the inductive hypothesis holds for all $2 \leqslant n \leqslant N$. The inductive step splits into two cases:

- If $m+N$ is odd, then $\left(\frac{1}{m}+\frac{1}{m+1}+\ldots+\frac{1}{m+N-1}\right)+\frac{1}{m+N} \in A$ by the inductive hypothesis and property (I).
- If $m+N$ is even, then separating terms with odd and even denominators shows that $\frac{1}{m}+\frac{1}{m+1}+\ldots+\frac{1}{m+N-1}+\frac{1}{m+N} = \sum\frac{1}{2r-1}+\sum\frac{1}{2s}$. By property (II), the first series on the right-hand side gives a fraction in B and, by the inductive hypothesis and property (III), the second sum is a fraction in A. Thus, by property (I) $\frac{1}{m}+\frac{1}{m+1}+\ldots+\frac{1}{m+N-1}+\frac{1}{m+N} \in A$.

Articles on whether series such as those above (and other closely related ones) have integer partial sums have appeared in the *Gazette* over the years: we list some recent references below.

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99.04 Another irrational number

For each natural number n, let $f(n) = \operatorname{lcm}\{1,2,3, \ldots, n\}$ denote the least common multiple of all the positive integers that are less than or equal to n. Thus $f(1) = 1$, $f(2) = 2$, $f(3) = 6$, $f(4) = 12$, $f(5) = f(6) = 60$, and so on. The arithmetic function $f(n)$ crops up in several places in mathematics. For example, it is the exponent of the symmetric group $\mathcal{S}(n)$, the elements of which are the $n!$ permutations on n objects. This means, for example, that for $n = 4$, since $f(4) = 12$, x^{12} is the identity for all x in $\mathcal{S}(4)$, and 12 is the least such positive integer with this property. We also find $f(n)$ appearing in the classic problem about the monkey and the coconuts.

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The function $f(n)$ has some nice properties. For example, we have

1. $2^n - 5 < f(n) < 3^n$ for all $n \in \mathbb{N}$ [1, 2];
2. There are arbitrarily long intervals on which $f(n)$ remains constant. To see this, note first that $f(n) = f(n-1)$ if, and only if, n has at least two distinct prime factors. Second, it is the case that for any fixed k there exists, by the Chinese remainder theorem, x such that $p_{2i-1}p_{2i}$ divides $x + i$ for $i = 1,2, \ldots k$. Furthermore, since prime powers are rather rare, long runs of consecutive composite numbers are often runs of constant $f(n)$. For example, none of the composite numbers between the primes 89 and 97 is a prime power, leading to the run $\{89, 90, 91, 92, 93, 94, 95, 96\}$. A considerably longer run is given by $\{9551, 9552, \ldots, 9586\}$. As will be seen, prime gaps are in fact relevant to the proof of the result given in this note.

Roughly speaking, one might describe $f(n)$ as 'n! without unnecessary repetitions', and since $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ is one of the most important numbers in mathematics, it is natural to consider the series $s = \sum_{n=1}^{\infty} \frac{1}{f(n)}$, where with a slight abuse of notation, we use s to denote the series and its sum.

That the series converges is a consequence of the bound $f(n) > 2^n - 5$ given above in point 1, although it already follows from the rather trivial bound $f(n) \geqslant n(n-1)$, which can also be used to show that $s \leqslant 2$. In fact, using computer algebra, we obtain $s = 1.7877804561$ correct to ten decimal places. Note, as is clear from the definition, that $s + 1$ is greater than e, but (perhaps surprisingly) not by very much. Since it is a famous result that e is an irrational number, we ask if the same is true for s? We now prove, by way of a contradiction, that the answer to this question is in the affirmative.

