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Lifting mixing properties by Rokhlin cocycles

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Abstract. We study the problem of lifting various mixing properties from a base automorphism $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ to skew products of the form $T_{\varphi,\mathcal{S}}$, where $\varphi : X \to G$ is a cocycle with values in a locally compact Abelian group G, $\mathcal{S} = (S_g)_{g\in G}$ is a measurable representation of G in $\mathrm{Aut}(Y, \mathcal{C}, \nu)$ and $T_{\varphi,\mathcal{S}}$ acts on the product space $(X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu)$ by

$$T_{\varphi,\mathcal{S}}(x, y) = (Tx, S_{\varphi(x)}(y)).$$

It is also shown that whenever T is ergodic (mildly mixing, mixing) but $T_{\varphi,\mathcal{S}}$ is not ergodic (is not mildly mixing, not mixing), then, on a non-trivial factor $\mathcal{A} \subset \mathcal{C}$ of $\mathcal{S}$, the corresponding Rokhlin cocycle $x \mapsto S_{\varphi(x)}|_{\mathcal{A}}$ is a coboundary (a quasi-coboundary).

1. *Introduction*

Given an ergodic automorphism T of a standard Borel space $(X, \mathcal{B}, \mu)$, we can study various extensions $\widetilde{T}$ of it. Among such extensions, a special role is played by so-called compact group extensions or, more generally, isometric extensions (see [**9, 12, 31**]). In particular, one can ask which ergodic properties of T are lifted by isometric extensions. Two papers by Dan Rudolph, [**26**] where it is proved that Bernoullicity is lifted whenever the extension is weakly mixing and [**27**] where it is shown that mixing (multiple mixing) lifts whenever the extension is weakly mixing, are beautiful examples of the mechanism whereby once the extension enjoys some 'minimal' ergodic property then it shares some strong ergodic properties assumed to hold for its base. By iterating the procedure of taking isometric extensions, we can hence lift ergodic properties of T to weakly mixing distal extensions of it.

The notion complementary to distality is relative weak mixing [**9, 12, 31**] and a natural question arises: what happens with lifting ergodic properties from T to $\widetilde{T}$ when $\widetilde{T}$ is relatively weakly mixing over the factor T? This, by Abramov–Rokhlin's theorem [**2**], leads to the study of so-called Rokhlin cocycle extensions, which are automorphisms of

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the form $\widetilde{T} = T_\Theta$ acting on $(X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu)$ by the formula

$$T_\Theta(x, y) = (Tx, \Theta_x(y)),$$

where the map $\Theta : X \to \mathrm{Aut}(Y, \mathcal{C}, \nu)$, often called a Rokhlin cocycle, is measurable. Since the above formula describes all possible (ergodic) extensions of T, it is hard to expect interesting theorems at such a level of generality—one has to specify subclasses of Rokhlin cocycles for which one can obtain some results. We will focus on the following class.

Let G be a second-countable locally compact Abelian (LCA) group. Assume that we have a measurable action $\mathcal{S}$ of this group given by $g \mapsto S_g \in \mathrm{Aut}(Y, \mathcal{C}, \nu)$. Let $\varphi : X \to G$ be a cocycle. The automorphism $T_{\varphi,\mathcal{S}}$ acting on $(X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu)$ given by

$$T_{\varphi,\mathcal{S}}(x, y) = (Tx, S_{\varphi(x)}(y))$$

will be called the *Rokhlin* $(\varphi, \mathcal{S})$*-extension*† of T.

A systematic study of the problem of lifting ergodic properties from T to $T_{\varphi,\mathcal{S}}$ was originated by Rudolph in [**28**]. Since then, extensions $T_{\varphi,\mathcal{S}} \to T$ have been studied in numerous papers; see [**5, 11–13, 22, 23, 25, 29**], for example.

The present paper is a continuation of investigations from [**22, 23**], and, as a result of the new approach presented here, makes them complete. This new approach is based on a harmonic analysis result from [**18**], and it consists of showing that, given an action $\mathcal{S} = (S_g)_{g\in G}$ of a second-countable LCA group G on a probability standard Borel space $(Y, \mathcal{C}, \nu)$ and a saturated Borel subgroup $\Lambda \subset \widehat{G}$, the spectral space of functions in $L^2(Y, \mathcal{C}, \nu)$ whose spectral measures are concentrated on Λ is the L^2-space of a $\mathcal{S}$-invariant sub-σ-algebra $\mathcal{A} \subset \mathcal{C}$ (a measure-theoretic factor of $\mathcal{S}$). This will systematically be used in our study because the group of L^∞-eigenvalues of the Mackey G-action associated to T and φ is saturated and hence yields a $\mathcal{S}$-factor.

Using this, we will prove natural necessary and sufficient conditions for weak mixing of $T_{\varphi,\mathcal{S}}$ and relatively weak mixing of $T_{\varphi,\mathcal{S}}$ over T. We also compute possible eigenvalues of $T_{\varphi,\mathcal{S}}$ and determine the relative Kronecker factor whenever $T_{\varphi,\mathcal{S}}$ is ergodic. The idea of a factor determined by a saturated group allows us to prove that if T is ergodic but $T_{\varphi,\mathcal{S}}$ is not, then the Rokhlin cocycle $x \mapsto S_{\varphi(x)}|_{\mathcal{A}}$ is a coboundary as a cocycle taking values in $\mathrm{Aut}(\mathcal{A})$, where $\mathcal{A}$ is the non-trivial factor of $\mathcal{S}$ corresponding to the above-mentioned eigenvalue group. Finally, by replacing coboundary by quasi-coboundary, a similar conclusion is achieved when T is mildly mixing but $T_{\varphi,\mathcal{S}}$ is not, and when T is mixing but $T_{\varphi,\mathcal{S}}$ is not.

Another tool explored here is use of mixing sequences of weighted unitary operators, that is, of operators on $L^2(X, \mathcal{B}, \mu)$, given by the formula

$$f \mapsto \xi \cdot f \circ T \quad \text{for each } f \in L^2(X, \mathcal{B}, \mu),$$

determined by a measurable $\xi : X \to \mathbb{T}$ and an automorphism T. This, in particular, will solve the problem of lifting the mild mixing property, and complete the picture from [**23**] of lifting mixing and multiple mixing.

† We would like to emphasize that, as noted in [**5**], if we admit G to be non-Abelian locally compact, then each ergodic extension $\widetilde{T} = T_\Theta$ is of the form $T_{\varphi,\mathcal{S}}$; more specifically, a general Rokhlin cocycle $x \mapsto \Theta(x)$ is cohomologous to a cocycle $x \mapsto S_{\varphi(x)}$ for some G, φ and $\mathcal{S}$.

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2. *Preliminaries*

We briefly recall basic definitions and some known results, and fix notation for the rest of the paper.

2.1. *Self-joinings of an automorphism; relative concepts.* Assume that T is an automorphism of a standard probability Borel space $(X, \mathcal{B}, \mu)$, which we denote by $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$. We shall also denote by T the unitary operator $f \mapsto f \circ T$ on $L^2(X, \mathcal{B}, \mu)$. Denote by $J(T)$ the set of self-joinings of T; this means the set of $T \times T$-invariant probability measures on $(X \times X, \mathcal{B} \otimes \mathcal{B})$ whose marginals are both equal to μ. To each self-joining $\eta \in J(T)$, one associates a Markov operator† Φ_η of $L^2(X, \mathcal{B}, \mu)$, given by

$$\int_X \Phi_\eta f(y) g(y)\, d\mu(y) = \int_{X \times X} f(x) g(y)\, d\eta(x, y),$$

for each $f, g \in L^2(X, \mathcal{B}, \mu)$. Moreover, the $T \times T$-invariance of η means that

$$\Phi_\eta \circ T = T \circ \Phi_\eta. \tag{1}$$

On the other hand, each Markov operator Φ on $L^2(X, \mathcal{B}, \mu)$ for which (1) holds determines a self-joining η_Φ by the formula

$$\eta_\Phi(A \times B) = \int_B \Phi(1_A)\, d\mu$$

for each $A, B \in \mathcal{B}$. Then

$$\Phi = \Phi_{\eta_\Phi} \quad \text{and} \quad \eta = \eta_{\Phi_\eta}. \tag{2}$$

Therefore the set $J(T)$ can naturally be identified with the set $\mathcal{J}(T)$ of Markov operators on $L^2(X, \mathcal{B}, \mu)$ satisfying (1). The set $\mathcal{J}(T)$ is a closed subset in the weak operator topology and hence it is compact. Thus

$$\Phi_n \to \Phi \text{ if and only if } \langle \Phi_n f, g\rangle \to \langle \Phi f, g\rangle \text{ for each } f, g \in L^2(X, \mathcal{B}, \mu).$$

By transferring the weak operator topology via (2), we obtain the weak topology on $J(T)$ and

$$\eta_n \to \eta \text{ if and only if } \eta_n(A \times B) \to \eta(A \times B) \text{ for each } A, B \in \mathcal{B}.$$

Since the composition of two Markov operators is Markov, $\mathcal{J}(T)$ is a compact semitopological semigroup (a semigroup is said to be *semitopological* if it is endowed with a topology under which the multiplication is separately continuous). By the same token, $J(T)$ is also a compact semitopological semigroup ($\eta_1 \circ \eta_2 := \eta_{\Phi_{\eta_1} \circ \Phi_{\eta_2}}$).

Given a *factor*‡, i.e., a T-invariant sub-σ-algebra $\mathcal{A} \subset \mathcal{B}$, let

$$\mu = \int_{X/\mathcal{A}} \delta_{\overline{x}} \otimes \mu_{\overline{x}}\, d\mu(\overline{x})$$

† A linear bounded operator Φ of $L^2(X, \mathcal{B}, \mu)$ is called *Markov* if $\Phi(1) = 1 = \Phi^*(1)$ and $\Phi f \geq 0$ whenever $f \geq 0$. Note also that we always have $\|\Phi_\eta f\| \leq \|f\|$ and thus $\|\Phi_\eta\| = 1$.

‡ With a little abuse of notation, we define the *factor system* $T|_{\mathcal{A}} : (X/\mathcal{A}, \mathcal{A}, \mu|_{\mathcal{A}}) \to (X/\mathcal{A}, \mathcal{A}, \mu|_{\mathcal{A}})$ in which cosets $\overline{x} \in X/\mathcal{A}$ are given by those points which cannot be distinguished by the sets from $\mathcal{A}$; then $T|_{\mathcal{A}}(\overline{x}) = \overline{Tx}$.

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be the disintegration of μ over the factor $\mathcal{A}$. By setting

$$\mu \otimes_{\mathcal{A}} \mu = \int_{X/\mathcal{A}} \delta_{\overline{x}} \otimes \mu_{\overline{x}} \otimes \mu_{\overline{x}} \, d\mu(\overline{x}),$$

we obtain a self-joining $\mu \otimes_{\mathcal{A}} \mu$, which is often called the *relative product over* $\mathcal{A}$. Note that $\mu \otimes_{\mathcal{A}} \mu|_{\mathcal{A}\otimes\mathcal{A}} = \Delta_{\mathcal{A}}$, where $\Delta_{\mathcal{A}}(A_1 \times A_2) = \mu(A_1 \cap A_2)$ for each $A_1, A_2 \in \mathcal{A}$. Moreover, we have $\Phi_{\mu\otimes_{\mathcal{A}}\mu} = E(\cdot|\mathcal{A})$.

Assume additionally that T is ergodic. Then we can speak about ergodic self-joinings of T and the set of such joinings will be denoted by $J^e(T)$. By $\mathcal{J}^e(T)$, we denote the subset of $\mathcal{J}(T)$ corresponding to $J^e(T)$. The elements of $\mathcal{J}^e(T)$ are exactly the extremal points in the natural simplex structure of $\mathcal{J}(T)$. Recall that T is said to be *relatively weakly mixing* over a factor $\mathcal{A}$ if $E(\cdot|\mathcal{A}) \in \mathcal{J}^e(T)$.

For more about joinings or relative concepts in ergodic theory, see [**9, 12, 20, 29, 31**], for example.

