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Envelopes and characteristics

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Envelopes of hypersurfaces arise in many different geometric situations. For example, the canal surfaces associated with a space curve can be obtained as the envelope of spheres of a fixed radius centred on that curve. (Other examples can be found in [**11**], chapter II and [**4**], chapter 5). In what follows we shall be entirely concerned with the local structure of such envelopes. So essentially we have a smooth map germ $F\colon (\mathbb{R}^n \times \mathbb{R}^p, 0) \to (\mathbb{R}, 0)$, with 0 a regular value of the restriction of F to $\mathbb{R}^n \times \{0\}$. For $t \in \mathbb{R}^p$ near 0 we have $F_t^{-1}(0)$ a smooth hypersurface near $0 \in \mathbb{R}^n$, where $F_t(x) = F(x, t)$. The *envelope* of this family of hypersurfaces is obtained by eliminating t from the equations $F = \partial F/\partial t_1 = \ldots = \partial F/\partial t_p = 0$. Moreover, it can be identified with the discriminant of F viewed as an unfolding, *by the x variables*, of the function germ $F^0 = F(0, -)\colon (\mathbb{R}^p, 0) \to (\mathbb{R}, 0)$. This observation, together with standard results on versal unfoldings of functions, allows one to determine the local structure of many 'generic' geometric envelopes. See [**2**] for details.

In this paper we investigate the distribution of the *characteristics* on this envelope. Corresponding to a parameter value $t \in \mathbb{R}^p$ the characteristic E_t is the set

$$\{x \in \mathbb{R}^n\colon F(x, t) = \partial F/\partial t_i(x, t) = 0,\ 1 \leqslant i \leqslant p\}.$$

If $p = 1$ the characteristic can be viewed as the intersection of 'consecutive' hypersurfaces of our family, and it is this case with which we shall be largely concerned here. As a consequence we shall only need to consider Arnold's A_r singularities. In § 1 we shall give a brief discussion of discriminants and normalizations, and their relationship with the problem at hand. In § 2 we shall describe the distribution of the characteristics at a generic (A_2) point of the cuspidal edge of the envelope and give some applications. In § 3 we shall deal with the general case $(A_r, r \geqslant 3)$; we do not obtain as complete information in this case, in fact only on the characteristic under study and not on nearby ones. This problem is, to some extent, overcome in § 4 for the case $r = 3$ of the swallowtail, where further applications are given.

Notation

In what follows $\mathbf{k}$ will denote the real or complex numbers, $(\mathbf{k}^n)^*$ the dual of the vector space $\mathbf{k}^n$. If (A, a) is the germ of a smooth manifold with local coordinates $x_1, \ldots, x_n$ at a, we shall write both $\mathcal{O}(A)$ and $\mathcal{O}(x)$ for the ring of smooth function germs $(A, a) \to \mathbf{k}$, and $\mathcal{M}(A)$ or $\mathcal{M}(x)$ for the maximal ideal of function germs in this ring which vanish at a. A smooth germ $f\colon (A, a) \to (B, b)$ induces a ring homomorphism, composition, which we denote by $f^*\colon \mathcal{O}(B) \to \mathcal{O}(A)$.

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1. *Discriminants and normalizations*

Let $F^0\colon (\mathbf{k}^p, 0) \to (\mathbf{k}, 0)$ be a germ of finite right codimension, which we can assume to be polynomial, and let $F\colon (\mathbf{k}^n \times \mathbf{k}^p, 0) \to (\mathbf{k}, 0)$ be a polynomial versal unfolding of F^0, so in particular $F(0, -) = F^0(-)$. If $J^1(p, 1)$ is the space of 1-jets with constants, that is essentially the product $\mathbf{k} \times (\mathbf{k}^p)^*$, the obvious jet-extension

$$jF\colon (\mathbf{k}^n \times \mathbf{k}^p, 0) \to J^1(p, 1),$$
$$(x, t) \mapsto (j^1F(x, -))(t)$$

is transverse to $\{(0, 0)\} \subset J^1(p, 1)$, the set of singular jets with value zero. Thus $C = (jF)^{-1}\{(0, 0)\}$, the set of points (x, t) with $F(x, -)$ singular at t and $F(x, t) = 0$, is a smooth submanifold (germ) of $\mathbf{k}^n \times \mathbf{k}^p$ of dimension $n - 1$. If $\Pi\colon (\mathbf{k}^n \times \mathbf{k}^p, 0) \to (\mathbf{k}^n, 0)$ is the natural projection then $(\Pi(C), 0)$ is the germ of the discriminant of F, which we denote by $(D, 0)$.

In ([9], § 5) Teissier proves that in the complex case the restriction of this projection $\Pi|C\colon (C, 0) \to (D, 0)$ is the normalization of D. Since we are considering *function* germs the proof is relatively easy ([9], p. 646, remark 2); in fact Teissier proves that the map $\Pi|C$ is the 'development' of D. We shall not use these results in any essential way in what follows, but the fact that $\Pi|C$ is the normalization of D is useful.

The 'foliation' of the discriminant or envelope by the characteristics is determined by the following diagram:

$$\begin{array}{ccc} (C, 0) \subset (\mathbf{k}^n \times \mathbf{k}^p, 0) & \xrightarrow{\mu} & (\mathbf{k}^p, 0) \\ \Pi\downarrow & & \\ (\mathbf{k}^n, 0), & & \end{array}$$

where Π and μ are the obvious projections. This is because the characteristics are the images under Π of the fibres of the restriction of μ to C. We wish to classify these foliations, at least in the generic cases, and so determine the distribution of the characteristics. The most obvious way of proceeding is as follows.

We first note that any two versal unfoldings of a singularity of the same dimension are equivalent. So if G is an unfolding of some singular function germ G^0 which is right equivalent to F^0, and the dimension of G coincides with that of F ($= n$), then there are smooth germs

$$\Psi\colon (\mathbf{k}^n \times \mathbf{k}^p, 0) \to (\mathbf{k}^p, 0), \quad \Phi\colon (\mathbf{k}^n, 0) \to (\mathbf{k}^n, 0)$$

with $\Psi(0, -)\colon (\mathbf{k}^p, 0) \to (\mathbf{k}^p, 0)$ and Φ germs of diffeomorphisms and

$$G(\Phi(x), \Psi(x, t)) = F(x, t).$$

Consequently we get the following commuting diagram of map germs with (Φ, Ψ) and Φ diffeomorphisms:

$$\begin{array}{ccc} (\mathbf{k}^p, 0) & & (\mathbf{k}^p, 0) \\ \mu_F\uparrow & & \uparrow\mu_G \\ (C(F), 0) & \xrightarrow{(\Phi, \Psi)} & (C(G), 0) \\ \Pi_F\downarrow & & \downarrow\Pi_G \\ (D(F), 0) & \xrightarrow{\Phi} & (D(G), 0) \end{array}$$

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If there is a germ of a diffeomorphism $\theta\colon (\mathbf{k}^p, 0) \to (\mathbf{k}^p, 0)$ such that

$$\mu_G \circ (\Phi, \Psi)\,(= \Psi) = \theta \circ \mu_F,$$

then clearly the foliations of the discriminants $(D(F), 0)$, $(D(G), 0)$ by characteristics are equivalent in a very natural sense: there is a diffeomorphism taking $D(F)$ to $D(G)$ which preserves characteristics. The approach we should like to adopt here is to consider $\mu_G \circ (\Phi, \Psi)|C(F) = f$ as a general map germ on $C(F)$ at 0 and classify such germs by declaring two such $g_1, g_2\colon (C(F), 0) \to (\mathbf{k}^p, 0)$ to be *equivalent* if there are germs of diffeomorphisms

$$\phi\colon (D(F), 0) \to (D(F), 0), \quad \tilde{\phi}\colon (C(F), 0) \to (C(F), 0). \quad \theta\colon (\mathbf{k}^p, 0) \to (\mathbf{k}^p, 0)$$

making the following diagram commute:

$$\begin{array}{ccc}
(\mathbf{k}^p, 0) & \xrightarrow{\theta} & (\mathbf{k}^p, 0) \\
\uparrow{\scriptstyle g_1} & & \uparrow{\scriptstyle g_2} \\
(C(F), 0) & \xrightarrow{\tilde{\phi}} & (C(F), 0) \\
\downarrow{\scriptstyle \Pi_F} & & \downarrow{\scriptstyle \Pi_F} \\
(D(F), 0) & \xrightarrow{\phi} & (D(F), 0)
\end{array}$$

This approach has the advantage that we need only work with standard versal unfoldings F of germs, e.g. the unfolding

$$F(x, t) = t^{k+1} + x_1 t^{k-1} + x_2 t^{k-2} + \ldots + x_{k-1} t + x_k$$

of an A_k singularity, where the remaining x variables $x_{k+1}, \ldots, x_n$ are redundant. For given any other versal unfolding G of dimension n we know that G and F are isomorphic and the projection $\mu_G\colon (C(G), 0) \to (k^p, 0)$ gives rise to a germ $f = \mu_G \circ (\Phi, \Psi)|C(F)$ on $C(F)$. Having, hopefully, classified the stable germs for this standard unfolding F, we can decide if f is stable, and the foliation of $(D(G), 0)$ by characteristics is naturally equivalent to the foliation of $(D(F), 0)$ determined by this stable germ f.

