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Möbius transformations in stability theory

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1. *Introduction.* Consider the system of ordinary differential equations

$$a_0 D^n x + a_1 D^{n-1} x + a_2 D^{n-2} x + \ldots + a_n x = 0, \tag{1}$$

where the unknown $x(t)$ is a complex m-vector, t is a real variable, D is the operator d/dt and $a_0, \ldots, a_n$ are complex $m \times m$ matrices whose elements are continuous functions of $t, x, Dx, \ldots, D^{n-1}x$. Furthermore, $\det a_0 \neq 0$. In the special case when $a_0, \ldots, a_n$ are constant matrices the trivial solution $x = 0$ is asymptotically stable if and only if all the roots of the characteristic equation $\det f(\zeta) = 0$ have negative real parts, where

$$f(\zeta) = a_0 \zeta^n + a_1 \zeta^{n-1} + a_2 \zeta^{n-2} + \ldots + a_n. \tag{2}$$

But when $a_0, \ldots, a_n$ are variable matrices, the roots of $\det f(\zeta) = 0$ do not determine the stability of $x = 0$, in general. In this case a sufficient condition for the Lyapunov stability of $x = 0$ is the existence of a positive definite hermitian form

$$V(x, Dx, \ldots, D^{n-1}x)$$

with constant coefficients such that $DV < 0$ for all non-zero solutions $x(t)$ of (1) (see (1), p. 14). Such a form V will be called an *autonomous quadratic Lyapunov function* (briefly, AQLF) of (1).

A *Möbius transformation* is a mapping $\zeta \to u(\zeta)$ of the closed complex ζ plane onto itself of the form $u(\zeta) = (\alpha\zeta + \beta)/(\gamma\zeta + \delta)$, where $\alpha, \beta, \gamma, \delta$ are complex constants satisfying $\alpha\delta \neq \beta\gamma$. Its inverse mapping is $u^{-1}(\zeta) = (\beta - \delta\zeta)/(\gamma\zeta - \alpha)$. It is assumed that $\alpha, \beta, \gamma, \delta$ are independent not only of ζ but also of the variables $t, x, Dx, \ldots, D^n x$ in (1) (except in section 2). From $f(\zeta)$ and $u(\zeta)$ we derive a new polynomial

$$g(\zeta) = (\alpha\delta - \beta\gamma)^{-\frac{1}{2}n} (\alpha - \gamma\zeta)^n f(u^{-1}(\zeta)), \tag{3}$$

which can be written as $g(\zeta) = b_0 \zeta^n + b_1 \zeta^{n-1} + \ldots + b_n$, where the matrices $b_0, \ldots, b_n$ are linear functions of $a_0, \ldots, a_n$ with scalar coefficients. Then $\det g(\zeta) = 0$ is the characteristic equation of the system of differential equations

$$b_0 D^n x + b_1 D^{n-1} x + b_2 D^{n-2} x + \ldots + b_n x = 0. \tag{4}$$

If $\lambda_1, \lambda_2, \ldots, \lambda_{mn}$ are the mn roots of $\det f(\zeta) = 0$ then (3) shows that

$$u(\lambda_1), u(\lambda_2), \ldots, u(\lambda_{mn})$$

are the mn roots of $\det g(\zeta) = 0$. Clearly $\det b_0 \neq 0$ when $u(\lambda_1), \ldots, u(\lambda_{mn})$ are all finite. Let Π be the set of all complex numbers with negative real parts. If Π contains all the roots of $\det f(\zeta) = 0$ and $u(\Pi) \subset \Pi$ then Π also contains all the roots of $\det g(\zeta) = 0$.

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In the special case when $a_0, \ldots, a_n$ are constant, it then follows that $x = 0$ is an asymptotically stable solution of (4) provided that it is an asymptotically stable solution of (1). When $a_0, \ldots, a_n$ are variable matrices, this argument is not valid but the following version of the result is still true:

THEOREM 1. *If* $u(\Pi) \subset \Pi$ *and* (1) *has an AQLF then* (4) *has an AQLF.*

