From these we see that:

(i) for all $\alpha > 0, h(0) = h(\frac{1}{2}) = 0$;

(ii) for small positive t, $h'(t) > 0$ if $0 < \alpha < 1$ or $\alpha > 2$ while $h'(t) < 0$ if $1 < \alpha < 2$;

(iii) for $0 < t < \frac{1}{2}$, $h''(t) < 0$ if $0 < \alpha < 1$ or $\alpha > 2$ while $h''(t) > 0$ if $1 < \alpha < 2$.

Between them, these suffice to establish the Lemma. For example, if $0 < \alpha < 1$, (i) and (ii) together show that $h(t)$ is positive for small positive values of t. Subsequently, $h(t)$ cannot take negative values otherwise there would be a local minimum, contradicting (iii).

Finally, it is worth noting that the special case $n = 4$ is an attractive one which can be dealt with algebraically. For in that case, (1) becomes

$$\cos A = \frac{1 + x^2 - \sqrt{1 + x^4}}{2x} = \frac{(1 + x^2)^2 - (1 + x^4)}{2x(1 + x^2 + \sqrt{1 + x^4})}$$

$$= \frac{x}{1 + x^2 + \sqrt{1 + x^4}} = \frac{1}{x + \frac{1}{x} + \sqrt{\left(x + \frac{1}{x}\right)^2 - 2}}.$$

Since, $x + \frac{1}{x}$ is a decreasing function for $0 < x \leqslant 1$, we see that A increases from $\cos^{-1}\left(\frac{1}{2 + \sqrt{2}}\right) = \cos^{-1}\left(1 - \frac{\sqrt[4]{4}}{2}\right)$ to $\frac{\pi}{2}$. Noting that $\cos^{-1}\left(1 - \frac{\sqrt[4]{4}}{2}\right) \approx 72.97°$, a nice (but cruel!) variation on our opening theme is to ask: Is there a triangle with $A = 72°$ and $a^4 = b^4 + c^4$?

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99.16 Conditionally convergent series

Introduction

In 1984, Rao and Subbarao [1] established the following identity

$$\sum_{j=1}^{\infty} a_j \sum_{k=1}^{j} b_k + \sum_{k=1}^{\infty} b_k \sum_{j=1}^{k} a_j = \left(\sum_{j=1}^{\infty} a_j\right)\left(\sum_{k=1}^{\infty} b_k\right) + \sum_{k=1}^{\infty} a_k b_k, \qquad (1)$$

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where both infinite series $\sum_{j=1}^{\infty} a_j$ and $\sum_{k=1}^{\infty} b_k$ of real numbers are assumed to be absolutely convergent. Since the authors' approach depends on the Rearrangement Theorem of absolutely convergent series (see, for example, [2, p. 269]), it is not obvious (at least to the present authors) that the identity (1) remains true for conditionally convergent series. In order to gain better insight into this problem, we use *Mathematica* 8 and *Maple* 16 to compute several sums numerically. Such computations seem to suggest that the identity (1) holds for each of the following cases:

(a) $a_k = b_k = \dfrac{(-1)^{k-1}}{2k-1}$ for $k = 1, 2, \ldots$.

(b) $a_k = \dfrac{(-1)^{k-1}}{k}$ and $b_k = \dfrac{(-1)^{k-1}}{2k-1}$ for $k = 1, 2, \ldots$..

(c) $a_k = \dfrac{\sin k}{k}$ and $b_k = \dfrac{\cos k}{2k}$ for $k = 1, 2, \ldots$..

Such special verifications then lead us to a surprising generalisation of (1).

A generalisation of the identity (1)

The following result is a generalisation of the identity (1).

Theorem: Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences of real numbers such that the infinite series $\sum_{k=1}^{\infty} a_k$, $\sum_{k=1}^{\infty} b_k$ and $\sum_{k=1}^{\infty} a_k b_k$ converge. Then

$$\lim_{n\to\infty}\left\{\sum_{k=1}^{n} b_k \sum_{j=1}^{k} a_j + \sum_{j=1}^{n-1} a_j \sum_{k=1}^{j} b_k\right\} = \left(\sum_{j=1}^{\infty} a_j\right)\left(\sum_{k=1}^{\infty} b_k\right) + \sum_{k=1}^{\infty} a_k b_k. \quad (2)$$

Proof: For each non-negative integer k, we set $A_k = \sum_{i=0}^{k} a_i$ and $B_k = \sum_{i=0}^{k} b_i$, where both a_0 and b_0 are defined to be zero. We claim that

$$\sum_{k=1}^{n} b_k A_{k-1} = A_{n-1}B_n - \sum_{k=1}^{n-1} a_k B_k \text{ for } n = 2, 3, \ldots .$$