As it turns out, the inclusion of the diffeomorphism θ causes severe problems, so for our classification, which only deals with the case $p = 1$ here, we shall fix θ to be the identity in the case of an unfolding of an A_2 singularity. For A_r singularities, $r \geqslant 2$, we use a *weaker* notion of equivalence than the one mooted above; see § 3 below. Now for some details.

Let us first of all consider the complex case. Given the germ of a diffeomorphism $\phi\colon (\mathbb{C}^n, D, 0) \to (\mathbb{C}^n, D, 0)$, it is natural to ask if there is a *lift* $\tilde{\phi}\colon (C, 0) \to (C, 0)$, that is a germ $\tilde{\phi}$ with $\Pi \circ \tilde{\phi} = \phi \circ \Pi$.

LEMMA 1·1. *Such a lift always exists, and moreover $\tilde{\phi}$ is the germ of diffeomorphism.*

Proof. Since $\Pi\colon (C, 0) \to (D, 0)$ is a normalization, there is a map $\psi\colon (C, 0) \to (D, 0)$ with $\phi \circ \psi = \Pi$. Similarly there is then a map $\tilde{\phi}\colon (C, 0) \to (C, 0)$ with $\psi \circ \tilde{\phi} = \Pi$. Since ϕ is an isomorphism it follows that $\tilde{\phi}$ is also.

The basic idea then is to classify germs $f\colon (C, 0) \to (\mathbb{C}^p, 0)$ up to changes of coordinates in the source and target, with those in the source being lifts of discriminant-preserving diffeomorphisms of $(\mathbb{C}^n, 0)$.

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In this paper we shall be concerned largely with the case $p = 1$.

Definition 1·2. As before let $\mathbf{k}$ denote the real or complex numbers and

$$\Pi: (C, 0) \to (D, 0)$$

be the natural projection. Given the germ of a diffeomorphism $\phi: (\mathbf{k}^n, D, 0) \to (\mathbf{k}^n, D, 0)$, a diffeomorphism $\tilde{\phi}: (C, 0) \to (C, 0)$ is a *lift* of ϕ, and called a lifted diffeomorphism if $\Pi \circ \tilde{\phi} = \phi \circ \Pi$. The set of such lifts forms a group denoted by Diff (C/D). Two function germs $g_1, g_2: (C, 0) \to (\mathbf{k}, 0)$ are *$\mathscr{R}(C)$-equivalent* if for some lifted diffeomorphism $\tilde{\phi}: (C, 0) \to (C, 0)$ we have $g_1 \circ \tilde{\phi} = g_2$.

As usual in singularity theory we shall employ the Thom–Mather method of obtaining our changes of coordinates by integrating vector fields. Arnold, Zakalyukin and Saito have constructed nice modules of vector fields tangent to discriminants ([**1**], [**12**], [**8**]; see also [**3**]). We now show that these fields are liftable. Given vector fields ξ on $(\mathbb{C}^n, 0)$, $\bar{\xi}$ on $(C, 0)$ we say that $\bar{\xi}$ is a *lift* of ξ if $d\Pi(\bar{\xi}) = \xi \circ \Pi$, where $d\Pi$ is the derivative of the projection Π.

PROPOSITION 1·3. *In the terminology of* [**7**], *the vector fields in* Derlog D *lift to vector fields on* C. (*A holomorphic vector field* ξ *on* $(\mathbb{C}^n, 0)$ *lies in* Derlog D *if it is tangent to the smooth part of* D.) *Conversely, any vector field on* $(\mathbb{C}^n, 0)$ *which lifts to* C *lies in* Derlog D.

Proof. The proofs of these facts are contained in the appendix to [**3**], but since they are relatively straightforward we repeat them. Let ξ be a vector field in Derlog D. If $x \in D$ is a smooth point of the discriminant there is a unique corresponding point (x, t) of C and $\Pi: (C, (x, t)) \to (D, x)$ is a local diffeomorphism. Thus at every smooth point of D the vector field ξ lifts uniquely. On the other hand, the singular points of D can be stratified into smooth codimension 1 manifolds of type A_1^2 and A_2 and a set of codimension 2 in D. If a point x lies on the stratum A_1^2 there are two corresponding points $(x, t_1), (x, t_2)$ of C, $\Pi: (C, (x, t_i)) \to (\mathbb{C}^n, x)$ are immersions and their images meet transversally in the A_1^2 stratum. Clearly there is no obstruction to lifting ξ on this stratum. Finally, if a point x lies on the A_2 stratum there is a unique corresponding point (x, t) of C and the projection $\Pi: (C, (x, t)) \to (\mathbb{C}^n, x)$ is locally equivalent to the map

$$(\lambda, \tau_1, \ldots, \tau_m) \mapsto (\lambda^2, \lambda^3, \tau_1, \ldots, \tau_m) = (x_1, \ldots, x_n).$$

At x the module Derlog D is generated by

$$2x_1 \partial/\partial x_1 + 3x_2 \partial/\partial x_2, \quad 2x_2 \partial/\partial x_1 + 3x_1^2 \partial/\partial x_2, \partial/\partial x_i \quad (3 \leqslant i \leqslant n),$$

and these all lift to C (to respectively $\lambda \partial/\partial\lambda$, $\lambda^2 \partial/\partial\lambda$, $\partial/\partial\tau_i$). Since ξ is an $\mathcal{O}(x)$-linear combination of these vector fields near x it also lifts. Thus we obtain a vector field $\bar{\xi}$ well defined and holomorphic outside a codimension 2 subset of C; by Hartog's theorem $\bar{\xi}$ extends holomorphically to all of C and this extension is clearly a lift of ξ.

Conversely any vector field which lifts to C is clearly tangent to D at its smooth points, and hence lies in Derlog D. |

The basic difficulty with these lifted vector fields is that they do *not* form an $\mathcal{O}(C)$-module, but rather a $\Pi^*\mathcal{O}(x)$-module. This creates very substantial problems in what follows.

Of course, we are primarily interested in the real case when considering applications. We can choose the vector fields which generate Derlog D in the complex case

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to be real when restricted to the real discriminant, and their lifts to the real critical set are also real. The key fact, that integrating these vector fields yields lifted diffeomorphisms, remains true in the real case.

2. *Lifted vector fields and generic functions*

In what follows we write $\mathrm{Der}(C/D)$ for the $\Pi^*\mathcal{O}(x)$-module of lifts of vector fields in Derlog D. We now compute this module in various cases.

Example 2·0. A_1 *singularities*

At a general smooth point of an envelope the corresponding function germ has a single A_1 singularity. Since the normalization $C \to D$ is a local diffeomorphism, the set of lifted germs of diffeomorphisms of C and lifted germs of vector fields on C coincide with the sets of germs of diffeomorphisms of C and germs of vector fields on C respectively; that is, there is no obstruction to 'lowering' diffeomorphisms or vector fields from C to D. The classification of generic functions $\mu: C \to \mathbb{R}$ up to $\mathscr{R}(C)$-equivalence consequently reduces simply to the classification up to $\mathscr{R}$-equivalence and so we will encounter germs of submersions and Morse singularities. The details are so straightforward and the examples so uninteresting we will from now on concentrate on the non-A_1 cases.

Example 2·1. A_2 *singularities*

(*a*) Consider the discriminant of a miniversal unfolding of an A_2 singularity: $F(x,t) = t^3 + x_1 t + x_2$. Solving $F = \partial F/\partial t = 0$ we find $x_1 = -3t^2$, $x_2 = 2t^3$, and the normalization $(C,0) \to (D,0)$ can be identified with the germ

$$\pi_1: (\mathbf{k},0) \to (\mathbf{k}^2,0), t \mapsto (-3t^2, 2t^3) = (x_1, x_2).$$

The discriminant is the cusp $27x_2^2 + 4x_1^3 = 0$ and Derlog D is generated by

$$2x_1\,\partial/\partial x_1 + 3x_2\,\partial/\partial x_2 \quad \text{and} \quad -9x_2\,\partial/\partial x_1 + 2x_1^2\,\partial/\partial x_2.$$

These lift respectively to $t\,\partial/\partial t$ and $3t^2\,\partial/\partial t$. Since $\pi_1^*\mathcal{O}(x) \supset \mathscr{M}(t)^2$ it is clear that the set of all lifted vector fields is $\mathcal{O}(t).t\,\partial/\partial t$.