Let E be the set of all polynomials of the form (2), with $m \times m$ matrix coefficients, and let $T_u: E \to E$ be the mapping defined by $T_u f(\zeta) = g(\zeta)$, where $g(\zeta)$ is the polynomial (3). If $v(\zeta) = (\theta\zeta + \phi)/(\psi\zeta + \chi)$ is another Möbius transformation then (3) gives

$$\begin{aligned} T_v g(\zeta) &= (\theta\chi - \phi\psi)^{-\frac{1}{2}n}(\theta - \psi\zeta)^n g(v^{-1}(\zeta)), \\ &= [(\theta\chi - \phi\psi)(\alpha\delta - \beta\gamma)]^{-\frac{1}{2}n}[(\gamma\phi + \alpha\theta) - (\gamma\chi + \alpha\psi)\zeta]^n f(u^{-1}(v^{-1}(\zeta))), \end{aligned}$$

from which it follows that $T_v g(\zeta) = T_{v \circ u} f(\zeta)$, where $v \circ u$ is the composition mapping $\zeta \to v(u(\zeta))$. Hence, $(T_u) \circ (T_v) = T_{u \circ v}$. This shows in particular that T_u is invertible with $(T_u)^{-1} = T_{u^{-1}}$. The systems (1), (4) are therefore reciprocally related and the following holds:

COROLLARY 1·1. *If* $u(\Pi) = \Pi$ *then* (4) *has an AQLF if and only if* (1) *has an AQLF.*

The simplest example with $u(\Pi) = \Pi$ is $u(\zeta) = \zeta^{-1}$ which gives:

COROLLARY 1·2. *The system* $\sum_{r=0}^{n} a_r D^r x = 0$ *has an AQLF if and only if* (1) *has an AQLF.*

In (1), the variables $Dx, D^2x, \ldots, D^nx$ can be replaced everywhere by

$$x(t+1), x(t+2), \ldots, x(t+n),$$

respectively to obtain a corresponding system of difference equations

$$a_0 x(t+n) + a_1 x(t+n-1) + \ldots + a_n x(t) = 0, \tag{5}$$

in which the matrices $a_0, \ldots, a_n$ are functions of $t, x(t), x(t+1), \ldots, x(t+n-1)$. Here t is still regarded as a real variable. In the special case when $a_0, \ldots, a_n$ are constant, the trivial solution of (5) is asymptotically stable if and only if all the roots of the characteristic equation $\det f(\zeta) = 0$ have modulus less than 1. When $a_0, \ldots, a_n$ are variable matrices, a sufficient condition for the Lyapunov stability of $x = 0$ is the existence of a positive definite Hermitian form $W(y_1, y_2, \ldots, y_n)$ with constant coefficients such that $w(t+1) < w(t)$ for all non-zero solutions $x(t)$ of (5), where

$$w(t) = W(x(t), x(t+1), \ldots, x(t+n-1)). \tag{6}$$

such a form W is called an *autonomous quadratic Lyapunov function* (AQLF) of (5). The replacement of $Dx, \ldots, D^nx$ in (4) by $x(t+1), \ldots, x(t+n)$ gives the corresponding system of difference equations

$$b_0 x(t+n) + b_1 x(t+n-1) + \ldots + b_n x(t) = 0, \tag{7}$$

whose characteristic equation is $\det g(\zeta) = 0$. Let Δ denote the set of all complex numbers ζ with $|\zeta| < 1$. If Δ contains all the roots $\lambda_1, \ldots, \lambda_{mn}$ of $\det f(\zeta) = 0$ then Δ also contains the roots $u(\lambda_1), \ldots, u(\lambda_{mn})$ of $\det g(\zeta) = 0$, provided that $u(\Delta) \subset \Delta$. In the

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special case when $a_0, \ldots, a_n$ are constant it then follows that $x = 0$ is an asymptotically stable solution of (7) provided that it is an asymptotically stable solution of (5). When $a_0, \ldots, a_n$ are variable matrices, this argument is not valid but the following version of the result is still true:

THEOREM 2. *If* $u(\Delta) \subset \Delta$ *and* (5) *has an AQLF then* (7) *has an AQLF.*

The above theorems are proved at the end of section 3 together with the following result which connects them:

THEOREM 3. *If* $u(\Pi) = \Delta$ *then* (7) *has an AQLF if and only if* (1) *has an AQLF.*

The property $DV < 0$ of an AQLF of (1) ensures the Lyapunov stability of the trivial solution but is not sufficient to establish its asymptotic stability (except in the special case when (1) is autonomous). However, global asymptotic stability is ensured if there exists a positive constant κ such that $DV < -2\kappa V$ for all non-zero solutions $x(t)$ of (1). When this is so, V is called a *strong* AQLF of (1). The existence of a strong AQLF of (1) is also of interest because it can be used to provide applications of the convergence theorem of Demidovič (see (2), section V). Analogously, W is said to be a *strong* AQLF of (5) if there exists a constant $\rho > 1$ such that $\rho^2 w(t+1) < w(t)$ for every non-zero solution $x(t)$ of (5), where $w(t)$ is given by (6). When 'AQLF' is replaced by 'strong AQLF', Theorem 2 remains true but Theorems 1 and 3 become false. Modified versions of Theorems 1 and 3 for the case of strong AQLF are discussed in section 4. Some practical applications of the above theorems are described in section 5.

