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Dynamical properties of shift maps on inverse limits with a set valued function

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Abstract. Set-valued functions from an interval into the closed subsets of an interval arise in various areas of science and mathematical modeling. Research has shown that the dynamics of a single-valued function on a compact space are closely linked to the dynamics of the shift map on the inverse limit with the function as the sole bonding map. For example, it has been shown that with Devaney's definition of chaos the bonding function is chaotic if and only if the shift map is chaotic. One reason for caring about this connection is that the shift map is a homeomorphism on the inverse limit, and therefore the topological structure of the inverse-limit space must reflect in its richness the dynamics of the shift map. In the set-valued case there may not be a natural definition for chaos since there is not a single well-defined orbit for each point. However, the shift map is a continuous single-valued function so it together with the inverse-limit space form a dynamical system which can be chaotic in any of the usual senses. For the set-valued case we demonstrate with theorems and examples rich topological structure in the inverse limit when the shift map is chaotic (on certain invariant sets). We then connect that chaos to a property of the set-valued function that is a natural generalization of an important chaos producing property of continuous functions.

1. *Introduction*

In [**SL**] Li showed that with a sole bonding map on a compact space the shift map on the inverse limit is chaotic in the sense of Devaney if and only if the single bonding map is chaotic. The shift map on the inverse limit with a single bonding map is a homeomorphism, and in [**CM**] Mouran has shown such an inverse limit that admits a positive-entropy homeomorphism must contain an indecomposable continuum. Taken

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together these results imply that the dynamics of a continuous function on an interval are linked to the dynamics of the shift map on the inverse limit and that the complexity of the dynamics will be reflected in the topological complexity of the inverse limit by the presence of indecomposable subcontinua.

Set-valued functions from an interval into the closed subsets of an interval do arise in various areas of science and mathematical modeling, most notably economics and game theory. The dynamics for set valued functions are a little problematic since the orbit of a point is not well defined, but the shift map, no longer a homeomorphism, is nonetheless a continuous function. So the shift map and the inverse limit form a dynamical system which can be chaotic in any of the usual senses.

In this paper we consider the appearance of invariant Cantor sets under the shift map on the inverse limit in which a sole bonding map is a surjective upper semicontinuous set-valued function with a connected graph from an interval into the closed subsets of the interval. This has resulted in a simple and natural method for constructing families of λ-dendroids and dendroids that have uncountably many mutually disjoint dense arc components, and dendroids where the complement of a single point has uncountably many dense arc components. These hereditarily decomposable tree-like continua have a topology that is similar to that of indecomposable continua, and the shift map resulting from the inverse-limit construction is monotone (i.e. preimages of continua are continua) and chaotic in the sense that the shift map has positive entropy.

While there are more general settings, we focus on the case in which we are interested. Suppose for each non-negative integer i, I_i is a closed interval and $f_{i+1} : I_{i+1} \to 2^{I_i}$, called a *bonding map*, is an upper semicontinuous mapping of I_{i+1} into the closed subsets of I_i. Then the *generalized inverse limit*, or the *inverse limit with set-valued mappings*, associated with these mappings is the set

$$M = \varprojlim(I_i,\ f_i) := \{(x_0, x_1, \ldots) : \text{for each } i \geq 1,\ x_{i-1} \in f_i(x_i)\},$$

which is a subspace of $\prod_{i\geq 0} I_i$ endowed with the product topology.

We make a further simplification: for us, each f_i is the same bonding map f, i.e. for each $i \geq 1$, $f_i = f$. Then the inverse limit space is $M = \varprojlim(I_i,\ f)$, which we abbreviate as $M = \varprojlim f$. In this setting a natural map σ, called the shift map, arises and is a continuous function from the space M onto itself. Thus, we now have the dynamical system (M, σ).

Generalized inverse limits represent a new topic of study in continuum theory. They first appeared in 2004 in a paper published by Mahavier [**M**]. Soon thereafter, Ingram began studying them (see [**IM2**, **I1**, **I2**, **I3**, **I4**]). The topic caught the interest of a number of continuum theorists, and is currently an intensely studied area, with papers from many authors at this point. See [**B**, **BCMM1**, **BCMM2**, **BCMM3**, **BK**, **CR**, **GK1**, **GK2**, **Ill**, **N1**, **N2**, **N3**, **N4**, **V**], for example. (This list is far from exhaustive, with the number of papers on generalized inverse limits now over 60.) Ingram and Mahavier included a chapter on these spaces in their book *Inverse Limits: from Continua to Chaos* [**IM1**], and since then Ingram has written another book on the topic, *An Introduction to Inverse Limits with Set-Valued Functions* [**I5**]. Most of the work done to-date on generalized inverse limits has focused on understanding the topology of the spaces. Our goal is to begin studying the dynamical properties that arise in these inverse limits, while continuing to investigate the

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topology. This relatively new area of research gives us: (1) a new tool for constructing examples (and we do that in this paper); and (2) a new way of considering some of the old unsolved problems from topology and dynamics.

Another important motivation for the study of generalized inverse limits is that they represent a completely new approach to studying multi-valued functions. Set-valued functions arise in various areas of science and mathematics. When these functions are 'iterated' *in the usual sense, information is lost—and often lost very fast*. For example, consider the set-valued function $f : I \to 2^I$ defined by $f(x) = \{1/2\}$ for $x \neq 1/2$, and $f(1/2) = I$. Next consider the set-valued function f^2 (where $f^2(x) = \bigcup_{y \in f(x)} f(y)$). Then, for each x, $f^2(x) = I = f^3(x) = \cdots$, and *each point maps to the entire space!*

On the other hand, for this particular function f, $M = \varprojlim f = \{(x_0, x_1, \ldots) : x_{i-1} \in f(x_i) \text{ for } i > 0\}$ contains a copy of the Hilbert cube: if $0 \leq x \leq 1$, then the set $\{1/2\} \times I \times \{1/2\} \times I \times \cdots$ is contained in M. In fact, this particular space is a continuum that contains many copies of the Hilbert cube.

Information is not lost with this approach to set-valued functions in the sense that *all possible orbits are kept*. That is, one doesn't just take an initial value, iterate a number of times, then keep only the end value, throwing away all the possibly large number of ways one got to that ending value.

However, this is no free lunch: now rather than just considering a point (initial value) in the interval and a set (the ending values) in the interval, there is a closed subspace of the Hilbert cube, and an initial value leads to a sequence of numbers, which is a point in the subspace of the Hilbert cube.

Hence, now one must understand the properties of the much more complicated space (than is the interval), and understand the dynamical behavior that occurs as a point (which is now an entire sequence) is iterated.

2. *Background and preliminaries*

Sometimes it is convenient to index our factor spaces, sometimes not. Suppose for each integer $i \geq 0$, $I_i = [0, 1]$. The Hilbert cube is $I^\infty = [0, 1]^\infty = \prod_0^\infty I_i$ with metric $d(x, y) = \sum_{i=0}^\infty (|\pi_i(x) - \pi_i(y)|/2^i)$, where x and y are points in I^∞, $x = (x_0, x_1, \ldots)$, $y = (y_0, y_1, \ldots)$, and for each $i \geq 0$, $\pi_i(z) = \pi_i((z_0, z_1, \ldots)) = z_i$ for $z = (z_0, z_1, \ldots)$ a point in I^∞.

We will need to talk about various projections from a subset of I^∞ into an interval or a product of intervals. Unless it leads to confusion, for a subset X of I^∞, and a point $x = (x_0, x_1, \ldots)$ in X, $\pi_i(x) = x_i$. (That is, we do not specify the momentary domain of π_i.) Likewise, if N is a positive integer, $x = (x_0, x_1, \ldots, x_N)$, $x \in X \subset I^{N+1}$, then $\pi_i(x) = x_i$ for $0 \leq i \leq N$. Also, we make following definitions.

- Let $m \geq 0$ be an integer less than the integer n. Then $\langle m, n\rangle = \{m, m + 1, \ldots, n\}$, and we call $\langle m, n\rangle$ the *integer interval from m to n*. If $x = (x_0, x_1, \ldots) \in I^\infty$, then $\pi_{\langle m,n\rangle}(x) = (x_m, x_{m+1}, \ldots, x_n)$. We define $\langle m, \infty\rangle$ to be the set $\{m, m + 1, \ldots\}$.
- Suppose $x = (x_0, x_1, \ldots, x_n)$ is a point in I^{n+1} and $(y_0, y_1, \ldots)$ is a point in I^∞. Then we define $x \oplus y$ to be the point $(x_0, \ldots, x_n, y_0, y_1, \ldots)$ in I^∞.

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- The *shift map* $\sigma : I^\infty \to I^\infty$ is defined by $\sigma((x_0, x_1, x_2, \ldots)) = (x_1, x_2, \ldots)$. The shift map takes I^∞ continuously onto itself. Also, if $f : I_i \to 2^{I_{i-1}}$ for each $i > 0$, and $M = \varprojlim f$, then $\sigma(M) = M$. Hence, M is invariant under the action of σ.
- The *graph* of the set-valued function $f : I \to 2^I$ is the set $\Gamma(f) = \{(x, y) : y \in f(x)\}$.
- The set-valued function $f : I \to 2^I$ is *upper semicontinuous at the point* x in I if for each open set V in I that contains $f(x)$, there is an open set U in I that contains x, and if $z \in U$, then the set $f(z) \subset V$. The function f is *upper semicontinuous* if it is upper semicontinuous at each point x in I. The function f is upper semicontinuous if and only if $\Gamma(f)$ is closed in $I \times I$. (See [**A**, **IM1**].)
- The set-valued function $f : I \to 2^I$ is called *surjective* if for each $y \in I$, there is $x \in I$ such that $y \in f(x)$.

If S and T are closed subsets of the compact metric space (X, d), the *Hausdorff distance* from S to T is defined as $d_H(S, T) = \max\{\sup\{d(s, T) : s \in S\}, \sup\{d(t, S) : t \in T\}\}$. Let 2^X denote the set of all non-empty closed subsets of X. The Hausdorff distance d_H gives a metric on the space 2^X, and under this *Hausdorff metric*, 2^X is a compact metric space. It is also true that the continua in 2^X form a closed subset of 2^X. That is if a sequence of compact and connected subsets in 2^X converges in the Hausdorff metric, then the sequence converges to a compact and connected subset in 2^X.

Let Z denote the Hilbert cube I^∞ or the finite product I^n for some positive integer $n > 1$. If C is a continuum in Z containing two points x and y, then there is a sequence $x = x_0, \ldots, x_n = y$ in C which is ϵ-dense in C and such that the distance from x_{i-1} to x_i is less than ϵ for $i = 1, \ldots, n$. We can then construct an arc from x_{i-1} to x_i which parameterized by a function $r_i : [0, 1] \to Z$ such that for each $t \in [0, 1]$, $r_i(t)$ is a point in Z whose jth coordinate is $\pi_j(x_{i-1}) + (\pi_j(x_i) - \pi_j(x_{i-1}))t$. Therefore we can construct a piecewise linear map $r : [0, 1] \to Z$ with $r(t) = r_i(nt - (i - 1))$ for $t \in [(i - 1)/n, i/n]$. The image $r([0, 1])$ is a continuum with Hausdorff distance from C at most ϵ. In this way we can construct a sequence $\{G_k\}$ of finite graphs whose Hausdorff limit is C. If $F \subset C$ is finite, we may assume that $F \subset G_k$ for each k. If A and B are points in G_k for each k, then for each k, there is a map f_k that maps $[0, 1]$ onto G_k such that $f_k(0) = A$ and $f_k(1) = B$ and $\pi_i \circ f_k$ is a piecewise linear map from $[0, 1]$ onto $[0, 1]$ for each (allowed) i. Therefore, if C is a continuum in Z, then there is a sequence $\{f_k\}$ of maps with each $f_k : [0, 1] \to Z$, $f_k(A) = 0$, $f_k(B) = 1$ for each k, and $\lim_H f_k([0, 1]) = C$.

The Mountain Climbing theorem is a useful tool in studying generalized inverse limits over intervals. We make use of a variation of this theorem called Mioduszewski's Uniformization theorem [**Mi**].

THEOREM 1. (Mioduszewski Uniformization theorem) *If $f, g : I \to I$ are piecewise linear mappings with f surjective, then there exist continuous mappings $a, b : I \to I$ such that $f \circ a = g \circ b$ and b is surjective. If, in addition, both f and g are surjective, then both a and b can be chosen to be surjective.*

The Mahavier product is a useful tool for studying subsets of a generalized inverse limit. (See [**GK2**, **BK**].) They have nice topological and algebraic properties. A closed subset of a standard inverse limit (where the bonding maps are single-valued functions—

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so actually a function from the space to itself) is a subinverse limit. *Unfortunately, a closed subset of a generalized inverse limit need not be a 'subgeneralized inverse limit'.* The Mahavier product allows one to consider closed subsets of the generalized inverse-limit space, whether or not they are subgeneralized inverse limits. It also makes it easy to consider 'finite' generalized inverse limits and their subsets. Suppose $n \geq 2$. Then $\{(x_0, x_1, \ldots, x_n) : x_{i-1} \in f(x_i) \text{ for } n \geq i > 0\}$ is what we mean by a *finite generalized inverse limit*. (For standard inverse limits over intervals, these sets are always arcs topologically, and therefore of limited interest. This is not the case for generalized inverse limits.)

Suppose X, Y, and Z are sets. Suppose $A \subset X \times Y$ and $B \subset Y \times Z$. Define

$$A \star B = \{(x, y, z) : (x, y) \in A, \text{ and } (y, z) \in B\}$$

to be the *Mahavier product* of A and B. Thus, $A \star B \subset X \times Y \times Z$. If $A = \{(a, b)\}$ and $B = \{(b, c)\}$, then we write $(a, b) \star (b, c)$ to mean $A \star B = \{(a, b)\} \star \{(b, c)\}$. Note that $A \star B = (A \times Z) \cap (X \times B)$. In the definition, the factor space Y is acting as the 'link up' factor; that is, if $a \in A$, then there is a point $c \in A \star B$ such that the projection of c to $X \times Y$ is a only if some point $b \in B$ has its projection to Y the same as the projection of a to Y. In particular, if the projection of A to Y does not intersect the projection of B to Y, then $A \star B = \emptyset$.