(*b*) Suppose that the unfolding of the A_2 singularity is no longer of minimal dimension, so $F(x_1, x_2, \tau_1, \ldots, \tau_m, t) = t^3 + x_1 t + x_2$ say, so $n = m+2$. (Here the τ_i are 'redundant' x-variables.) Derlog D is generated by $2x_1\,\partial/\partial x_1 + 3x_2\,\partial/\partial x_2$, $-9x_2\,\partial/\partial x_1 + 2x_1^2\,\partial/\partial x_2$, $\partial/\partial\tau_i$, $1 \leqslant i \leqslant m$, and these lift to $t\,\partial/\partial t$, $3t^2\,\partial/\partial t$, $\partial/\partial\tau_i$, $1 \leqslant i \leqslant m$. Since the normalization is now given by $\Pi_1(t, \tau_1, \ldots, \tau_m) = (-3t^2, 2t^3, \tau_1, \ldots, \tau_m)$ these lifted vector fields form a module over the ring

$$\Pi_1^*\mathcal{O}(x) = \mathcal{O}(\tau_1, \ldots, \tau_m).(\{1\} + \mathscr{M}^2(t)) = R.$$

Example 2·2. A_3 *singularities*

Consider the discriminant of a miniversal unfolding of an A_3 singularity:

$$F(x,t) = t^4 + x_1 t^2 + x_2 t + x_3.$$

Solving $F = \partial F/\partial t = 0$ we find $x_2 = -4t^3 = 2x_1 t$, $x_3 = 3t^4 + x_1 t^2$ and the normalization is determined by the germ $\Pi_1: (\mathbf{k}^2,0) \to (\mathbf{k}^3,0)$ given by

$$\Pi_1(x_1, t) = (x_1, -4t^3 - 2x_1 t, 3t^4 + x_1 t^2).$$

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The discriminant is the swallowtail and Derlog D is generated by the following three vector fields:

$$2x_1\,\partial/\partial x_1 + 3x_2\,\partial/\partial x_2 + 4x_3\,\partial/\partial x_3,$$
$$6x_2\,\partial/\partial x_1 + (8x_3 - 2x_1^2)\,\partial/\partial x_2 - x_1x_2\,\partial/\partial x_3,$$
$$(16x_3 - 4x_1^2)\,\partial/\partial x_1 - 8x_1x_2\,\partial/\partial x_2 - 3x_2^2\,\partial/\partial x_3.$$

(See [3], § 1, for this particular result.)

It is now a straightforward matter to compute the corresponding lifts to the normalization. They are respectively

$$\delta_1 = 2x_1\,\partial/\partial x_1 + t\,\partial/\partial t,$$
$$\delta_2 = (2t^2 + x_1).(-12t\,\partial/\partial x_1 + \partial/\partial t),$$
$$\delta_3 = 4(2t^2 + x_1).((6t^2 - x_1)\,\partial/\partial x_1 - t\,\partial/\partial t).$$

Note that these three vector fields span a space of dimension $\leqslant 1$ precisely when $2t^2 + x_1 = 0$ or $6t^2 + x_1 = 0$; note also that the third vector is an $\mathcal{O}(x_1, t)$ linear combination of the first two. The locus $2t^2 + x_1 = 0$ is easily seen to correspond to the double line of the swallowtail, while $6t^2 + x_1 = 0$ is the lift of the cuspidal edge.

We now wish to classify generic functions and maps on these normalizations by using the vector fields in $\mathrm{Der}(C/D)$. From now on we shall deal exclusively with functions, i.e. maps with target dimension $p = 1$.

Examples 2·3

(*a*) *Miniversal unfolding of an A_2 singularity.* Here we are classifying function germs $f\colon (C, 0) \to (\mathbf{k}, 0)$ up to changes of coordinates in the source which preserve the origin. Clearly there is one stable such germ, namely $f(t) = t$, and any germ with $\partial f/\partial t(0) \neq 0$ is equivalent to this. (We include this case in 2·4(i) below.)

(*b*) *General versal unfolding of an A_2 singularity.* Here we state the main result of this section.

THEOREM 2·4. *Let $F(x_1, x_2, \tau_1, \ldots, \tau_m, t) = t^3 + x_1t + x_2$ be a versal unfolding of the A_2 singularity $F(0, t) = t^3$, and let C denote the critical set of F.*

(i) *If $f\colon (C, 0) \to (\mathbf{k}, 0)$ is a smooth germ with $\partial f/\partial t(0) \neq 0$ and $\partial f/\partial \tau_i(0) \neq 0$ for some i, then f is $\mathscr{R}(C)$-equivalent to the germ $t + \tau_1$. That is, there is an automorphism*

$$\tilde{\phi}\colon (C, 0) \to (C, 0)$$

of the critical set C which is a lift of an automorphism ϕ of the discriminant $(D, 0)$ with $f \circ \phi = t + \tau_1$. If $m = 0$ and $\partial f/\partial t(0) \neq 0$ then f is $\mathscr{R}(C)$ equivalent to t.

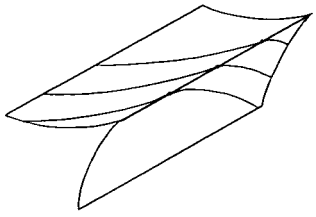
(ii) *If $\partial f/\partial t(0) \neq 0$, $\partial f/\partial \tau_1(0) = \ldots = \partial f/\partial \tau_m(0) = 0$, $\det(\partial^2 f/\partial \tau_i\,\partial \tau_j(0)) \neq 0$, then f is $\mathscr{R}(C)$-equivalent to $t + \sum_{i=1}^{m} \epsilon_i \tau_i^2$ where each $\epsilon_i = \pm 1$.*

(iii) *If $\partial f/\partial t(0) = 0$, and the τ-linear parts of f and $\partial f/\partial t$ are linearly independent when $n > 3$ then f is $\mathscr{R}(C)$-equivalent to $\tau_1 + \tau_2 t$. In the case $n = 3$ when $\partial f/\partial t(0) = 0$ and $\partial^2 f/\partial t\,\partial \tau_1(0).\partial f/\partial \tau_1(0) \neq 0$ then f is $\mathscr{R}(C)$-equivalent to $\tau_1 + \tau_1 t$.*

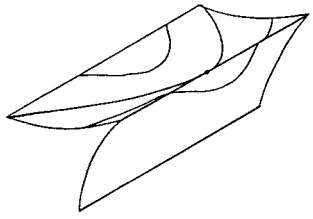
These are the only generic cases which can occur.

In the cases $m = 1$ we refer to the three types (i), (ii) and (iii) classified in Theorem 2·4 as the standard characteristic, tangent characteristic and cuspidal characteristic respectively. The characteristics on D are illustrated in Figure 1.

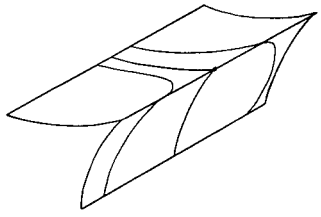
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standard characteristic



tangent characteristic



cuspidal characteristic

Fig. 1

To prove the theorem above we shall need the following discussion. In what follows Δ will denote a finitely-generated $\Pi_1^* \mathcal{O}(x) = R$-submodule of Der (C/D). The ring R is, of course, a local ring with maximal ideal $\mathcal{M}^2(t).\mathcal{O}(\tau) + \mathcal{M}(\tau)$ denoted by $\mathcal{M}(R)$. We shall obtain our normal forms by integrating time dependent vector fields, so writing $\tilde{R} = \mathcal{O}(\lambda).R$ we shall need to consider the $\tilde{R}$-module $\tilde{R}.\Delta = \tilde{\Delta}$; $\tilde{R}$ is also a local ring, and again we denote its maximal ideal by $\mathcal{M}(\tilde{R})$.

Given $H = f + \lambda h$ we shall say that H is an $\mathcal{R}(C)$-*trivial family* if there is a germ $\Phi\colon (C \times \mathbf{k}, 0) \to (C, 0)$ with $\Phi(0, \lambda) = 0$, $\Phi(-, \lambda)\colon (C, 0) \to (C, 0)$ a lifted diffeomorphism for each small λ, and $H(\Phi(-, \lambda), \lambda) = f(-)$. We shall need our vector fields to vanish at $(0, \lambda) \in C \times k$, of course, and shall suppose that the fields in Δ all vanish at $0 \in C$.