2. *Lyapunov and Stein matrices.* The companion matrix $C[f]$ of (2) is the block matrix

$$C[f] = \left(\begin{array}{cccc:c} 0 & I & 0 & & 0 \\ 0 & 0 & I & & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & & I \\ \hdashline -a_0^{-1}a_n & -a_0^{-1}a_{n-1} & -a_0^{-1}a_{n-2} & & -a_0^{-1}a_1 \end{array}\right),$$

where 0, I are zero and unit matrices. The system (1) is equivalent to the $mn \times mn$ system $DX = AX$, where $A = C[f]$ and $X = \mathrm{col}\,(x, Dx, D^2x, \ldots, D^{n-1}x)$. The Hermitian form $V = X^*PX$ is an AQLF of (1) if and only if P is a constant $mn \times mn$ matrix satisfying $P > 0$ and

$$A^*P + PA < 0, \tag{8}$$

for all t, X, where A^* is the conjugate transpose of A and $P > 0$ is understood to mean that P is Hermitian positive definite. Any matrix P with these properties is called a *Lyapunov matrix* of A. If A has a Lyapunov matrix then all the eigenvalues of A lie in Π (but not conversely when A is variable). Similarly, (5) is equivalent to the $mn \times mn$ system of difference equations $X(t+1) = AX(t)$, where $A = C[f]$ and $X(t) = \mathrm{col}\,(x(t), x(t+1), \ldots, x(t+n-1))$. The Hermitian form $W = X^*QX$ is an AQLF of (5) if and only if Q is a constant Hermitian $mn \times mn$ matrix satisfying $Q > 0$ and

$$Q - A^*QA > 0, \tag{9}$$

for all t, X. Any matrix Q with these properties is called a *Stein matrix* of A (see (6)). If A has a Stein matrix then all the eigenvalues of A lie in Δ (but not conversely when A is variable). In the remainder of section 2, A is understood to be any square matrix

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whose elements are functions of t, X and is not necessarily of companion form. Also, the coefficients α, β, γ, δ of $u(\zeta)$ are allowed to be functions of t, X in section 2. The matrix $u(A) = (\alpha A + \beta I)/(\gamma A + \delta I)$ exists provided that $u(\zeta)$ is finite at all the eigenvalues of A. In the proofs which follow, this is not mentioned because its verification is trivial in all cases.

LEMMA 1. *If $u(\Delta) = \Delta$ then Q is a Stein matrix of $u(A)$ if and only if Q is a Stein matrix of A.*

Proof. The most general Möbius transformation with $u(\Delta) = \Delta$ is

$$u(\zeta) = d(\zeta - c)/(\bar{c}\zeta - 1),$$

where c, d are complex numbers with $|c| < 1$, $|d| = 1$. This gives the identity

$$Q - u(A)^* Q u(A) = (1 - |c|^2)(cA^* - I)^{-1}(Q - A^*QA)(\bar{c}A - I)^{-1},$$

which shows that $Q - u(A)^* Q u(A) > 0$ if and only if (9) holds.

LEMMA 2. *If $u(\Pi) = \Delta$ then P is a Stein matrix of $u(A)$ if and only if P is a Lyapunov matrix of A.*

Proof. For the special case $u_1(\zeta) = (\zeta + 1)/(\zeta - 1)$ the identity

$$P - u_1(A)^* P u_1(A) = -2(A^* - I)^{-1}(A^*P + PA)(A - I)^{-1}$$

gives the result at once, as was shown by Taussky(6). The most general Möbius transformation with $u(\Pi) = \Delta$ can be expressed as the composition mapping $u_2 \circ u_1$, where u_2 is any Möbius transformation with $u_2(\Delta) = \Delta$. Lemma 2 now follows by replacing A by $u_1(A)$ in Lemma 1.