2.2. *G-actions.* Assume that G is a second-countable LCA group. By a G-action $\mathcal{S} = (S_g)_{g\in G}$ we mean a *measurable representation* of G on a probability standard Borel space $(Y, \mathcal{C}, \nu)$, that is, a group homomorphism $g \mapsto S_g$, $G \to \mathrm{Aut}(Y, \mathcal{C}, \nu)$. Then we also denote by $\mathcal{S} = (S_g)_{g\in G}$ the associated unitary representation of G on $L^2(Y, \mathcal{C}, \nu)$, which is continuous. For each $f \in L^2(Y, \mathcal{C}, \nu)$, by $\sigma_{f,\mathcal{S}}$ (or σ_f if $\mathcal{S}$ is understood), we denote the *spectral measure* of f, i.e., the measure on the character group $\widehat{G}$ determined by the Fourier transform

$$\widehat{\sigma}_{f,\mathcal{S}}(g) := \int_{\widehat{G}} \chi(g) \, d\sigma_{f,\mathcal{S}}(\chi) = \int_Y f \circ S_g \cdot \overline{f} \, d\nu$$

(recall that $\widehat{G}$ is still a second-countable LCA group and, by the Pontryagin Duality theorem, the character group of $\widehat{G}$ has a natural identification with G). We denote $G(f) = \overline{\mathrm{span}}\{S_g f : g \in G\}$. Then the correspondence $f \to 1_{\widehat{G}}$ yields the canonical isomorphism of $\mathcal{S}|_{G(f)}$ with the representation $\mathcal{V}_{\sigma_f} = (V_g^{\sigma_f})_{g\in G}$ of $L^2(\widehat{G}, \mathcal{B}(\widehat{G}), \sigma_f)$, where $V_g^{\sigma_f} j(\chi) = \chi(g) j(\chi)$. The maximal spectral type of $\mathcal{S}$ on $L^2_0(Y, \mathcal{C}, \nu)$ (the subspace of zero mean functions in $L^2(Y, \mathcal{C}, \nu)$) will be denoted by $\sigma_{\mathcal{S}}$. Formally speaking, it is the class of equivalence of measures which are maximal spectral measures but in what follows we abuse the vocabulary and often speak about a given measure as the maximal spectral type.

For more about the spectral theory of G-actions, see [**20, 21**], for example.

Suppose that $\mathcal{S}_i = (S_g^{(i)})_{g\in G}$ is a G-action on $(Y_i, \mathcal{C}_i, \nu_i)$, $i = 1, 2$. By a *joining* of these two G-actions we mean an $(S_g^{(1)} \times S_g^{(2)})_{g\in G}$-invariant measure on $(Y_1 \times Y_2, \mathcal{C}_1 \otimes \mathcal{C}_2)$ with projections ν_1 and ν_2, respectively. Slightly generalizing §2.1, η determines a Markov intertwining operator $\Phi_\eta : L^2(Y_1, \mathcal{C}_1, \nu_1) \to L^2(Y_2, \mathcal{C}_2, \nu_2)$; a correspondence similar to (2) also takes place. Recall that $\mathcal{S}_1$ and $\mathcal{S}_2$ are called *disjoint* (in the sense of Furstenberg [**8**]) if the only possible joining between them is product measure. We then write $\mathcal{S}_1 \perp \mathcal{S}_2$. It is well known [**15**] that

$$\sigma_{\mathcal{S}_1} \perp \sigma_{\mathcal{S}_2} \Rightarrow \mathcal{S}_1 \perp \mathcal{S}_2.$$

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Denote by $M(\widehat{G})$ the convolution Banach algebra of all complex Borel measures on $\widehat{G}$†. Let $M^+(\widehat{G}) \subset M(\widehat{G})$ (respectively $M^{+,1}(\widehat{G}) \subset M(\widehat{G})$) consist of non-negative members of $M(\widehat{G})$ (of all probability measures in $M(\widehat{G})$).

Assume that G is not compact. Recall that $\sigma \in M^+(\widehat{G})$ is called *Dirichlet* if $\limsup_{g\to\infty} |\widehat{\sigma}(g)| = \sigma(\widehat{G})$ or, equivalently, if there exists a sequence $g_n \to \infty$ in G such that $\widehat{\sigma}(g_n) \to \sigma(\widehat{G})$.

If $\sigma = \sigma_{\mathcal{S}}$ is Dirichlet, then $\mathcal{S}$ is called *rigid*, and $\widehat{\sigma}(g_n) \to \sigma(\widehat{G})$ is equivalent to $S_{g_n} \to \mathrm{Id}$ in the strong (or, which here is the same, in the weak) operator topology. In this case, (g_n) is called a *rigidity sequence* for $\mathcal{S}$.

2.3. *G-valued cocycles for an ergodic automorphism.* Assume that $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$. Let G be a second-countable LCA group. Here and throughout the paper, we use multiplicative notation. Let $\varphi : X \to G$ be measurable. This determines a cocycle $\varphi(n, x) = \varphi^{(n)}(x)$‡ by the following formula

$$\varphi^{(n)}(x) = \begin{cases} \varphi(x) \cdot \varphi(Tx) \cdot \ldots \cdot \varphi(T^{n-1}x) & \text{if } n > 0, \\ 1 & \text{if } n = 0, \\ (\varphi(T^n x) \cdot \ldots \cdot \varphi(T^{-1}x))^{-1} & \text{if } n < 0. \end{cases}$$

Let us recall now the definitions of two groups related to T and φ that play basic roles in the study of Rokhlin cocycle extensions (studied in §4). We denote $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

The group Λ_φ: this is a Borel subgroup of $\widehat{G}$, defined as

$$\Lambda_\varphi = \{\chi \in \widehat{G} : \chi \circ \varphi = \xi/\xi \circ T \text{ for a measurable } \xi : X \to \mathbb{T}\}.$$

This group turns out to be the group of L^∞-eigenvalues of the Mackey action (of G) associated to the cocycle φ (see [**1, 16, 19, 23**], for example).

The group Σ_φ: this is a Borel subgroup of $\widehat{G}$ defined as

$$\Sigma_\varphi = \{\chi \in \widehat{G} : \chi \circ \varphi = c \cdot \xi/\xi \circ T \text{ for a measurable } \xi : X \to \mathbb{T} \text{ and } c \in \mathbb{T}\}.$$

3. *Tools*

In this section, we will present tools that will be needed to prove lifting of various properties by Rokhlin cocycles (see §4). Some of the results that will be presented here are new and seem to be of independent interest (see §3.2).

3.1. *Idempotents in $J(T)$.* Assume that $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$. Then the closure of the group $\{T^j : j \in \mathbb{Z}\}$ in $\mathcal{J}(T)$, denoted by $\overline{\{T^j : j \in \mathbb{Z}\}}$, is a closed subsemigroup of $\mathcal{J}(T)$, and therefore

$$\overline{\{T^j : j \in \mathbb{Z}\}} \text{ is a semitopological compact semigroup.} \tag{3}$$

† Since $\widehat{G}$ is Polish, all members of $M(\widehat{G})$ are regular measures.

‡ $\varphi(\cdot, \cdot)$ satisfies the cocycle identity $\varphi(m + n, \cdot) = \varphi(m, \cdot) \cdot \varphi(n, T^m \cdot)$; it is often φ itself which is called a cocycle. A cocycle $\varphi : X \to G$ is called a *coboundary* if $\varphi = f/f \circ T$ for a measurable $f : X \to G$. If two cocycles differ by a coboundary, then they are called *cohomologous*. A cocycle is said to be a *quasi-coboundary* if it is cohomologous to a constant cocycle.

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Given a factor $\mathcal{A} \subset \mathcal{B}$, we have $E(\cdot|\mathcal{A}) = \Phi_{\mu\otimes_{\mathcal{A}}\mu}$. Note that given $\Phi \in \mathcal{J}(T)$,

$$\Phi \circ E(\cdot|\mathcal{A}) = E(\cdot|\mathcal{A}) \text{ if and only if } \Phi f = f \text{ for each } f \in L^2(\mathcal{A}). \tag{4}$$

It follows that

$$\Phi \circ E(\cdot|\mathcal{A}) = E(\cdot|\mathcal{A}) \text{ if and only if } \eta_\Phi|_{\mathcal{A}\otimes\mathcal{A}} = \Delta_{\mathcal{A}}. \tag{5}$$

Indeed, assume $\eta_\Phi|_{\mathcal{A}\otimes\mathcal{A}} = \Delta_{\mathcal{A}}$, fix $f \in L^2(\mathcal{A})$ and let $g \in L^2(\mathcal{A})$ be arbitrary. Then

$$\int_X \Phi f(y)g(y)\,d\mu(y) = \int_{X\times X} f(x)g(y)\,d\eta_\Phi(x, y) = \int_X fg\,d\mu.$$

Since g was arbitrary in $L^2(\mathcal{A})$, $\Phi f - f$ is orthogonal to $L^2(\mathcal{A})$. But we must have $\|\Phi f\| \le \|f\|$, and thus $\Phi f = f$. Conversely, if $\Phi f = f$ for each $f \in L^2(\mathcal{A})$, we get the same equalities for all $f, g \in L^2(\mathcal{A})$, whence $\eta_\Phi|_{\mathcal{A}\otimes\mathcal{A}} = \Delta_{\mathcal{A}}$. Now (5) follows from (4).

In view of (5), we obtain that

$$\{\Phi \in \mathcal{J}(T) : \eta_\Phi|_{\mathcal{A}\otimes\mathcal{A}} = \Delta_{\mathcal{A}}\} \text{ is a compact semitopological semigroup.} \tag{6}$$

LEMMA 1. *Assume that $\eta \in J(T)$. Then*

$$\begin{aligned} &L^2(\mathcal{B}\otimes\{\emptyset, X\}, \eta) \cap L^2(\{\emptyset, X\}\otimes\mathcal{B}, \eta) \\ &\quad = \{f\otimes 1 : f \in L^2(X, \mathcal{B}, \mu),\ \|\Phi_\eta f\| = \|f\|\}. \end{aligned} \tag{7}$$

Proof. Suppose that $f(x) = g(y)$ for η-a.e. $(x, y) \in X \times X$ (a.e. denotes almost every). Then we have $\Phi_\eta f = g$ and

$$\|f\|^2 = \int_{X\times X} |f(x)|^2\,d\eta(x, y) = \int_{X\times X} |g(y)|^2\,d\eta(x, y) = \|g\|^2,$$

so $\|\Phi_\eta f\| = \|f\|$.

On the other hand, take $f \in L^2(X, \mathcal{B}, \mu)$ satisfying $\|\Phi_\eta f\| = \|f\|$. Then, similarly,

$$\int_{X\times X} |f(x)|^2\,d\eta(x, y) = \int_{X\times X} |\Phi_\eta f(y)|^2\,d\eta(x, y).$$

But, immediately from the definition, the function $(x, y) \mapsto \Phi_\eta f(y)$ is the orthogonal projection of $(x, y) \mapsto f(x)$ on the subspace $L^2(\{\emptyset, X\}\otimes\mathcal{B})$ of $L^2(X\times X, \mathcal{B}\otimes\mathcal{B}, \eta)$. It follows that $f(x) = \Phi_\eta f(y)$ for η-a.e. (x, y). □

The σ-algebra $\mathcal{B}\otimes\{\emptyset, X\} \cap \{X, \emptyset\}\otimes\mathcal{B}$ (modulo η) can be seen on one hand as a factor $\mathcal{B}_1(\eta)\otimes\{\emptyset, X\}$ of $\mathcal{B}\otimes\{\emptyset, X\}$, and on the other hand as a factor $\{\emptyset, X\}\otimes\mathcal{B}_2(\eta)$ of $\{\emptyset, X\}\otimes\mathcal{B}$. This defines two factors $\mathcal{B}_1(\eta)$, $\mathcal{B}_2(\eta)$ of $(X, \mathcal{B}, \mu)$, the largest factors identified by the joining η.

We can see factors of the form $\mathcal{B}_1(\eta)$ in a different way. Indeed, given $\eta \in J(T)$, define

$$\mathcal{B}(\eta) := \{A \in \mathcal{B} : \eta((A\times X)\triangle(X\times A)) = 0\}.$$

Then $L^2(\mathcal{B}(\eta)) = \{f \in L^2(X, \mathcal{B}, \mu) : \Phi_\eta f = f\}$. Indeed, from the von Neumann theorem for contractions,

$$\frac{1}{N}\sum_{n=0}^{N-1} \Phi_\eta^n \to \mathrm{proj}_{\mathrm{Fix}(\Phi_\eta)}, \tag{8}$$

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and since the limit is an idempotent and a Markov operator, it is the orthogonal projection on the L^2-space of a factor†.

Recall also that if W is a contraction of a Hilbert space H, then so is its adjoint, and then $W^*Wf = f$ if and only if $\|Wf\| = \|f\|$. Hence, for any $\eta \in J(T)$,

$$\mathcal{B}_1(\eta) = \mathcal{B}(\eta^* \circ \eta),$$

where $\eta^* := \eta_{\Phi^*}$.