If, in addition, W is a set and $C \subset Z \times W$, we can form the Mahavier products $(A \star B) \star C$ and $A \star (B \star C)$. One of the nice algebraic properties of Mahavier products is that they are associative. Hence, $(A \star B) \star C = A \star (B \star C)$, and we can just write $A \star B \star C$ without ambiguity.

Now suppose that for each $i \geq 0$, X_i is a set, and $A_i \subset X_i \times X_{i-1}$. Then, using induction, we can form the Mahavier products

$$A_1 \star A_2 \star \cdots \star A_n \subset X_0 \times \cdots X_n$$

for each positive integer n. We often write $\star_{i=1}^{n} A_i$ for $A_1 \star A_2 \star \cdots \star A_n$. We also define $A_1 \star A_2 \star \cdots$ to be the set of points $x = (x_0, x_1, \ldots) \in \prod_{i=0}^{\infty} X_i$ such that $(x_{i-2}, x_{i-1}, x_i) \in A_{i-1} \star A_i$ for each $i \geq 2$, and we write $\star_{i=1}^{\infty} A_i$ for $A_1 \star A_2 \star \cdots$.

Note that if $f : I \to 2^I$ is a bonding function with graph $\Gamma(f)$, and $G := (\Gamma(f))^{-1} = \{(y, x) : (x, y) \in \Gamma(f)\}$, then

$$M = \varprojlim f = G \star G \star G \star \cdots = \star_{i=1}^{\infty} G.$$

Mahavier products make it easy to discuss finite generalized inverse limits and their subsets without introducing more notation. If n is a positive integer, and $f : I \to 2^I$ is a bonding function with graph $\Gamma(f)$, and $G := (\Gamma(f))^{-1} = \{(y, x) : (x, y) \in \Gamma(f)\}$, then

$$\overbrace{G \star G \star G \star \cdots \star G}^{n} = \star_{i=1}^{n} G = \{(x_0, x_1, \ldots, x_n) : x_{i-1} \in f(x_i) \text{ for } n \geq i > 0\}.$$

If $X = I^{k+1}$, $Y = I^{j+1}$, then $X \star Y = I^{k+j+1}$ as the 'link up' factor is the last interval factor of I^{k+1} and the first interval factor of I^{j+1}. (We think of the factors of I^{k+1} as running from 0 to k and the factors of I^{j+1} as running from 0 to j. Hence the last factor of I^{k+1} is the kth factor and the first factor of I^{j+1} is the 0th factor.) Suppose

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$f : [0, 1] \to X$ and $g : [0, 1] \to Y$ are continuous maps such that $\pi_k \circ f = \pi_0 \circ g$. Define $f \star g : [0, 1] \to X \star Y$ to be the function defined by $f(w) \star g(w)$ for each $w \in [0, 1]$. Then $f \star g$ is continuous. If f and g are piecewise linear, then $f \star g$ is piecewise linear. Also, if $Z = I^{m+1}$ and $h : [0, 1] \to Z$ is continuous such that $\pi_0 \circ h = \pi_j \circ g$, then $(f \star g) \star h = f \star (g \star h) = f \star g \star h$ is continuous, $f \star g \star h : [0, 1] \to X \star Y \star Z$, and if f, g, h are all piecewise linear, then so is $f \star g \star h$.

For the reader accustomed to inverse limits we introduce a new convention. If for $i > 0$, $f_i : I_i \to 2^{I_{i-1}}$ with the bonding map $f = f_i$ for each i, then if $x = (x_0, x_1, \ldots) \in M = \varprojlim f$, one usually thinks of $x_0 \in f(x_1)$, $x_1 \in f(x_2)$, and so forth; and (x_1, x_0), (x_2, x_1), $\ldots$ are points in $\Gamma(f)$. Now if $G \subset [0, 1] \times [0, 1]$, whether or not G is the graph of some bonding function, or the inverse of a bonding function, or neither, it makes sense, and is more natural, to think of a point $x = (x_0, x_1, \ldots)$ in $\star_{i=1}^{\infty} G$ as, coordinate by coordinate, $(x_0, x_1) \in G$, $(x_1, x_2) \in G$, and so forth.

Let Σ_m denote the set of all sequences of with members from the set $\{0, 1, \ldots, m-1\}$, i.e. $\Sigma_m = \{(s_1, s_2, \ldots) : s_i \in \{0, 1, \ldots, m-1\} \text{ for each } i \geq 1\}$. The *shift on m symbols* is the map $\sigma_m : \Sigma_m \to \Sigma_m$ defined by

$$\sigma_m(s_1, s_2, s_3, \ldots) = (s_2, s_3, \ldots).$$

The map σ_m takes Σ_m continuously onto itself, and is the exemplar of chaos. (For more details about its dynamical properties, see [**R**, **D**].) In this paper, m only takes the value 2, although our results generalize easily to other values of m in some cases.

If X is a space, and f is a map from X to itself, by the *dynamical system* (X, f), we mean the action of f on X. A set A in X is *invariant* (*strictly invariant*) if $f(A) \subset A$ ($f(A) = A$). If $f : X \to X$ and $g : Y \to Y$ are dynamical systems, f is *conjugate* to g if there is a homeomorphism $h : X \to Y$ such that $h \circ f(x) = g \circ h(x)$ for each $x \in X$. If two dynamical systems are conjugate, they are dynamically equivalent. If $f : X \to X$ and $g : Y \to Y$ are dynamical systems, f is *semiconjugate* to g, or more specifically, *f factors over g*, if there is a continuous surjection $h : X \to Y$ such that $h \circ f(x) = g \circ h(x)$ for each $x \in X$.

In this paper we find a number of strictly invariant Cantor sets C (contained in spaces X) such that $f|C$ is conjugate to a shift on 2 symbols, or $f|C$ is conjugate to a subshift of the shift on 2 symbols. Only one subshift appears in this paper, so we define the one we use: let

$$C' = \{(x_0, x_1, \ldots) \in \Sigma_2 : x_i = 0 \text{ or } x_i = 1, \text{ and if } x_i = 1, \text{ then } x_{i+1} = 0\}.$$

Then C' is a closed subset of Σ_2, is itself a Cantor set, and $\sigma_2(C') = C'$. This subshift of the full 2-shift is also chaotic.

3. *Shifts in generalized inverse limits*

Unlike standard inverse limits over intervals with bonding map f, which are always connected, generalized inverse limits over intervals need not be connected, not even when the maps are surjective with connected graphs. Of course, the same holds for Mahavier products of continua. Consider the following example.

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Example 1. (Ingram's example [**IM1**]) Let G be the closed subset of $[0, 1] \times [0, 1]$ that is the union of four line segments L_i, $1 \leq i \leq 4$, where (recall our new convention):

(1) L_1 is the line segment from $(0, 0)$ to $(\frac{1}{4}, \frac{1}{4})$;
(2) L_2 is the horizontal line segment from $(0, 0)$ to $(0, 1)$;
(3) L_3 is the vertical line segment from $(0, 1)$ to $(1, 1)$; and
(4) L_4 is the line segment from $(1, 1)$ to $(\frac{1}{4}, \frac{3}{4})$.

Then G is topologically just an arc with both its projections mapping onto $[0, 1]$, but $G \star G$ is not connected. $G \star G$ has two components with one component a tree and the other a single point.

Proof. It is all about how G 'lines up' with itself. Note that $G \star G = \bigcup\{L_i \star L_j : 1 \leq i, j \leq 4\}$. Next, $L_3 \star L_1$, $L_3 \star L_2$, $L_4 \star L_1$, and $L_4 \star L_2$ are all empty. The set $L_1 \star L_4 = \{(\frac{1}{4}, \frac{1}{4}, \frac{3}{4})\}$ is degenerate.

The remaining products are all arcs and together form a tree: $L_1 \star L_1 = \{(x, x, x) : x \in [0, \frac{1}{4}]\}$, $L_1 \star L_2 = L_2 \star L_2 = \{(0, 0, x) : x \in [0, 1]\}$, $L_1 \star L_3 = \{(x, x, 1) : x \in [0, \frac{1}{4}]\}$, $L_2 \star L_1 = \{(0, x, x) : x \in [0, \frac{1}{4}]\}$, $L_2 \star L_3 = \{(0, x, 1) : x \in [0, 1]\}$, $L_2 \star L_4 = \{(0, x, (x + 2)/3) : x \in [\frac{1}{4}, 1]\}$, $L_3 \star L_3 = L_3 \star L_4 = \{(x, 1, 1) : x \in [0, 1]\}$, $L_4 \star L_3 = \{(x, (x + 2)/3, 1) : x \in [\frac{1}{4}, 1]\}$, and $L_4 \star L_4 = \{(x, (x + 2)/3, (x + 8)/9) : x \in [\frac{1}{4}, 1]\}$ are the nine arcs comprising the tree. Since $(L_1 \star L_2) \cap (L_2 \star L_1) \cap (L_1 \star L_1) = \{(0, 0, 0)\}$, $(L_2 \star L_4) \cap (L_2 \star L_3) \cap (L_3 \star L_3) = \{(0, 1, 1)\}$, $(L_1 \star L_2) \cap (L_2 \star L_3) \cap (L_1 \star L_3) = \{(0, 0, 1)\}$, and $(L_3 \star L_3) \cap (L_4 \star L_4) \cap (L_4 \star L_3) = \{(1, 1, 1)\}$, the union of these nine arcs is connected. There are no other intersections, so the result is indeed a tree and a component of $G \star G$, with the other component consisting of the singleton $L_1 \star L_4 = \{(\frac{1}{4}, \frac{1}{4}, \frac{3}{4})\}$. □

However, the next lemma shows that there are continua contained in Mahavier products of continua

LEMMA 1. *Suppose $G \subset [0, 1] \times [0, 1]$ such that for some $j > 0$ (where j is either a positive integer or ∞), $\star_{i=1}^{j} G$ contains a continuum J such that $\pi_0(J) = [0, 1]$, and K is any continuum in $\star_{i=1}^{k} G$ (where k is a positive integer). Then there is a continuum $K' \subset \star_{i=1}^{j+k} G$ such that $\pi_{\langle 0,k\rangle}(K') = K$, and $\pi_{\langle k,j+k\rangle}(K') \subset J$. If $\pi_k(K) = [0, 1]$, then $\pi_{\langle k,j+k\rangle}(K') = J$. (Note that $K' \subset K \star J$.)*

Proof. By assumption, J contains two points P_0 and P_1 with $\pi_0(P_0) = 0$ and $\pi_0(P_1) = 1$. There exists a sequence of maps $f_n : [0, 1] \to I^{j+1}$ such that for each positive integer n, $f_n(0) = P_0$ and $f_n(1) = P_1$, f_n is piecewise linear, and $J = \lim_H f_n([0, 1])$. Also, there exists a sequence of maps $g_n : [0, 1] \to I^{k+1}$ such that $K = \lim_H g_n([0, 1])$, and each g_k is piecewise linear.

Then $\pi_0 \circ f_n : [0, 1] \to [0, 1]$ and $\pi_k \circ g_n : [0, 1] \to [0, 1]$ are piecewise linear maps with $\pi_0 \circ f_n$ onto. Therefore, according to Mioduszewski's Uniformization theorem [**Mi**], there are maps $\alpha_n, \beta_n : [0, 1] \to [0, 1]$ with β_n onto such that $\pi_0 \circ f_n \circ \alpha_n = \pi_k \circ g_n \circ \beta_n$.

Define the map $\Psi_n : [0, 1] \to I^{j+k+1}$ by $\Psi_n = (g_n \circ \beta_n) \star (f_n \circ \alpha_n)$. Then Ψ_n is piecewise linear and continuous. Since $\{\Psi_n([0, 1])\}$ is a bounded collection of continua,

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some subsequence of $\{\Psi_n([0, 1])\}$ converges in the Hausdorff metric to a continuum K'. Without loss of generality, assume that $K' = \lim_H \Psi_n([0, 1])$.

Let $x \in K'$. Then there is a sequence $\{x_n\}$ converging to x such that $x_n \in \Psi_n([0, 1])$ for each n. So for each n, there is a $t_n \in [0, 1]$ such that $x_n = [(g_n \circ \beta_n) \star (f_n \circ \alpha_n)](t_n) = \Psi_n(t_n)$. Since $\{x_n\}$ converges to x, $\pi_{\langle 0,k\rangle}(x) = \lim_{n\to\infty}(g_n \circ \beta_n)(t_n)$ and $\pi_{\langle k,k+j\rangle}(x) = \lim_{n\to\infty}(f_n \circ \alpha_n)(t_n)$. Let $x' = \pi_{\langle 0,k\rangle}(x) = \lim_{n\to\infty}(g_n \circ \beta_n)(t_n)$, and $x'' = \pi_{\langle k,k+j\rangle}(x) = \lim_{n\to\infty}(f_n \circ \alpha_n)(t_n)$. Then $x' \in K$, $x'' \in J$, and $x = x' \star x''$. We have shown that $K' \subset \star_{i=1}^{j+k} G$, $\pi_{\langle 0,k\rangle}(K') \subset K$, and $\pi_{\langle k,k+j\rangle}(K') \subset J$. Also, $K' \subset K \star J$. We next show that $K \subset \pi_{\langle 0,k\rangle}(K')$.

Let $x \in K$. Since $K = \lim_H g_n([0, 1])$, there is a sequence $\{t_n\}$ in $[0, 1]$ such that $x = \lim_{n\to\infty} g_n(t_n)$. Since $\beta_n : [0, 1] \to [0, 1]$ is onto for each n, for each n there is an $s_n \in [0, 1]$ such that $\beta_n(s_n) = t_n$. Then some subsequence $\{\Psi_{n_i}(s_{n_i})\}$ of $\{\Psi_n(s_n)\}$ converges to a point $x' \in K'$. So $\pi_{\langle 0,k\rangle}(x') = \lim_{i\to\infty} \pi_{\langle 0,k\rangle}(\Psi(s_{n_i})) = \lim_{i\to\infty} g_{n_i} \circ \beta_{n_i}(s_{n_i}) = x$. Hence, $K = \pi_{\langle 0,k\rangle}(K')$.