THEOREM 4. *If Q is a Stein matrix of A and $u(\Delta) \subset \Delta$ then Q is a Stein matrix of $u(A)$.*

Proof. The frontier of the open disc Δ is unit circle Ω in the complex plane and its image $\Gamma = u(\Omega)$ is also a circle. If $u(\Delta) \subset \Delta$ then either $\Gamma \subset \Delta$ or Γ touches Ω internally or $\Gamma = \Omega$. The case $\Gamma = \Omega$ is treated by Lemma 1. In the case $\Gamma \subset \Delta$, the two circles Γ, Ω belong to a coaxial family of circles having one limit point c inside Γ and the other limit point $1/\bar{c}$ outside Ω. These limit points are mapped by the Möbius transformation $u_3(\zeta) = (\zeta - c)/(\bar{c}\zeta - 1)$ onto the points 0, ∞ and the coaxial family is mapped onto a concentric family with centre at the origin. Clearly $u_3(\Omega) = \Omega$ and $u_3(\Gamma)$ is a circle $|\zeta| = \eta$ with $\eta < 1$. Since the Möbius transformation $u_4(\zeta) = \eta\zeta$ maps the interior of $u_3(\Omega)$ onto the interior of $u_3(\Gamma)$, the most general Möbius transformation of Δ onto the interior of Γ can be expressed as a composition mapping $u_3^{-1} \circ u_4 \circ u_3 \circ u_2$, where u_2 is any Möbius transformation with $u_2(\Delta) = \Delta$. Lemma 1 shows that Q remains a Stein matrix under each of the transformations u_2, u_3, u_3^{-1}. Furthermore, (9) shows that if Q is a Stein matrix of A then Q is also a Stein matrix of $u_4(A) = \eta A$ because $\eta < 1$. Thus Q remains a Stein matrix under the composition transformation $u_3^{-1} \circ u_4 \circ u_3 \circ u_2$ of A and Theorem 4 is established for the case $\Gamma \subset \Delta$. It only remains to consider the case when Γ touches Ω internally. If $\zeta = d$ is the point of contact of these circles then $u_5(\zeta) = (d + \zeta)/(d - \zeta)$ maps d onto ∞ and Δ onto Π. Clearly $u_5(\Omega)$ is the imaginary axis and $u_5(\Gamma)$ is a parallel line $\Re\zeta = -\delta$ with $\delta > 0$. Since the transformation $u_6(\zeta) = \zeta - \delta$ maps Π onto the half plane $\Re\zeta < -\delta$, the most general Möbius

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transformation of Δ onto the interior of Γ can be expressed as the composition $u_5^{-1} \circ u_6 \circ u_5 \circ u_2$, where $u_2(\Delta) = \Delta$. It is obvious from (8) that if Q is a Lyapunov matrix of A then Q is also a Lyapunov matrix of $u_6(A) = A - \delta I$. Since $u_5(\Delta) = \Pi$, Lemma 2 shows that Q remains a Stein matrix under the transformation $u_5^{-1} \circ u_6 \circ u_5$ of A. Then Lemma 1 shows that Q remains a Stein matrix under the transformation $u_5^{-1} \circ u_6 \circ u_5 \circ u_2$ of A. Because the given transformation u is of this form, the proof of Theorem 4 is finished. By means of Lemma 2 we readily deduce from Theorem 4 the following:

COROLLARY 4·1. *If P is a Lyapunov matrix of A and $u(\Pi) \subset \Pi$ then P is a Lyapunov matrix of $u(A)$.*

3. *On companion matrices.* To deduce Theorem 2 from Theorem 4 we need the following relation between the companion matrices $C[f]$, $C[g]$:

THEOREM 5. *If $g(\zeta) = T_u f(\zeta)$ and $u(\zeta)$ is finite at all the eigenvalues of $C[f]$ then $u(C[f]) = MC[g]M^{-1}$, where M is an invertible matrix which depends only on the coefficients $\alpha, \beta, \gamma, \delta$ of $u(\zeta)$.*

Proof. If the result is valid for two particular Möbius transformations u, v then it is also valid for their composition $v \circ u$ because

$$v(u(C[f])) = v(MC[g]M^{-1}) = Mv(C[g])M^{-1} = MNC[h]N^{-1}M^{-1},$$

where $h = T_v g = T_{v \circ u} f$ and N is the similarity matrix corresponding to v. Since any Möbius transformation can be expressed as a composition of transformations of the three special types $\rho\zeta$, ζ^{-1}, $\zeta + \kappa$, it is sufficient to verify the theorem for each of these types.