Whenever $\mathcal{A} \subset \mathcal{B}$ is a factor of T, the relative product $\mu \otimes_{\mathcal{A}} \mu$ is an idempotent in $J(T)$. The following result states that this is the only way to obtain idempotents in $J(T)$ (compare to [**12**, Theorem 6.9], where self-adjoint idempotents of $\mathcal{J}(T)$ are shown to correspond to factors).

PROPOSITION 1. *Assume that η is an idempotent in $J(T)$. Then there exists a factor $\mathcal{A}$ of T such that $\eta = \mu \otimes_{\mathcal{A}} \mu$.*

Proof. In view of (8), Φ_η must be an orthogonal projection and it is an isometry exactly on its range. Now, by Lemma 1, Φ_η is an isometry exactly on $L^2(\mathcal{B}_1(\eta))$. Therefore it is the orthogonal projection onto $L^2(\mathcal{B}_1(\eta))$, that is, $\Phi_\eta = E(\cdot|\mathcal{B}_1(\eta))$, and the result follows. □

3.2. *Canonical factor of a G-action associated to a saturated Borel subgroup.* Assume that Λ is a Borel subgroup of $\widehat{G}$. Let us recall (see [**18**]) that if $\sigma, \tau \in M^{+,1}(\widehat{G})$, then τ *sticks to* σ if

$$\widehat{\sigma}(g_j) \to 1 \Longrightarrow \widehat{\tau}(g_j) \to 1$$

for any sequence $(g_j)_{j\geq 1}$ in G going to infinity. Following [**18**] (see [**18**, Theorem 2.2]), one says that Λ is *saturated* if, for any $\sigma, \tau \in M^{+,1}(\widehat{G})$,

$$\sigma(\Lambda) = 1 \text{ and } \tau \text{ sticks to } \sigma \implies \tau(\Lambda) = 1.$$

THEOREM 1. [**18**, Theorem 3.4] *Every group Λ_φ is saturated.*

Remark 1. As noted, for example in [**23**], every subgroup Σ_φ is also of the form Λ_ψ, whence Σ_φ is also a saturated subgroup.

We will also need the following characterization of saturated groups.

THEOREM 2. [**18**, Theorem 2.1] *A Borel subgroup $\Lambda \subset \widehat{G}$ is saturated if and only if, for any $\tau \in M^+(\widehat{G})$, the indicator function 1_Λ belongs to the closed convex hull in $L^1(\widehat{G}, \tau)$ of the characters of $\widehat{G}$.*

The following corollary describes a dynamical consequence of Theorem 2, which seems to appear for the first time in the literature. Given a Borel subset Λ of $\widehat{G}$, we denote by H_Λ the spectral subspace corresponding to Λ, i.e., the space of those elements in $L^2(Y, \mathcal{C}, \nu)$ whose spectral measures are concentrated on Λ. We denote by $\tilde{g}$ the character of $\widehat{G}$ associated by Pontryagin duality to $g \in G : \tilde{g}(\chi) := \chi(g)$ for $\chi \in \widehat{G}$.

† The limit is still a Markov operator, say $\Phi_{\eta'}$, and also an orthogonal projection. Therefore $\|\Phi_{\eta'} f\| = \|f\|$ implies $\Phi_{\eta'} f = f$ and thus $\Phi_\eta f = f$. It then follows from Lemma 1 that $\mathrm{Fix}(\Phi_\eta)$ is the L^2-space of a factor; see also [**6**, Theorem A, p. 80].

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COROLLARY 1. *Let $\mathcal{S}$ be an action of G on $(Y, \mathcal{C}, \nu)$ and Λ be a saturated subgroup of $\widehat{G}$. Then $H_\Lambda = L^2(\mathcal{A})$, where $\mathcal{A} \subset \mathcal{C}$ is a factor of $\mathcal{S}$.*

Proof. First note that in Theorem 2, L^1-convergence can be replaced by L^2-convergence; in particular,

$$\sum_{k=1}^{N_n} a_k^{(n)} \tilde{g}_k^{(n)}(\chi) \to 1_\Lambda \quad \text{in } L^2(\widehat{G}, \mathcal{B}(\widehat{G}), \sigma_{\mathcal{S}})$$

for some $a_k^{(n)} \geq 0$ with $\sum_{k=1}^{N_k} a_k^{(n)} = 1$ and some $g_k^{(n)} \in G$. Then, for each $\tau \in M^+(\widehat{G})$, $\tau \ll \sigma_{\mathcal{S}}$ we still have

$$\sum_{k=1}^{N_n} a_k^{(n)} \tilde{g}_k^{(n)}(\chi) \to 1_\Lambda \quad \text{in } L^2(\widehat{G}, \mathcal{B}(\widehat{G}), \tau). \tag{9}$$

Consider $f \in L^2(Y, \mathcal{C}, \nu)$. In the canonical representation of $G(f)$, the function $\sum_{k=1}^{N_n} a_k^{(n)} \tilde{g}_k^{(n)} \in L^2(\widehat{G}, \mathcal{B}(\widehat{G}), \sigma_f)$ corresponds to $\sum_{k=1}^{N_n} a_k^{(n)} S_{g_k^{(n)}} f$ and the subspace $1_\Lambda \cdot L^2(\widehat{G}, \mathcal{B}(\widehat{G}), \sigma_f)$ corresponds to $H_\Lambda \cap G(f)$. So, by taking $\tau = \sigma_f$ in (9),

$$\sum_{k=1}^{N_n} a_k^{(n)} S_{g_k^{(n)}} f \to \operatorname{proj}_{H_\Lambda \cap G(f)} f.$$

Now, since H_Λ is a spectral subspace, $\operatorname{proj}_{H_\Lambda} f = \operatorname{proj}_{H_\Lambda \cap G(f)} f$. It follows that the sequence $(\sum_{k=1}^{N_n} a_k^{(n)} S_{g_k^{(n)}})_{n \geq 1}$ of Markov operators of $L^2(Y, \mathcal{C}, \nu)$ converges weakly to $\operatorname{proj}_{H_\Lambda}$. Therefore the latter projection is a Markov operator and the result follows from Proposition 1. □

The following lemma allows us to localize some eigenvalues of $\mathcal{S}$.

LEMMA 2. *Let $\mathcal{S}$ be a G-action on $(Y, \mathcal{C}, \nu)$ and Λ be a saturated subgroup of $\widehat{G}$. Assume that $\sigma_{\mathcal{S}}(\Lambda) = 0$ and that $\sigma_{\mathcal{S}}(\chi_0 \Lambda) > 0$ for some $\chi_0 \in \widehat{G}$. Then $\mathcal{S}$ has an eigenvalue in $\chi_0 \Lambda$. More precisely, there exists exactly one eigenvalue of $\mathcal{S}$ in $\chi_0 \Lambda$, and $H_{\chi_0 \Lambda}$ is the eigenspace corresponding to that eigenvalue.*

Proof. Denote by Γ the cyclic group $\{\chi_0^n : n \in \mathbb{Z}\}$, considered with the discrete topology. Let Z be the dual group of Γ. Hence we obtain a probability space $(Z, \mathcal{D}, \eta)$, where $\mathcal{D} = \mathcal{B}(Z)$ and η is the normalized Haar measure on Z. Given $g \in G$, we define $\tilde{g} \in Z$ by $\tilde{g}(\chi) = \chi(g)$ for each $\chi \in \Gamma$, and $R_g : Z \to Z$ by $R_g(z) = \tilde{g} \cdot z$. In this way, we obtain an ergodic discrete spectrum G-action $\mathcal{R} = (R_g)_{g \in G}$ on $(Z, \mathcal{D}, \eta)$, whose point spectrum is equal to Γ. Let us consider the diagonal G-action $\mathcal{S} \times \mathcal{R} = (S_g \times R_g)_{g \in G}$ on $(Y \times Z, \mathcal{C} \otimes \mathcal{D}, \nu \otimes \eta)$. The maximal spectral type $\sigma_{\mathcal{S} \times \mathcal{R}}$ of the associated unitary G-action is equal to

$$\left(\sum_{n \in \mathbb{Z}} \frac{1}{2^{|n|}} \delta_{\chi_0^n} \right) * \sigma_{\mathcal{S}} + \sum_{n \neq 0} \frac{1}{2^{|n|}} \delta_{\chi_0^n}.$$

By the assumption, we have $\sigma_{\mathcal{S} \times \mathcal{R}}(\Lambda) > 0$. In view of Corollary 1, there exists a non-trivial $\mathcal{S} \times \mathcal{R}$-factor $\mathcal{A} \subset \mathcal{C} \otimes \mathcal{D}$ for which

$$L^2(\mathcal{A}) = \{F \in L^2(Y \times Z, \nu \otimes \eta) : \sigma_{F, \mathcal{S} \times \mathcal{R}} \text{ is concentrated on } \Lambda\}.$$

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Now fix a non-zero $f \in H_{\chi_0\Lambda}$ and let h be the eigenfunction $z \mapsto \overline{z(\chi_0)}$ of $\mathcal{R}$, corresponding to the eigenvalue $\overline{\chi}_0$. Then $\sigma_{f\otimes h,\mathcal{S}\times\mathcal{R}} = \delta_{\overline{\chi_0}} * \sigma_{f,\mathcal{S}}$, hence

$$\sigma_{f\otimes h,\mathcal{S}\times\mathcal{R}} \text{ is concentrated on } \Lambda \tag{10}$$

and $f \otimes h \in L^2_0(\mathcal{A})$. Consider the function $|f|^2 \otimes h$. First note that

$$|f|^2 \otimes h = (f \otimes h) \cdot (\overline{f} \otimes 1),$$

so the function $|f|^2 \otimes h$ is measurable with respect to $\mathcal{A} \vee (\mathcal{C} \otimes \{\emptyset, Z\})$.

The two G-actions $(\mathcal{S} \times \mathcal{R})|_{\mathcal{A}}$ and $\mathcal{S}$ are spectrally disjoint since, by assumption, $\sigma_{\mathcal{S}}(\Lambda) = 0$. Hence they are disjoint. In particular, $|f|^2 \otimes h$ is in $L^2(Y \times Z, \nu \otimes \eta)$ and

$$\sigma_{|f|^2\otimes h,\mathcal{S}\times\mathcal{R}} = \sigma_{f\otimes h,(\mathcal{S}\times\mathcal{R})|_{\mathcal{A}}} * \sigma_{\overline{f},\mathcal{S}}. \tag{11}$$

At the same time, since $\int_Y |f|^2\, d\nu > 0$, we have $\delta_1 \ll \sigma_{|f|^2,\mathcal{S}}$, and therefore

$$\sigma_{|f|^2\otimes h,\mathcal{S}\times\mathcal{R}} = \sigma_{|f|^2,\mathcal{S}} * \sigma_{h,\mathcal{R}} \gg \delta_1 * \delta_{\overline{\chi}_0} = \delta_{\overline{\chi}_0}.$$

It now follows directly from (11) that $\sigma_{\overline{f},\mathcal{S}}$ is not a continuous measure. More precisely, in view of (10), $\sigma_{\overline{f},\mathcal{S}}$ must have a point mass at some $\chi \in \widehat{G}$ such that $\overline{\chi}_0 \in \chi\Lambda$, and f cannot be orthogonal to some eigenfunctions corresponding to an eigenvalue of $\mathcal{S}$ in $\chi_0\Lambda$. Since f is an arbitrary element of $H_{\chi_0\Lambda}$, the space $H_{\chi_0\Lambda}$ is spanned by eigenfunctions. Finally, since $\sigma_{\mathcal{S}}(\Lambda) = 0$, no two different eigenvalues of $\mathcal{S}$ can be in the coset $\chi_0\Lambda$ and the proof is complete. □

Remark 2. Note that $\sigma_{\mathcal{S}}(\Lambda) = 0$ implies that $\mathcal{S}$ is ergodic. It follows that under the assumptions of the above lemma, $H_{\chi_0\Lambda}$ is moreover one-dimensional.

Although the result below (Proposition 2) will not be used in what follows, we bring it up, as it is another example of applications of saturated groups in (non-singular) ergodic theory.

Assume that T is a non-singular ergodic automorphism of a standard probability space $(X, \mathcal{B}, \mu)$ and that $S : (Y, \mathcal{C}, \nu) \to (Y, \mathcal{C}, \nu)$ is another non-singular automorphism. Let $\pi : (X, \mathcal{B}, \mu) \to (Y, \mathcal{C}, \nu)$ settle a homomorphism between T and S. Following [**5**], (S, π) is called a *relatively finite measure-preserving* (rfmp) factor of T if $(d\mu \circ T)/d\mu$ is $\pi^{-1}(\mathcal{C})$-measurable.