Finally, suppose $\pi_k(K) = [0, 1]$. We must show that $\pi_{\langle k,j+k\rangle}(K') = J$. We have already shown that $\pi_{\langle k,k+j\rangle}(K') \subset J$, so we must show that $J \subset \pi_{\langle k,j+k\rangle}(K')$. Since $\pi_k(K) = [0, 1]$, each g_n and α_n from above can be chosen so α_n and $\pi_k \circ g_n$ are onto. Suppose $x \in J$. Since $J = \lim_H f_n([0, 1])$, there is a sequence $\{x_n\}$ converging to x such that $x_n \in f_n([0, 1])$ for each n. For each n, there is $t_n \in [0, 1]$ such that $x_n = f_n(t_n)$. Furthermore, for each n, there is some $s_n \in [0, 1]$ such that $\alpha_n(s_n) = t_n$. Then $(g_n \circ \beta_n) \star (f_n \circ \alpha_n)(s_n) \in \Psi_n([0, 1])$. Let $z_n = (g_n \circ \beta_n) \star (f_n \circ \alpha_n)(s_n)$. Without loss of generality, assume that: (1) the sequence $\{s_n\}$ converges to $s \in [0, 1]$; and (2) the sequence $\{z_n\}$ converges to $z \in K'$. Then the sequence $\{\pi_{\langle k,k+j\rangle}(z_n)\}$ converges to $\pi_{\langle k,k+j\rangle}(z)$. However, $\pi_{\langle k,k+j\rangle}(z_n) = f_n \circ \alpha_n(s_n) = x_n$, so $\pi_{\langle k,k+j\rangle}(z) = x$, and it follows that $J \subset \pi_{\langle k,j+k\rangle}(K')$. □

Before proceeding to the next lemma, let us introduce some new notation and make some remarks. In the next lemma and a number of the results that follow in this section, we are making the assumption that G is a closed subset of $[0, 1] \times [0, 1]$ such that for some positive integer k, $\star_{i=1}^{k} G$ contains continua J_0 and J_1 such that $\pi_0(J_0) = \pi_0(J_1) = [0, 1]$. Let $\mathcal{A}$ denote the collection of all Mahavier products $A_1 \star A_2 \star A_3 \star \cdots$ such that each member of the sequence $A_1, A_2, A_3, \ldots$ is either J_0 or J_1. Note that because of the assumption that $\pi_0(J_0) = \pi_0(J_1) = [0, 1]$, when each member of the sequence $A_1, A_2, A_3, \ldots$ is either J_0 or J_1, then the Mahavier product $A_1 \star A_2 \star A_3 \star \cdots$ is closed and non-empty.

If, in addition, J_0 and J_1 are disjoint, then $\mathcal{A}$ is a Cantor set in the hyperspace 2^M (recall, 2^M is the space of all non-empty closed subsets of M, endowed with the Hausdorff metric), where $M = \star_{i=1}^{\infty} G$. We say that $\mathcal{A}$ is a *Cantor set of closed sets*. If each member of $\mathcal{A}$ is a continuum, we say that $\mathcal{A}$ is a *Cantor set of continua*. However, recall that even if J and K are continua it may not be the case that $J \star K$ is connected, as Ingram's example, Example 1 above, shows.

LEMMA 2. *If G is a closed subset of $[0, 1] \times [0, 1]$ such that for some positive integer k, $\star_{i=1}^{k} G$ contains continua J_0 and J_1 such that $\pi_0(J_0) = \pi_0(J_1) = [0, 1]$, then for each*

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$s = (s_1, s_2, \ldots)$ in Σ_2, the set $A_s = J_{s_1} \star J_{s_2} \star J_{s_3} \star \cdots$ contains a continuum A'_s such that $\pi_0(A'_s) = [0, 1]$. Moreover, we have the following.

(1) *If $\pi_k(J_0) = \pi_k(J_1) = [0, 1]$, then for each n, $\pi_{\langle nk,(n+1)k\rangle}(A_s) = J_{s_n}$.*

(2) *If J_0 and J_1 are disjoint, then there is a pairwise disjoint collection $\mathcal{A}_C$ of continua A'_s such that for each s in Σ_2, A'_s is a continuum contained in A_s such that $\pi_0(A'_s) = [0, 1]$.*

Proof. Suppose $s = (s_1, s_2, \ldots) \in \Sigma_2$. Apply Lemma 1 repeatedly to produce a continuum H_n in $J_{s_1} \star J_{s_2} \star J_{s_3} \star \cdots \star J_{s_n} \subset \star_{i=1}^{nk} G$ such that $\pi_0(H_n) = [0, 1]$. Then $\{H_n \times I^\infty\}$ is a collection of continua in the Hilbert cube. Without loss of generality, we can assume that $\{H_n \times I^\infty\}$ converges in the Hausdorff metric to the continuum H. Then H is a subcontinuum of $A_s = J_{s_1} \star J_{s_2} \star J_{s_3} \star \cdots$ such that $\pi_0(H) = [0, 1]$. Let $A'_s = H$.

(1) If $\pi_k(J_0) = \pi_k(J_1) = [0, 1]$, then the continuum A_s can be chosen so that $\pi_{\langle ik,(i+1)k\rangle}(A_s) = J_{s_i}$ for $1 \le i \le n$. Then A'_s has the property that $\pi_{\langle nk,(n+1)k\rangle}(A_s) = J_{s_n}$ for each n.

(2) If J_0 and J_1 are disjoint, then for $s = (s_1, s_2, \ldots)$ and $t = (t_1, t_2, \ldots)$ in Σ_2, with $s \ne t$, there is some n such that $s_n \ne t_n$, and this means that $\pi_{\langle n,(n+1)k\rangle}(A_s) \subset J_{s_n}$, $\pi_{\langle n,(n+1)k\rangle}(A_t) \subset J_{t_n}$ and $J_{s_n} \cap J_{t_n} = \emptyset$. Hence $A_s \cap A_t = \emptyset$ and $A'_s \cap A'_t = \emptyset$. □

Notation and Remarks.

- If we define $p : \Sigma_2 \to \mathcal{A}$ by $p(s) = A_s = J_{s_1} \star J_{s_2} \star J_{s_3} \star \cdots$ for each $s = (s_1, s_2, \ldots)$ in Σ_2, then p is continuous, and onto.
- If J_0 and J_1 are disjoint, then p is also one-to-one, so it is a homeomorphism.
- Let $\mathbf{A} = \bigcup \mathcal{A}$. Then $\mathbf{A}$ is a closed set in $\star_{i=1}^{\infty} G$, and if J_0 and J_1 are disjoint, $\mathbf{A}$ is a Cantor set of closed sets.
- Furthermore, $\sigma^k(J_{s_1} \star J_{s_2} \star J_{s_3} \star \cdots) \subset J_{s_2} \star J_{s_3} \star J_{s_4} \star \cdots$, so $\sigma^k \circ p(s) \subset p \circ \sigma_2(s)$.
- If $\pi_k(J_0) = \pi_k(J_1) = [0, 1]$, then $\sigma^k \circ p(s) = p \circ \sigma_2(s)$. Hence, if p is continuous and surjective (*and* $\pi_k(J_0) = \pi_k(J_1) = [0, 1]$), then σ_2 factors over σ^k; and if p is a homeomorphism, then σ_2 is conjugate to σ^k.

We can actually strengthen parts of Lemma 2 above. If we choose the continuum A'_s in the closed set A_s carefully, there is a continuous map p_C from Σ_2 onto $\mathcal{A}_C := \{A'_s : A_s \in \mathcal{A}\}$, and $\bigcup \mathcal{A}_C$ is a closed set in M. We can do this by requiring in the construction in Lemma 2 that if $s = (s_1, s_2, \ldots)$ and $t = (t_1, t_2, \ldots)$ are points in Σ_2 that agree in their first n components, then at step n in the induction, H_n is chosen to be the same set for s and t. If J_0 and J_1 are disjoint, then p_C is a homeomorphism, and the collection $\mathcal{A}_C$ is a Cantor set of continua in M. Thus, we have the following lemma.

LEMMA 3. *Suppose G is a closed subset of $[0, 1] \times [0, 1]$ such that for some positive integer k, $\star_{i=1}^{k} G$ contains continua J_0 and J_1 such that $\pi_0(J_0) = \pi_0(J_1) = [0, 1]$. Then we may choose the continua A'_s (guaranteed by Lemma 2 to exist in A_s) so that $\bigcup \mathcal{A}_C = \bigcup\{A'_s : s \in \Sigma_2\}$ is a closed set in M, $\pi_{\langle 0,k\rangle}(A'_s) = J_{s_1}$, $\pi_{\langle (n-1)k,nk\rangle}(A'_s) \subset J_{s_n}$ for $n > 1$, and $\sigma^k(A'_s) \subset A'_{\sigma_2(s)}$. Furthermore, we have the following.*

(1) *The function $p_C : \Sigma_2 \to \mathcal{A}_C$ defined by $p_C(s) = A'_s$ for $s \in \Sigma_2$ is continuous and surjective, and for each s in Σ_2, $A'_s \subset A_s$.*

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(2) *If J_0 and J_1 are disjoint, then the collection $\mathcal{A}_C$ is pairwise disjoint, and p_C is a homeomorphism.*

(3) *If $\pi_k(J_0) = \pi_k(J_1) = [0, 1]$, then for each $n \geq 0$, $\pi_{\langle nk,(n+1)k\rangle}(A'_s) = J_{s_n}$ for $s = (s_1, s_2, \ldots)$ in Σ_2.*

LEMMA 4. *Suppose G is a closed subset of $[0, 1] \times [0, 1]$ such that for some positive integer k, $\star_{i=1}^{k} G$ contains two disjoint continua J_0 and J_1 such that $\pi_0(J_0) = \pi_0(J_1) = [0, 1]$, and $\mathcal{A}_C$ is the uncountable pairwise disjoint collection of continua constructed in the previous lemma. (Hence, for each s in Σ_2, $A'_s \subset A_s$, $\pi_0(A'_s) = [0, 1]$.) Then for each continuum $W \subset \star_{i=1}^{\infty} G$, each positive integer m, and each $s \in \Sigma_2$, there is a continuum $W_{m,s} \subset \star_{i=1}^{\infty} G$ such that $\pi_{\langle 0,m\rangle}(W_{m,s}) = \pi_{\langle 0,m\rangle}(W)$ and $\sigma^m(W_{m,s}) \subset A'_s$. If $\pi_m(W) = [0, 1]$, then $\pi_{\langle m,\infty\rangle}(A_s) = A'_s$.*

Proof. This result follows from combining the previous lemmas. Note that it helps a bit here to index intervals. We are assuming that $\pi_{\langle 0,m\rangle}(W) \subset \prod_{i=0}^{m} I_i$ and $A'_s \subset \prod_{i=m}^{\infty} I_i$, and we apply Lemma 1 to obtain the continuum $W_{m,s}$. The continuum $W_{m,s} \subset \pi_{\langle 0,m\rangle}(W) \star A'_s$. □

Summarizing these lemmas, we have the following theorem.

THEOREM 2. *Suppose G is a closed subset of $[0, 1] \times [0, 1]$ such that for some positive integer k, $\star_{i=1}^{k} G$ contains continua J_0 and J_1 such that $\pi_0(J_0) = \pi_0(J_1) = [0, 1]$. (We use the notation from the previous lemmas.) Then the following hold.*

(1) *There is a continuous, surjective map $p : \Sigma_2 \to \mathcal{A}$ such that for each s in Σ_2, $p(s) = A_s$. If J_0 and J_1 are disjoint, p is a homeomorphism from Σ_2 onto $\mathcal{A}$.*

(2) *There is a continuous, surjective map $p_C : \Sigma_2 \to \mathcal{A}_C$ such that for each s in Σ_2, $p_C(s) = A'_s \subset A_s$ and A'_s is a continuum. If J_0 and J_1 are disjoint, p_C is a homeomorphism from Σ_2 onto $\mathcal{A}_C$. For each s in Σ_2, $\pi_0(A'_s) = [0, 1]$.*

(3) *For s in Σ_2, $p \circ \sigma_2(s) \subset \sigma^k \circ p(s)$. If $\pi_k(J_0) \cup \pi_k(J_1) \neq [0, 1]$, then $\sigma^k(\bigcup \mathcal{A}) \neq \bigcup \mathcal{A}$. If $\pi_k(J_0) = \pi_k(J_1) = [0, 1]$, $\mathcal{A}$ is invariant under σ^k, and $p \circ \sigma_2(s) = \sigma^k \circ p(s)$ for s in Σ_2, or, in other words, σ_2 factors over σ^k.*

(4) *For each continuum $W \subset \star_{i=1}^{\infty} G$ and positive integer m and $s \in \Sigma_2$, there is a continuum $W_{m,s} \subset \star_{i=1}^{\infty} G$ such that $\pi_{\langle 0,m\rangle}(W_{m,s}) = \pi_{\langle 0,m\rangle}(W)$ and $\sigma^m(W_{m,s}) \subset A'_s \subset A_s$. If $\pi_m(W) = [0, 1]$, $\sigma^m(W_{m,s}) = A'_s \subset A_s$.*

Proof. This essentially follows by combining the previous results. That $\sigma^k(\bigcup \mathcal{A})$ need not equal $\bigcup \mathcal{A}$ is easy to see if $\pi_k(J_0) \cup \pi_k(J_1) \neq [0, 1]$. Suppose $\pi_k(J_0) \cup \pi_k(J_1) \neq [0, 1]$. Suppose also $x_0 \in [0, 1] \backslash (\pi_k(J_0) \cup \pi_k(J_1))$. Then there is a point $x \in \bigcup \mathcal{A}$ such that $\pi_0(x) = x_0$ and $\pi_{\langle 0,k\rangle}(x) = (x_0, \ldots, x_k)$ is in either J_0 or J_1. Suppose $y \in \sigma^k(\bigcup \mathcal{A})$. Then there is $z \in \bigcup \mathcal{A}$ such that $\sigma^k(z) = y$. If $z = (z_0, z_1, \ldots)$, then $\sigma^k(z) = y = (z_k, z_{k+1}, \ldots)$, but $z_k \in \pi_k(J_0) \cup \pi_k(J_1)$, so $z_k \neq x_0$, and $x \notin \sigma^k(\bigcup \mathcal{A})$. □

THEOREM 3. *Let $\mathbf{A} = \bigcup \mathcal{A}$ and $\mathbf{A}_C = \bigcup \mathcal{A}_C$. Then $\mathbf{A}$ and $\mathbf{A}_C$ are closed subsets of M, $\sigma^k(\mathbf{A}) \subset \mathbf{A}$, and $\mathbf{A}_C \subset \mathbf{A}$. Let $A^* = \bigcap_{n=0}^{\infty} \sigma^{nk}(\mathbf{A})$. Then:*

(1) *$\sigma^k(A^*) = A^*$;*

(2) *$A^* \cap A \neq \emptyset$ for each $A \in \mathcal{A}$;*

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(3) *$A^* \subset \mathbf{A}$, but if $\pi_k(J_0) \cup \pi_k(J_1) \neq I_k$, then $A^* \neq \mathbf{A}$.*

Proof. Since $\bigcap_{n=0}^{\infty} \sigma^{nk}(\mathbf{A}) \subset \bigcap_{n=1}^{\infty} \sigma^{nk}(\mathbf{A}) \subset \cdots$, and $\mathbf{A} \supset \sigma^k(\mathbf{A}) \supset \sigma^{2k}(\mathbf{A}) \supset \cdots$, then $\sigma^k(A^*) = A^*$. Clearly $A^* \subset \mathbf{A}$.