PROPOSITION 2·5. *Let H be as above. If* (i) $h \in \Delta.f$ *and* (ii) $\Delta.h \subset \Delta.f$ *then H is an $\mathcal{R}(C)$-trivial family.*

Proof. By [6], p. 36, the family is trivial if for some $v \in \tilde{\Delta}$ we have $\partial H/\partial\lambda = h = v.H$. Since $\Delta.f \subset \tilde{\Delta}.f$, it is enough to prove that $\tilde{\Delta}.f \subset \tilde{\Delta}.H$. Clearly $\tilde{\Delta}.f$ is a finitely generated $\tilde{R}$-module, so by Nakayama's lemma we need to prove that

$$\tilde{\Delta}.H + \mathcal{M}(\tilde{R}).(\tilde{\Delta}.f) \supset \tilde{\Delta}.f.$$

But given $v \in \tilde{\Delta}$ we have $v.f = v.(H - \lambda h) = v.H - \lambda v.h$. Writing $v = \Sigma \alpha_i v_i$ with $\alpha_i \in \tilde{R}$, $v_i \in \Delta$, we see that $v.f = v.H - \Sigma\lambda\alpha_i v_i.h$. Since $v_i.h \in \Delta.f \subset \tilde{\Delta}.f$ by (ii), clearly $\Sigma\lambda\alpha_i v_i.h \in \mathcal{M}(\tilde{R}).\tilde{\Delta}.f$ and the inclusion above follows. ∎

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Proof of Theorem 2·4

(i) Write $f = \alpha t + \Sigma\beta_i\tau_i + O(2)$ where here and below $O(r)$ denotes terms of order $\geqslant r$. Without loss of generality we may suppose that $\alpha\beta_1 \neq 0$. Let $\Delta \subset \mathrm{Der}\,(C/D)$ denote the R-submodule of vector fields which vanish at $0 \in \mathbf{k}^n$. Put $f_0 = \alpha t + \beta_1\tau_1$, $h = f - f_0$ and set $H_{\lambda_0} = f_0 + \lambda_0 h + \lambda h$. If we show that H_{λ_0} is $\mathscr{R}(C)$-trivial for each λ_0, clearly it follows that f is equivalent to f_0 and the result follows. Consequently, by Proposition 2·5 above, we need to show that $h \in \Delta.(f_0 + \lambda_0 h)$ and $\Delta.h \subset \Delta.(f_0 + \lambda_0 f)$. In fact we claim that $\Delta.(f_0 + \lambda_0 h) = \mathscr{M}(t,\tau)$, the maximal ideal in $\mathscr{O}(t,\tau)$, from which both of these conditions follow.

By Nakayama's lemma we need only show that

$$\mathscr{M}(t,\tau) = \Delta.(f_0 + \lambda_0 h) + \mathscr{M}(R).\mathscr{M}(t,\tau) \tag{1}$$

for $\mathscr{M}(t,\tau)$ is a finitely generated R-module, generated in fact by $t, t^2, \tau_1, \ldots, \tau_m$. Since $\mathscr{M}(R) = \mathscr{M}^2(t).\mathscr{O}(\tau) + \mathscr{M}(\tau)$, the second term on the right hand side of (1) above contains $\mathscr{M}^3(t).\mathscr{O}(\tau)$, so we need only consider elements in $\mathscr{O}(\tau).t^2$, $\mathscr{O}(\tau).t$ and $\mathscr{M}(\tau)$. However, since R contains $\mathscr{O}(\tau)$ and $\tau_i\,\partial/\partial\tau_1$, $1 \leqslant i \leqslant m$, $t\,\partial/\partial t$, $t^2\,\partial/\partial t$ lie in Δ the result follows.

(ii) Write $f = \alpha t + \beta t^2 + \Sigma\gamma_{ij}\tau_i\tau_j + \Sigma\delta_i t\tau_i + 0(3)$, $\alpha \neq 0$, $\det(\gamma_{ij}) \neq 0$. By a preliminary linear change of coordinates in the τ_i only, we may suppose that

$$\Sigma\gamma_{ij}\tau_i\tau_j = \sum_{i=1}^{m} \epsilon_i\tau_i^2$$

with $\epsilon_i \pm 1$ for each i. Set $f_0 = \alpha t + \Sigma\epsilon_i\tau_i^2$, $h = f - f_0$ and $H_{\lambda_0} = f_0 + \lambda_0 h + h$. We take Δ to be the R-module of vector fields generated by $t\,\partial/\partial t$, $t^2\,\partial/\partial t$, $t\,\partial/\partial\tau_i$, $\tau_j\,\partial/\partial\tau_i$, $1 \leqslant i,j \leqslant m$. Clearly we need only show

$$\Delta.(f_0 + \lambda_0 h) \supset t.\mathscr{M}(\tau) + t^2.\mathscr{O}(\tau) + \mathscr{M}^3(t,\tau) = M,$$

for then we can apply Proposition 2·4, since $h \in M$ and $\Delta.h \subset M$. Again M is a finitely generated R-module, so by Nakayama's lemma we need to prove that

$$M_1 = \Delta.(f_0 + \lambda_0 h) + \mathscr{M}(R).M \supset M; \tag{2}$$

but this can be shown as follows.

Since $R \supset \mathscr{O}(\tau)$ we can show

(*a*) $t^3.\mathscr{O}(\tau)$ is contained in M_1. For $t^3\,\partial/\partial t(f_0 + \lambda_0 h) = \alpha t^3 + 2\beta t^4 + \Sigma\delta_i t^3\tau_i + \phi$, where $\phi \in \mathscr{M}^2(t).\mathscr{M}^3(t,\tau) \subset \mathscr{M}(R).M$, and as t^4, $t^3\tau_i$ also lie in $\mathscr{M}(R).M$ we deduce the result.

(*b*) $t^2.\mathscr{M}(\tau)$ is contained in M_1. For

$$\tau_j t^2\,\partial/\partial t(f_0 + \lambda_0 h) = \alpha t^2\tau_j + 2\beta t^3\tau_j + \Sigma\delta_i t^2\tau_i\tau_j + \phi,$$

where $\phi \in \mathscr{M}^2(t).\mathscr{M}^3(t,\tau) \subset \mathscr{M}(R)\,M$, and as $t^3\tau_j$ lie in M_1 by (*a*) and

$$\Sigma\delta_i t^2\tau_i\tau_j \in \mathscr{M}(\tau).\mathscr{M}^3(t,\tau) \subset \mathscr{M}(R).M$$

the inclusion follows.

(*c*) t^2 belongs to M_1. For $t^2\,\partial/\partial t(f_0 + \lambda_0 h) = \alpha t^2 + 2\beta t^3 + \Sigma\delta_i t^2\tau_i + \phi$ where

$$\phi \in t^2.\mathscr{M}^2(t,\tau) \subset \mathscr{M}^4(t).\mathscr{O}(\tau) + \mathscr{M}(\tau).\mathscr{M}^3(t,\tau) \subset \mathscr{M}(R).M.$$

The inclusion follows from (*a*) and (*b*).

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(*d*) $t\mathscr{M}^2(\tau)$ is contained in M_1. Here

$$\tau_j\tau_k t\,\partial/\partial t(f_0+\lambda_0 h) = \alpha t\tau_j\tau_k + 2\beta t^2\tau_j\tau_k + \Sigma\delta_i t\tau_i\tau_j\tau_k + \phi$$

where $\phi, t\tau_i\tau_j\tau_k \in \mathscr{M}(\tau).\mathscr{M}^3(t,\tau) \subset \mathscr{M}(R).M$, and since $t^2\tau_j\tau_k \in M_1$ by (*b*) the inclusion follows.

(*e*) $t\tau_j$ belongs to M_1, $1 \leqslant j \leqslant m$. Here

$$\tau_j t\,\partial/\partial t(f_0+\lambda_0 h) = \alpha t\tau_i + 2\beta t^2\tau_j + \Sigma\delta_j t\tau_i\tau_j + \phi \quad \text{and} \quad \phi \in \mathscr{M}(\tau).\mathscr{M}^3(t,\tau) \subset \mathscr{M}(R).M$$

while $t^2\tau_j$, $t\tau_i\tau_j \in M_1$ by (*b*) and (*d*) respectively.

(*f*) Finally we need to show that $\mathscr{M}^3(t,\tau)$ is contained in M_1. However,

$$\mathscr{M}^3(t,\tau) = \mathscr{M}^3(\tau) + t.\mathscr{M}^2(\tau) + t^2.\mathscr{M}(\tau) + t^3.\mathscr{O}(\tau) + \mathscr{M}^4(t).\mathscr{O}(\tau),$$

so we need only prove that $\mathscr{M}^3(\tau) \subset M_1$ since we have dealt with the second, third and fourth terms above and $\mathscr{M}^4(t).\mathscr{O}(\tau) \subset \mathscr{M}(R).M$. But

$$\tau_k\tau_j\,\partial/\partial\tau_i(f_0+\lambda_0 h) = 2\epsilon_i\tau_k\tau_j\tau_i + \delta_i t\tau_k\tau_j + \phi$$

where $\phi \in \mathscr{M}^2(\tau).\mathscr{M}^2(t,\tau) \subset \mathscr{M}(R).M$. The result now follows.