PROPOSITION 2. *Assume that T is a non-singular ergodic automorphism and S is an rfmp factor of it. Let R be a weakly mixing probability-preserving automorphism of a standard probability Borel space $(Z, \mathcal{D}, \eta)$. Assume that $R \times S$ is ergodic. Then $R \times T$ is also ergodic*†.

Proof. We need to show that $\sigma_R(e(T)) = 0$, where $e(T)$ stands for the group of L^∞ eigenvalues ($c \in e(T) \subset \mathbb{T}$ if for some $f \in L^\infty(X, \mathcal{B}, \mu)$, $f \circ T = c \cdot f$); see [**24**], for example. In view of Theorem 2 in [**4**], $e(T)$ is the union of countably many cosets $c \cdot e(S)$, $e(T) = \bigcup_{i=1}^{\infty} c_i \cdot e(S)$. On the other hand, $\sigma_R(e(S))$=0 and $e(S)$ is saturated (see [**18**]). Since R is weakly mixing, in view of Lemma 2, $\sigma_R(c \cdot e(S)) = 0$ for each $c \in \mathbb{T}$. Therefore $\sigma_R(e(T)) = 0$ and the result follows. □

† Usually, to ensure ergodicity one has to assume mild mixing of R, a property stronger than weak mixing. As a matter of fact, R is mildly mixing if and only if $R \times T$ is ergodic for each non-singular ergodic T; see [**10**]. By definition [**10**], mild mixing means that R has no non-trivial rigid factors.

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Following [**5**], given a non-singular automorphism S of $(Y, \mathcal{C}, \nu)$, to obtain T and π so that (S, π) is an rfmp factor of T, we must take any ergodic skew product $T = S_\Theta$ on $(X, \mathcal{B}, \mu) = (Y, \mathcal{C}, \nu) \otimes (X', \mathcal{B}', \mu')$, where $(X', \mathcal{B}', \mu')$ is another probability standard Borel space and $\Theta : X \to \mathrm{Aut}(X', \mathcal{B}', \mu')$ is measurable.

3.3. *Lemma on mixing times of weighted unitary operators.* In the study of mixing properties of automorphisms of the form $T_{\varphi,S}$, unitary operators V_ξ, defined below, will play a crucial role.

Let T be an automorphism of $(X, \mathcal{B}, \mu)$. Let $\xi : X \to \mathbb{T}$ be a cocycle. We define a unitary operator V_ξ on $L^2(X, \mathcal{B}, \mu)$ by setting

$$V_\xi(f)(x) = \xi(x) \cdot f(Tx)$$

for each $f \in L^2(X, \mathcal{B}, \mu)$. A sequence $(n_i) \subset \mathbb{N}$, $n_i \to \infty$, is said to be a *mixing sequence* for V_ξ if $V_\xi^{n_i} \to 0$ in the weak operator topology (while (n_i) is a mixing sequence for T if T^{n_i} restricted to $L^2_0(X, \mathcal{B}, \mu)$ goes to 0, that is, $T^{n_i} \to \Phi_{\mu\otimes\mu}$ in $\mathcal{J}(T)$).

Denote by T_ξ the Anzai skew product corresponding to T and ξ, i.e., the automorphism of $(X \times \mathbb{T}, \mathcal{B} \otimes \mathcal{B}(\mathbb{T}), \mu \otimes \lambda)$ given by $T_\xi(x, z) = (Tx, \xi(x)z)$, where λ stands for the Lebesgue measure of the circle.

We assume now that T is ergodic.

PROPOSITION 3. *Assume that $T^{n_i} \to \Phi$ in $\mathcal{J}(T)$, where $\Phi = \Phi_\rho \in \mathcal{J}^e(T)$. Suppose that the $(T \times T, \rho)$-cocycle $\xi \otimes \overline{\xi}$ is not a coboundary. Then (n_i) is a mixing sequence for V_ξ.*

Proof. We can moreover assume that $(T_\xi)^{n_i} \to \widetilde{\Phi} = \Phi_{\tilde\rho}$ in $\mathcal{J}(T_\xi)$. Given $f_1, f_2 \in L^2(X, \mathcal{B}, \mu)$, we define $F_i \in L^2(X \times \mathbb{T}, \mu \otimes \lambda)$ by $F_i(x, z) = f_i(x)z$, $i = 1, 2$. Set also $J(x_1, z_1, x_2, z_2) = z_1\overline{z}_2$, so $J \in L^2((X \times \mathbb{T}) \times (X \times \mathbb{T}), \tilde\rho)$. Moreover, let, with some abuse of notation, H denote $E^{\tilde\rho}(J|\mathcal{B} \otimes \mathcal{B})$. We have

$$\begin{aligned}
\int_X (V_\xi)^{n_i} f_1 \cdot \overline{f}_2 \, d\mu &= \int_X \xi^{(n_i)} \cdot f_1 \circ T^{n_i} \cdot \overline{f}_2 \, d\mu \\
&= \int_{X\times\mathbb{T}} F_1 \circ T_\xi^{n_i} \cdot \overline{F}_2 \, d\mu \, d\lambda \\
&\longrightarrow \int_{(X\times\mathbb{T})\times(X\times\mathbb{T})} F_1(x_1, z_1)\overline{F_2(x_2, z_2)} \, d\widetilde{\rho}((x_1, z_1), (x_2, z_2)) \\
&= \int_{(X\times\mathbb{T})\times(X\times\mathbb{T})} f_1(x_1) f_2(x_2) z_1\overline{z}_2 \, d\widetilde{\rho}((x_1, z_1), (x_2, z_2)) \\
&= \int_{X\times X} f_1 \otimes f_2 \cdot H \, d\rho.
\end{aligned}$$

We claim now that, for ρ-a.a. $(x_1, x_2) \in X \times X$ (a.a. denotes almost all),

$$\xi(x_1)\overline{\xi(x_2)}H(x_1, x_2) = H(Tx_1, Tx_2). \tag{12}$$

Indeed, $J \circ (T_\xi \times T_\xi) = (\xi \otimes \overline{\xi}) \cdot J$, so

$$\begin{aligned}
E^{\widetilde{\rho}}(J|\mathcal{B} \otimes \mathcal{B}) \circ (T \times T) &= E^{\widetilde{\rho}}(J \circ (T_\xi \times T_\xi)|\mathcal{B} \otimes \mathcal{B}) \\
&= \xi \otimes \overline{\xi} \cdot E^{\widetilde{\rho}}(J|\mathcal{B} \otimes \mathcal{B})
\end{aligned}$$

and (12) follows.

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Now, ergodicity of ρ implies that H is of constant modulus. If $H \neq 0$, then, from (12), it follows that $\xi \otimes \overline{\xi}$ is a $(T \times T, \rho)$-coboundary. Otherwise, $\int_X (V_\xi)^{n_i} f_1 \cdot \overline{f}_2 \, d\mu \to 0$ for all $f_1, f_2 \in L^2(X, \mathcal{B}, \mu)$, so (n_i) is a mixing sequence for V_ξ. □

COROLLARY 2. *If T is weakly mixing and (n_i) is a mixing sequence for T, then (n_i) is also a mixing sequence for V_ξ whenever ξ is not a quasi-coboundary.*

Proof. We have then $T^{n_i} \to \Phi_{\mu\otimes\mu}$ in $\mathcal{J}(T)$. As T is weakly mixing, $\mu \otimes \mu \in \mathcal{J}^e(T)$, and it is well known that $\xi \otimes \overline{\xi}$ is a $(T \times T, \mu \otimes \mu)$-coboundary if and only if ξ is a quasi-coboundary (see [**23**, Appendix], for example). □

4. *Lifting mixing properties to Rokhlin cocycle extensions*

In this section, we will present a systematic study of mixing properties of automorphisms of the form $T_{\varphi,\mathcal{S}}$. Throughout, T is assumed to be an automorphism of $(X, \mathcal{B}, \mu)$, $\varphi : X \to G$ a cocycle and $\mathcal{S} = (S_g)_{g\in G}$ a G-action acting on $(Y, \mathcal{C}, \nu)$.

4.1. *Maximal spectral type of $T_{\varphi,\mathcal{S}}$.* Let $\{f_n\}_{n\geq 0}$ and $\{g_n\}_{n\geq 0}$ be orthonormal bases in $L^2(X, \mathcal{B}, \mu)$ and $L^2(Y, \mathcal{C}, \nu)$, respectively, where $f_0 = g_0 = 1$. For the maximal spectral type $\sigma_{T_{\varphi,\mathcal{S}}}$ of $T_{\varphi,\mathcal{S}}$ on $L^2_0(X \times Y, \mu \otimes \nu)$, we take

$$\sigma_{T_{\varphi,\mathcal{S}}} = \sum_{(m,n)\neq(0,0)} 2^{-(m+n)} \sigma_{f_n\otimes g_m, T_{\varphi,\mathcal{S}}}. \tag{13}$$

According to the notation of §3.3, given $\chi \in \widehat{G}$, we denote by $V_{\chi\circ\varphi}$ the unitary operator on $L^2(X, \mathcal{B}, \mu)$, which acts by the formula

$$(V_{\chi\circ\varphi} f)(x) = \chi(\varphi(x)) f(Tx).$$

Its maximal spectral type, on $L^2(X, \mathcal{B}, \mu)$, is equal to

$$\sigma_{V_{\chi\circ\varphi}} = \sum_{n\geq 0} \frac{1}{2^n} \sigma_{f_n, V_{\chi\circ\varphi}}.$$

Note also that the maximal spectral type of $\mathcal{S}$ on $L^2_0(Y, \mathcal{C}, \nu)$ is given by

$$\sigma_{\mathcal{S}} = \sum_{m\geq 1} \frac{1}{2^m} \sigma_{g_m, \mathcal{S}}$$

and σ_T, the maximal spectral type of T on $L^2_0(X, \mathcal{B}, \mu)$, is equal to $\sum_{n\geq 1}(1/2^n)\sigma_{f_n,T}$.

LEMMA 3. *We have $\sigma_{T_{\varphi,\mathcal{S}}} = \sigma_T + \int_{\widehat{G}} \sigma_{V_{\chi\circ\varphi}} \, d\sigma_{\mathcal{S}}(\chi)$. Moreover,*

$$\sigma_{T_{\varphi,\mathcal{S}}|_{L^2(X\times Y,\mu\otimes\nu)\ominus L^2(X,\mu)\otimes 1_Y}} = \int_{\widehat{G}} \sigma_{V_{\chi\circ\varphi}} \, d\sigma_{\mathcal{S}}(\chi).$$

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Proof. First, we calculate the spectral measure of $f \otimes g$ for $f \in L^2(X, \mathcal{B}, \mu)$, $g \in L^2(Y, \mathcal{C}, \nu)$. For each $k \in \mathbb{Z}$, we have

$$\begin{aligned}
\widehat{\sigma}_{f\otimes g, T_{\varphi,\mathcal{S}}}(k) &= \int_{X\times Y} (f\otimes g)\circ (T_{\varphi,\mathcal{S}})^k \cdot \overline{f\otimes g}\, d(\mu\otimes\nu)\\
&= \int_X f(T^k x)\overline{f(x)} \left(\int_Y g(S_{\varphi^{(k)}(x)}(y))\overline{g(y)}\, d\nu(y)\right) d\mu(x)\\
&= \int_X f(T^k x)\overline{f(x)} \left(\int_{\widehat{G}} \chi(\varphi^{(k)}(x))\, d\sigma_{g,\mathcal{S}}(\chi)\right) d\mu(x)\\
&= \int_{\widehat{G}} \left(\int_X \chi(\varphi^{(k)}(x)) f(T^k x)\overline{f(x)}\, d\mu(x)\right) d\sigma_{g,\mathcal{S}}(\chi).\\
&= \int_{\widehat{G}} \widehat{\sigma}_{f, V_{\chi\circ\varphi}}(k)\, d\sigma_{g,\mathcal{S}}(\chi).
\end{aligned}$$

It follows that

$$\sigma_{f\otimes g, T_{\varphi,\mathcal{S}}} = \int_{\widehat{G}} \sigma_{f, V_{\chi\circ\varphi}}\, d\sigma_{g,\mathcal{S}}(\chi). \tag{14}$$