Suppose $B_0, B_{-1}, \ldots$ is a sequence each member of which is either J_0 or J_1. Let $A = A_1 \star A_2 \star \cdots$ be a member of $\mathcal{A}$. Then $B_0 \star A \in \mathcal{A}$, and, since $\sigma^k(B_0 \star A) \subset A$, there is some point $z_{-1} \in \sigma^k(B_0 \star A) \cap A$, so $z_{-1} \in \sigma^k(\mathbf{A}) \cap A$. Next, $B_{-1} \star B_0 \star A \in \mathcal{A}$, and since $\sigma^k(B_{-1} \star B_0 \star A) \subset B_0 \star A$ then $\sigma^{2k}(B_{-1} \star B_0 \star A) \subset \sigma^k(B_0 \star A) \subset A$. Hence there is a point $z_{-2} \in \sigma^{2k}(\mathbf{A}) \cap \sigma^k(\mathbf{A}) \cap A$. We continue this process, finding for each n a point $z_{-n} \in A \cap \sigma^k(\mathbf{A}) \cap \cdots \cap \sigma^{nk}(\mathbf{A})$. Without loss of generality, we can assume that the sequence $z_{-1}, z_{-2}, \ldots$ converges to a point z in A. Then, however, by construction z must be in $\sigma^{nk}(\mathbf{A})$ for each n. So $z \in A \cap A^*$.

Now suppose $\pi_k(J_0) \cup \pi_k(J_1) \neq I_k$. Then for each $A \in \mathcal{A}$, while $\pi_0(A) = I_0$, $\pi_0(\sigma^k(A)) \subset \pi_k(J_0) \cup \pi_k(J_1)$, but now this set is in I_0, rather than I_k, and it is not equal to I_0. Then $\pi_0(\sigma^k(\mathbf{A})) \neq I_0$, and $A^* \neq \mathbf{A}$. □

Remarks. Hence, we have the appearance of our invariant Cantor sets carrying shift dynamics.

- For $s = (s_1, s_2, \ldots) \in \Sigma_2$, let $A_s = J_{s_1} \star J_{s_2} \star \cdots$ and let $A_s^* = A^* \cap A_s \neq \emptyset$. Note that $A^* = \bigcup\{A_s^* : s \in \Sigma_2\}$. Let $p^* : \Sigma_2 \to A^*$ be defined by $p^*(s) = A_s^*$. If J_0 and J_1 are disjoint, since σ_2 on Σ_2 is conjugate to σ^k on $\mathcal{A}^*$, $\mathcal{A}^*$ is a Cantor set in the space 2^M, σ^k is a map from the space 2^M to itself, and $\sigma^k|\mathcal{A}^*$ is conjugate to σ_2.
- If J_0 and J_1 are disjoint, and $\pi_k(J_0) = \pi_k(J_1) = [0, 1]$, then σ_2 on Σ_2 is conjugate to σ^k on $\mathcal{A}$, $\mathcal{A}$ is a Cantor set in the space 2^M, σ^k is a map from the space 2^M to itself, and $\sigma^k|\mathcal{A}$ is conjugate to σ_2.
- Note that if $k = 1$, it is not possible for the closed set G to satisfy both: (1) J_0 and J_1 are disjoint; and (2) $\pi_k(J_0) = \pi_k(J_1) = [0, 1]$. If $k \geq 2$, it is possible to satisfy both conditions non-trivially. For example, let $H_1 = \{(1/3, x) : 0 \leq x \leq 1\}$, $H_2 = \{(2/3, x) : 0 \leq x \leq 1\}$, $L_1 = \{(x, 1/3) : 0 \leq x \leq 1\}$, $L_2 = \{(x, 2/3) : 0 \leq x \leq 1\}$, and $G = H_1 \cup H_2 \cup L_1 \cup L_2$. Then $J_0 = \{(x, 1/3, y) : 0 \leq x, y \leq 1\}$ and $J_1 = \{(x, 2/3, y) : 0 \leq x, y \leq 1\}$ are disjoint closed subsets of $G \star G$ that satisfy both conditions.

COROLLARY 1. *Suppose, in addition, that J_0 and J_1 are disjoint. (Again, use the notation of the previous results.) Then $\mathbf{A}$ is a Cantor set of closed sets in M, and $\mathbf{A}_C$ is a Cantor set of continua in M. Furthermore A^* is a Cantor set of closed sets with $\sigma^k(A^*) = A^*$.*

It may or may not be the case that A^* is a Cantor set as the following examples demonstrate.

Example 2. Suppose $G \subset [0, 1] \times [0, 1]$ is closed, $0 \leq p < q \leq 1$, and $J_0 = [0, 1] \times \{p\} \subset G$, $J_1 = [0, 1] \times \{q\} \subset G$. (Recall our new convention of listing points in G with zeroth coordinate first.) Then $\mathcal{A} = \{I_0 \times \{t_1\} \times \{t_2\} \times \cdots$: for each $i \geq 1$, $t_i \in \{p, q\}\} = \mathcal{A}_C$, so $\mathcal{A} = \mathcal{A}_C$ is a Cantor set of arcs. Furthermore, $A^* = A_C^* = \{(t_1, t_2, \ldots)$: for each $i \geq 1$, $t_i \in \{p, q\}\}$ is a Cantor set. (Note that $k = 1$ here.)

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Example 3. Suppose $G \subset [0, 1] \times [0, 1]$ is closed, $0 < p < q < 1$, $K_0 = [0, p]$, and $K_1 = [q, 1]$. Let $J_0 = I_0 \times K_0$ be a subset of G, and let $J_1 = I_0 \times K_1$ be a subset of G. Then $\mathcal{A} = \mathcal{A}_C$ is a Cantor set of Hilbert cubes, as each A_s is a copy of the Hilbert cube. Suppose $s = (s_1, s_2, \ldots) \in \Sigma_2$. Then $I_0 \times K_{s_1} \times K_{s_2} \times K_{s_3} \times \cdots = A_s$. Furthermore, $\sigma(\mathbf{A}) = A^* = A^*_C$, in this example. Hence, A^* also consists of a Cantor set of Hilbert cubes, none of which projects onto I_0.

4. *A two-pass condition*

In [**I1**] Ingram defined a two-pass map from a graph G to itself as a continuous function $f : G \to G$ such that there exist two non-overlapping closed subgraphs H and K of G such that $f(H) = f(K) = G$. In this section we will recycle and generalize this term for use with set-valued functions, and present a number of results about the topological structure of the inverse limit when the bonding map has a two pass. These results are in continuum theory, so we begin with some definitions.

- A *continuum* is a compact, connected metric space.
- A *tree* is a finite connected graph that contains no simple closed curve.
- A continuum is *decomposable* if it can be written as the union of two proper subcontinua. If a continuum is not decomposable, it is *indecomposable*. A continuum is *hereditarily decomposable* if every non-degenerate subcontinuum of it is decomposable.
- A continuum X is *treelike* if, for each $\epsilon > 0$, there are a tree T and a surjective continuous map $f : X \to T$ such that, for each $t \in T$, the set $f^{-1}(t)$ has diameter less than ϵ.
- A continuum is $\frac{1}{2}$*-indecomposable* if there do not exist two disjoint subcontinua with non-empty interior in the continuum.
- A continuum is a λ*-dendroid* if it is a hereditarily decomposable treelike continuum.
- A continuum is a *dendroid* if it is a treelike arcwise connected continuum.
- Suppose x is a point of the space X. The *arc component* C_x of x is the union of all arcs in X that contain the point x. If $x \neq y$ are points in X, then either $C_x \cap C_y = \emptyset$ or $C_x = C_y$.
- A continuum X is *unicoherent* provided it is true that when X is the union of two subcontinua H and K, then $H \cap K$ is connected. A continuum is *hereditarily unicoherent* provided each subcontinuum of it is unicoherent. If X is a treelike continuum, then it is hereditarily unicoherent.
- Suppose X and Y are continua. A continuous function $f : X \to Y$ is *weakly confluent* if, for each continuum C in $f(X)$, at least one component of the preimage of C is a continuum that maps onto C.
- A continuum C is *irreducible between the points a and b* (with $a \neq b$) in the space X if C contains a and b, but no proper subcontinuum C' of C contains both a and b. A continuum X is *uniquely irreducible* if for each pair a, b of distinct points there is exactly one subcontinuum C in X which is irreducible between a and b. Treelike continua are uniquely irreducible.
- Suppose n is a positive integer greater than 2. A continuum X is an n*-od* if there exists a subcontinuum M of X such that $X \backslash M$ has at least n components.

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LEMMA 5. *If $G_1, G_2, \ldots, G_n$ is a finite sequence of closed sets such that $G_i \subset I_{i-1} \times I_i$ and $\pi_{i-1}(G_i) = I_{i-1}$, and there is a component K of G_n such that $\pi_{n-1}(K) = I_{n-1}$, then $\pi_{\langle 0,n-1\rangle} : \star_{i=1}^{n} G_i \to \star_{i=1}^{n-1} G_i$ is weakly confluent.*

Proof. Let J be a continuum in $\star_{i=1}^{n-1} G_i$. The method used in Lemma 4 is easily adapted to produce a continuum J' in $\star_{i=1}^{n} G_i$ such that $\pi_{\langle 0,n-1\rangle}(J') = J$ and $\pi_{\langle n-1,n\rangle}(J') \subset K$. □

THEOREM 4. *If, for each positive integer i, G_i is a closed subset of $I_{i-1} \times I_i$, and there is a component K_i of G_i such that $\pi_{i-1}(K_i) = I_{i-1}$, then for each positive integer j and each k either a positive integer or ∞, where $j \leq k \leq \infty$, the projection $\pi_{\langle 0,j\rangle} : \star_{i=1}^{k} G_i \to \star_{i=1}^{j} G_i$ is weakly confluent.*

Proof. If k is a positive integer, then the projection $\pi_{\langle 0,j\rangle} : \star_{i=1}^{k} G_i \to \star_{i=1}^{j} G_i$ is the composition of a finite number of projections that satisfy the conditions of Lemma 5, and a map that is the composition of weakly confluent maps is also weakly confluent.

Suppose $k = \infty$. Let J be a continuum in $\star_{i=1}^{j} G_i$. Then again using Lemma 5, for each positive integer n, there is a continuum $J_n \subset \star_{i=1}^{j+n} G_i$ such that $\pi_{\langle 0,j\rangle}(J_1) = J$ and $\pi_{\langle 0,j+i-1\rangle}(J_i) = J_{i-1}$ for $1 < i \leq n$. Let $H_n = J_n \times I^{\infty}$ for each n. Then $\{H_n\}$ is a nested collection of continua in the Hilbert cube whose intersection is a continuum $J' \subset \star_{i=1}^{\infty} G_i$ such that $\pi_{\langle 0,j\rangle}(J') = J$. □

THEOREM 5. *If, for each positive integer i, G_i is a closed subset of $I_{i-1} \times I_i$, and there is a component J_i of G_i such that $\pi_i(J_i) = I_i$, then for each positive integer j and each k either a positive integer or ∞, where $j < k \leq \infty$, the projection $\pi_{\langle j-1,k\rangle} : \star_{i=1}^{k} G_i \to \star_{i=j}^{k} G_i$ is weakly confluent.*

COROLLARY 2. *If G is a closed subset of $[0, 1] \times [0, 1]$, and there is a component J of G such that $\pi_0(J) = [0, 1]$, then the shift map $\sigma : \star_{i=1}^{\infty} G \to \star_{i=1}^{\infty} G$ is weakly confluent.*

A set-valued function $f : I \to 2^I$ with graph G has a *two pass* $\{J_0, J_1\}$ if, for some natural number m, there are disjoint continua J_0 and J_1 in $\star_{i=1}^{m} G^{-1}$ such that $\pi_0(J_0) = \pi_0(J_1) = [0, 1]$.

THEOREM 6. *For a set-valued function $f : I \to 2^I$ with graph G the following are equivalent.*

(i) *f has a two pass.*

(ii) *There is an integer m such that $\star_{i=1}^{m} G^{-1}$ contains disjoint continua K_0 and K_1 such that $\pi_0(K_0) = [0, 1]$ and $\pi_m(K_0) \subset \pi_0(K_1)$.*

(iii) *$\star_{i=1}^{\infty} G^{-1}$ contains disjoint continua L_0 and L_1 such that $\pi_0(L_0) = \pi_0(L_1) = [0, 1]$*

Proof. It follows easily from Lemma 2 that (i) implies (iii). It is also easy to see that (i) implies (ii).

Assume (iii). That is, assume $\star_{i=1}^{\infty} G^{-1}$ contains disjoint continua L_0 and L_1 such that $\pi_0(L_0) = \pi_0(L_1) = [0, 1]$. Then there is an integer m such that $\pi_{\langle 0,m\rangle}(L_0) \cap \pi_{\langle 0,m\rangle}(L_1) = \emptyset$. It follows that $\{\pi_{\langle 0,m\rangle}(L_0), \pi_{\langle 0,m\rangle}(L_1)\}$ is a two pass for f. Therefore (iii) implies (i).