Indeed, for any integer $n \geqslant 2$, we have

$$\begin{aligned}
&\sum_{k=1}^{n} b_k A_{k-1} \\
&= \sum_{k=1}^{n} (B_k - B_{k-1}) A_{k-1} \text{ (since } b_k = B_k - B_{k-1} \text{ for } k = 1, 2, \ldots) \\
&= \sum_{k=1}^{n} B_k A_{k-1} - \sum_{k=2}^{n} B_{k-1} A_{k-1} \text{ (because } A_0 = B_0 = 0) \\
&= \sum_{k=1}^{n} B_k A_{k-1} - \sum_{j=1}^{n-1} B_j A_j \text{ (where } j = k-1 \text{ for } k = 2, \ldots, n)
\end{aligned}$$

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$$= \left(A_{n-1}B_n + \sum_{k=1}^{n-1} B_k A_{k-1}\right) - \sum_{k=1}^{n-1} B_k A_k$$

$$= A_{n-1}B_n - \sum_{k=1}^{n-1} a_k B_k \text{ (because } a_k = A_k - A_{k-1} \text{ for } k = 1, 2, \ldots).$$

Since both series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are assumed to be convergent and $n \geqslant 2$ is arbitrary, we let $n \to \infty$ in (3}) to conclude that

$$\lim_{n\to\infty}\left\{\sum_{k=1}^{n} b_k \sum_{j=1}^{k-1} a_j + \sum_{j=1}^{n-1} a_j \sum_{k=1}^{j} b_k\right\} = \left(\sum_{j=1}^{\infty} a_j\right)\left(\sum_{k=1}^{\infty} b_k\right). \tag{4}$$

Finally, we invoke the convergent series $\sum_{k=1}^{\infty} a_k b_k$ and (4) to obtain the desired result.

We may be very tempted to rewrite (2) like (1). Unfortunately, the following example shows that the left-hand side of (2) cannot be improved.

Example: For each integer $n \geqslant 1$, we set $a_{2n-1} = 0$, $a_{2n} = \dfrac{(-1)^{n-1}}{\sqrt{n+1}}$, $b_{2n-1} = \dfrac{(-1)^{n-1}}{\sqrt{n}} + \dfrac{(-1)^{n-1}}{\sqrt{n+1}}$ and $b_{2n} = 0$. Then we have

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{2k} \text{ and } \sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} b_{2k-1}$$

that is, both series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are conditionally convergent. Since $a_k b_k = 0$ for $k = 1, 2, \ldots$, the series $\sum_{k=1}^{\infty} a_k b_k$ converges absolutely. However, since the series $\sum_{k=1}^{\infty} \dfrac{(-1)^{k-1}}{\sqrt{k+1}}$ converges and the series $\sum_{k=1}^{\infty} \dfrac{1}{k+1}$ diverges, the series $\sum_{j=1}^{\infty} a_j \sum_{k=1}^{j} b_k$ diverges:

$$\sum_{j=1}^{\infty} a_j \sum_{k=1}^{j} b_j = \sum_{j=1}^{\infty} a_{2j} \sum_{k=1}^{2j} b_k \qquad (\text{since } a_{2k-1} = 0 \text{ for } k = 1, 2, \ldots)$$

$$= \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{\sqrt{j+1}} \sum_{r=1}^{j} b_{2r-1} \qquad (\text{since } b_{2k} = 0 \text{ for } k = 1, 2, \ldots)$$

$$= \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{\sqrt{j+1}} \cdot \left(1 + \frac{(-1)^{j-1}}{\sqrt{j+1}}\right).$$

Thus, the left-hand side of (2) cannot be improved.

In conclusion, the Theorem is a proper generalisation of the identity (1).

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We remark that (2) can be rewritten as (1) if at least one of the convergent series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ is absolutely convergent.

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99.17 Irrational sums from reciprocals of factorials

Introduction

That e is irrational is reasonably straightforward to prove. A proof by contradiction appears on page 300 of [1], for example. We extend this approach here in order to show that each of the members of a particular class of numbers, some of which are associated with e, are irrational.

An initial result

Theorem 1: Let $f(n)$ denote a strictly increasing function, for which the domain is the set of non-negative integers and the range is some infinite subset of the positive integers. Then

$$S_f = \sum_{k=0}^{\infty} \frac{1}{f(k)!}$$

is irrational.

Proof: Suppose that $S_f = a/b$ for some $a, b \in \mathbb{N}$. Choose $M \in \mathbb{N}$ such that $f(M) > b$; then $b \mid f(m)!$ for all $m \geqslant M$. Let us now consider the number

$$Q = f(M)!\left(\frac{a}{b} - \sum_{k=0}^{M} \frac{1}{f(k)!}\right). \tag{1}$$

Since $b \mid f(M)!$ and $f(k)! \mid f(M)!$ for $k = 0, 1, 2, \ldots, M$, it must be the case that Q is an integer. Therefore, as

$$\frac{a}{b} = S_f = \sum_{k=0}^{\infty} \frac{1}{f(k)!} = \sum_{k=0}^{M} \frac{1}{f(k)!} + \sum_{k=M+1}^{\infty} \frac{1}{f(k)!},$$

we have, from (1) and the formula for the sum to infinity of a convergent

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