Therefore, in view of (13) and (14),

$$\begin{aligned}
\sigma_{T_{\varphi,\mathcal{S}}} &= \sum_{(n,m)\neq(0,0)} \frac{1}{2^{n+m}} \sigma_{f_n\otimes g_m, T_{\varphi,\mathcal{S}}}\\
&= \int_{\widehat{G}} \sum_{(n,m)\neq(0,0)} \frac{1}{2^{n+m}} \sigma_{f_n, V_{\chi\circ\varphi}}\, d\sigma_{g_m,\mathcal{S}}(\chi)\\
&= \sum_{m\geq 1} \frac{1}{2^m} \int_{\widehat{G}} \sum_{n\geq 0} \frac{1}{2^n} \sigma_{f_n, V_{\chi\circ\varphi}}\, d\sigma_{g_m,\mathcal{S}}(\chi) + \int_{\widehat{G}} \sum_{n\geq 1} \frac{1}{2^n} \sigma_{f_n, V_{\chi\circ\varphi}}\, d\sigma_{g_0,\mathcal{S}}(\chi)\\
&= \sum_{m\geq 1} \frac{1}{2^m} \int_{\widehat{G}} \sigma_{V_{\chi\circ\varphi}}\, d\sigma_{g_m,\mathcal{S}}(\chi) + \sum_{n\geq 1} \frac{1}{2^n} \sigma_{f_n, T}\\
&= \int_{\widehat{G}} \sigma_{V_{\chi\circ\varphi}}\, d\sigma_{\mathcal{S}}(\chi) + \sigma_T.
\end{aligned}$$

The result immediately follows. □

4.2. *Maximal spectral type of $T_{\varphi,\mathcal{S}}$ on subspaces of the form $L^2(X, \mathcal{B}, \mu) \otimes G(g)$.* Assume now that $g \in L^2_0(Y, \mathcal{C}, \nu)$. Recall that by $G(g)$ we denote the cyclic space generated by g, i.e.,

$$G(g) = \overline{\operatorname{span}}\{g \circ S_h : h \in G\}.$$

The space $L^2(X, \mathcal{B}, \mu) \otimes G(g)$ is $T_{\varphi,\mathcal{S}}$-invariant. Indeed, we can naturally identify $L^2(X \times Y, \mathcal{B}\otimes\mathcal{C}, \mu\otimes\nu)$ with the space $L^2\big(X, \mathcal{B}, \mu;\ L^2(Y, \mathcal{C}, \nu)\big)$ of square-integrable $L^2(Y, \mathcal{C}, \nu)$-valued functions on $(X, \mathcal{B}, \mu)$, and then $L^2(X, \mathcal{B}, \mu) \otimes G(g)$ becomes the subspace of functions taking μ-almost everywhere their values in $G(g)$. Now, if $F(x, \cdot) \in G(g)$, then $(F \circ T_{\varphi,\mathcal{S}})(x, \cdot) = F(Tx, \cdot) \circ S_{\varphi(x)} \in G(g)$.

For each $h \in G$, as $\sigma_{g\circ S_h,\mathcal{S}} = \sigma_{g,\mathcal{S}}$, we have, from (14),

$$\sigma_{f\otimes g, T_{\varphi,\mathcal{S}}} = \sigma_{f\otimes(g\circ S_h), T_{\varphi,\mathcal{S}}}.$$

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Let $\{f_n\}_{n\geq 0}$ be an orthonormal base in $L^2(X, \mathcal{B}, \mu)$ with $f_0 = 1$. It is then clear that

$$\sigma_{T_{\varphi,\mathcal{S}}|_{L^2(X,\mu)\otimes G(g)}} = \sum_{n\geq 0} 2^{-n}\sigma_{f_n\otimes g, T_{\varphi,\mathcal{S}}}. \tag{15}$$

Therefore, by the proof of Lemma 3, we obtain the following.

LEMMA 4. *We have*

$$\sigma_{T_{\varphi,\mathcal{S}}|_{L^2(X,\mu)\otimes G(g)}} = \int_{\widehat{G}} \sigma_{V_{\chi\circ\varphi}}\, d\sigma_{g,\mathcal{S}}(\chi). \qquad \square$$

4.3. *Ergodicity of $T_{\varphi,\mathcal{S}}$.* We assume here that T is ergodic. Let us first note that then

$$\chi \in \Lambda_\varphi \text{ if and only if } \sigma_{V_{\chi\circ\varphi}}(\{1\}) > 0 \tag{16}$$

or, more generally, that

$$\textit{the cocycle } \chi\circ\varphi \textit{ is cohomologous to } e^{2\pi it} \textit{ if and only if } \sigma_{V_{\chi\circ\varphi}}(\{e^{2\pi it}\}) > 0. \tag{17}$$

Indeed, $\sigma_{V_{\chi\circ\varphi}}(\{e^{2\pi it}\}) > 0$ if and only if $e^{2\pi it}$ is an eigenvalue of $V_{\chi\circ\varphi}$ and any eigenfunction corresponding to this eigenvalue will have constant modulus and so, up to normalization, be a transfer function j in the cohomology equation $\chi\circ\varphi = e^{2\pi it}\cdot j/j\circ T$.

The result below has already been proved in [**22**]. However, we give a shorter proof.

PROPOSITION 4. [**22**] *$T_{\varphi,\mathcal{S}}$ is ergodic if and only if T is ergodic and $\sigma_{\mathcal{S}}(\Lambda_\varphi) = 0$.*

Proof. It is clearly necessary that T be ergodic. Then, by Lemma 3, $\sigma_{T_{\varphi,\mathcal{S}}}(\{1\}) = 0$ if and only if $\sigma_{V_{\chi\circ\varphi}}(\{1\}) = 0$ for $\sigma_{\mathcal{S}}$-a.e. $\chi \in \widehat{G}$, and therefore, in view of (16), if and only if $\sigma_{\mathcal{S}}(\Lambda_\varphi) = 0$. $\square$

Remark 3. Let us note that $\sigma_{\mathcal{S}}(\Lambda_\varphi) = 0$ implies that $\sigma_{\mathcal{S}}(\{1\}) = 0$. Indeed, a necessary condition for ergodicity of $T_{\varphi,\mathcal{S}}$ is the ergodicity property of $\mathcal{S}$ itself.

4.4. *Eigenvalues of $T_{\varphi,\mathcal{S}}$.* Assume now that $T_{\varphi,\mathcal{S}}$ is ergodic. We will determine its eigenvalues (and eigenfunctions). Let us fix $t \in [0, 1)$ and set

$$A_t = \{\chi \in \widehat{G} : \chi\circ\varphi \text{ is cohomologous to } e^{2\pi it}\}.$$

Note that $A_t \subset \Sigma_\varphi$ and that if $\chi \in A_t$ and $\chi_1 \in \Lambda_\varphi$, then $\chi\chi_1 \in A_t$. Moreover, if $\chi_1, \chi_2 \in A_t$ belong to A_t, then $\chi_1\overline{\chi}_2 \in \Lambda_\varphi$. It follows that A_t is a coset of Λ_φ whenever it is not empty.

Suppose that $e^{2\pi it}$ is an eigenvalue of $T_{\varphi,\mathcal{S}}$, i.e., $\sigma_{T_{\varphi,\mathcal{S}}}(\{e^{2\pi it}\}) > 0$, and let F be a corresponding eigenfunction. We will assume that F is not a function of x alone (otherwise F is an eigenfunction of T and the result below is trivial). Then there exists $g \in L^2_0(Y, \mathcal{C}, \nu)$ such that F is not orthogonal to $L^2(X, \mathcal{B}, \mu)\otimes G(g)$. Since the spectral measure of F is the Dirac measure at $e^{2\pi it}$, it follows from Lemma 4 that

$$\int_{\widehat{G}} \sigma_{V_{\chi\circ\varphi}}\, d\sigma_{g,\mathcal{S}}(\chi)(\{e^{2\pi it}\}) > 0.$$

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The latter occurs if and only if $\sigma_{g,\mathcal{S}}(\{\chi \in \widehat{G} : \sigma_{V_{\chi\circ\varphi}}(\{e^{2\pi it}\}) > 0\}) > 0$, that is, by (17), if and only if $\sigma_{g,\mathcal{S}}(A_t) > 0$. Now, A_t is a coset $\chi_0\Lambda_\varphi$ of Λ_φ and there exists then a non-zero $g_1 \in G(g) \cap H_{\chi_0\Lambda_\varphi}$. According to Lemma 2 (and Theorem 1), since $T_{\varphi,\mathcal{S}}$ is ergodic and thus $\sigma_{\mathcal{S}}(\Lambda_\varphi) = 0$, it follows that g_1 is an eigenfunction corresponding to an eigenvalue $\chi \in A_t$.

Now let f be a measurable function of modulus 1 satisfying $\chi \circ \varphi \cdot f \circ T = e^{2\pi it} \cdot f$ μ-almost everywhere. As $g_1 \circ S_{\varphi(x)} = \chi(\varphi(x)) \cdot g_1$, we get

$$(f \otimes g_1) \circ T_{\varphi,\mathcal{S}} = (\chi \circ \varphi \cdot f \circ T) \otimes g_1 = e^{2\pi it} \cdot (f \otimes g_1).$$

So, $f \otimes g_1$ is an eigenfunction of $T_{\varphi,\mathcal{S}}$ corresponding to $e^{2\pi it}$ and, since $T_{\varphi,\mathcal{S}}$ is ergodic, F can be different from $f \otimes g_1$ only by a multiplicative constant. Therefore we have proved the following.

PROPOSITION 5. *Assume that $T_{\varphi,\mathcal{S}}$ is ergodic. Then the eigenfunctions of $T_{\varphi,\mathcal{S}}$ are functions of the form $f \otimes g$, where $\chi \circ \varphi = e^{2\pi it} \cdot f/f \circ T$, χ is an eigenvalue of $\mathcal{S}$ and g is an eigenfunction corresponding to χ. In particular, $e^{2\pi it}$ ($t \in [0, 1)$) is an eigenvalue of $T_{\varphi,\mathcal{S}}$ if and only if there exists an eigenvalue of $\mathcal{S}$ in A_t.*

4.5. *Weak mixing and relative weak mixing.* A characterization of the weak mixing property for $T_{\varphi,\mathcal{S}}$ is a direct corollary of Proposition 5.

COROLLARY 3. *$T_{\varphi,\mathcal{S}}$ is weakly mixing if and only if it is ergodic, T is weakly mixing and $\mathcal{S}$ has no eigenvalues in Σ_φ.*

Remark 4. Note that this corollary generalizes the well-known criterion for the weak mixing property of Abelian compact group extensions.

Let us pass to a characterization of the relative weak mixing property. We still assume that $T_{\varphi,\mathcal{S}}$ is ergodic.

Let us first note that the relative product of $T_{\varphi,\mathcal{S}}$ with itself over the factor T is isomorphic to $T_{\varphi,\mathcal{S}\times\mathcal{S}}$, where $\mathcal{S} \times \mathcal{S}$ stands for the diagonal action $g \mapsto S_g \times S_g$ of G on $(Y \times Y, \mathcal{C} \otimes \mathcal{C}, \nu \otimes \nu)$. So, $T_{\varphi,\mathcal{S}}$ is relatively weakly mixing over T if and only if $T_{\varphi,\mathcal{S}\times\mathcal{S}}$ is ergodic.

Since $\sigma_{\mathcal{S}\times\mathcal{S}} = \sigma_{\mathcal{S}} + \sigma_{\mathcal{S}} * \sigma_{\mathcal{S}}$, it follows from Proposition 4 that $T_{\varphi,\mathcal{S}}$ is relatively weakly mixing over T if and only if $\sigma_{\mathcal{S}}(\Lambda_\varphi) + \sigma_{\mathcal{S}} * \sigma_{\mathcal{S}}(\Lambda_\varphi) = 0$. The latter statement is equivalent to saying that $\sigma_{\mathcal{S}}(\chi\Lambda_\varphi) = 0$ for each $\chi \in \widehat{G}$. This has already been proved in [**22**] but now we have Lemma 2 at our disposal, which finally improves and clarifies the result: since $T_{\varphi,\mathcal{S}}$ is ergodic, we have $\sigma_{\mathcal{S}}(\Lambda_\varphi) = 0$, and $\sigma_{\mathcal{S}}(\chi_0\Lambda_\varphi) > 0$ for some χ_0 if and only if $\mathcal{S}$ has an eigenvalue.