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Now assume (ii). That is, assume there is an integer m such that $\star_{i=1}^{m} G^{-1}$ contains disjoint continua K_0 and K_1 such that $\pi_0(K_0) = [0, 1]$ and $\pi_m(K_0) \subset \pi_0(K_1)$. Then by a slight modification of Lemma 1 (because rather than $\pi_0(K_1) = [0, 1]$ we have only that $\pi_m(K_0) \subset \pi_0(K_1)$), $\star_{i=1}^{2m} G^{-1}$ contains a continuum $J_0 \subset K_0 \star K_1$ such that $\pi_{\langle 0,m\rangle}(J_0) = K_0$, and $\star_{i=1}^{2m} G^{-1}$ contains a continuum $J_1 \subset K_0 \star K_0$ such that $\pi_{\langle 0,m\rangle}(J_1) = K_0$. Thus $J_0 \cap J_1 = \emptyset$ and $\pi_0(J_0) = \pi_0(J_1)$. Therefore $\{J_0, J_1\}$ is a two pass for f. Therefore (ii) implies (i). □

For inverse limits with one continuous single-valued function from [0, 1] to [0, 1] there is a well-studied connection between chaos in the shift map and the presence of an indecomposable continuum in the inverse limit [**I1**]. Such inverse-limit spaces are chainable, and therefore cannot contain a simple triod, much less an n-od for every n. In the case of a set-valued bonding map with a two pass the inverse limit can be hereditarily decomposable, even hereditarily arcwise connected, and the shift map can be chaotic, but there must be an n-od for each n.

THEOREM 7. *Suppose $f : [0, 1] \to 2^{[01]}$ is an upper semicontinuous function with graph G such that f has a two pass and $\star_{i=1}^{\infty} G^{-1}$ is a decomposable continuum. Then $\star_{i=1}^{\infty} G^{-1}$ contains an n-od for each $n \in \mathbb{N}$.*

Proof. Assume $M = \star_{i=1}^{\infty} G^{-1}$ is decomposable. Then there is a proper subcontinuum K of M with non-empty interior. Also M contains an uncountable pairwise disjoint collection of continua each having Hausdorff distance from M so small that each of these continua intersects K and $M \setminus K$. It follows that M contains and n-od for each n. □

Even when the existence of a two pass does not lead to an indecomposable continuum, it does, in important cases such as treelike continua, lead to $\frac{1}{2}$-indecomposability.

THEOREM 8. *Suppose $f : [0, 1] \to 2^{[01]}$ is an upper semicontinuous function with graph G such that f has a two pass and $\star_{i=1}^{\infty} G^{-1}$ is a hereditarily unicoherent continuum. Then $\star_{i=1}^{\infty} G^{-1}$ is $\frac{1}{2}$-indecomposable.*

Proof. Assume G is a closed subset of $[0, 1] \times [0, 1]$ for which there is a k such that $\star_{i=1}^{k} G^{-1}$ contains disjoint continua J_0 and J_1 such that $\pi_0(J_0) = \pi_0(J_1) = [0, 1]$. Also suppose $M = \star_{i=1}^{\infty} G^{-1}$ is a continuum but not $\frac{1}{2}$-indecomposable. Thus M contains disjoint continua H and K each having non-empty interior. For each $\epsilon > 0$, there is an m such that $1/2^m < \epsilon$, and if $L \subset I^{\infty}$ is a continuum such that $\pi_0(L) = [0, 1]$, then $d_H((\star_{i=1}^{m} G^{-1}) \star L, \star_{i=1}^{\infty} G^{-1}) \leq 1/2^m$. Then by Lemma 4, M contains an uncountable pairwise disjoint collection of continua such that the Hausdorff distance from each of them to M is less than ϵ. So there are disjoint continua C and D in M such that $C \cap H \neq \emptyset$, $C \cap K \neq \emptyset$, $D \cap H \neq \emptyset$, and $D \cap K \neq \emptyset$. It follows that $H \cup C \cup D$ and $K \cup C \cup D$ are both subcontinua of M, and $(H \cup C \cup D) \cap (K \cup C \cup D) = C \cup D$ is not connected. Thus M is not hereditarily unicoherent. □

5. *Examples*

Suppose $f : I \to 2^I$ is a surjective upper semicontinuous bonding map such that the graph G of f is composed of two line segments J_0 and J_1 such that $\pi_0(J_0) = \pi_0(J_1) = I$

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and $J_0 \cap J_1$ consists of exactly one point. Let $M = \varprojlim f$. We consider the topology and dynamics of some families of inverse limits satisfying these requirements. It follows from results from the literature that each member of this family of inverse limits is connected, treelike, and one-dimensional. (In [**CC**] it was proven that a one-dimensional continuum with trivial shape is treelike. In [**CR**] it was proven that when the factor spaces have trivial shape and the graphs of the bonding maps have trivial shape, then the inverse limit has trivial shape. In [**N2**], it was shown that if the graph of a set-valued function on an interval does not contain a vertical line and a horizontal line then the inverse limit is one-dimensional.) There are many cases to consider (and we do not consider them all). First, however, a proposition.

PROPOSITION 1. *The space M consists of the union of all arcs $A_1 \star A_2 \star A_3 \star \cdots$ such that each member of the sequence $A_1, A_2, A_3, \ldots$ is either J_0 or J_1.*

Remark. This proposition suggests that M always consists of a Cantor set of arcs. This is not the case. In some of the cases for this family many of the arcs $A_1 \star A_2 \star A_3 \star \cdots$ turn out to represent the same arc. That is, it can happen that $A_1, A_2, A_3, \ldots$ and $B_1, B_2, B_3, \ldots$ are *different* sequences each member of which is either J_0 or J_1, and $A_1 \star A_2 \star A_3 \star \cdots = B_1 \star B_2 \star B_3 \star \cdots$.

Our findings suggest that: (1) if for some $k > 1$, $\star_{i=1}^{k} G^{-1}$ contains two disjoint continua each of which projects onto I_0, then the dynamics and topology of M are quite interesting; and (2) if, for each $k > 1$, $\star_{i=1}^{k} G^{-1}$ does not contain two disjoint continua each of which projects onto I_0, then the dynamics and topology of M are relatively simple and not so interesting. With Theorem 6 in the previous section, we have already turned this into a mathematical statement.

Example 4. For each c in $[0, 1]$, let $f_c : [0, 1] \to 2^{]0,1[}$ be the upper semicontinuous, surjective map whose graph G_c is the union of (1) the straight line with endpoints $(0, 0)$ and $(c, 1)$, and (2) the straight line with endpoints $(c, 1)$ and $(1, 0)$.

(1) Consider the case where $0 < c < 1$, first. (We note that each f_c in this case is just an asymmetric version of the tent map.) For c in $(0, 1)$, G_c is the graph of the continuous function f_c, where $f_c(x) = x/c$ for $x \in [0, c]$, and $f_c(x) = 1 + (c - x)/(1 - c)$ for $x \in [c, 1]$. Clearly, G_c does not contain two disjoint continua whose projection onto the 0-coordinate is $[0, 1]$. However, consider the two sets $J_0 = \{(x_0, x_1, x_2) \in \star_{i=1}^{2} G_c^{-1} : x_2 \in [0, c^2]\}$ and $J_1 = \{(x_0, x_1, x_2) \in \star_{i=1}^{2} G_c^{-1} : x_2 \in [c + (1 - c)^2, 1]\}$.

Since f_c is a continuous function, J_0 and J_1 are continua. Since $c^2 < c < c + (1 - c)^2$, $J_0 \cap J_1 = \emptyset$. Since $f_c^2(0) = 0$ and $f_c^2(c^2) = 1$, $\pi_0(J_0) = [0, 1]$. Since $f_c^2(c + (1 - c)^2) = 1$ and $f_c^2(1) = 0$, $\pi_0(J_1) = [0, 1]$. So even though G_c does not contain disjoint continua whose projection onto the 0-coordinate is $[0, 1]$, $G_c^{-1} \star G_c^{-1}$ does contain two such disjoint continua. That is, f_c has a two pass. It is well known that $\star_{i=1}^{\infty} G_c^{-1} = M = \varprojlim f_c$ is indecomposable, homeomorphic to the Knaster bucket handle continuum, and that the dynamics of both f_c and σ are chaotic.

So here we have two disjoint continua $G_c^{-1} \star G_c^{-1}$ whose projection onto the 0-coordinate is $[0, 1]$, and both the dynamics and topology are interesting.

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(2) Now suppose $c = 0$. In this case, $G_c = G_0$ is not the graph of a continuous (single-valued) function. Let $J_0 = \{(x, 0, 0) \in \star_{i=1}^{2} G_c^{-1} : x \in [0, 1]\}$ and $J_1 = \{(x, 0, 1) \in \star_{i=1}^{2} G_c^{-1} : x \in [0, 1]\}$.

These are two disjoint arcs in $G_c^{-1} \star G_c^{-1}$ both of whose projections onto the 0-coordinate is [0, 1]. So again we have disjoint continua at stage two (i.e. in $G_c^{-1} \star G_c^{-1}$) with projections onto the 0-coordinate being [0, 1], that is, f_0 has a two pass.

Indeed, the continuum $\star_{i=1}^{\infty} G_0^{-1} = M = \varprojlim f_0$ has been given the name the 'Monster' by Banič. Several authors have studied it, including Banič [**BCMM1**] and Ingram [**I5**]. It contains an invariant Cantor set, is treelike, and has both complicated dynamics and topology. (We discuss in detail more interesting properties of an entire family of maps of which it is a member later.)

(3) For $c = 1$ and $n > 1$, $\star_{i=1}^{n} G_1^{-1}$ consists of the union of a collection of $n + 1$ arcs L_j, whose first j coordinates are the same x in [0, 1] and whose remaining coordinates are equal to 1. Also, each continuum C in $\star_{i=1}^{n} G_1^{-1}$ such that $\pi_0(C) = [0, 1]$ contains the point each of whose coordinates is 1. So there are *no disjoint continua* projecting onto I_0 here.

$\star_{i=1}^{\infty} G_1^{-1}$ consists of the union of a countably infinite collection of arcs A_j, whose first j coordinates are the same x in [0, 1] and whose remaining coordinates are equal to 1 together with the arc $A_\infty = \{(x, x, x, \ldots) : 0 \le x \le 1\}$. In fact, this inverse limit is homeomorphic to a harmonic fan with vertex $(1, 1, 1, \ldots)$. Under the shift, A_1 maps onto the point $(1, 1, 1, \ldots)$; for $j > 1$, A_j maps onto A_{j-1}; and A_∞ maps onto itself with each point fixed.

So, in this case with no two pass we have no invariant Cantor set, and simple topology and simple dynamics. For this family of functions interesting topology and interesting dynamics both depend on having a two pass.

Next we turn to the exploration of another family of graphs composed of two line segments J_0 and J_1 such that $\pi_0(J_0) = \pi_0(J_1) = [0, 1]$ and $J_0 \cap J_1$ consists of exactly one point. This family overlaps with the family above, as it contains G_0 from the previous family.

Suppose $0 \le p \le 1$, and the line J_0 is a vertical line in $[0, 1] \times [0, 1]$ with $\pi_1(J_0) = \{p\}$. Suppose J_1 is another line in $[0, 1] \times [0, 1]$ and $G = J_0 \cup J_1$. Then, in order for $J_0 \cup J_1$ to be the graph of an upper semicontinuous function, J_1 must be either the diagonal line given by $y = x$ or the line $y = 1 - x$, or with our order convention the line $x = y$ or the line $x = 1 - y$. Let $H = G^{-1}$. Then $M = \star_{i=1}^{\infty} H$. If J_1 is the diagonal line, then both the topology and dynamics are simple, and we leave the investigation of this case to the reader. The case where J_1 is the line $x = 1 - y$ is more interesting. The dynamics and topology are quite different for $p = 1/2$ and $p \neq 1/2$.

Example 5. ($\mathbf{p = 1/2}$) Suppose the line J_0 is a vertical line with $\pi_1(J_0) = \{p\} = \{1/2\}$, and the line J_1 is the line given by $x = 1 - y$. The topology and dynamics are relatively simple for the case when $p = 1/2$, but this case is the limiting case for the topology and dynamics when $p \neq 1/2$, as it is a 'collapsed' version of the topology and dynamics that occur when $p \neq 1/2$.

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Note that if $x = (x_0, x_1, \ldots)$ is a point in M with $n \geq 0$ such that $x_n \neq p$, but $x_{n+1} = p$, then $x_i = 1 - x_{i-1}$ for $0 < i \leq n$ and $x_i = p$ for each $i > n$. For each $n > 0$, $x \in [0, 1]$, let $\mathbf{x}_n = \overbrace{(x, 1-x, x, 1-x, \ldots)}^{n}$ and let $\mathbf{x}$ denote the point $(x, 1-x, x, 1-x, \ldots)$ in M. For $n > 0$, let $T_n = \{\mathbf{x}_n \oplus \mathbf{p} : x \in [0, 1]\}$ and let $T_\infty = \{\mathbf{x} : x \in [0, 1]\}$. Then $M = \bigcup\{T_n : n \in \mathbb{N} \text{ or } n = \infty\}$, so M is again a countably infinite union of arcs, each of which contains the point $\mathbf{p}$. In fact, the space M in this example is homeomorphic to the space obtained by using the vertical line from $(0, q)$ to $(1, q)$, $q \neq 0, 1$, together with the diagonal line to make the closed set G: in this example, M is a 'double' fan. The shift map σ maps each point of T_1 to $\mathbf{p}$, and for $\infty > n > 1$, maps T_n onto T_{n-1}, but here the points are 'flipped' since a point $\mathbf{x}_n \oplus \mathbf{p}$ is mapped to the point $(\mathbf{1} - \mathbf{x})_{n-1} \oplus \mathbf{p}$. The arc T_∞ is mapped onto itself with each point $\mathbf{x}$ mapped to $(\mathbf{1} - \mathbf{x})$. Hence the dynamics and the topology are simple, and there is a 'flip' of the space under the action of σ.

It is also important to note that in this case the function does not have a two pass since every continuum in the inverse limit with first coordinate $\frac{1}{2}$ contains the point each of whose coordinates is $\frac{1}{2}$.

Example 6. ($\mathbf{p} \neq \mathbf{1/2}$) Suppose $p \neq 1/2$, but as before J_0 is the vertical line with $\pi_0(J_0) = \{p\}$ and J_1 is the line given by $x = 1 - y$. Note that when $p = 0$, we get the inverse limit M that has been called the 'Monster' by Banič [**BCMM1**]. Now the topology and dynamics become more complicated. Indeed, if $K_0 = J_0$ and $K_1 = \{(p, 1-p)\}$, then K_0 and K_1 satisfy Theorem 6 (part (ii)), so these functions do have a two pass. Again we find that for this family having interesting topology and interesting dynamics depends on having a two pass. However, in the case of these functions, the inverse limits are all hereditarily decomposable treelike continua, illustrating that at least in this family interesting dynamics and topology are not related to indecomposability.