In order to obtain the proof, we modify the one used to show that e is irrational by bracketing the tail-end sum over prime gaps. To this end, suppose that

$$s = \frac{a}{b}$$

for some $a, b \in \mathbb{N}$. It then follows that $f(n)s \in \mathbb{N}$ for any $n \geqslant b$. We now choose $k \in \mathbb{N}$ such that the following conditions are simultaneously satisfied:

(i) $p_k - 1 \geqslant b$, where p_i denotes the i th prime;

(ii) for all $i \geqslant k$,

$$\frac{p_{i+1} - p_i}{p_i} < \frac{1}{3}.$$

Note that for any $b \in \mathbb{N}$ it is indeed possible to find $k \in \mathbb{N}$ such that (i) and (ii) are satisfied simultaneously. In particular, that (ii) may be met follows from the prime number theorem.

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Our choice of k guarantees that $f(p_k - 1)s$ is an integer and that each of the first $p_k - 1$ terms of the sum

$$\sum_{j=1}^{\infty} \frac{f(p_k - 1)}{f(j)}$$

is an integer (noting though that that none of the later terms in the sum are integers). This implies that the sum of the series

$$t = \sum_{j=p_k}^{\infty} \frac{f(p_k - 1)}{f(j)},$$

the terms of which form a non-increasing sequence of rational numbers each lying strictly between 0 and 1, is an integer.

Let us now consider the terms of t in a little more detail. Since $f(p_k) = p_k f(p_k - 1)$, the first is given by

$$\frac{1}{p_k}.$$

Note also that

$$\frac{f(p_k - 1)}{f(j)} \leqslant \frac{1}{p_k}$$

for $p_k \leqslant j \leqslant p_{k+1} - 1$. Next, as

$$\begin{aligned} f(p_{k+1}) &= p_{k+1} f(p_{k+1} - 1) \\ &\geqslant p_{k+1} f(p_k) \\ &= p_{k+1} p_k f(p_{k+1} - 1), \end{aligned}$$

we have

$$\frac{f(p_k - 1)}{f(p_{k+1})} \leqslant \frac{1}{p_k p_{k+1}},$$

and in fact

$$\frac{f(p_k - 1)}{f(j)} \leqslant \frac{1}{p_k p_{k+1}}$$

for $p_{k+1} \leqslant j \leqslant p_{k+2} - 1$, and so on. From this we see that

$$\begin{aligned} t &\leqslant \frac{p_{k+1} - p_k}{p_k} + \frac{p_{k+2} - p_{k+1}}{p_k p_{k+1}} + \frac{p_{k+3} - p_{k+2}}{p_k p_{k+1} p_{k+2}} + \ldots \\ &< \frac{1}{3}\left(1 + \frac{1}{p_k} + \frac{1}{p_k p_{k+1}} + \ldots\right) \\ &< \frac{1}{3}\left(1 + \frac{1}{p_k} + \frac{1}{p_k^2} + \ldots\right) \end{aligned}$$

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$$= \frac{p_k}{3(p_k - 1)}$$

$$< 1$$

where we have used the formula for the sum to infinity of a convergent geometric series. This provides a contradiction, and our assumption that s is rational is therefore false, thereby completing the proof.

We make the final point here that the above argument does not actually require the prime number theorem, which is really rather a deep result. The number of terms in each section of the tail-end is equal to the prime gap $p_{k+j+1} - p_{k+j}$, and this can be bounded by a Chebyshev-type elementary estimate such as $3p_{i+1} < 4p_i$ for all sufficiently large i [2].

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99.05 An inequality for Si(x)

The 'sine integral' is

$$\mathrm{Si}\,(x) = \int_0^x \frac{\sin t}{t} dt.$$

The aim of this note is to establish the pair of inequalities

$$\tan^{-1} x < \mathrm{Si}\,(x) < \pi - \tan^{-1} x \qquad \text{for } x > 0. \tag{1}$$

These bounds imply the well-known integral

$$\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}.$$

Our proof assumes this result; of course, it would be more satisfying to find a proof that does not rely on it.

An unusual feature of the bounds is that the upper and lower bounds are monotone functions (whose graphs are symmetrical about the line $y = \pi/2$), whereas Si (x) is an oscillatory function: see Figure 1.

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