(i) *In the case* $u(\zeta) = \rho\zeta$, (3) gives $g(\zeta) = \rho^{\frac{1}{2}n} f(\zeta/\rho)$. It is trivial to verify that if $M = \operatorname{diag}(I, \rho^{-1}I, \rho^{-2}I, \ldots, \rho^{1-n}I)$ then

$$MC[g]M^{-1} = \rho C[f] = u(C[f]). \tag{10}$$

Theorem 5 is therefore true in the case $u(\zeta) = \rho\zeta$.

(ii) *In the case* $u(\zeta) = \zeta^{-1}$, (3) gives $g(\zeta) = i^n(a_0 + a_1\zeta + \ldots + a_n\zeta^n)$. It is trivial to verify by direct multiplication that

$$C[f]^{-1} = \begin{pmatrix} -a_n^{-1}a_{n-1} & -a_n^{-1}a_{n-2} & \vdots & -a_n^{-1}a_1 & -a_n^{-1}a_0 \\ I & 0 & \vdots & 0 & 0 \\ 0 & I & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & I & 0 \end{pmatrix}.$$

By inspection, this matrix can be reduced to $C[g]$ by reversing the order of its rows and columns. More precisely, $MC[g]M^{-1} = C[f]^{-1} = u(C[f])$, where M is the following block matrix:

$$M = \begin{pmatrix} 0 & \vdots & 0 & 0 & I \\ 0 & \vdots & 0 & I & 0 \\ 0 & \vdots & I & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ I & \vdots & 0 & 0 & 0 \end{pmatrix}.$$

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(iii) *In the case* $u(\zeta) = \zeta + \kappa$ tedious computation is avoided by using the equivalence of (1) to the system $DX = C[f]X$, where

$$X = \operatorname{col}(x, Dx, \ldots, D^{n-1}x).$$

If $Y = \operatorname{col}(y, Dy, \ldots, D^{n-1}y)$ with $y = e^{\kappa t}x$ then the well-known operator relation $D^r y = e^{\kappa t}(D+\kappa)^r x$ gives $Y = e^{\kappa t}KX$, where

$$K = \begin{pmatrix} I & 0 & 0 & 0 & \vdots \\ \kappa I & I & 0 & 0 & \vdots \\ \kappa^2 I & 2\kappa I & I & 0 & \vdots \\ \kappa^3 I & 3\kappa^2 I & 3\kappa I & I & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

When $x = e^{-\kappa t}y$ is substituted in (1) it becomes

$$0 = \sum_{r=0}^{n} a_{n-r}(D-\kappa)^r y = \sum_{r=0}^{n} c_r D^r y,$$

which is equivalent to the system $DY = C[g_0]\,Y$ with $g_0(\zeta) = \Sigma c_r \zeta^r = f(\zeta - \kappa)$. Since $u(\zeta) = \zeta + \kappa$, (3) gives $g_0 = T_u f$. If $Y = e^{\kappa t}KX$ is now substituted in $DY = C[g_0]\,Y$ it reduces to $DX = -\kappa X + K^{-1}C[g_0]KX$. Since this is the same as the system $DX = C[f]X$ it follows that

$$C[f] = -\kappa I + K^{-1}C[g_0]K. \tag{11}$$

This proves that Theorem 5 holds in the case $u(\zeta) = \zeta + \kappa$ with $M = K^{-1}$. The proof of Theorem 5 is now finished.

Proof of Theorem 3. For any matrix Q, Theorem 5 gives the identity

$$Q - u(C[f])^* Qu(C[f]) = (M^{-1})^* \{R - C[g]^* RC[g]\} M^{-1},$$

where $R = M^*QM$. Since M depends only on $\alpha, \beta, \gamma, \delta$, it is independent of t, x, etc. This identity shows that R is a Stein matrix of $C[g]$ if and only if Q is a Stein matrix of $u(C[f])$. That is, (7) has an AQLF if and only if $u(C[f])$ has a Stein matrix. Since $u(\Pi) = \Delta$, Lemma 2 shows that $u(C[f])$ has a Stein matrix if and only if $C[f]$ has a Lyapunov matrix. That is, (7) has an AQLF if and only if (1) has an AQLF. This establishes Theorem 3.