(iii) Let us consider the case $n > 3$ first. So we can write

$$f = \Sigma\alpha_i\tau_i + \Sigma\beta_i t\tau_i + \Sigma\gamma_{ij}\tau_i\tau_j + \delta t^2 + O(3)$$

where the vectors $(\alpha_1, \ldots, \alpha_m)$, $(\beta_1, \ldots, \beta_m)$ are linearly independent. By a change of coordinates in the τ-variables, we may reduce f to $\tau_1 + t\tau_2 + \delta t^2 + O(3)$. We now set $f_0 = \tau_1 + t\tau_2$, $h = f - f_0$ and $H_{\lambda_0} = f_0 + \lambda_0 h + \lambda h$. We take Δ to be the R-module of vector fields generated by $\tau_i\tau_j\,\partial/\partial\tau_1$, $t^2\,\partial/\partial\tau_1$, $t^3\,\partial/\partial\tau_1$, $\tau_i\,\partial/\partial\tau_2$. We claim that

$$\Delta(f_0+\lambda_0 h) \supset \mathscr{M}^2(t,\tau),$$

and by Nakayama's lemma need to show that

$$\mathscr{M}^2(t,\tau) \subset \Delta(f_0+\lambda_0 h) + \mathscr{M}(R).\mathscr{M}^2(t,\tau)$$

(note that $\mathscr{M}(R).\mathscr{M}^2(t,\tau) = (\mathscr{M}^2(t).\mathscr{O}(\tau) + \mathscr{M}(\tau)).\mathscr{M}^2(t,\tau) \supset \mathscr{M}^4(t,\tau)$). Since

$$\tau_i\tau_j\,\partial/\partial\tau_1(f_0+\lambda_0 h) = \tau_i\tau_j + \phi$$

with $\phi \in \mathscr{M}^2(\tau).\mathscr{M}^2(t,\tau)$, the right-hand side above contains $\mathscr{M}^2(\tau)$, and as

$$\tau_i\,\partial/\partial\tau_2(f_0+\lambda_0 h) = t\tau_i + \phi$$

with $\phi \in \mathscr{M}(\tau).\mathscr{M}^2(t,\tau)$ it also contains $t.\mathscr{M}(\tau)$. This only leaves $t^2.\mathscr{O}(\tau)$ and $t^3.\mathscr{O}(\tau)$ terms, but $t^r\,\partial/\partial\tau_1(f_0+\lambda_0 h) = t^r + \phi$ with $\phi \in \mathscr{M}^r(t).\mathscr{M}^2(t,\tau) \subset \mathscr{M}(R).\mathscr{M}^2(t,\tau)$ for $r \geqslant 2$, and since R contains $\mathscr{O}(\tau)$ the result follows. Clearly $\Delta.h \subset \mathscr{M}^2(t,\tau)$, so an application of Proposition 2·4 yields the required result. The case $n = 3$ follows similarly.

Finally, these are the only generic cases to arise, for if all of the above sets of inequalities fail we have imposed $n-1$ conditions on the (t,τ) variables, and since we already have the condition that the envelope has an A_2 singularity ($t = 0$) this is non-generic. ∎

Having dealt fairly conclusively with A_2 points of the envelope, it is natural to turn to the general case, and in particular to consider the distribution of characteristics at a swallowtail point. If δ_1, δ_2, δ_3 are the three vector fields of Example 2·2, then a function $f\colon (C,0) \to (\mathbf{k},0)$ determining the distribution will be stable if the R-module generated

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by $\delta_1 f$, $\delta_2 f$, $\delta_3 f$ should coincide with $\mathscr{M}(C)$. But by the Preparation Theorem ([6], p. 134), this will be so if and only if $\delta_1 f, \delta_2 f, \delta_3 f$ span $\mathscr{M}(C)/\mathscr{M}(R).\mathscr{M}(C)$ as a $\mathbf{k}$-vector space. However, $\mathscr{M}(C)/\mathscr{M}(R).\mathscr{M}(C)$ clearly has dimension 4 (with basis x_1, t, t^2, t^3) so no stable function exists. To obtain normal forms in the general case we shall need to broaden our notion of equivalence. Before doing so we give some geometric applications of our Theorem 2·4.

First some general remarks. In geometric applications we have a versal unfolding G of some function germ G^0, and a standard or model versal unfolding F of the same dimension of an equivalent germ F^0. When F^0 is of type A_2, we have classified function germs up to $\mathscr{R}(C)$-equivalence for the model versal unfolding $F(x,t,\tau) = t^3 + x_1 t + x_2$. We have from § 1 a diagram

$$\begin{array}{ccc} & & \mathbf{k} \\ & & \uparrow{\scriptstyle \mu_G} \\ (C(F),0) & \xrightarrow{(\Phi,\Psi)} & (C(G),0) \\ \downarrow{\scriptstyle \Pi_F} & & \downarrow{\scriptstyle \Pi_G} \\ (D(F),0) & \xrightarrow{\Phi} & (D(G),0) \end{array}$$

and seek to determine the $\mathscr{R}(C)$-type of $\mu_G \circ (\Phi, \Psi)$. But the composite $\mu_G \circ (\Phi, \Psi)$ is Ψ, the parametrized change of coordinates used in the equivalence of the unfoldings of F and G, so since $\partial(\Psi(0,0))/\partial t \neq 0$ only the normal forms of 2·4 (i) and (ii) are relevant. We also mention the fact that the conditions $\partial f/\partial t(0) \neq 0$, $\partial f/\partial \tau(0) \neq 0$ have coordinate-free interpretations. The second is clearly that the restriction of f to the cuspidal edge is a submersion. For the first we note that at 0 the kernel of the derivative of the normalization is spanned by $\partial/\partial t$ and so the condition $\partial f/\partial t(0) \neq 0$ is equivalent to the derivative of f restricted to this kernel being injective.

Example 2·6

Here we consider the envelope of spheres of a fixed radius r centred on a space curve $\gamma(s)$, parametrized by arc length s. So we take $F\colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ to be

$$F(u,s) = \|\gamma(s) - u\|^2 - r^2,$$

where $\| \ \|^2$ is the square of the Euclidean norm. The envelope is given by the equations $F = 0$ and $\partial F/\partial s = 2(\gamma(s) - u).T(s) = 0$ where $T(s)$ is the unit tangent to the curve at $\gamma(s)$. Consequently the characteristics are circles of radius r centred at $\gamma(s)$ in the normal plane at $\gamma(s)$. The edge of regression is determined by the additional condition that $\partial^2 F/\partial s^2 = 2\{1 + (\gamma(s) - u).\kappa(s)\,N(s)\} = 0$ where $\kappa(s)$ is the curvature, $N(s)$ the principal normal at $\gamma(s)$, and is the locus of points

$$\gamma(s) + \sigma(s)\,N(s) \pm \sqrt{(r^2 - \sigma^2(s))}\,B(s), \quad (r^2 - \sigma^2 \geqslant 0),$$

where $\sigma(s) = 1/\kappa(s)$, $B(s)$ is the binormal at $\gamma(s)$. The condition for the function F to have an A_2 at this point of regression is $\kappa(s)\,\sigma(s) \pm \sqrt{(r^2 - \sigma(s)^2)}\,\kappa(s)\,\tau(s) \neq 0$ where $\tau(s)$ is the torsion at $\gamma(s)$. One easily checks that F always versally unfolds such A_2 points, so that the edge of regression is a genuine cuspidal edge.

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To determine the distribution of the characteristics it remains, by Theorem 2·4, to determine the nature of the projection of this cuspidal edge to the s-parameter. More precisely in $\mathbb{R}^3 \times \mathbb{R}$ we consider the A_2 locus $F = \partial F/\partial s = \partial^2 F/\partial s^2 = 0$ and its projection to the s-coordinate. If this projection is a local submersion we have the normal form of 2·4 (i); if it is a Morse singularity we have the normal form of 2·4 (ii). It is not difficult to see that this projection is a submersion if and only if the restriction of $(F, \partial F/\partial s, \partial^2 F/\partial s^2)$ to $s =$ constant is a submersion and has a Morse singularity if and only if this restriction is of type $\Sigma^{1,0}$ ([6], p. 196). If we take the relevant parameter value to be $s = 0$, $\gamma(0) = 0$, $T(0) = e_1$, $N(0) = e_2$, $B(0) = e_3$ (where e_j is the jth unit vector) and $u = \sigma(0)\, e_2 \pm \sqrt{(r^2 - \sigma(0)^2)}\, e_3$, the germ $(F, \partial F/\partial s, \partial^2 F/\partial s^2)\colon (\mathbb{R}^3 \times \{0\}, (u, 0)) \to (\mathbb{R}^3, 0)$ is given by $(u_1, u_2, u_3) \mapsto (u_1^2 + u_2^2 + u_3^2 - r^2, -u_1, -\kappa(0)\, u_2 + 1)$. Setting

$$v_2 = u_2 - \sigma(0), \quad v_3 = u_3 \mp \sqrt{(r^2 - \sigma(0)^2)},$$

this becomes

$$(u_1, v_2, v_3) \mapsto (u_1^2 + v_2^2 + 2\sigma(0)\, v_2 + v_3^2 \pm 2\sqrt{(r^2 - \sigma(0)^2)}\, v_3, -u_1, -\kappa(0)\, v_2)$$

which is the germ of a submersion if and only if $\kappa(0)\sqrt{(r^2 - \sigma(0)^2)} \neq 0$. Assuming κ does not vanish on our curve (a generic condition) we consequently have a distribution of type 2·4 (i) if $\kappa^2 r^2 \neq 1$ and of type 2·4 (ii) if $\kappa^2 r^2 = 1$, for then the germ above is clearly of type $\Sigma^{1,0}$.