PROPOSITION 6. *$T_{\varphi,\mathcal{S}}$ is relatively weakly mixing over T if and only if it is ergodic and $\mathcal{S}$ is weakly mixing.*

4.6. *Relative Kronecker factor of $T_{\varphi,\mathcal{S}}$ over T.* The notion which is complementary to relative weak mixing is the concept of the relative Kronecker factor [**9, 31**]. More precisely, assume that R is an automorphism acting on $(Z, \mathcal{D}, \eta)$ and $\mathcal{E} \subset \mathcal{D}$ is its factor. By a *relative*

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(with respect to $\mathcal{E}$) *eigenvalue* of R one means an $\mathcal{E}$-measurable map $c : (Z/\mathcal{E}, \mathcal{E}, \eta|_{\mathcal{E}}) \to U(n)$ for which there is $M : (Z, \mathcal{D}, \eta) \to \mathbb{C}^n$ satisfying the following:

$$c(\overline{z}) \begin{pmatrix} M_1(z) \\ \cdots \\ M_n(z) \end{pmatrix} = \begin{pmatrix} M_1(Rz) \\ \cdots \\ M_n(Rz) \end{pmatrix} \text{ for a.e. } z \in Z, \tag{18}$$

$$M_i \perp_{\mathcal{E}} M_j \text{ for } i \neq j \quad \text{and} \quad E(|M_i|^2|\mathcal{E}) = 1,\ i,\ j = 1, \ldots, n. \tag{19}$$

The map M satisfying (18) and (19) is called a *relative eigenfunction corresponding to* c. Then the *relative Kronecker factor* $\mathcal{K}(\mathcal{E})$ (of R over $R|_{\mathcal{E}}$) is the smallest σ-algebra making all relative eigenfunctions measurable ($\mathcal{E} \subset \mathcal{K}(\mathcal{E})$).

Denote by $\mathcal{K}(\mathcal{S}) \subset \mathcal{C}$ the Kronecker factor of $\mathcal{S}$, i.e., the factor generated by the eigenfunctions of the unitary action of $\mathcal{S}$. If g, h are eigenfunctions of $\mathcal{S}$ (corresponding to χ and χ', respectively) then

$$(g \otimes h) \circ T_{\varphi,\mathcal{S}\times\mathcal{S}}(x,\ y,\ y') = \chi(\varphi(x))\chi'(\varphi(x)) \cdot (g \otimes h)(y,\ y').$$

It follows that $\mathcal{B} \otimes \mathcal{K}(\mathcal{S})$ is contained in the relative Kronecker factor of $T_{\varphi,\mathcal{S}}$ over T. In fact, we have the following.

PROPOSITION 7. *Assume that $T_{\varphi,\mathcal{S}}$ is ergodic. The relative Kronecker factor of $T_{\varphi,\mathcal{S}}$ over T is equal to $\mathcal{B} \otimes \mathcal{K}(\mathcal{S})$.*

Proof. We will establish condition C_1 on p. 131 of [**9**] for $\mathcal{B} \otimes \mathcal{K}(\mathcal{S})$, which implies the result due to Theorem 6.13 from [**9**].

Assume that $F \in L^2(X \times Y \times Y, \mu \otimes \nu \otimes \nu)$ is a $T_{\varphi,\mathcal{S}\times\mathcal{S}}$-invariant function. Take $g, h \in L^2(Y, \mathcal{C}, \nu)$ and suppose that F is not orthogonal to $L^2(X, \mathcal{B}, \mu) \otimes G(g \otimes h)$. Then, proceeding as in the proof of Proposition 5, we obtain that $\sigma_{g\otimes h,\mathcal{S}\times\mathcal{S}}(\Lambda_\varphi) = \sigma_{g,\mathcal{S}} * \sigma_{h,\mathcal{S}}(\Lambda_\varphi) > 0$. Therefore $\sigma_{g,\mathcal{S}}(\chi\Lambda_\varphi) > 0$ for some $\chi \in \widehat{G}\backslash\{1\}$ (and the same holds for h). By Lemma 2, remembering that $T_{\varphi,\mathcal{S}}$ is ergodic, g is not orthogonal to an eigenfunction of $\mathcal{S}$ from $H_{\chi\Lambda}$. It follows that if $\{g_i\}_{i\geq 0}$ stands for an orthonormal base of $L^2(\mathcal{K}(\mathcal{S}))$, where each g_i is an eigenfunction corresponding to χ_i, $i \geq 0$, then

$$F(x,\ y,\ y') = \sum_{i,j\geq 0} a_{ij}(x) g_i(y) g_j(y'),$$

where $\chi_i \cdot \chi_j \in \Lambda_\varphi$ whenever $a_{ij} \neq 0$ (in fact $\chi_j = \overline{\chi_i}$ since there is at most one eigenvalue in a coset $\chi\Lambda_\varphi$). Fix any function $J = J(x,\ y') \in L^2(X \times Y, \mu \otimes \nu)$. Then, for each $i,\ j \geq 0$, the function given by

$$((a_{ij} \otimes g_i \otimes g_j) * J)(x,\ y) := \int_Y a_{ij}(x) g_i(y) g_j(y') J(x,\ y')\, d\nu(y')$$

is of the form $A(x)g_i(y)$, so it is measurable with respect to $\mathcal{B} \otimes \mathcal{K}(\mathcal{S})$. □

Remark 5. Proposition 7 yields another proof of the result about eigenfunctions of $T_{\varphi,\mathcal{S}}$ when $T_{\varphi,\mathcal{S}}$ is ergodic. Indeed, eigenfunctions are measurable with respect to the relative Kronecker factor. If, as before, $\{g_i\}$ stands for an orthonormal base of eigenfunctions in $L^2(\mathcal{K}(\mathcal{S}))$ and $F \circ T_{\varphi,\mathcal{S}} = c \cdot F$, then

$$F(x,\ y) = \sum_{i\geq 0} a_i(x) g_i(y), \qquad F \circ T_{\varphi,\mathcal{S}}(x,\ y) = \sum_{i\geq 0} a_i(Tx)\chi_i(\varphi(x)) g_i(y),$$

so $c \cdot a_i(x) = a_i(Tx) \cdot \chi_i(\varphi(x))$ for each $i \geq 0$.

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4.7. *Regularity of Rokhlin cocycles.* When the group Λ_φ of a cocycle $\varphi : X \to G$ is not trivial then, obviously, the skew product $T_\varphi : (X \times G, \mu \otimes \lambda_G) \to (X \times G, \mu \otimes \lambda_G)$, $T_\varphi(x, g) = (Tx, \varphi(x) \cdot g)$, is not ergodic, but φ need not be a coboundary. We will show, however, in this section that on the level of Rokhlin cocycles, that is when considering the cocycle $x \mapsto S_{\varphi(x)}$, it must be a coboundary as soon as $\sigma_\mathcal{S}$ is concentrated on Λ_φ†. In the general case, we denote by $\mathcal{A}_{\Lambda_\varphi}$ the $\mathcal{S}$-factor corresponding to Λ_φ according to Corollary 1, i.e., $L^2(\mathcal{A}_{\Lambda_\varphi}) = H_{\Lambda_\varphi}$, and we will show that $x \mapsto S_{\varphi(x)}|_{\mathcal{A}_{\Lambda_\varphi}}$ is a coboundary.

We first show that, when T is ergodic, $\mathcal{B} \otimes \mathcal{A}_{\Lambda_\varphi}$ contains the factor of $T_{\varphi,\mathcal{S}}$-invariant sets.

LEMMA 5. *Assume that T is ergodic. Every $T_{\varphi,\mathcal{S}}$-invariant function F in $L^2(X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu)$ is $\mathcal{B} \otimes \mathcal{A}_{\Lambda_\varphi}$-measurable.*

Proof. Given any $g \in L^2_0(Y, \mathcal{C}, \nu)$, the projection of F on the $T_{\varphi,\mathcal{S}}$-invariant subspace $L^2(X, \mathcal{B}, \mu) \otimes G(g)$ is still $T_{\varphi,\mathcal{S}}$-invariant. If this projection is non-zero, the maximal spectral type of $T_{\varphi,\mathcal{S}}$ on $L^2(X, \mathcal{B}, \mu) \otimes G(g)$ must have an atom at 1. Then, by Lemma 4, $\sigma_{g,\mathcal{S}}(\{\chi \in \widehat{G} : \sigma_{V_{\chi\circ\varphi}}(\{1\}) > 0\}) > 0$, and so, since T is ergodic, $\sigma_{g,\mathcal{S}}(\Lambda_\varphi) > 0$, in view of (16). Since g was arbitrary, it follows that $F \in L^2(X, \mathcal{B}, \mu) \otimes H_{\Lambda_\varphi} = L^2(X \times Y, \mathcal{B} \otimes \mathcal{A}_{\Lambda_\varphi}, \mu \otimes \nu)$. □

We give now a short description of the action of $T_{\varphi,\mathcal{S}}$ on $L^2(X, \mathcal{B}, \mu) \otimes G(g)$ for a non-zero $g \in L^2(X, \mathcal{B}, \mu)$, which will shed some light on the proof of Proposition 8 below. We identify naturally $L^2(X, \mathcal{B}, \mu) \otimes G(g)$ to $L^2(X, \mathcal{B}, \mu; G(g))$. We may furthermore replace $G(g)$ by $L^2(\widehat{G}, \mathcal{B}(\widehat{G}), \sigma_{g,\mathcal{S}})$ through the canonical spectral isomorphism, which we denote by I. We will determine the unitary operator V_φ on $L^2(X, \mathcal{B}, \mu; L^2(\widehat{G}, \mathcal{B}(\widehat{G}), \sigma_{g,\mathcal{S}})) = L^2(X \times \widehat{G}, \mathcal{B} \otimes \mathcal{B}(\widehat{G}), \mu \otimes \sigma_{g,\mathcal{S}})$ which corresponds to $T_{\varphi,\mathcal{S}}$ acting on $L^2(X, \mathcal{B}, \mu) \otimes G(g)$.

Let $\widetilde{F} \in L^2(X \times \widehat{G}, \mathcal{B} \otimes \mathcal{B}(\widehat{G}), \mu \otimes \sigma_{g,\mathcal{S}})$ correspond to $F \in L^2(X, \mathcal{B}, \mu) \otimes G(g)$, i.e., $\widetilde{F}(x, \cdot) = I(F(x, \cdot))$ for μ-a.e. x. We have, for μ-a.e. x,

$$(V_\varphi \widetilde{F})(x, \cdot) = I(F(Tx, S_{\varphi(x)}(\cdot))).$$

Now, if $h \in G$, the image of $g \circ S_h$ under I is $\tilde{h}$, where $\tilde{h}(\chi) = \chi(h)$ for $\chi \in \widehat{G}$. If we take F of the form $F = f \otimes g \circ S_h$, then $\widetilde{F} = f \otimes \tilde{h}$ and

$$V_\varphi(f \otimes \tilde{h})(x, \cdot) = f(Tx) \cdot I(g \circ S_{\varphi(x)\cdot h}) = \chi(\varphi(x)) \cdot (f \otimes \tilde{h})(Tx, \cdot).$$

It follows that for each $\widetilde{F} \in L^2(X \times \widehat{G}, \mu \otimes \sigma_{g,\mathcal{S}})$, $\mu \otimes \sigma_{g,\mathcal{S}}$-a.e. $(x, \chi) \in X \times \widehat{G}$

$$(V_\varphi \widetilde{F}), (x, \chi) = \chi(\varphi(x)) \widetilde{F}(Tx, \chi). \tag{20}$$

We come to the result announced at the beginning of this section.

PROPOSITION 8. *The Rokhlin cocycle $x \mapsto S_{\varphi(x)}|_{\mathcal{A}_{\Lambda_\varphi}}$ is a coboundary. In other words, $T_{\varphi,\mathcal{S}}|_{\mathcal{B}\otimes\mathcal{A}_{\Lambda_\varphi}}$ is relatively isomorphic to $T \times \mathrm{id}_Y|_{\mathcal{B}\otimes\mathcal{A}_{\Lambda_\varphi}}$.*

† When Λ_φ is uncountable, then there is always a weakly mixing Gaussian action $\mathcal{S}$ such that $\sigma_\mathcal{S}$ is concentrated on Λ_φ.

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Proof. Let $\sigma_{\mathcal{S}}|_{\Lambda_\varphi}$ be the spectral type of $\mathcal{S}|_{L^2(\mathcal{A}_{\Lambda_\varphi})}$ (i.e., $\sigma_{\mathcal{S}}|_{\Lambda_\varphi}$ is $\sigma_{\mathcal{S}}$ restricted to Λ_φ). When $\chi \in \Lambda_\varphi$, the cocycle $\chi \circ \varphi$ is a coboundary, that is, $\chi \circ \varphi = f/f \circ T$ μ-almost everywhere for some measurable f of modulus 1. In fact, there exists a measurable selector of transfer functions defined as $\sigma_{\mathcal{S}}|_{\Lambda_\varphi}$-almost everywhere (see [**18**, §§3.1 and 3.2], for example). This means that there exists a measurable function $\widetilde{F}$ of modulus 1 on $X \times \Lambda_\varphi$ such that

$$\chi(\varphi(x)) = \widetilde{F}(x, \chi)/\widetilde{F}(Tx, \chi) \quad \text{for } \mu \otimes \sigma_{\mathcal{S}}|_{\Lambda_\varphi}\text{-a.a. } (x, \chi) \tag{21}$$

(so, for every $g \in L^2(\mathcal{A}_{\Lambda_\varphi})$, the function $F \in L^2(X, \mathcal{B}, \mu) \otimes G(g)$ corresponding to $\widetilde{F}$ is $T_{\varphi,\mathcal{S}}$-invariant).