Proof of Theorem 2. It was shown in the proof of Theorem 3 that (7) has an AQLF if and only if $u(C[f])$ has a Stein matrix. Since $u(\Delta) \subset \Delta$, Theorem 4 shows that $u(C[f])$ has a Stein matrix provided that $C[f]$ has a Stein matrix. That is, (7) has an AQLF provided that (5) has an AQLF. This establishes Theorem 2.

Proof of Theorem 1. For any matrix P, Theorem 5 gives

$$M^*\{u(C[f])^*P + Pu(C[f])\}M = C[g]^*R + RC[g],$$

where $R = M^*PM$. This shows that R is a Lyapunov matrix for $C[g]$ if and only if P is a Lyapunov matrix for $u(C[f])$. That is, (4) has an AQLF if and only if $u(C[f])$ has a Lyapunov matrix. Since $u(\Pi) \subset \Pi$, Corollary 4·1 shows that if $C[f]$ has a Lyapunov matrix then so does $u(C[f])$. That is, if (1) has an AQLF then so does (4). This establishes Theorem 1.

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4. *Strong AQLF*. The Hermitian form $V = X^*PX$ in section 2 satisfies the strong condition $DV < -2\kappa V$ if and only if (8) is replaced by $A^*P + PA + 2\kappa P < 0$. That is, V is a strong AQLF of (1) if and only if P is a Lyapunov matrix of $\kappa I + C[f]$. From (11) we get

$$C[f]^*P + PC[f] + 2\kappa P = K^*\{C[g_0]^*R + RC[g_0]\}K,$$

where $R = (K^{-1})^* PK^{-1}$ and $g_0(\zeta) = f(\zeta - \kappa)$. This shows that P is a Lyapunov matrix of $\kappa I + C[f]$ if and only if R is a Lyapunov matrix of $C[f(\zeta - \kappa)]$. Hence, (1) *has a strong AQLF if and only if, for some constant* $\kappa > 0$, *there exists an ordinary AQLF for the system of differential equations of type* (1) *whose characteristic polynomial is* $f(\zeta - \kappa)$. This reduction of a strong AQLF to an ordinary AQLF is used to prove the following modification of Theorem 1:

THEOREM 6. *If* $u(\Pi) \subset \Pi$, $u(\infty) \in \Pi$ *and* (1) *has a strong AQLF then* (4) *has a strong AQLF*.

Proof. Given an ordinary AQLF for the system of differential equations with characteristic polynomial $f(\zeta - \kappa)$ we must prove that, for some constant $\sigma > 0$, there exists an ordinary AQLF for the system of differential equations with characteristic polynomial $g(\zeta - \sigma)$. Here $g(\zeta) = T_u f(\zeta)$, $f(\zeta - \kappa) = T_k f(\zeta)$, $g(\zeta - \sigma) = T_s g(\zeta)$, where $k(\zeta) = \zeta + \kappa$, $s(\zeta) = \zeta + \sigma$. Since $g(\zeta - \sigma) = (T_s) \circ (T_u) \circ (T_k)^{-1} f(\zeta - \kappa) = T_{s \circ u \circ (k^{-1})} f(\zeta - \kappa)$, the required result follows at once from Theorem 1 if we can find $\sigma > 0$ such that $s \circ u \circ (k^{-1})$ maps Π into itself. The mapping k^{-1} sends Π onto the half plane $\Re\, \zeta < -\kappa$. Since $u(\Pi) \subset \Pi$ and $u(\infty) \in \Pi$, the half plane $\Re\, \zeta < -\kappa$ is mapped by u onto a circular disc whose boundary Γ lies wholly in Π. The mapping s displaces Γ through a distance σ and it remains in the open set Π provided that σ is sufficiently small. Then $s \circ u \circ (k^{-1})$ maps Π into itself. Theorem 6 is now proved. The purpose of the condition $u(\infty) \in \Pi$ is to prevent Γ from touching the imaginary axis because the above argument would then fail. To see that the condition is relevant to Theorem 6 (and not merely to this method of proof) consider the case when $u(\zeta) = \zeta^{-1}$ and (1) is the system

$$Dx + (1 + e^t)x = 0.$$

Then $V = x^*x$ is a strong AQLF of (1), $u(\Pi) \subset \Pi$ but $u(\infty) \notin \Pi$. The corresponding system (4) is $(1 + e^t)Dx + x = 0$ whose general solution is $x(t) = \frac{1}{2}(1 + e^{-t})x(0)$. Since $x = 0$ is not an asymptotically stable solution of (4), this system cannot have a strong AQLF. Theorem 6 therefore becomes false when the condition $u(\infty) \in \Pi$ is deleted. However, Theorem 6 remains true if the condition $u(\infty) \in \Pi$ is replaced by $u(\infty) = \infty$. The proof of this is omitted because it is only a slight modification of that above.