Example 2·7

Consider the envelope of spheres centred on a space curve $\gamma(s)$, passing through a fixed point, say 0. We take

$$F(u, s) = u \,.\, (u - 2\gamma(s))$$

and $\partial F/\partial s = -2u \,.\, T(s)$, so the characteristics are the intersections of these spheres with planes parallel to the normal plane and through the origin. It is not difficult to check ([2], p. 45) that the edge of regression is given by $2(B(s) \,.\, \gamma(s))\, B(s)$ and away from 0 a point on this edge of regression is of type A_2 if $\tau(s) \neq 0$. Moreover the unfolding F always versally unfolds this A_2. Using the same methods as in the previous example, one now easily checks that we have a distribution of characteristics of type 2·4 (i) if the osculating plane at $\gamma(s)$ does not pass through the origin. If it does, however, the corresponding point on the edge of regression *is* the origin, and we can say nothing about even the local structure of the envelope here.

Example 2·8

Finally, we consider the envelope of osculating planes to a space curve. So we take

$$F(u, s) = (u - \gamma(s)) \,.\, B(s)$$

and $\partial F/\partial s = -\tau(s)\,(u - \gamma(s)) \,.\, N(s)$; consequently the characteristics are the tangent lines to the curve. To find the edge of regression we need

$$\partial^2 F/\partial s^2 = (u - \gamma(s)) \,.\, (-\tau(s)\, N(s) - \tau^2(s)\, B(s) + \tau(s)\, \kappa(s)\, T(s)) = 0.$$

A short computation shows that the edge of regression is the curve itself and the relevant function has an A_2 singularity provided $\kappa \neq 0$ (a generic condition) and $\tau \neq 0$. Moreover this singularity is versally unfolded by F. Again the distribution of characteristics at these points is easily seen to be of type 2·4 (i).

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We note in passing that the local models of Theorem 2·4 do not apply to all the generic geometrical situations that they might. For example, the foliation of a surface by its lines of curvature near a ridge point gives rise to a foliation of the corresponding pieces of focal set by geodesics, but *each* geodesic has a cusp on the cuspidal edge so 2·4 does not apply (see [**11**], chapter VIII).

3. *The general case:* A_n, $n \geqslant 3$

In the general case we have an A_n singularity f: $(\mathbf{k}, 0) \to (\mathbf{k}, 0)$ given by $f(t) = t^{n+1}$ and, we shall suppose initially, a miniversal unfolding F: $(\mathbf{k}^n \times \mathbf{k}, 0) \to (\mathbf{k}, 0)$ given by

$$F(x, t) = t^{n+1} + x_1 t^{n-1} + \ldots + x_{n-1} t + x_n.$$

The set germ $(C, 0) \subset (\mathbf{k}^n \times \mathbf{k}, 0)$ is given by the equations $F = \partial F/\partial t = 0$ and is clearly smooth. Indeed, writing

$$G = t^{n+1} + x_1 t^{n-1} + \ldots + x_{n-2} t^2,$$

the map Φ: $(\mathbf{k}^{n-1}, 0) \to (\mathbf{k}^{n+1}, 0)$ given by

$$\Phi(x_1, \ldots, x_{n-2}, t) = (x_1, \ldots, x_{n-2}, -\partial G/\partial t, t\,\partial G/\partial t - G, t)$$

is a parametrization of C. Thus we can use $x_1, \ldots, x_{n-2}, t$ as coordinates on C, and in what follows shall identify C with the $(x_1, \ldots, x_{n-2}, t)$ space. The normalization map Π: $(\mathbf{k}^{n-1}, 0) \equiv (C, 0) \to (\mathbf{k}^n, 0)$ is given by

$$\Pi(x_1, \ldots, x_{n-2}, t) = (x_1, \ldots, x_{n-2}, -\partial G/\partial t, t\,\partial G/\partial t - G) = (x_1, \ldots, x_n).$$

Again we will denote the ring $\Pi^* \mathcal{O}(x)$ by R; it is a local ring with maximal ideal $\mathcal{M}(R) = \Pi^* \mathcal{M}(x)$. The lifted vector fields Der (C/D) form an R-module, not an $\mathcal{O}(C)$-module. We note however that $\mathcal{O}(C)$ is a finitely generated R-module, indeed generated by $1, t, t^2, \ldots, t^{n-1}$. For $\mathcal{O}(C)/\mathcal{M}(R).\mathcal{O}(C)$ is a $\mathbf{k}$-vector space with these elements as a basis, and the result follows by the Preparation Theorem. Our first task is to obtain further information concerning the generators of Der (C/D).

PROPOSITION 3·1. *Let $\delta_1, \ldots, \delta_n$ be lifts of the n generators of Derlog D to Der (C/D), and let $\bar{x}$ denote the coordinates $x_1, \ldots, x_{n-2}$. Then modulo terms in*

$$\mathcal{M}(C).(\mathcal{M}^n(t) + \mathcal{M}(\bar{x})).\{\partial/\partial x_i\}$$

we have

$$\delta_1 \equiv 2x_1\,\partial/\partial x_1 + \ldots + (n-1)\,x_{n-2}\,\partial/\partial x_{n-2} + t\,\partial/\partial t,$$

$$\delta_2 \equiv 3x_2\,\partial/\partial x_1 + \ldots + (n-1)\,x_{n-2}\,\partial/\partial x_{n-3} - n(n+1)\,t^n\,\partial/\partial x_{n-2} + \left(t^2 + \frac{2}{(n+1)}\,x_1\right)\partial/\partial t,$$

$$\delta_r \equiv (r+1)\,x_r\,\partial/\partial x_1 + \ldots + (n-1)\,x_{n-2}\,\partial/\partial x_{n-r-1} - n(n+1)\,t^n\,\partial/\partial x_{n-r} + \left(t^r + \frac{r}{(n+1)}\,x_{r-1}\right)\partial/\partial t \quad \text{for} \quad 3 \leqslant r \leqslant n-2,$$

$$\delta_{n-1} \equiv -n(n+1)\,t^n\,\partial/\partial x_1 + \left(t^{n-1} + \frac{(n-1)}{(n+1)}\,x_{n-2}\right)\partial/\partial t,$$

and

$$\delta_n \equiv (1-n)\,t^n\,\partial/\partial t.$$

Proof. We recall the construction of the vector fields tangent to the discriminant of an A_n singularity given for the first time by Zakalyukin in [**12**] and also in [**7**] (see [**10**],

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[3] for proofs and [1] for an alternative method). For each r with $0 \leqslant r \leqslant n-1$, we consider the product $F.t^r$ and write it as a sum

$$F.t^r = \sum_{s=0}^{n-1} a_{rs}(x)\, t^s + \alpha_r(x,t)\, \partial F/\partial t,$$

for some a_{rs} in $\mathcal{O}(x)$, α_r in $\mathcal{O}(x,t)$. The vector fields $\theta_r = \Sigma a_{rs}\, \partial/\partial x_{n-s}$ then generate Derlog D as a free $\mathcal{O}(x)$-module. Moreover the vector field $\theta_r + \alpha_r\, \partial/\partial t = \tilde{\delta}_r$ is tangent to $(C,0) \subset (\mathbf{k}^n \times \mathbf{k}, 0)$. To see this, we note that

$$\tilde{\delta}_r(F) = \Sigma a_{rs}(x)\, t^s + \alpha_r(x,t)\, \partial F/\partial t = t^r F$$

and

$$\begin{aligned}\tilde{\delta}_r(\partial F/\partial t) &= \Sigma a_{rs}(x)\, s t^{s-1} + \alpha_r(x,t)\, \partial^2 F/\partial t^2 \\ &= \partial/\partial t(t^r F) - \partial \alpha_r/\partial t(x,t)\, \partial F/\partial t \\ &= r t^{r-1} F + (t^r - \partial \alpha_r/\partial t(x,t))\, \partial F/\partial t,\end{aligned}$$

so both $\tilde{\delta}_r(F)$ and $\tilde{\delta}_r(\partial F/\partial t)$ vanish on C.

