)\Big)\frac{1}{\lfloor\frac{N}{d^m l^k r}\rfloor}\sum_{t\le\frac{N}{d^m l^k r}}b(d^m l^k rt)+O\Big(\frac{N}{H^{k-1}}\Big)\\
&=N\sum_{d\le H}\frac{\mu(d)}{d^m}\sum_{\substack{l\le H\\(l,d)=1}}\frac{\mu(l)}{l^k}\sum_{r|d}\frac{\mu(r)}{r}\frac{1}{\lfloor\frac{N}{d^m l^k r}\rfloor}\sum_{t\le\frac{N}{d^m l^k r}}b(d^m l^k rt)+O(H^2\log H)+O\Big(\frac{N}{H^{k-1}}\Big),
\end{aligned}$$

where $\lfloor x\rfloor := \max\{n\in\mathbb{Z}: n\le x\}$ denotes the integral part of a real number x. It follows that

$$\begin{aligned}
\frac{1}{N}\sum_{n\le N}\mu_{k,m}(n)b(n) &= \sum_{d\le H}\frac{\mu(d)}{d^m}\sum_{\substack{l\le H\\(l,d)=1}}\frac{\mu(l)}{l^k}\sum_{r|d}\frac{\mu(r)}{r}\frac{1}{\lfloor\frac{N}{d^m l^k r}\rfloor}\sum_{t\le\frac{N}{d^m l^k r}}b(d^m l^k rt)\\
&\quad+O\Big(\frac{H^2\log H}{N}\Big)+O\Big(\frac{1}{H^{k-1}}\Big).
\end{aligned} \tag{5.2}$$

Fix H, taking $N\to\infty$ in (5.2), by (5.1), we get that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu_{k,m}(n)b(n) = \beta\sum_{d\le H}\frac{\mu(d)}{d^m}\sum_{\substack{l\le H\\(l,d)=1}}\frac{\mu(l)}{l^k}\sum_{r|d}\frac{\mu(r)}{r}+O\Big(\frac{1}{H^{k-1}}\Big). \tag{5.3}$$

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Then taking $H \to \infty$ gives us that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu_{k,m}(n) b(n) = \beta \sum_{d=1}^{\infty} \frac{\mu(d)}{d^m} \sum_{\substack{l=1 \\ (l,d)=1}}^{\infty} \frac{\mu(l)}{l^k} \sum_{r|d} \frac{\mu(r)}{r}, \tag{5.4}$$

which is equal to $A_{k,m} \cdot \beta$ by (3.8), as desired. This completes the proof of Theorem 1.2. ∎

6 Proof of Theorem 1.4

In this section, we use (3.2) of Proposition 3.1 and the Siegel–Walfisz theorem to prove Theorem 1.4. For integers $1 \le a \le q$ with $(a,q) = 1$, let $\pi(x;q,a) := \#\{p \le x : p \equiv a \pmod q\}$ be the prime-counting function in arithmetic progressions.

Theorem 6.1 (The Siegel–Walfisz theorem, [17, Corollary 11.21]) *There is a constant $c_1 > 0$ such that for any given $A > 0$, if $q \le (\log x)^A$ and $(a,q) = 1$, then*

$$\pi(x;q,a) = \frac{\mathrm{li}(x)}{\varphi(q)} + O_A\left(x \exp(-c_1\sqrt{\log x})\right). \tag{6.1}$$

Proof of Theorem 1.4. Let $\mathbb{P}$ be the set of all primes. Denote by $1_{\mathbb{P}}$ the characteristic function of primes defined by

$$1_{\mathbb{P}}(n) = \begin{cases} 1 & \text{if } n \in \mathbb{P}, \\ 0 & \text{if } n \notin \mathbb{P}. \end{cases}$$

Then

$$\sum_{p\le x} \mu_{k,m}(p-1) = \sum_{n\le x-1} \mu_{k,m}(n) 1_{\mathbb{P}}(n+1) \text{ and } \sum_{t\le \frac{x}{q}} 1_{\mathbb{P}}(qt+1) = \pi(x+1;q,1).$$

Taking $g(n) = 1_{\mathbb{P}}(n+1)$ in (3.2) of Proposition 3.1, for $H \le (x-1)^{\frac{1}{m+k}}$, we have

$$\begin{aligned} \sum_{p\le x} \mu_{k,m}(p-1) &= \sum_{d\le H} \mu(d) \sum_{\substack{l\le H \\ (l,d)=1}} \mu(l) \sum_{r|d} \mu(r) \sum_{t\le \frac{x}{d^m l^k r}} g(d^m l^k r t) + O\left(\frac{x}{H^{k-1}}\right) \\ &= \sum_{d\le H} \mu(d) \sum_{\substack{l\le H \\ (l,d)=1}} \mu(l) \sum_{r|d} \mu(r) \pi(x+1; d^m l^k r, 1) + O\left(\frac{x}{H^{k-1}}\right). \end{aligned} \tag{6.2}$$

Since $r \mid d$, we have $\varphi(d^m r) = r\varphi(d^m)$. Moreover, by $(l,d) = 1$, we get that $\varphi(d^m l^k r) = \varphi(d^m r)\varphi(l^k) = r\varphi(d^m)\varphi(l^k)$. Clearly, $\varphi(d^m) = d^{m-1}\varphi(d)$. Take $H = (\log x)^{\frac{A}{k-3/2}}$, then $d^m l^k r \le H^{m+k+1}$. We invoke the Siegel–Walfisz theorem in (6.2) and the upper bound in (4.3) to deduce that