Now consider the analogous problem for difference equations. The Hermitian form $W = X^*QX$ in section 2 satisfies the strong condition $\rho^2 w(t+1) < w(t)$ if and only if (9) is replaced by $Q - \rho^2 A^*QA > 0$. That is, W is a strong AQLF of (5) if and only if Q is a Stein matrix of $\rho C[f]$. From (10) we get

$$Q - \rho^2 C[f]^* QC[f] = (M^{-1})^* \{R - C[g]^* RC[g]\} M^{-1},$$

where $R = M^*QM$ and $g(\zeta) = \rho^{\frac{1}{2}n} f(\zeta/\rho)$. This shows that Q is a Stein matrix of $\rho C[f]$ if and only if R is a Stein matrix of $C[\rho^{\frac{1}{2}n} f(\zeta/\rho)]$. Hence, (5) *has a strong AQLF if and only if, for some constant* $\rho > 1$, *there exists an ordinary AQLF for the system of difference*

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equations of type (5) *whose characteristic polynomial is* $\rho^{\frac{1}{2}n}f(\zeta/\rho)$. This fact is assumed in the proof of the following:

THEOREM 7. *If* $u(\Delta) \subset \Pi$ *and* (5) *has a strong AQLF then* (4) *has a strong AQLF.*

Proof. Given an ordinary AQLF for the system of difference equations with characteristic polynomial $\rho^{\frac{1}{2}n}f(\zeta/\rho)$ we must prove that, for some constant $\sigma > 0$ there exists an ordinary AQLF for the system of differential equations with characteristic polynomial $g(\zeta-\sigma)$. Here $g(\zeta) = T_u f(\zeta)$, $\rho^{\frac{1}{2}n}f(\zeta/\rho) = T_r f(\zeta)$, $g(\zeta-\sigma) = T_s g(\zeta)$, where $r(\zeta) = \rho\zeta$, $s(\zeta) = \zeta+\sigma$. Since $g(\zeta-\sigma) = T_{s\circ u\circ(r^{-1})}\rho^{\frac{1}{2}n}f(\zeta/\rho)$, the required result follows at once from Theorems 3 and 1 provided that $s\circ u\circ(r^{-1})$ maps Δ into Π for some $\sigma > 0$. The frontier of $r^{-1}(\Delta)$ is the circle $|\zeta| = \rho^{-1}$ which lies in Δ because $\rho > 1$. Since $u(\Delta) \subset \Pi$, this circle is mapped by u onto a circle Γ which lies in Π. When displaced through a distance σ by the mapping s the circle remains in Π provided that σ is sufficiently small. That is, $s\circ u\circ(r^{-1})$ maps Δ into Π for small $\sigma > 0$. This establishes Theorem 7. The next two theorems can be derived in a similar way from Theorems 2 and 3. Their proofs are omitted because they follow closely the pattern of the proofs of Theorems 6 and 7.

THEOREM 8. *If* $u(\Pi) \subset \Delta$, $u(\infty)\in\Delta$ *and* (1) *has a strong AQLF then* (7) *has a strong AQLF.*

THEOREM 9. *If* $u(\Delta) \subset \Delta$ *and* (5) *has a strong AQLF then* (7) *has a strong AQLF.*

5. *Applications.* Consider the case when (4) is the real scalar equation

$$b_0 D^2x + b_1 Dx + b_2 x = 0, \tag{12}$$

where b_0, b_1, b_2 are continuous functions of t, x, Dx. Then (1) is a constant multiple of the equation

$$a_0 D^2x + a_1 Dx + a_2 x = 0, \tag{13}$$

with coefficients given by $a_0 = \alpha^2b_0+\alpha\gamma b_1+\gamma^2b_2$, $a_1 = 2\alpha\beta b_0+(\alpha\delta+\beta\gamma)b_1+2\gamma\delta b_2$, $a_2 = \beta^2b_0+\beta\delta b_1+\delta^2b_2$. Starzhinskii (3) showed that (12) has an AQLF if

$$\lambda \leqslant b_1/b_0 \leqslant \Lambda, \quad \mu \leqslant b_2/b_0 \leqslant M, \tag{14}$$

for all t, x, Dx, where λ, Λ, μ, M are any positive constants satisfying

$$0 \leqslant \sqrt{M}-\sqrt{\mu} < \lambda < \Lambda < [M+5\mu+2\sqrt{(M\mu)}]/(\sqrt{M}-\sqrt{\mu}).$$