We shall see that an invariant Cantor set arises, and M is connected, but in a more complicated way. First there are two fixed points: $\mathbf{1/2} = (1/2, 1/2, \ldots)$ on the arc $J_1 \star J_1 \star J_1 \star \cdots$, and $\mathbf{p} = (p, p, p, \ldots)$ on the arc $J_0 \star J_0 \star J_0 \star \cdots$.

Now consider

$$C = \{(x_0, x_1, \ldots) \in M : x_i = p \text{ or } x_i = 1 - p, \text{ and if } x_i = 1 - p, \text{ then } x_{i+1} = p\}.$$

Thus, C is a Cantor set contained in M, and $\sigma(C) = C$, so it is a strictly invariant Cantor set contained in M, with the action of σ on C conjugate to the subshift of finite type of the full shift on two symbols discussed in the introduction.

The dynamics on σ on M are also fairly easy to describe. Let $\overline{K}$ denote the arc $J_1 \star J_1 \star J_1 \star \cdots$. Clearly, $\sigma(\overline{K}) = \overline{K}$, with σ 'flipping' points on $\overline{K}$. More precisely, each point of $\overline{K}$ is a period 2 point. (The point $\mathbf{1/2}$ is fixed, but all other points on this arc have prime period 2.) For any point $x = (x_0, x_1, \ldots)$ in $M \backslash \overline{K}$ there is least n such that $x_n \in \{p, 1-p\}$. Then for $m > n$, we also have $x_m \in \{p, 1-p\}$. Hence, $\sigma^m(x) \in C$ for $m \geq n$. Thus, all points of $M \backslash \overline{K}$ are eventually pulled into C and, once pulled in, they stay there.

In the results that follow, we investigate the interesting topology of this family of spaces.

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Let $\mathcal{M} = \{A_1 \star A_2 \star A_3 \star \cdots : A_i \in \{J_0, J_1\}, i \geq 1\}$. For each positive integer n, let $\mathcal{M}_n = \{A_1 \star A_2 \star A_3 \star \cdots \star A_n : A_i \in \{J_0, J_1\}, n \geq i \geq 1\}$. Also, let $M_n = \bigcup \mathcal{M}_n = \star_{i=1}^{n} G$. Note that the arc $J_0 \star J_1 \star J_0$ is the same arc as $J_0 \star J_1 \star J_1$. Both arcs consist of the points of the form $(x, p, 1-p, p)$ for $0 \leq x \leq 1$. It follows that if $A := A_1 \star A_2 \star A_3 \star \cdots \in \mathcal{M}$ and there is some n such that $A_n = J_0$, $A_{n+1} = J_1$, $A_{n+2} = J_0$, and $B := B_1 \star B_2 \star B_3 \star \cdots \in \mathcal{M}$ such that $B_i = A_i$ for $i \neq n+2$, and $B_{n+2} = J_1$, then $B = A$. Hence, many sequences of continua J_0 and J_1 result in the same arc in $\mathcal{M}$.

LEMMA 6. *There is a decomposition $\mathcal{G}$ of the inverse-limit space M into an uncountable, mutually disjoint collection of dense, arcwise connected sets.*

Proof. For ease of notation, let $J_0 = L$ and $J_1 = K$. Let $\overline{K}$ denote the arc $K \star K \star K \star \cdots \in \mathcal{M}$. For each positive integer n, let $\mathcal{G}_{\overline{K},n} = \{x = (x_0, x_1, \ldots) : \text{ for each } i \geq 1, x_{n+i} = 1 - x_{n+i-1}\}$. Let $\mathcal{G}_{\overline{K}} = \bigcup \mathcal{G}_{\overline{K},n}$. Note the following.

- For each positive integer n, $\mathcal{G}_{\overline{K},n} \subset \mathcal{G}_{\overline{K},n+1}$.
- For each positive integer n, $\mathcal{G}_{\overline{K},n}$ is arcwise connected since $\mathcal{G}_{\overline{K},n} = M_n \star \overline{K}$. Hence, $\mathcal{G}_{\overline{K},n}$ is a tree in M. (This is because $\psi : M_n \to M_n \star \overline{K}$ defined by $\psi((x_0, x_1, \ldots, x_n)) = (x_0, x_1, \ldots, x_n, 1-x_n, x_n, 1-x_n, \ldots)$ is a homeomorphism.)
- Let $\overline{(p, 1-p)} = (p, 1-p, p, 1-p, p, \ldots)$ and $\overline{(1-p, p)} = (1-p, p, 1-p, p, 1-p, \ldots)$. The points $\overline{(p, 1-p)}$ and $\overline{(1-p, p)}$ are in $\mathcal{G}_{\overline{K},n}$ for each n. Thus, $\mathcal{G}_{\overline{K}}$ is connected and arcwise connected.
- Since $\bigcup(M_n \star \overline{K}) = \bigcup \mathcal{G}_{\overline{K},n} = \mathcal{G}_{\overline{K}}$, $\mathcal{G}_{\overline{K}}$ is dense in M.

Now for each $x = (x_0, x_1, \ldots) \in M \backslash \mathcal{G}_{\overline{K}}$, there is least $N = N_x$ such that $x_N = p$, and there is a sequence $N < n_1 < n_2 < \cdots$ such that $(x_{n_i}, x_{n_i+1}) = (p, p)$.

For each positive integer $n > N_x$, let $\mathcal{G}_{x,n} = \{z = (z_0, z_1, \ldots) : z_m = x_m \text{ for each } m > n\}$. Let $\mathcal{G}_{x,n} = \{x\}$ for $1 < n \leq N_x$. Let $\mathcal{G}_x = \bigcup \mathcal{G}_{x,n}$. Note the following.

- For each positive integer n, $\mathcal{G}_{x,n} \subset \mathcal{G}_{x,n+1}$.
- For each positive integer n such that $x_n = x_{n+1} = p$, $\mathcal{G}_{x,n}$ is arcwise connected since $\mathcal{G}_{x,n} = M_n \oplus \sigma^{n+1}(x) = \{(z_0, z_1, \ldots) \in M : (z_{n+1}, z_{n+2}, \ldots) = (x_{n+1}, x_{n+2}, \ldots)\}$, and $\mathcal{G}_{x,n}$ is a tree in M. Since, for infinitely many n, $(x_n, x_{n+1}) = (p, p)$, then, for infinitely many n, $\mathcal{G}_{x,n}$ is arcwise connected.
- The point x is in $\mathcal{G}_{x,n}$ for each n. Thus, $\mathcal{G}_x$ is arcwise connected, since $\mathcal{G}_x = \bigcup \mathcal{G}_{x,n} = \bigcup\{\mathcal{G}_{x,n} : (x_n, x_{n+1}) = (p, p)\}$.
- Since $\bigcup\{M_n \oplus \sigma^{n+1}(x) : (x_n, x_{n+1}) = (p, p)\} = \bigcup \mathcal{G}_{x,n} = \mathcal{G}_x$, $\mathcal{G}_x$ is dense in M.

Now suppose $x \notin \mathcal{G}_{\overline{K}}$. If $z \in \mathcal{G}_{\overline{K}} \cap \mathcal{G}_x$, then there is sufficiently large n such that $z \in \mathcal{G}_{\overline{K},n} \cap \mathcal{G}_{x,n}$. Further, there are infinitely many m such that $(z_m, z_{m+1}) = (p, p) = (x_m, x_{m+1})$, since $z \in \mathcal{G}_x$. However, since $z \in \mathcal{G}_{\overline{K}}$, $(z_m, z_{m+1}) = (z_m, 1 - z_m)$ for sufficiently large m. This is a contradiction, so $\mathcal{G}_{\overline{K}} \cap \mathcal{G}_x = \emptyset$.

Suppose x, y are points in $M \backslash \mathcal{G}_{\overline{K}}$, and $z \in \mathcal{G}_x \cap \mathcal{G}_y$. There is N such that if $n > N$, $z \in \mathcal{G}_{x,n} \cap \mathcal{G}_{y,n}$. Then for each $n > N$, $z_n = x_n = y_n$. Then $\mathcal{G}_x = \mathcal{G}_y$.

Let $\mathcal{G} = \{\mathcal{G}_{\overline{K}}\} \cup \{\mathcal{G}_x : x \notin \mathcal{G}_{\overline{K}}\}$. Then $\mathcal{G}$ is a mutually disjoint, uncountable collection of dense, arcwise connected sets whose union is all of M. □

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Next we consider the structure of certain basic closed sets. Specifically, suppose n is a positive integer, and $A \in \mathcal{M}_n$. (We continue to use the notation developed previously for the case $p \neq 1/2$.) The properties of $(A \times I^\infty) \cap M$ are very different for $A = \overbrace{K \star K \star \cdots \star K}^{n}$ and $A \neq \overbrace{K \star K \star \cdots \star K}^{n}$, as in the following lemma.

LEMMA 7.

(1) *Suppose* $A = A_1 \star A_2 \star \cdots \star A_n \in \mathcal{M}_n$, *and, for some* $1 \leq i \leq n$, $A_i = L$. *Then* $(A \times I^\infty) \cap M$ *consists of a Cantor set of arcs.*

(2) *For n a positive integer, let* $\overline{K}^n = \star_{i=1}^n K$. *Then* $\overline{K}^n \star M$ *is the closure of an open set in* M. *Furthermore,* $\overline{K}^n \star M$ *is a continuum homeomorphic to* M, *and* $M \supset \overline{K}^1 \star M \supset \overline{K}^2 \star M \supset \cdots$.

Proof. (1) Suppose $A = A_1 \star A_2 \star \cdots \star A_n \in \mathcal{M}_n$, and for some $1 \leq i \leq n$, $A_i = L$. Then either (1) for each $x = (x_0, \ldots, x_n) \in A$, $x_n = p$, or (2) for each $x = (x_0, \ldots, x_n) \in A$, $x_n = 1 - p$. Let $C_0 = \{s = (s_1, s_2, \ldots) \in C : s_1 = p\}$ and let $C_1 = \{s = (s_1, s_2, \ldots) \in C : s_1 = 1 - p\}$.

If for each $x = (x_0, \ldots, x_n) \in A$, $x_n = p$, then $(A \times I^\infty) \cap M = \bigcup\{A \star s : s \in C_0\}$ and $(A \times I^\infty) \cap M$ consists of a mutually disjoint Cantor set of arcs, as follows. If $z = (z_0, z_1, \ldots) \in (A \times I^\infty) \cap M$, then $(z_0, \ldots, z_n) \in A$ and $z_n = p$. Then $(z_n, z_{n+1}, \ldots) \in C_0$. Let $t = (z_n, z_{n+1}, \ldots)$. Then $z \in A \star t$, and $z \in \bigcup\{A \star s : s \in C_0\}$. If, on the other hand, $z \in \bigcup\{A \star s : s \in C_0\}$, there is some $s \in C_0$ such that $z \in A \star s$. Then $z \in M$, $z_n = p$, $(z_0, z_1, \ldots, z_n) \in A$, and $z \in (A \times I^\infty) \cap M$. Thus, $(A \times I^\infty) \cap M = \bigcup\{A \star s : s \in C_0\}$. Since $A \star s \cap A \star t = \emptyset$ for $s \neq t$ with both $s, t \in C_0$, the result follows.

The proof is similar if for each $x = (x_0, \ldots, x_n) \in A$, $x_n = 1 - p$, so we omit it. In this case, $(A \times I^\infty) \cap M = \bigcup\{A \star s : s \in C_1\}$ and again $(A \times I^\infty) \cap M$ consists of a mutually disjoint Cantor set of arcs.

(2) Since $\overline{K}^n \setminus (\bigcup\{A : A \in \mathcal{M}_n, A \neq \overline{K}^n\}) = \overline{K}^n \setminus \{\pi_{\langle 0,n \rangle}(\overline{p, 1-p}), \pi_{\langle 0,n \rangle}(\overline{1-p, p})\}$ is open in M_n, $\overline{K}^n \star M$ is the closure of an open set. Next we show that $\overline{K}^n \star M$ is a continuum homeomorphic to M, and $M \supset \overline{K}^1 \star M \supset \overline{K}^2 \star M \supset \cdots$.

For a point $x = (x_0, x_1, \ldots)$ in M, define $\phi(x)$ to be the point $z = (z_0, z_1, \ldots)$ in $\overline{K}^n \star M$ such that $(z_n, z_{n+1}, \ldots) = (x_0, x_1, \ldots)$ and $z_{n-1} = 1 - x_0$, $z_{n-2} = x_0$, $z_{n-3} = 1 - x_0, \ldots$, until the process ends at z_{n-n}. Then ϕ is a homeomorphism from M onto $\overline{K}^n \star M$. (This wouldn't work with any other arc than $\overline{K}^n$ in M_n.) Then $\overline{K}^n \star M$ is a continuum homeomorphic to M.

In fact, it is easy to check that $(\overline{K}^n \times I^\infty) \cap M = \overline{K}^n \star M$.

Then $\overline{K}^n \star M$ is a continuum, and it is a copy of M. Also, we have $M \supset \overline{K}^1 \star M \supset \overline{K}^2 \star M \supset \cdots$. □

LEMMA 8. *M is decomposable, and*

$$M \supsetneq \overline{K}^1 \star M \supsetneq \overline{K}^2 \star M \supsetneq \overline{K}^3 \star M \cdots .$$

However, $(\overline{K}^n \star M)^\circ \not\supseteq \overline{K}^m \star M$ *for* $m > n$.

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Proof. It suffices to show that M contains a proper subcontinuum with interior. For each positive integer n, let $\overline{K}^n \star M$ be defined as in the previous lemma. Then for each n, $\overline{K}^n \star M$ is a continuum homeomorphic to M (from the previous lemma). So $\overline{K}^1 \star M$ is a proper subcontinuum of M, since $\overline{K}^1 \star M$ does not contain the point $\mathbf{p}$. Since $\overline{K}^1 \star M$ contains the open set $(\pi_1^{-1}([0, 1]\backslash\{p\}) \cap M)$, it has a a non-empty interior. Hence, M is decomposable.