Given $x \in X$, for every $g \in L^2(\mathcal{A}_{\Lambda_\varphi})$, the action of $S_{\varphi(x)}$ on $G(g)$ corresponds through the spectral isomorphism to multiplication by the function $\chi \mapsto \chi(\varphi(x))$. On the other hand, by the canonical action of $L^\infty(\widehat{G}, \mathcal{B}(\widehat{G}), \sigma_{\mathcal{S}|_{\Lambda_\varphi}})$ on $L^2(\mathcal{A}_{\Lambda_\varphi})$, there also exists a unitary operator W_x on $L^2(\mathcal{A}_{\Lambda_\varphi})$ whose restriction to each $G(g)$, for $g \in L^2(\mathcal{A}_{\Lambda_\varphi})$, corresponds to multiplication by the unit-modulus function $\widetilde{F}(x, \cdot)$. Then equality (21) yields

$$S_{\varphi(x)}|_{L^2(\mathcal{A}_{\Lambda_\varphi})} = W_x W_{Tx}^{-1} \quad \text{for } \mu\text{-a.a. } x.$$

So, the cocycle $x \mapsto S_{\varphi(x)}|_{L^2(\mathcal{A}_{\Lambda_\varphi})}$ is a T-coboundary as a cocycle taking values in $\mathcal{U}(L^2(\mathcal{A}_{\Lambda_\varphi}))$.

Finally, $\mathrm{Aut}(\mathcal{A}_{\Lambda_\varphi})$ is naturally identified to a closed subgroup of the Polish group $\mathcal{U}(L^2(\mathcal{A}_{\Lambda_\varphi}))$. Since $x \mapsto S_{\varphi(x)}|_{L^2(\mathcal{A}_{\Lambda_\varphi})}$ takes its values in $\mathrm{Aut}(\mathcal{A}_{\Lambda_\varphi})$, it is still a coboundary as an $\mathrm{Aut}(\mathcal{A}_{\Lambda_\varphi})$-valued cocycle. □

In view of Proposition 4, we obtain the following result.

COROLLARY 4. *If T is ergodic and $T_{\varphi,\mathcal{S}}$ is not ergodic, then there is a non-trivial factor $\mathcal{A}$ of $\mathcal{S}$ such that $T_{\varphi,\mathcal{S}}|_{\mathcal{B}\otimes\mathcal{A}}$ is relatively isomorphic to $T \times \mathrm{id}_Y|_{\mathcal{B}\otimes\mathcal{A}}$.*

4.8. *Lifting mild mixing property.* In this section, we will show that the triviality of the Rokhlin cocycle described in Proposition 8 also takes place when dealing with the mild mixing property, and we give necessary and sufficient conditions in order that the mild mixing property lift from T to $T_{\varphi,\mathcal{S}}$. Recall that T is mildly mixing if T has no non-trivial rigid factors and that a factor $\mathcal{A}$ of T is rigid if and only if the spectral type of $T|_{\mathcal{A}}$ is a Dirichlet measure.

Suppose now that $\mathcal{A}$ is a non-trivial factor of T and suppose moreover that (q_n) is a rigidity sequence for $T|_{\mathcal{A}}$. Consider

$$I(\mathcal{A}) := \{\Phi \in \mathcal{J}(T) : \eta_\Phi|_{\mathcal{A}\otimes\mathcal{A}} = \Delta_{\mathcal{A}} \text{ and } \Phi \text{ is a limit point of } \{T^j : j \in \mathbb{Z}\}\}. \tag{22}$$

Note that $I(\mathcal{A})$ is non-empty since any limit Markov operator of the set $\{T^{q_n} : n \geq 1\}$ belongs to it. It follows from (5) and (6) that $I(\mathcal{A})$ is a closed subsemigroup of $\mathcal{J}(T)$, hence a semitopological compact semigroup. We recall (see [**14**, p. 6, Lemma 2.2], for example) that each compact semitopological semigroup contains an idempotent. Now, using Proposition 1, we obtain the following.

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PROPOSITION 9. *Assume that T is an automorphism of $(X, \mathcal{B}, \mu)$ and let $\mathcal{A} \subset \mathcal{B}$ be a non-trivial rigid factor of T. Then there exist a factor $\mathcal{A}'$ containing $\mathcal{A}$ and a rigidity sequence (q_n) for $T|_{\mathcal{A}'}$ such that $T^{q_n} \to E(\cdot|\mathcal{A}')$.*

We will also need the following.

LEMMA 6. *Assume that T is mildly mixing. If $\xi : X \to \mathbb{T}$ is a cocycle and T_ξ has a non-trivial rigid factor $\mathcal{A} \subset \mathcal{B} \otimes \mathcal{B}(\mathbb{T})$, then there exist a factor $\mathcal{A}'$ of T_ξ containing $\mathcal{A}$ and a mixing sequence (q_n) for T such that $(T_\xi)^{q_n} \to E(\cdot|\mathcal{A}')$.*

Proof. From Proposition 9, there exist a factor $\mathcal{A}'$ of T_ξ containing $\mathcal{A}$ and a rigid sequence (q_n) for $T_\xi|_{\mathcal{A}'}$ such that

$$(T_\xi)^{q_n} \to E(\cdot|\mathcal{A}'). \tag{23}$$

It remains to show that (q_n) is a mixing sequence for T. But, since we assume that T is mildly mixing, no spectral measure of a function in $L^2_0(X, \mathcal{B}, \mu)$ is a Dirichlet measure, while the maximal spectral type of T_ξ on $L^2(\mathcal{A}')$ is a Dirichlet measure. Thus $L^2_0(\mathcal{B} \otimes \{\emptyset, \mathbb{T}\}) \perp L^2(\mathcal{A}')$ and the result follows from (23). □

PROPOSITION 10. *Assume that T is mildly mixing. If $\sigma_{\mathcal{S}}(\Sigma_\varphi) = 0$, then $T_{\varphi,\mathcal{S}}$ is also mildly mixing.*

Proof. First, we claim that, if some positive measure $\sigma_1 \ll \sigma_{V_{\chi\circ\varphi}}$ is a Dirichlet measure, then $\chi \in \Sigma_\varphi$.

Indeed, then there exists $f \in L^2(X, \mathcal{B}, \mu)$ such that $\sigma_{f, V_{\chi\circ\varphi}} = \sigma_1$. If we consider $F = f \otimes z \in L^2(X \times \mathbb{T}, \mu \otimes \lambda)$ (here z denotes the identity function from $\mathbb{T}$ to $\mathbb{C}$), we have $F \circ T^n_{\chi\circ\varphi} = V^n_{\chi\circ\varphi} f \otimes z$ for all $n \in \mathbb{Z}$ and it follows that $\sigma_{F, T_{\chi\circ\varphi}} = \sigma_1$. Since σ_1 is a Dirichlet measure, F is measurable with respect to some rigid factor $\mathcal{A}$ of $T_{\chi\circ\varphi}$.

In view of Lemma 6, there exist a factor $\mathcal{A}'$ of $T_{\chi\circ\varphi}$ containing $\mathcal{A}$ and a mixing sequence (q_n) for T such that $T^{q_n}_{\chi\circ\varphi} \to E(\cdot|\mathcal{A}')$. In particular, $F \circ T^{q_n}_{\chi\circ\varphi} \to F$ and $V^{q_n}_{\chi\circ\varphi} f \to f$, so (q_n) is not a mixing sequence for $V_{\chi\circ\varphi}$. But Corollary 2 implies then that the cocycle $\chi \circ \varphi$ is a quasi-coboundary, in other words, $\chi \in \Sigma_\varphi$, which proves our claim.

It remains to show that there is no positive measure $\sigma \ll \sigma_{T_{\varphi,\mathcal{S}}}$ which is a Dirichlet measure. Suppose the contrary; then, for some sequence $n_j \to \infty$, $z^{n_j} \to 1$ σ-almost everywhere. It follows that if we set $A = \{z \in \mathbb{T} : z^{n_j} \to 1\}$, then $\sigma_{T_{\varphi,\mathcal{S}}}(A) > 0$. Since T is mildly mixing and thus $\sigma_T(A) = 0$, it follows, by Lemma 3, that $\sigma_{\mathcal{S}}(\{\chi \in \widehat{G} : \sigma_{V_{\chi\circ\varphi}}(A) > 0\}) > 0$.

Clearly, if $\sigma_{V_{\chi\circ\varphi}}(A) > 0$, then the positive measure $\sigma_{V_{\chi\circ\varphi}}|_A$ is Dirichlet, hence χ must belong to Σ_φ by the first part of the proof. Since $\sigma_{\mathcal{S}}(\Sigma_\varphi) = 0$, we obtain a contradiction. □

Remark 6. Supposing only that T is mildly mixing, we get that each rigid function of $T_{\varphi,\mathcal{S}}$ belongs to $L^2(X, \mathcal{B}, \mu) \otimes H_{\Sigma_\varphi}$. Indeed, if $\sigma_{g,\mathcal{S}}(\Sigma_\varphi) = 0$, in view of Lemma 4, the same proof gives that there is no Dirichlet measure $\sigma \ll \sigma_{T_{\varphi,\mathcal{S}}|_{L^2(X,\mathcal{B},\mu)\otimes G(g)}}$.

Let us denote by $\mathcal{A}_{\Sigma_\varphi}$ the factor of $\mathcal{S}$ corresponding to the saturated group Σ_φ according to Corollary 1, so that $H_{\Sigma_\varphi} = L^2(\mathcal{A}_{\Sigma_\varphi})$. In other words,

$$\textit{each rigid factor of } T_{\varphi,\mathcal{S}} \textit{ is contained in } \mathcal{B} \otimes \mathcal{A}_{\Sigma_\varphi}. \tag{24}$$

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PROPOSITION 11. *Assume that T is weakly mixing (and $\sigma_{\mathcal{S}}(\Sigma_\varphi) > 0$). Then there exists an automorphism U of $(Y|_{\mathcal{A}_{\Sigma_\varphi}}, \mathcal{A}_{\Sigma_\varphi}, \nu|_{\mathcal{A}_{\Sigma_\varphi}})$ such that $T_{\varphi,\mathcal{S}}|_{\mathcal{B}\otimes\mathcal{A}_{\Sigma_\varphi}}$ is isomorphic to $T \times U$.*

Proof. By definition, if $\chi \in \Sigma_\varphi$, the cocycle $\chi \circ \varphi$ is cohomologous to a constant $e^{2\pi i t}$. However, T is weakly mixing, so this constant is unique, hence we can write $t = t(\chi)$ ($t(\chi) \in [0, 1)$). Then, as in the case of Λ_φ, there exists a measurable selector of transfer functions $\chi \mapsto \widetilde{F}(\cdot, \chi)$ defined $\sigma_{\mathcal{S}}|_{\Sigma_\varphi}$-almost everywhere; equivalently, there is a measurable function $\widetilde{F}$ of modulus 1 on $X \times \Sigma_\varphi$ such that

$$\chi(\varphi(x)) = e^{2\pi i t(\chi)} \widetilde{F}(x, \chi)/\widetilde{F}(Tx, \chi) \quad \text{for } \mu \otimes \sigma_{\mathcal{S}}|_{\Sigma_\varphi}\text{-a.a. } (x, \chi). \tag{25}$$

In particular, $e^{2\pi i t(\chi)} = u(\chi)$, $\sigma_{\mathcal{S}}|_{\Sigma_\varphi}$-almost everywhere, where u is a measurable function of modulus 1 on Σ_φ.