If α, β, γ, δ are real constants such that $u(\Pi) \subset \Pi$ then Theorem 1 shows that the application of the Starzhinskii conditions (14) to equation (13) gives a new set of sufficient conditions for (12) to have an AQLF, namely

$$\left.\begin{aligned} \lambda &\leqslant [\alpha^2b_0+\alpha\gamma b_1+\gamma^2b_2]^{-1}[2\alpha\beta b_0+(\alpha\delta+\beta\gamma)b_1+2\gamma\delta b_2] \leqslant \Lambda, \\ \mu &\leqslant [\alpha^2b_0+\alpha\gamma b_1+\gamma^2b_2]^{-1}[\beta^2b_0+\beta\delta b_1+\delta^2b_2] \leqslant M. \end{aligned}\right\} \tag{15}$$

The condition $u(\Pi) \subset \Pi$ is satisfied if $0 > u(-1) > -\infty$ and $0 \geqslant u(0)$, $u(\infty) \geqslant -\infty$. That is, $u(\Pi) \subset \Pi$ if the real constants α, β, γ, δ satisfy $0 > (\beta-\alpha)/(\delta-\gamma) > -\infty$, $0 \geqslant \beta/\delta$, $\alpha/\gamma \geqslant -\infty$ and $\alpha\delta \neq \beta\gamma$. *From the Starzhinskii condition* (14) *we thus obtain an infinite number of other conditions corresponding to other Lyapunov functions of* (12). Furthermore, Theorem 3 shows that conditions (15) are sufficient for the scalar difference equation

$$b_0 x(t+2) + b_1 x(t+1) + b_2 x(t) = 0 \tag{16}$$

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to have an AQLF provided that α, β, γ, δ are real constants satisfying $u(\Pi) = \Delta$. In fact, Theorem 2 shows that it is sufficient to have only $u(\Pi) \subset \Delta$ and this is so if $-1 < u(-1) < 1$ and $-1 \leqslant u(0), u(\infty) \leqslant 1$. That is, *if* α, β, γ, δ *are any real constants satisfying*

$$-1 < (\beta-\alpha)/(\gamma-\delta) < 1, \quad -1 \leqslant \beta/\delta, \quad \alpha/\gamma \leqslant 1, \quad \alpha\delta \neq \beta\gamma,$$

and b_0, b_1, b_2 *are any functions of* t, $x(t)$, $x(t+1)$ *satisfying* (15) *then* (16) *has an AQLF.*

In (2) the Starzhinskii conditions (14) are generalized so as to apply to vector systems of the form (12). These can be made to yield other conditions generalizing (15) and conditions for vector systems of the form (16) to have an AQLF. Furthermore, Starzhinskii (4), (5) has obtained conditions similar to (14) which are sufficient for certain differential equations of the third and fourth orders to have an AQLF. These can be made to yield new conditions analogous to (15), and similar conditions for difference equations of the third and fourth orders.

REFERENCES

(1) Hahn, W. *Theory and application of Liapunov's direct method* (Prentice-Hall; Englewood Cliffs, 1963).

(2) Smith, R. A. Stability of certain systems of second order differential equations. *Ann. Mat. Pura Appl.* (4) 82 (1969), 257–266.

(3) Starzhinskii, V. M. Sufficient conditions for stability of a mechanical system with one degree of freedom (Russian). *Prikl. Mat. Meh.* **16** (1952), 369–374.

(4) Starzhinskii, V. M. On stability of unsteady motions in one case (Russian). *Prikl. Mat. Meh.* **16** (1952), 500–504.

(5) Starzhinskii, V. M. On stability of unsteady motions in a special case (Russian). *Prikl. Mat. Meh.* **19** (1955), 471–480.

(6) Taussky, O. Matrices C with $C^n \to 0$. *J. Algebra* **1** (1964), 5–10.

https://doi.org/10.1017/S0305004100001158 Published online by Cambridge University Press