For a point $x = (x_0, x_1, \ldots)$ in M, define $\psi(x)$ to be the point $z = (1 - x_0, x_0, x_1, \ldots)$ in $\overline{K}^1 \star M$. Then ψ is a homeomorphism from M onto $\overline{K}^1 \star M$. Also, $\psi(\overline{K}^n \star M) = \overline{K}^{n+1} \star M$ for each n. For each n, $\psi^n(\mathbf{p})$ is in $\overline{K}^n \star M$ but not in $\overline{K}^{n+1} \star M$, so $\overline{K}^{n+1} \star M$ is a proper subset of $\overline{K}^n \star M$. Since $(L \star \overline{K}) \cap \overline{K} = \{\overline{(1 - p, p)}\}$, but no other point of $L \star \overline{K}$ is in $\overline{K}^m \star M$ for any m, any neighborhood u of $\overline{(1 - p, p)}$ must contain a point in $L \star \overline{K}$ that is not in $\overline{K}^m \star M$ for each m. Hence, $(\overline{K}^n \star M)^\circ \not\supseteq \overline{K}^m \star M$ for $m > n$. □

LEMMA 9. *For each $x \in M$ such that $x \notin \mathcal{G}_{\overline{K}}$, $\mathcal{G}_x$ is an arc component of M. The set $\mathcal{G}_{\overline{K}}$ is the only other arc component of M.*

Proof. Each $\mathcal{G}_x$, for $x \in M\backslash\mathcal{G}_{\overline{K}}$, is arcwise connected and dense, as is $\mathcal{G}_{\overline{K}}$. To finish the proof it must be shown that each arc in M is contained either in $\mathcal{G}_x$, for some $x \in M\backslash\mathcal{G}_{\overline{K}}$, or it is contained in $\mathcal{G}_{\overline{K}}$. So assume x and y are elements of M that are contained in distinct members of $\mathcal{G}$.

First suppose neither x nor y is contained in $\mathcal{G}_{\overline{K}}$. Then $x \in \mathcal{G}_x$ and $y \in \mathcal{G}_y$ and $\mathcal{G}_x \cap \mathcal{G}_y = \emptyset$. Let $x = (x_0, x_1, \ldots)$. There is a subsequence $n_1, n_2, \ldots$ of the non-negative integers such that for each n_i, $(x_{n_i}, x_{n_{i+1}}) = (p, p)$, and for m not in this subsequence, $(x_m, x_{m+1}) \neq (p, p)$. Let $S = J_{s_1} \star J_{s_2} \star \cdots$ be a member of $\mathcal{M}$ that contains x (so $(s_1, s_2, \ldots)$ is a sequence such that each $s_i \in \{0, 1\}$). Then let x^1 be the following point: $x^1_{n_1} = 1 - p$, $x^1_i = 1 - x^1_{i+1}$ for $i < n_1$, and $x^1_i = x_i$ for $i > n_1$. Then $x^1 \in \mathcal{G}_x$, so there is an irreducible arc B_0 from x to x^1 contained in $\mathcal{G}_x$. Note that $x^1 \in \overline{K}^{n_2} \star J_{s_{n_2+1}} \star J_{s_{n_2+2}} \star \cdots$ and $x^1 \in C$. The point $x^2 \in C$ defined by $x^2_{n_2} = 1 - p$, $x^2_i = 1 - x^2_{i+1}$ for $i < n_2$, and $x^2_i = x_i$ for $i > n_2$ is also in $C \cap \overline{K}^{n_2} \star J_{s_{n_2+1}} \star J_{s_{n_2+2}} \star \cdots$ and is different from x^1. Thus, there is an irreducible subarc B_1 in $\overline{K}^{n_2} \star J_{s_{n_2+1}} \star J_{s_{n_2+2}} \star \cdots$ from x^1 to x^2. Then $x^2 \in \overline{K}^{n_3} \star J_{s_{n_3+1}} \star J_{s_{n_3+2}} \star \cdots$ and so is the point x^3 defined by $x^3_{n_3} = 1 - p$, $x^3_i = 1 - x^3_{i+1}$ for $i < n_3$, and $x^3_i = x_i$ for $i > n_3$. We can continue, obtaining a sequence of subarcs $B_1, B_2, \ldots$ intersecting only at their endpoints, and whose union is (topologically) a ray in $\mathcal{G}_x$ with endpoint x^1. Furthermore, $\overline{\bigcup_{i>0} B_i} = (\bigcup_{i>0} B_i) \cup \overline{K}'$, where $\overline{K}'$ is the irreducible arc with endpoints $\overline{(p, 1 - p)}$ and $\overline{(1 - p, p)}$ in $\overline{K}$.

Similarly, there is a subsequence $m_1, m_2, \ldots$ of the nonnegative integers such that for each m_i, $(y_{m_i}, y_{m_{i+1}}) = (p, p)$, and for k not in this subsequence, $(y_k, y_{k+1}) \neq (p, p)$. Let $T = J_{t_1} \star J_{t_2} \star \cdots$ be a member of $\mathcal{M}$ that contains y (so, again, $(t_1, t_2, \ldots)$ is a sequence each member of which is in $\{0, 1\}$). The point y^1 defined by $y^1_{m_1} = 1 - p$, $y^1_i = 1 - y^1_{i+1}$ for $i < m_1$, and $y^1_i = y_i$ for $i > m_1$ is in $\mathcal{G}_y$, so there is an irreducible arc D_0 from y to y^1 in $\mathcal{G}_y$. Note that $y^1 \in \overline{K}^{m_2} \star J_{t_{m_2+1}} \star J_{t_{m_2+2}} \star \cdots$ and $y^1 \in C$. The point $y^2 \in C$ defined by $y^2_{m_2} = 1 - p$, $y^2_i = 1 - y^2_{i+1}$ for $i < m_2$, and $y^2_i = y_i$ for $i > m_2$ is also in $C \cap \overline{K}^{m_2} \star J_{t_{m_2+1}} \star J_{t_{m_2+2}} \star \cdots$ and is different from y^1. Thus, there is an irreducible subarc D_1 in

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$\overline{K}^{m_2} \star J_{t_{m_2+1}} \star J_{t_{m_2+2}} \star \cdots$ from y^1 to y^2. Then $y^2 \in \overline{K}^{m_3} \star J_{t_{m_3+1}} \star J_{t_{m_3+2}} \star \cdots$ and so is the point y^3 defined by $y^3_{m_3} = 1 - p$, $y^3_i = 1 - y^3_{i+1}$ for $i < m_3$, and $y^3_i = y_i$ for $i > m_3$. We can continue, obtaining a sequence of subarcs $D_1, D_2, \ldots$ intersecting only at their endpoints, and whose union is (topologically) a ray in $\mathcal{G}_y$ with endpoint y^1. Furthermore, $\overline{\bigcup_{i>0} D_i} = (\bigcup_{i>0} D_i) \cup \overline{K}'$, where $\overline{K}'$ is the irreducible arc with endpoints $(p, 1-p)$ and $(1-p, p)$ in $\overline{K}$.

Then $B_0 \cup (\bigcup_{i>0} B_i) \cup \overline{K}' \cup (\bigcup_{i>0} D_i) \cup D_0$ is a continuum in M from x to y, and it contains an irreducible subcontinuum from x to y which must be topologically the union of a ray in $\mathcal{G}_x$, the arc $\overline{K}'$ in $\mathcal{G}_{\overline{K}}$, and a ray in $\mathcal{G}_y$. Since M is treelike, M is uniquely irreducible, and thus M does not contain an arc that contains x and y.

Finally suppose $y \in \mathcal{G}_{\overline{K}}$, and $x \notin \mathcal{G}_{\overline{K}}$. Then by the construction above for x, there is a topological ray in $\mathcal{G}_x$ with endpoint x and whose closure is the topological ray union the arc $\overline{K}'$. Since y is in $\mathcal{G}_{\overline{K}}$, there is an irreducible arc from y to some point of $\overline{K}'$ in $\mathcal{G}_{\overline{K}}$. Thus, the unique irreducible continuum from x to y in M is not an arc in this case either. So each arc in M is contained in some member of $\mathcal{G}$. □

LEMMA 10. *M is hereditarily decomposable and $1/2$-indecomposable.*

Proof. If M contains a non-degenerate indecomposable continuum D, then D is irreducible between any two points in different composants of D. This contradicts the fact that for any two elements x and y of M there is a hereditarily decomposable continuum that is irreducible from x to y, and since M is tree-like this irreducible continuum is unique.

That M is $\frac{1}{2}$-indecomposable follows from Theorem 8 and the fact that treelike continua are hereditarily unicoherent. □

Summarizing these lemmas, we have the following theorem.

THEOREM 9. *The continuum M is hereditarily decomposable, $\frac{1}{2}$-indecomposable, treelike, and has uncountably many distinct dense arc components which are permuted under the shift map σ on M, and M is a λ-dendroid.*

Remark. In fact we have an entire parameterized family of these continua. If $0 \le p \le 1$, let line J_0^p denote the vertical line segment in $[0, 1] \times [0, 1]$ with $\pi_1(J_0^p) = \{p\}$. Suppose $J_1 = K$ is the line $y = 1 - x$, or with our order convention, the line $x = 1 - y$. Let $G^p = J_0^p \cup J_1$. Let $\mathsf{M}_p = \star_{i=1}^{\infty} G_p$. Then for $p \neq \frac{1}{2}$, M_p is a λ-dendroid, is $\frac{1}{2}$-indecomposable, is hereditarily decomposable, has uncountably many dense arc components, and has a strictly invariant Cantor set under the shift map σ on which the dynamics are chaotic. For $p = \frac{1}{2}$, $\mathsf{M}_p = \mathsf{M}_{1/2}$ is a double fan with relatively simple topology and dynamics. So it is again the case with this family of set-valued functions that complicated dynamics and complicated topology depend on having a two pass.

Let $\mathfrak{M} = \{\mathsf{M}_p : 0 \le p \le 1\}$.

THEOREM 10. *There are at most three topologically different continua in the family $\mathfrak{M}$.*

Proof. Clearly, $\mathsf{M}_{1/2}$ is not homeomorphic to M_p for $p \neq \frac{1}{2}$. Suppose $0 < p < \frac{1}{2}$ and $0 < q < \frac{1}{2}$. There is a homeomorphism $h : [0, 1] \to [0, 1]$ such that $h(0) = 0$,

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$h(p) = q$, $h(1-p) = 1-q$, $h(1) = 1$; h is piecewise linear; and $h(1-x) = 1 - h(x)$ for $x \in [0, 1]$. Define the transformation $H : \mathsf{M}_p \to \mathsf{M}_q$ by $H((x_0, x_1, \ldots)) = (h(x_0), h(x_1), \ldots)$, for $(x_0, x_1, \ldots) \in \mathsf{M}_p$. It is easy to check that $H(C_p) = C_q$, where $C_p = \{(s_1, s_2, \ldots) : s_i \in \{p, 1-p\}$, and $s_{i+1} = p$ if $s_i = 1-p\}$ and $C_q = \{(s_1, s_2, \ldots) : s_i \in \{q, 1-q\}$, and $s_{i+1} = q$ if $s_i = 1-q\}$, that $H(\overline{K}) = \overline{K}$, and that H is a homeomorphism.

If $0 \le p < \frac{1}{2}$, then the transformation that sends $(x_0, x_1, \ldots) \in \mathsf{M}_p$ to $(1-x_0, 1-x_1, \ldots) \in \mathsf{M}_{1-p}$ is a homeomorphism. Then each member of $\mathfrak{M}$ is homeomorphic to either M_0, $\mathsf{M}_{1/4}$, or $\mathsf{M}_{1/2}$. □

Finally, we consider a related family of generalized inverse limits. This time one of the two continua forming the closed set G_c for $0 < c < 1$ in $I_0 \times I_1$ does not project onto I_0, and the resulting inverse limits are dendroids. There is again an invariant Cantor set, and the dynamics of σ on this Cantor set are equivalent to the dynamics of the subshift of the full shift on two symbols discussed in the introduction.

Example 7. We consider set-valued functions whose graph D_c for $0 < c < 1$ is the union of the vertical line $\{(0, y) \mid 0 \le y \le 1\} \subset I_1 \times I_0$ and the graph of $L_c(x) = c(1-x)$ (again, where we consider this graph as a subset of $I_1 \times I_0$—so temporarily we go back to the old order convention). Let $G_c = (D_c)^{-1} = \{(x_0, x_1) : (x_1, x_0) \in D_c\}$. Note that $L_c(c/(1+c)) = c/(1+c)$, and so the point $\mathbf{z_c} = (c/(1+c), c/(1+c), c/(1+c), \ldots) \in \star_{i=1}^{\infty} G_c$.

We discuss below the properties of this family of spaces and systems. However, first notice that if K_0 is the vertical line $\{(0, y) \mid 0 \le y \le 1\}$ and $K_1 = \{(0, 1)\}$, then K_0 and K_1 satisfy the conditions of Theorem 6 part (ii), so for all values of $c \in [0, 1]$ the set-valued function with graph D_c has a two pass.

(1) Consider the function $L_c : [0, 1] \to [0, c]$. As a dynamical system on $[0, 1]$, every point in $[0, 1]$ is attracted to the fixed point $c/(1+c)$, that is, for $0 \le x \le 1$, $\lim_{n\to\infty} L_c^n(x) = c/(1+c)$, with $L_c(x) > c/(1+c)$ if $x < c/(1+c)$, and vice versa. Then for each $0 \le x \le c$, $x \ne c/(1+c)$, there is $M \in \mathbb{N}$ such that $(L_c^{-1})^M(x) \notin [0, c]$, but $(L_c^{-1})^n(x) \in [0, c]$ for $1 \le n < M$.

(2) Now D_c is the graph of an upper semicontinuous set valued function f_c from $[0, 1]$ to $2^{[0,1]}$. Again, we know from [**CR**] that $N_c := \star_{i=1}^{\infty}(D_c)^{-1} = \star_{i=1}^{\infty} G_c$ has a trivial shape. We know from [**CR**] that N_c is one-dimensional, and therefore we know from [**CC**] that N_c is treelike. Since $f_c(x)$ is connected for each $x \in [0, 1]$, we know N_c is a continuum. We will show that N_c is hereditarily decomposable, and N_c is arcwise connected and therefore N_c is a dendroid. We will also see that $N_c \setminus \{\mathbf{z_c}\}$ has uncountably many arc components each of which is dense in N_c.