We want to lift the vector fields $\tilde{\delta}_r$ via Φ to the parameter space $\mathbf{k}^{n-1}$, but first we compute the coefficients of the $\tilde{\delta}_r$ modulo terms in $\mathcal{M}(x,t).(\mathcal{M}^n(t) + \mathcal{M}(x)) = M$. Note that since

$$(n+1)\, F.1 = \sum_{i=1}^{n} (i+1)\, x_i t^{n-i} + t\, \partial F/\partial t \tag{1}$$

the first field (the so-called Euler field) is given by

$$\tilde{\delta}_1 = 2x_1\, \partial/\partial x_1 + 3x_2\, \partial/\partial x_2 + \ldots + (n+1)\, x_n\, \partial/\partial x_n + t\, \partial/\partial t.$$

Multiplying both sides of (1) by t and replacing the t^n term by

$$1/(n+1)\,(\partial F/\partial t - \sum_{i=1}^{n} (n-i)\, x_i t^{n-i})$$

we find that, modulo terms in M, we have

$$\tilde{\delta}_2 \equiv 3x_2\, \partial/\partial x_1 + 4x_3\, \partial/\partial x_2 + \ldots + (n+1)\, x_n\, \partial/\partial x_{n-1} + \left(t^2 + \frac{2}{(n+1)}\, x_1\right) \partial/\partial t.$$

Successively multiplying (1) by t and replacing the t^n term as above we find that, modulo terms in M,

$$\tilde{\delta}_r = (r+1)\, x_r\, \partial/\partial x_1 + \ldots + (n+1)\, x_n\, \partial/\partial x_{n-r+1} + \left(t^r + \frac{r}{(n+1)}\, x_{r-1}\right) \partial/\partial t$$

for $1 \leqslant r \leqslant n$.

To compute the lifts of δ_r of $\tilde{\delta}_r$ to the $(t, \bar{x})$-space C we ignore the $\partial/\partial x_{n-1}$ and $\partial/\partial x_n$ terms and replace x_{n-1}, x_n by $-\partial G/\partial t,\ t\, \partial G/\partial t - G$ respectively. Since we are working modulo terms in $\mathcal{M}(C).(\mathcal{M}^n(t) + \mathcal{M}(\bar{x}))\{\partial/\partial x_i\}$, we can replace x_{n-1} by $-(n+1)\, t^n$ and x_n by 0. The result now follows. |

We now wish to use the vector fields to classify generic functions on C. As we have seen, however, the analogue of right equivalence will not yield stable functions. We shall weaken our notion of equivalence as follows.

Definitions 3·2. Let g_1, g_2: $(C,0) \to (\mathbf{k},0)$ be two smooth function germs. We shall say that g_1 and g_2 are $\mathscr{K}(C)$-*equivalent* if for some germ $\alpha \in \mathcal{O}(C)$ with $\alpha(0) \neq 0$ and a lifted diffeomorphism $\tilde{\phi} \in \mathrm{Diff}(C/D)$ we have $\alpha.(g_1 \circ \tilde{\phi}) = g_2$. Note that $\tilde{\phi}$ takes the zero set of g_2 to that of g_1. Consequently there is an automorphism of the discriminant taking

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the characteristic through $0 \in D$ corresponding to g_1 to that through $0 \in D$ corresponding to g_2. Unfortunately $\mathscr{K}(C)$-equivalence does not preserve nearby characteristics; however, some information can be deduced about these in the case of the swallowtail (see § 4).

As in § 2, we shall obtain our changes of coordinates by integrating time dependent vector fields. So again Δ will denote a finitely generated $\Pi^* \mathscr{O}(x) = R$-submodule of $\mathrm{Der}(C/D)$. (Note all vector fields in $\mathrm{Der}(C/D)$ vanish at $0 \in C$). Writing $\tilde{R} = \mathscr{O}(\lambda).R$ we shall consider the $\tilde{R}$-module $\tilde{R}.\Delta = \tilde{\Delta}$.

Given $H = f + \lambda h$ we shall say that H is a *$\mathscr{K}(C)$-trivial family* if there is a germ $\Phi\colon (C \times \mathbf{k}, 0) \to (C, 0)$ and a germ $\alpha\colon (C \times \mathbf{k}, 0) \to (\mathbf{k}, 0)$ with

$$\Phi(0, \lambda) = 0, \Phi(-, \lambda)\colon (C, 0) \to (C, 0)$$

a lifted diffeomorphism for each small $\lambda, \alpha(0, 0) \neq 0$ and $\alpha(-, \lambda).H(\Phi(-, \lambda), \lambda) = f(-)$.

In what follows, let $B_0 = \Delta.f + f^* \mathscr{M}(y).\mathscr{O}(C)$ (where y is the target variable) and $B = \tilde{\Delta}.H + H^* \mathscr{M}(y).\mathscr{O}(C \times \mathbf{k})$.

PROPOSITION 3·3. *Let H be as above; if* (i) $\mathscr{O}(C).H \subset B_0$ *and* (ii) $\Delta.h \subset B_0$ *then H is $\mathscr{K}(C)$-trivial.*

Proof. By ([5], p. 20) the family will be $\mathscr{K}(C)$-trivial if for some $v \in \tilde{\Delta}, \alpha \in \mathscr{O}(C \times \mathbf{k})$ we have $\partial F/\partial \lambda = h = v.H + \alpha H$, i.e. $h \in B$. We claim that

$$B \supset \tilde{\Delta}.f + f^* \mathscr{M}(y).\mathscr{O}(C \times \mathbf{k}) = M.$$

We first note that M is a finitely generated $\tilde{R}$-module. Since $\tilde{\Delta}.f$ clearly is, it remains to show that $f^* \mathscr{M}(y).\mathscr{O}(C \times \mathbf{k})$ is also, and hence that $\mathscr{O}(C \times \mathbf{k})$ is finitely generated over $\tilde{R}$. However, $\mathscr{O}(C \times \mathbf{k})/\mathscr{M}(\tilde{R}).\mathscr{O}(C \times \mathbf{k}) \cong \mathscr{O}(C)/\mathscr{M}(R).\mathscr{O}(C)$, and by the Preparation Theorem the result follows. We may now use Nakayama's lemma to prove the inclusion $B \supset M$, so it suffices to show that

$$B + \mathscr{M}(\tilde{R}).M \supset M.$$

Given $v \in \tilde{\Delta}, \alpha \in \mathscr{O}(C \times \mathbf{k})$ we have

$$v.f + \alpha f = v.(H - \lambda h) + \alpha(H - \lambda h) = v.H + \alpha H - \lambda(v.h + \alpha h).$$

Since $v.h, \alpha h \in B_0 \subset M$ by hypothesis and $\lambda \in \mathscr{M}(\tilde{R})$, the claim follows. But by hypothesis $h \in B_0 \subset M$, and we have just proved that $M \subset B$ and the result is established. ∎

We now state and prove our main classification result.

THEOREM 3·4. *Let $f\colon (C, 0) \to (\mathbf{k}, 0)$ be a smooth germ on the critical set $(C, 0)$ of the unfolding $F(x_1, t) = t^{n+1} + x_1 t^{n-1} + \ldots + x_{n-1} t + x_n$, with $\partial f/\partial t(0) \neq 0$. Then f is $\mathscr{K}(C)$-equivalent to the germ t. In particular, there is an automorphism ϕ of the discriminant $(D, 0)$ taking the image of $f^{-1}(0)$ under the natural projection to D to the image of $t = 0$.*

Proof. Without loss of generality we may suppose that $\partial f/\partial t(0) = 1$, write $h = f - t$ and set $H_{\lambda_0} = t + \lambda_0 h + \lambda h$. If we show that H_{λ_0} is a locally $\mathscr{K}(C)$-trivial family for each λ_0 the result will follow. Consequently, by Proposition 3·3 above, we need to show that $\mathscr{O}(C).h$ and $\Delta.h$ are both contained in

$$B_0 = \Delta(t + \lambda_0 h) + (t + \lambda_0 h)^* \mathscr{M}(y).\mathscr{O}(C).$$

Write $A = \mathscr{M}(C)/(t + \lambda_0 h)^* \mathscr{M}(y).\mathscr{O}(C)$: this is a finitely generated R-module since $\mathscr{M}(C)$ is. We want to show that $\mathscr{M}(C) \subset B_0$, so we need to prove that the elements of