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$$\begin{aligned}
&\sum_{p\le x}\mu_{k,m}(p-1)\\
&=\sum_{d\le H}\mu(d)\sum_{\substack{l\le H\\(l,d)=1}}\mu(l)\sum_{r\mid d}\mu(r)\Big(\frac{\mathrm{li}(x)}{\varphi(d^m l^k r)}+O\big(x\exp(-c_1\sqrt{\log x})\big)\Big)+O\left(\frac{x}{H^{k-1}}\right)\\
&=\mathrm{li}(x)\sum_{d\le H}\frac{\mu(d)}{\varphi(d^m)}\sum_{\substack{l\le H\\(l,d)=1}}\frac{\mu(l)}{\varphi(l^k)}\sum_{r\mid d}\frac{\mu(r)}{r}+O\Big(x\exp(-c_1\sqrt{\log x})H^2\log H\Big)\\
&\quad+O\left(\frac{x}{H^{k-1}}\right).
\end{aligned}$$

Since $c_1>0$, we have $\exp(c_1\sqrt{\log x})\gg(\log x)^{AC/(k-3/2)}=H^C$ for any positive constant $C>0$. So the error term $O\Big(x\exp(-c_1\sqrt{\log x})H^2\log H\Big)$ can be absorbed in $O\left(x/H^{k-1}\right)$, and

$$\sum_{p\le x}\mu_{k,m}(p-1)=\mathrm{li}(x)\sum_{d\le H}\frac{\mu(d)}{d^m}\sum_{\substack{l\le H\\(l,d)=1}}\frac{\mu(l)}{\varphi(l^k)}+O\left(\frac{x}{H^{k-1}}\right). \tag{6.3}$$

Now, we estimate the first term of (6.3). By [11, Theorem 327], we have $\frac{l}{\varphi(l)}\ll l^{1/2}$, so

$$\Big|\sum_{\substack{l>H\\(l,d)=1}}\frac{\mu(l)}{\varphi(l^k)}\Big|\le\sum_{l>H}\frac{1}{l^{k-1}\varphi(l)}\ll\sum_{l>H}\frac{1}{l^{k-\frac12}}\ll\frac{1}{H^{k-\frac32}}. \tag{6.4}$$

Then

$$\begin{aligned}
\sum_{d\le H}\frac{\mu(d)}{d^m}\sum_{\substack{l\le H\\(l,d)=1}}\frac{\mu(l)}{\varphi(l^k)}&=\sum_{d\le H}\frac{\mu(d)}{d^m}\Big(\sum_{\substack{l=1\\(l,d)=1}}^{\infty}\frac{\mu(l)}{\varphi(l^k)}-\sum_{\substack{l>H\\(l,d)=1}}\frac{\mu(l)}{\varphi(l^k)}\Big)\\
&=\sum_{d\le H}\frac{\mu(d)}{d^m}\sum_{\substack{l=1\\(l,d)=1}}^{\infty}\frac{\mu(l)}{\varphi(l^k)}+O\left(\frac{1}{H^{k-\frac32}}\right)\\
&=\sum_{d=1}^{\infty}\frac{\mu(d)}{d^m}\sum_{\substack{l=1\\(l,d)=1}}^{\infty}\frac{\mu(l)}{\varphi(l^k)}+O\left(\sum_{d>H}\frac{1}{d^m}\right)+O\left(\frac{1}{H^{k-\frac32}}\right)\\
&=\sum_{d=1}^{\infty}\frac{\mu(d)}{d^m}\sum_{\substack{l=1\\(l,d)=1}}^{\infty}\frac{\mu(l)}{\varphi(l^k)}+O\left(\frac{1}{H^{k-\frac32}}\right)
\end{aligned} \tag{6.5}$$

due to the fact that $\sum_{d>H}1/d^m\ll 1/H^{m-1}<1/H^{k-3/2}$ for $m\ge k\ge 2$.

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As regards the double summation over d and l in (6.5), we have

$$\begin{aligned}
&\sum_{d=1}^{\infty}\frac{\mu(d)}{d^m}\sum_{\substack{l=1\\(l,d)=1}}^{\infty}\frac{\mu(l)}{\varphi(l^k)}\\
&=\sum_{d=1}^{\infty}\frac{\mu(d)}{d^m}\prod_{p\nmid d}\Big(1-\frac{1}{p^{k-1}(p-1)}\Big)\\
&=\prod_{p}\Big(1-\frac{1}{p^{k-1}(p-1)}\Big)\cdot\sum_{d=1}^{\infty}\frac{\mu(d)}{d^m}\prod_{p\mid d}\Big(1-\frac{1}{p^{k-1}(p-1)}\Big)^{-1}\\
&=\prod_{p}\Big(1-\frac{1}{p^{k-1}(p-1)}\Big)\cdot\prod_{p}\Big(1-\frac{1}{p^m}\Big(1-\frac{1}{p^{k-1}(p-1)}\Big)^{-1}\Big)\\
&=\prod_{p}\Big(1-\frac{1}{p^{k-1}(p-1)}-\frac{1}{p^m}\Big),
\end{aligned}\tag{6.6}$$

which is equal to $C_{k,m}$. It follows by the trivial estimate $\mathrm{li}(x)\ll x$ that

$$\mathrm{li}(x)\sum_{d\le H}\frac{\mu(d)}{d^m}\sum_{\substack{l\le H\\(l,d)=1}}\frac{\mu(l)}{\varphi(l^k)}=C_{k,m}\,\mathrm{li}(x)+O\Big(\frac{x}{H^{k-\frac{3}{2}}}\Big).\tag{6.7}$$

Thus, (1.13) follows by plugging $H=(\log x)^{\frac{A}{k-3/2}}$ into (6.3) and (6.7). This completes the proof of Theorem 1.4. ∎

Remark 6.2 One can also apply the Bombieri–Vinogradov theorem in (6.2) to prove Theorem 1.4.

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