(3) Let $V = I_0 \times \{0\} \subset I_0 \times I_1$ denote the vertical line, and $S_c = \{(x, 1 - x/c) \in I_0 \times I_1 : 0 \le x \le c\}$ denote the graph of L_c^{-1}. Let $\mathcal{N}_c = \{A_1 \star A_2 \star \cdots : A_i \in \{V, S_c\}$ for each $i\}$. Note that, because the projection of S_c to I_0 is not equal to I_0, we are not guaranteed that each $A_1 \star A_2 \star \cdots \ne \emptyset$. Indeed, if $A_i = V$ and there is $j > i$ such that $(A_j, A_{j+1}) = (S_c, S_c)$, then $A_1 \star A_2 \star \cdot = \emptyset$. However, $N_c := \star_{i=1}^{\infty} G_c = \bigcup \mathcal{N}_c$, with some members of $\mathcal{N}_c$ empty, some members of $\mathcal{N}_c$ consisting of one point, and the remainder of the members being arcs.

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(4) Let $C = \{(s_1, s_2, \ldots) : s_i \in \{0, 1\} \text{ for each } i, \text{ and if } s_i = 1, \text{ then } s_{i+1} = 0\}$. Then $C \subset N_c$, and under the shift σ on N_c, C is strictly invariant and conjugate to the subshift of the full shift on two symbols discussed in the introduction. So again we have chaotic dynamics on this set. Also, unless $x = \mathbf{z}_c$, there is M_x such that if $m > M_x$, then $\sigma^m(x) \in C$. So C is not only an attracting invariant set, it actually 'sucks' all points into it, except for the repelling fixed point $\mathbf{z}_c$.

(5) Notice that for each $x \in N_c$, if $\pi_n(x) \in \{0, 1\}$, then $\pi_m(x) \in \{0, 1\}$ for each $m > n$, and if $\pi_n(x) = t \in (0, 1)$, then $\pi_{\langle 0,n\rangle}(x) = (L_c^n(t), \ldots, L_c(t), t)$. Moreover the only element of N_c with $\pi_n(x) \in (0, 1)$ for each n is $\mathbf{z}_c = (c/(c+1), c/(c+1), \ldots)$.

(6) Perhaps it is worth mentioning the two limiting cases, when $c = 0$ and $c = 1$. If $c = 1$, then $N_1 = \mathsf{M}_0$ is Banič's 'Monster', and it actually occurred in both the previous families of systems. If $c = 0$, then $N_c = N_0$ is an infinite-dimensional continuum containing many copies of the Hilbert cube. For $0 \le c \le 1$, the Cantor set C is an invariant subset of N_c on which the dynamics are chaotic. For $c > 0$, this Cantor set is attracting, but for $c = 0$, this Cantor set is not attracting. Also, for $c = 0$, the fixed point $\mathbf{z}_0$ has 'merged' with the fixed point $(0, 0, 0, \ldots)$ and is no longer repelling. So, the dynamics and topology change a great deal from the case where $0 < c < 1$ to the case where $c = 0$ or $c = 1$.

(7) Suppose $c \in (0, 1)$ and $x \in N_c$. Then either $x = \mathbf{z}_c = (c/(c+1), c/(c+1), c/(c+1), \ldots)$ or there is some $s = (s_1, s_2, \ldots) \in C$, $t \in [0, 1]$, and $n \in \mathbb{N}$ such that $x = (L_c^n(t), \ldots, L_c(t), t) \oplus \sigma^n(s)$. If $t \neq 0$, then $s_{n+1} = 0$.

(8) Suppose $c \in (0, 1)$ and $d \in (0, 1)$. The transformation ϕ that sends $(L_c^n(t), \ldots, L_c(t), t, s_1, s_2, \ldots)$ to $(L_d^n(t), \ldots, L_d(t), t, s_1, s_2, \ldots)$ and $\mathbf{z}_c$ to $\mathbf{z}_d$ is a homeomorphism from N_c onto N_d, and it is also a conjugacy. (That is, for $x \in N_c$, $\phi \circ (\sigma|_{N_c})(x) = (\sigma|_{N_d}) \circ \phi(x)$.) *From here on we will consider only the case where* $c = \frac{1}{2}$. Let $N = N_{1/2}$ and $L = L_{1/2}$, and note that the fixed point of L is $\frac{1}{3}$. Let $\mathbf{z} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$.

(9) For $t \in [0, 1]$ and $n \in \mathbb{N} \cup \{0\}$, let $\mathbf{L}_{t,n} = (L^n(t), \ldots, L(t), t)$. If $s = (s_1, s_2, \ldots) \in C$, then $\mathbf{L}_{t,n} \oplus \sigma^n(s) \in N$ provided that if $t \neq 0$, then $s_{n+1} = 0$. For $s \in C$ and $n \in \mathbb{N}$, let $\mathcal{G}_{s,n} = \{\mathbf{L}_{t,n} \oplus \sigma^n(s) \in N : t \in [0, 1]\}$. Each $\mathcal{G}_{s,n}$ is either an arc (if $s_{n+1} = 0$) or a point (if $s_{n+1} = 1$). For $s \in C$ let $\mathcal{G}_s = \bigcup\{\mathbf{L}_{t,n} \oplus \sigma^n(r) \in N : t \in [0, 1], n \in \mathbb{N} \cup \{0\}, r \in C, \text{ and for some } m \in \mathbb{N}, \sigma^m(r) = \sigma^m(s)\}$. Note that if $r, s \in C$, then either $\mathcal{G}_s = \mathcal{G}_r$ or $\mathcal{G}_s \cap \mathcal{G}_r = \emptyset$. Also, $N = \{\mathbf{z}\} \cup (\bigcup_{s \in C} \mathcal{G}_s)$.

LEMMA 11. *If $n \in \mathbb{N}$ and $s = (s_1, s_2, \ldots) \in C$, then $\mathcal{G}_{s,n} \cap \mathcal{G}_{s,n+1} \neq \emptyset$.*

Proof. There are three cases: $s_{n+1} = 0$ and $s_{n+2} = 0$; $s_{n+1} = 0$ and $s_{n+2} = 1$; and $s_{n+1} = 1$ and $s_{n+2} = 0$.

Suppose $s_{n+1} = 0$ and $s_{n+2} = 0$. Then $\mathcal{G}_{s,n} = \{\mathbf{L}_{t,n} \oplus \sigma^n(s) \in N : t \in [0, 1]\}$, $\mathcal{G}_{s,n+1} = \{\mathbf{L}_{t,n+1} \oplus \sigma^{n+1}(s) \in N : t \in [0, 1]\}$, and both are non-degenerate arcs contained in N. In this case $\mathcal{G}_{s,n} \cap \mathcal{G}_{s,n+1} = \{\mathbf{L}_{1/2,n} \oplus \sigma^n(s)\} = \{\mathbf{L}_{0,n+1} \oplus \sigma^{n+1}(s)\}$.

Suppose $s_{n+1} = 0$ and $s_{n+2} = 1$. Then $\mathcal{G}_{s,n} = \{\mathbf{L}_{t,n} \oplus \sigma^n(s) \in N : t \in [0, 1]\}$ is an arc, and $\mathcal{G}_{s,n+1} = \{\mathbf{L}_{0,n+1} \oplus \sigma^{n+1}(s)\} = \{\mathbf{L}_{1/2,n} \oplus \sigma^n(s)\} \subset \mathcal{G}_{s,n}$.

Finally, if $s_{n+1} = 1$ and $s_{n+2} = 0$, then $\mathcal{G}_{\sigma(s),n+1} = \{\mathbf{L}_{t,n+1} \oplus \sigma^{n+1}(s) \in N : t \in [0, 1]\}$ is an arc in N, and $\mathcal{G}_{s,n} = \{\mathbf{L}_{0,n} \oplus \sigma^n(s)\} = \{\mathbf{L}_{1,n+1} \oplus \sigma^{n+1}(s)\} \subset \mathcal{G}_{s,n+1}$. □

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So far the situation is quite similar to that of the λ-dendroids in the previous example (e.g. with $c = 1$). The next lemma points out the major difference.

LEMMA 12. *For each $n \in \mathbb{N}$ and each $s \in C$, $\operatorname{diam}(\mathcal{G}_{s,n}) < n/2^n$, and $B := \bigcup_{m\geq 0} \mathcal{G}_{s,n+m}$ contains a topological ray (i.e. a set homeomorphic to $[0, 1)$), call it B', and the closure of B' equals $B' \cup \{\mathbf{z}\}$ and is an arc.*

Proof. Note that if $0 \leq a < b \leq 1$, then $L([a, b])$ is the interval $[\frac{1}{2}(1 - b), \frac{1}{2}(1 - a)]$. Thus the length of the interval $L([a, b])$ is $\frac{1}{2}(b - a)$. So in the case where $\mathcal{G}_{s,n}$ is a non-degenerate arc, $\pi_n(\mathcal{G}_{s,n}) = [0, 1]$, and $\pi_{n-j}(\mathcal{G}_{s,n}) = L(\pi_{n-j+1}(\mathcal{G}_{s,n}))$ for $j = 1, 2, \ldots, n$. Thus $\operatorname{diam}(\mathcal{G}_{s,n}) \leq \sum_{i=1}^{n} ((1/2)^{n-i}/2^i) = (\frac{1}{2})^n \sum_{i=1}^{n} 1 = n/2^n$. Therefore the diameters of the sets $\mathcal{G}_{s,n}$ have limit zero as n goes to infinity. Also, in the case where $\mathcal{G}_{s,n}$ is a non-degenerate arc, if $(\overline{\frac{1}{3}})^{n+1}$ denotes the point with $n + 1$ coordinates, each of which is $\frac{1}{3}$, then the point $(\overline{\frac{1}{3}})^{n+1} \oplus \sigma^n(s) \in \mathcal{G}_{s,n}$. For these reasons, if $x \in \mathcal{G}_{s,n}$ for some n and some s, then there is a sequence of arcs in $\bigcup_{i=n}^{\infty} \mathcal{G}_{s,i}$ whose union is homeomorphic to $[0, 1)$ with the one endpoint equal to x and whose closure is an arc with endpoints x and $\mathbf{z}$. □

LEMMA 13. *For each $s \in C$ the set $\mathcal{G}_s$ is an arc-component of $N\backslash\{\mathbf{z}\}$.*

Proof. Assume $s \in C$ and x and y are elements of $\mathcal{G}_s$. Let m, n, r, q and M be such that $x \in \mathcal{G}_{r,m}$, $y \in \mathcal{G}_{q,n}$, and $M \in \mathbb{N}$ such that $\sigma^M(r) = \sigma^M(q) = \sigma^M(s)$. It can be assumed that $M > \max\{m, n\}$. By Lemma 11 the sets $\mathcal{G}_{q,n} \cup \mathcal{G}_{q,n+1} \cup \cdots \mathcal{G}_{q,M}$, $\mathcal{G}_{r,m} \cup \mathcal{G}_{r,m+1} \cup \cdots \mathcal{G}_{r,M}$ are arcwise connected. Moreover $\mathcal{G}_{q,M} = \mathcal{G}_{r,M} = \mathcal{G}_{s,M}$. Therefore $\mathcal{G}_s$ is arcwise connected for each $s \in C$.

To finish the proof it must be shown that if there is an arc from $x \in \mathcal{G}_s$ to $y \in \mathcal{G}_r$ and that arc does not contain $\mathbf{z}$, then $\mathcal{G}_s = \mathcal{G}_r$. So assume x and y are elements of N such that $x \in \mathcal{G}_s$ and $y \in \mathcal{G}_r$ where $\mathcal{G}_s \neq \mathcal{G}_r$. Suppose $x \in \mathcal{G}_{s',n}$ such that $\mathcal{G}_{s',n} \subset \mathcal{G}_s$. Then $\mathbf{z}$ is not contained in $\bigcup_{i=n}^{\infty} \mathcal{G}_{s',i}$, $(\bigcup_{i=n}^{\infty} \mathcal{G}_{s',i}) \cap \mathcal{G}_r = \emptyset$, and $\mathcal{G}_s$ contains a ray R such that the closure of R which equals $R \cup \{\mathbf{z}\}$ is an arc with endpoints $\mathbf{z}$ and x. Similarly, $\mathcal{G}_r$ contains a ray R' such that the closure of R' which equals $R' \cup \{\mathbf{z}\}$ is an arc with endpoints y and $\mathbf{z}$. So there is an arc in N that is irreducible from x to y which contains $\mathbf{z}$. Since N is treelike, N is uniquely irreducible, and thus $N\backslash\{\mathbf{z}\}$ does not contain an arc that contains x and y. □

LEMMA 14. *$N\backslash\{\mathbf{z}\}$ has uncountably many arc components, each arc component of $N\backslash\{\mathbf{z}\}$ is dense in N, and the shift map σ permutes the arc components of $N\backslash\{\mathbf{z}\}$.*

Proof. That N has uncountably many arc components follows immediately from Lemma 13.

Let x be an element of $N\backslash\{\mathbf{z}\}$. Let $s \in C$. For each $m \in \mathbb{N}$, let y_m be the element of $\mathcal{G}_s$ such that $\pi_{\langle 0,m\rangle}(y_m) = \pi_{\langle 0,m\rangle}(x)$ and $\pi_{m+i}(y_m) = s_{m+i}$ for each $i \in \mathbb{N}$. The sequence $\{y_m\}$ converges to x. Therefore $\mathcal{G}_s$ is dense in $N\backslash\{\mathbf{z}\}$ for each $s \in C$.

For $s \in C$, it is easy to verify that $\sigma(\mathcal{G}_s) = \mathcal{G}_{\sigma(s)}$, and that if $\mathcal{G}_s \neq \mathcal{G}_r$, then $\mathcal{G}_{\sigma(s)} \neq \mathcal{G}_{\sigma(r)}$. Also, $\sigma(\mathbf{z}) = \mathbf{z}$. It follows that σ permutes the arc components of $N\backslash\{\mathbf{z}\}$. □

LEMMA 15. *N is hereditarily decomposable and $\frac{1}{2}$-indecomposable.*

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Proof. If N contains a non-degenerate indecomposable continuum K, then K is irreducible between any two points in different composants of K. This contradicts the fact that for any two elements x and y of M there is a hereditarily decomposable continuum that is irreducible from x to y, and since N is tree-like this irreducible continuum is unique.

That N is $\frac{1}{2}$-indecomposable follows from Theorem 8. □

Summarizing these results, we have the following theorem.

THEOREM 11. *For each $c \in (0, 1)$, the continuum N_c is a hereditarily decomposable, $\frac{1}{2}$-indecomposable dendroid such that $N\backslash\{\mathbf{z}\}$ has uncountably many arc components, each arc component of $N\backslash\{\mathbf{z}\}$ is dense in N, and the shift map σ permutes the arc components of $N\backslash\{\mathbf{z}\}$.*

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