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$\Delta(t+\lambda_0 h)$ generate A as an R-module and hence, by the Preparation Theorem, it suffices to prove that they generate

$$A/\mathscr{M}(R).A \equiv \mathscr{M}(C)/(\mathscr{M}(R).\mathscr{M}(C)+(t+\lambda_0 h)^*\mathscr{M}(y).\mathscr{O}(C))$$

as a $\mathbf{k}$-vector space. Clearly $\mathscr{M}(R).\mathscr{M}(C) \supset \mathscr{M}^{n+1}(C)+\mathscr{M}(\bar{x}).\mathscr{M}(C)$ and modulo terms in $\mathscr{M}^{n+1}(C)+\mathscr{M}(\bar{x}).\mathscr{M}(C)$ we have, writing $j^1(t+\lambda_0 h) = t+\sum_{i=1}^{n-2}\alpha_i x_i$,

$$\delta_1(t+\lambda_0 h) = t+\sum_{i=1}^{n-2}(i+1)\alpha_i x_i,$$

$$\delta_2(t+\lambda_0 h) = t^2+\sum_{i=1}^{n-3}(i+2)\alpha_i x_{i+1}+2/(n+1)x_1-n(n+1)t^n\alpha_{n-2},$$

$$\delta_r(t+\lambda_0 h) = t^r+\sum_{i=1}^{n-r-1}(i+r)\alpha_i x_{i+r-1}$$
$$+\frac{r}{(n+1)}x_{r-1}-n(n+1)t^n\alpha_{n-r}, \quad \text{for} \quad 3 \leqslant r \leqslant n-2,$$

$$\delta_{n-1}(t+\lambda_0 h) = t^{n-1}+(n-1)/(n+1)x_{n-2}-n(n+1)t^n.\alpha_1,$$

$$\delta_n(t+\lambda_0 h) = (1-n)t^n,$$

while

$$t^r(t+\lambda_0 h) = t^{r+1}+\phi_{r+2} \quad \text{for} \quad r \geqslant 1$$
$$= t+\sum_{i=1}^{n-2}\alpha_i x_i+\phi_2 \quad \text{for} \quad r = 0,$$

where ϕ_j is a polynomial in t of degree $\geqslant j$. Clearly $\delta_i(t+\lambda_0 h)$, $2 \leqslant i \leqslant n-1$, generate $A/\mathscr{M}(R).A$; thus $\mathscr{M}(C) \subset B_0$. Since $\mathscr{O}(C).h$, $\Delta.h \subset \mathscr{M}(C)$, we may now apply Proposition 3.3 to deduce the result. |

We now briefly review the situation when F is no longer a miniversal unfolding, that is there are m redundant unfolding terms, so

$$F(x_1, \ldots, x_n, \tau_1, \ldots, \tau_m, t) = t^{n+1}+x_1 t^{n-1}+\ldots+x_n.$$

Just as in § 2, the $R = \Pi^*\mathscr{O}(x,\tau)$-module $\mathrm{Der}(C/D)$ now contains the vector fields $\partial/\partial\tau_i$ for $1 \leqslant i \leqslant m$. Using the same methods as in the proofs of Theorems 2·4 and 3·4, one might conjecture that the analogue of Theorem 2·4 holds in the general case of an A_n for $n \geqslant 3$, provided we replace $\mathscr{R}(C)$-equivalence by $\mathscr{K}(C)$-equivalence. Thus one would hope that there are three generic functions up to $\mathscr{K}(C)$-equivalence, namely $t+\tau_1$, $t+\sum_{i=1}^m \epsilon_i\tau_i^2$, $\epsilon_i = \pm 1$ and $\tau_1+\tau_2 t$ (or $\tau_1+\tau_1 t$ in the case $m = 1$). The main problem is dealing with the corresponding ring R.

4. *The swallowtail*

We finish the paper by considering the special case of an A_3 singularity, the swallowtail $F^0(t) = t^4$ with miniversal unfolding

$$F(x,t) = t^4+x_1 t^2+x_2 t+x_3.$$

Recall from Example 2·2 that the normalization of the discriminant is given by

$$\Pi\colon (\mathbf{k}^2, 0) \to (\mathbf{k}^3, 0),\ \Pi(x_1, t) = (x_1, -4t^3-2x_1 t, 3t^4+x_1 t^2)$$

and $\mathrm{Der}(C/D)$ is generated by the vector fields δ_1, δ_2, δ_3 which span a space of dimen-

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sion $\leqslant 1$ precisely when $g = (2t^2+x_1)(6t^2+x_1) = 0$, i.e. at points of the lift of the double line or cuspidal edge of the swallowtail. If $X \subset C$ is given by $g = 0$ it follows from [7], p. 270, that the set X is a *free divisor*, that is the $\mathcal{O}(x_1, t)$-module of vector fields tangent to X, Derlog (X), is free, generated by

$$2x_1\,\partial/\partial x_1 + t\,\partial/\partial t \quad \text{and} \quad -(8x_1t + 24t^3)\,\partial/\partial x_1 + (x_1 + 4t^2)\,\partial/\partial t.$$

Note that the second of these vectors does not lie in Der (C/D). Instead of integrating vector fields in Der (C/D) to obtain our changes of coordinates, we can integrate fields in Derlog (X). The resulting changes of coordinates will at least preserve the intersections of the characteristics with the cuspidal edge and double line. It is not difficult to prove the following.

PROPOSITION 4·1. *Let* $f\colon (\mathbf{k}^2, 0) \to (\mathbf{k}, 0)$ *be a smooth germ with* $\partial f/\partial t(0) \neq 0$. *Then* f *is right equivalent to* t *via a diffeomorphism* $\phi\colon (\mathbf{k}^2, 0) \to (\mathbf{k}^2, 0)$ *which preserves* X.

One simply employs the arguments used in § 2 with Der (C/D) replaced by Derlog (X). As a consequence we can deduce that, given any germ $f\colon (\mathbf{k}^2, 0) \to (\mathbf{k}, 0)$ with $\partial f/\partial t(0) \neq 0$ arising in a geometrical application, there is a diffeomorphism of the swallowtail taking the characteristic through 0 to the standard one ($x_2 = x_3 = 0$), and nearby characteristics are smooth and meet the cuspidal edge and double line once. It is also not difficult to see that in this case the form of the characteristics along the cuspidal edge near the swallowtail point is as described in Theorem 2·4 (i). To prove this, we note that the normalization map is a local diffeomorphism except over the cuspidal edge of the swallowtail, that is on the locus $x_1 = -6t^2$. Along this locus the kernel of the normalization is spanned by the vector $\partial/\partial t$, as a short computation shows.

Since $\partial f/\partial t(0) \neq 0$, it follows that at points of the cuspidal locus near the swallowtail point the derivative of f is injective on the kernel of the normalization. Clearly if $\partial f/\partial t(0) \neq 0$ then the restriction of f to the cuspidal locus is also a submersion near the swallowtail point, and the result follows from the remarks following the proof of Theorem 2·4.

In Figure 2 (a) we illustrate the characteristic through 0 on the swallowtail, while in 2 (b) we have a typical foliation of the swallowtail in the generic case – only half of the characteristics are drawn in.

As previously mentioned, the condition $\partial f/\partial t(0) \neq 0$ holds automatically in all of the applications, so these results can be used to obtain geometrical information on the distribution of characteristics near any A_3 point of a generic envelope. Here is an example not quite in keeping with those previously discussed.

Example 4·2

Let $\gamma(s)$ be a space curve as before, and consider the family of height functions

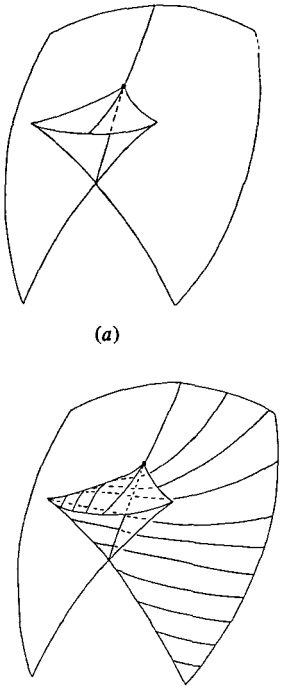
$$H\colon S^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

defined by $H(u, \alpha, s) = u.\gamma(s) - \alpha$, where S^2 is the unit sphere in $\mathbb{R}^3$. The germ $H_{(u_0, s_0, -)}\colon \mathbb{R} \to \mathbb{R}$ will have an A_3 singularity at s_0 if and only if $u_0 = \pm B(s_0)$, $\tau(s_0) = 0$ and $\dot{\tau}(s_0) \neq 0$. The set of points

$$\Delta = \{(u, \alpha) \in S^2 \times \mathbb{R}\colon \partial H/\partial s(u, \alpha, s) = H(u, \alpha, s) = 0\}$$

can be thought of as the dual of γ and near the point $(u_0, u_0.\gamma(s_0))$ this will be a swallow-

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(*a*)

(*b*)

Fig. 2

tail (H automatically versally unfolds A_3 singularities). For a fixed value of s the corresponding points of Δ are those of the form $(u, u.\gamma(s))$ where u is a unit normal vector to γ at $\gamma(s)$. These sets Δ_s then are smooth compact curves 'foliating' Δ. We now ask how these curves are distributed along Δ.

If $\Sigma = \{(u, \alpha, s)\colon H(u, \alpha, s) = \partial H/\partial S(u, \alpha, s) = 0\}$, then the projection of Σ to the (u, α)-space is the normalization of the discriminant Δ. Again, we consider the projection of Σ to the s-coordinate. Pulled back to the standard unfolding $t^4 + x_1 t^2 + x_2 t + x_3$ of an A_3 singularity, this projection, now f, will have $\partial f/\partial t(0) \neq 0$ and again we can deduce that the Δ_s are distributed around the swallowtail as in Diagram 2.

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