Then, as in the proof of Proposition 8, we deduce that there exist unitary operators U and W_x ($x \in X$) of $L^2(\mathcal{A}_{\Sigma_\varphi})$ corresponding to multiplication by u and $\widetilde{F}(x, \cdot)$, respectively, so that

$$S_{\varphi(x)}|_{\Sigma_\varphi} = U W_x W_{Tx}^{-1} \quad \text{for } \mu\text{-a.e. } x. \tag{26}$$

We have to show that U corresponds to an automorphism. Let us consider the $(T \times T, \mu \otimes \mu)$-cocycle $(x_1, x_2) \mapsto S_{\varphi(x_2)} S_{\varphi(x_1)}^{-1}|_{\Sigma_\varphi}$. In view of (26) (and the fact that the operators under consideration commute), it is a coboundary with the transfer operator map $(x_1, x_2) \mapsto W_{x_2} W_{x_1}^{-1}$, whence, as in the proof of Proposition 8, it is also a coboundary as a cocycle with values in $\mathrm{Aut}(\mathcal{A}_{\Sigma_\varphi})$. Thus there exists a measurable map $(x_1, x_2) \mapsto V_{x_1,x_2} \in \mathrm{Aut}(\mathcal{A}_{\Sigma_\varphi})$ with

$$S_{\varphi(x_2)} S_{\varphi(x_1)}^{-1}|_{\Sigma_\varphi} = V_{x_1,x_2} V_{Tx_1,Tx_2}^{-1} \quad \text{for } \mu \otimes \mu\text{-a.a. } (x_1, x_2).$$

Since T is weakly mixing, $T \times T$ is ergodic and therefore the two transfer operator maps must coincide up to a constant. More precisely, $W_{x_1} W_{x_2}^{-1} V_{x_1,x_2}$ is $T \times T$-invariant, so there exists a unitary operator V of $L^2(\mathcal{A}_{\Sigma_\varphi})$ such that

$$W_{x_1} W_{x_2}^{-1} V_{x_1,x_2} = V \quad \text{for } \mu \otimes \mu\text{-a.a. } (x_1, x_2).$$

By selecting x_1 so that the above equality is true for μ-a.e. x_2, we obtain

$$V_{x_1,x} = W_x W_{x_1}^{-1} V \quad \text{for } \mu\text{-a.a. } x.$$

Then the map $x \mapsto V_{x_1,x} \in \mathrm{Aut}(\mathcal{A}_{\Sigma_\varphi})$ is also a transfer operator map for equation (26):

$$U V_{x_1,x} V_{x_1,Tx}^{-1} = U W_x W_{Tx}^{-1} = S_{\varphi(x)}|_{\Sigma_\varphi} \quad \text{for } \mu\text{-a.e. } x.$$

Therefore $U \in \mathrm{Aut}(\mathcal{A}_{\Sigma_\varphi})$, $x \mapsto S_{\varphi(x)}$ is cohomologous to the constant U in $\mathrm{Aut}(\mathcal{A}_{\Sigma_\varphi})$, and the result follows. □

COROLLARY 5. *Assume that T is mildly mixing. Then $T_{\varphi,\mathcal{S}}$ is not mildly mixing if and only if there exists a non-trivial factor $\mathcal{A}$ of $\mathcal{S}$ and an automorphism U of $(Y|_{\mathcal{A}}, \mathcal{A}, \nu|_{\mathcal{A}})$ which is not mildly mixing such that $T_{\varphi,\mathcal{S}}|_{\mathcal{B}\otimes\mathcal{A}}$ is isomorphic to $T \otimes U$.*

Proof. In view of (24), if $T_{\varphi,\mathcal{S}}$ is not mildly mixing, neither is $T_{\varphi,\mathcal{S}}|_{\mathcal{B}\otimes\mathcal{A}_{\Sigma_\varphi}}$, and we apply Proposition 11. Then $T \times U$ is not mildly mixing and, as T is mildly mixing, U cannot be mildly mixing. The other direction is clear. □

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Remark 7. It turns out that in Corollary 5, we can replace U non-mildly mixing by U rigid. Indeed, in the proof of Proposition 11, U corresponds to multiplication by $u(\chi)$, and, given a rigid factor $\mathcal{A}$ of U defined by $U^{n_j}h \to h$ for some sequence (n_j), we have that $L^2(\mathcal{A})$ is the spectral subspace of $\mathcal{S}|_{\mathcal{A}_{\Sigma_\varphi}}$ corresponding to $\{\chi \in \Sigma_\varphi : u(\chi)^{n'_j} \to 1\}$ for some subsequence; in particular, $\mathcal{A}$ is also $\mathcal{S}$-invariant. Moreover, the cocycle $S_{\varphi(x_2)}S^{-1}_{\varphi(x_1)}|_{\Sigma_\varphi}$ is still cohomologous to the constant U in the closed subgroup of $\mathrm{Aut}(\mathcal{A}_{\Sigma_\varphi})$ of all automorphisms corresponding to multiplications by unit-modulus functions (in the spectral representation of $\mathcal{S}|_{\mathcal{A}_{\Sigma_\varphi}}$). So, the automorphisms V_{x_1,x_2} can be taken in this subgroup so that they preserve the invariant subspaces of $\mathcal{S}$. Then $\mathcal{B}\otimes\mathcal{A}$ is preserved by the conjugation automorphism and we have the relative isomorphism of $T_{\varphi,\mathcal{S}}|_{\mathcal{B}\otimes\mathcal{A}}$ with $T\otimes U|_{\mathcal{B}\otimes\mathcal{A}}$.

We now show that under the recurrence property of φ, the converse of Proposition 10 holds. For that, we need some elementary remarks about recurrent cocycles with values in Polish Abelian groups. They are taken directly from the theory of cocycles with values in LCA groups [**30**].

Let T be an ergodic automorphism of $(X, \mathcal{B}, \mu)$. Assume that A is an Abelian Polish group. Let $\omega : X \to A$ be a cocycle. The cocycle ω is said to be *recurrent* if, for each $\varepsilon > 0$, $B \in \mathcal{B}$ of positive measure and each neighbourhood V of 1, there exists a positive integer N such that

$$\mu(B \cap T^{-N}B \cap [\omega^{(N)} \in V]) > 0. \tag{27}$$

Suppose that $\omega' : X \to A$ is another cocycle. Then we have the following.

If ω and ω' are cohomologous and ω is recurrent, then so is ω'. (28)

Assume now that $a \in A$ and let $\widetilde{a}$ denote the corresponding constant cocycle $\widetilde{a}(x) = a$. Then the following holds.

$\tilde{a}$ is recurrent if and only if there exists $n_j \to \infty$ such that $a^{n_j} \to 1$ in A. (29)

Indeed, fix a neighbourhood V of 1 and apply (27) with $B = X$ to obtain that $\mu([\tilde{a}^{(N)} \in V]) > 0$ for some positive integer N. It follows that $a^N \in V$. Letting $V \to \{1\}$, either $N = N(a, V) \to \infty$ and we are done, or N stays bounded. In the latter case, we have $a^N = 1$ for some $N \geq 1$, and by taking multiples of this N, (29) also follows. On the other hand, if $a^{n_j} \to 1$, then the sequence of differences $n_i - n_j$ is a sequence universally good for the Poincaré recurrence and clearly $a^{n_i - n_j} \to 1$ when $i, j \to \infty$. Therefore $\widetilde{a}$ is recurrent.

Let $\sigma \in M^+(\widehat{G})$. Denote by $\mathcal{U}(\sigma)$ the group of measurable functions of modulus 1 defined on $\widehat{G}$, modulo equality σ-almost everywhere. We endow $\mathcal{U}(\sigma)$ with the $L^2(\sigma)$-topology, which makes it a Polish (Abelian) group. Given $g \in G$, we still denote by $\tilde{g}$ the function $\chi \mapsto \chi(g)$ taken as an element of $\mathcal{U}(\sigma)$. Then we define φ_σ from X to $\mathcal{U}(\sigma)$ by setting

$$\varphi_\sigma(x)(\chi) = \chi(\varphi(x)) \quad \text{for each } \chi \in \widehat{G},$$

i.e., φ_σ is the composition of φ and of the map $g \mapsto \tilde{g}$. As the latter map is a continuous group homomorphism, it is clear from definition (27) that

if φ is recurrent, then so is φ_σ. (30)

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PROPOSITION 12. *If T is mildly mixing, then $T_{\varphi,\mathcal{S}}$ is mildly mixing if and only if $\sigma(\Sigma_\varphi) = 0$ for each positive measure $\sigma \ll \sigma_{\mathcal{S}}$ such that the cocycle φ_σ is recurrent.*

In particular, if φ is recurrent and $T_{\varphi,\mathcal{S}}$ is mildly mixing, then $\sigma_{\mathcal{S}}(\Sigma_\varphi) = 0$.

Proof. We keep the notation as in the proof of Proposition 11: U is the automorphism of $(Y|_{\mathcal{A}_{\Sigma_\varphi}}, \mathcal{A}_{\Sigma_\varphi}, \nu|_{\mathcal{A}_{\Sigma_\varphi}})$ corresponding to the unit-modulus function u on Σ_φ, and equation (25) may now be written as

$$\varphi_\sigma(\chi) = u(\chi)\widetilde{F}(x, \chi)/\widetilde{F}(Tx, \chi) \quad \text{for } \mu \otimes \sigma_{\mathcal{S}}|_{\Sigma_\varphi}\text{-a.a. } (x, \chi).$$

Suppose that φ_σ is recurrent for some positive measure $\sigma \ll \sigma_{\mathcal{S}}$ with $\sigma(\Sigma_\varphi) > 0$. We can then assume that $0 < \sigma \ll \sigma_{\mathcal{S}}|_{\Sigma_\varphi}$. Then the constant cocycle u restricted to $\mathcal{U}(\sigma)$ is cohomologous to φ_σ and it is also recurrent by (28). Thus, in view of (29), there is a sequence (n_j) with $u^{n_j} \to 1$ in $\mathcal{U}(\sigma)$, whence $U^{n_j}h \to h$ for each function h such that $\sigma_{h,\mathcal{S}} \ll \sigma$. It follows that U is not mildly mixing.

For the other direction, if $T_{\varphi,\mathcal{S}}$ is not mildly mixing, then U is not mildly mixing and we find conversely that u restricted to $\mathcal{U}(\sigma_{h,\mathcal{S}})$ is recurrent, for some non-zero $h \in L^2_0(\mathcal{A}_{\Sigma_\varphi})$. Then $\varphi_{\sigma_{h,\mathcal{S}}}$ is also recurrent and $\sigma_{h,\mathcal{S}}(\Sigma_\varphi) > 0$.

The second assertion then follows from (30): if φ is recurrent, then the cocycle $\varphi_{\sigma_{\mathcal{S}}}$ is also recurrent. □

4.9. *Lifting mixing and multiple mixing.* We give here two corollaries of results from [**23**].

PROPOSITION 13. [**23**] *Assume that T is mixing. If $\sigma_{\mathcal{S}}(\Sigma_\varphi) = 0$, then $T_{\varphi,\mathcal{S}}$ is mixing. Conversely, if $\sigma_{\mathcal{S}}(\Sigma_\varphi) > 0$ and φ is recurrent, then $T_{\varphi,\mathcal{S}}$ is not mixing.*

COROLLARY 6. *Assume that T is mixing. Then $T_{\varphi,\mathcal{S}}$ is not mixing if and only if there exists a non-trivial factor $\mathcal{A}$ of $\mathcal{S}$ and an automorphism U of $(Y|_{\mathcal{A}}, \mathcal{A}, \nu|_{\mathcal{A}})$ which is not mixing, such that $T_{\varphi,\mathcal{S}}|_{\mathcal{B}\otimes\mathcal{A}}$ is isomorphic to $T \otimes U$.*

Proof. The proof of Proposition 13 in [**23**] (Theorem 7.1) actually shows that if T is mixing and $\sigma_{h,\mathcal{S}}(\Sigma_\varphi) = 0$, then $\hat{\sigma}_{f\otimes h, T_{\varphi,\mathcal{S}}}(n) \to 0$ for each $f \in L^2(X, \mathcal{B}, \mu)$. Therefore if $T_{\varphi,\mathcal{S}}$ is not mixing, we must have a function $F \in L^2_0(X, \mathcal{B}, \mu) \otimes H_{\Sigma_\varphi}$ whose spectral measure does not vanish at infinity, whence $T_{\varphi,\mathcal{S}}|_{\mathcal{B}\otimes\mathcal{A}_{\Sigma_\varphi}}$ is not mixing. Then we can apply Proposition 11 and the result follows exactly as for Corollary 5. □

PROPOSITION 14. [**23**] *Assume that T is r-fold mixing and that φ is recurrent. If $T_{\varphi,\mathcal{S}}$ is mildly mixing, then it is also r-fold mixing.*

Now, the corollary below directly follows from Propositions 14 and 12.

COROLLARY 7. *Assume that T is r-fold mixing and φ is recurrent. Then $T_{\varphi,\mathcal{S}}$ is r-fold mixing if and only if $\sigma_{\mathcal{S}}(\Sigma_\varphi) = 0$.*

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