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RANDOM SPHERICAL DISC–POLYGONS AND A DUALITY

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Abstract

We consider the asymptotic behaviour of the expectation of the perimeter deviation of a uniform random spherical disc–polygon in a spherical spindle convex disc with smooth boundary. We also introduce the notion of duality on the sphere, define a model of random circumscribed disc–polygons, and determine some asymptotic results about them.

Keywords: Spindle convexity; expected area; spherical duality; random polygon

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1. Introduction

With their 1963 paper [17], Rényi and Sulanke launched the investigation of the now well-studied topic of random polytopes. For a convex body K in $\mathbb{R}^d$, the convex hull of n independently chosen points from K according to the uniform probability distribution is called a *uniform random polytope* in K. The questions of interest are mainly focused on the asymptotic behaviour of this random object, for example the expectation of the volume of K missed by the random polytope.

More recently, similar problems started to arise in different settings. Consider the following notion of convexity in the plane. Let $r > 0$ be fixed. The *r-spindle convex hull* of a set $H \subset \mathbb{R}^2$ is the intersection of all circular discs of radius r containing H, and we say that H is *r-spindle convex* if it coincides with its r-spindle convex hull. We refer to r as the *radius* of the spindle convexity. The spindle convex hull of finitely many points is called a *disc–polygon*, which can be obtained as the intersection of finitely many circular discs. In higher dimensions, the spindle convex hull of finitely many points is called a *ball–polytope*; however, in dimensions higher than two, it is no longer the case that this is the intersection of finitely many balls. The notion of spindle convexity, including its higher-dimensional form, was studied in [3]; see also [7]. Using this form of convexity, we can define a model similarly to the classical one. Consider the spindle convex hull K_n of finitely many points $X_n = \{P_1, \dots, P_n\}$ chosen uniformly and independently from a fixed spindle convex disc K. The goal is again to investigate the asymptotic properties of the random disc–polygon obtained in this way. In dimension two, such results on the expected number of vertices $f_0(K_n)$, the expectation of the missed area $\mathrm{Area}(K \setminus K_n)$, and the perimeter deviation $\mathrm{Per}(K) - \mathrm{Per}(K_n)$ were given in [6], in two different settings: either

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choosing points from an r_0-spindle convex disc with some value $0 < r_0 < r$, or from a circular disc of radius r. In the former, the expected number of vertices has order of magnitude $n^{1/3}$, similarly to the linear convex case, but in the latter, the expectation tends to a finite constant, which has no analogue in linear convexity. In [5], Fodor considered the expected number of facets in the higher-dimensional analogue of the above spindle convex model, and showed a similar distinction between the two choices of K. In dimension two, the series expansion of the expectation of the vertices and the missed area is also known [8].

In a recent paper by Marynych and Molchanov [13], it was shown that the f-vector of a uniform random r ball–polytope in a ball of radius r converges in distribution, and in moments of all orders, to the f-vector of the polar body of the zero cell of a stationary isotropic Poisson hyperplane tessellation, or equivalently, the convex hull of a Poisson point process having intensity measure with density proportional to $\|x\|^{-(d+1)}$ (see [13], Corollary 6.7). Specifically, as the expected number of facets of the convex hull of the above point process is known, it follows that the expected number of facets of a uniform random ball–polytope tends to a finite number, namely $2^{-d}d!\kappa_d^2$, where κ_d denotes the volume of the unit ball of $\mathbb{R}^d$. This result is a special case of a much broader theorem, which is set in a more general notion of strong convexity with respect to an arbitrary convex body.

Another approach to random polytopes was introduced by Bárány, Hug, Reitzner, and Schneider in [1]. They considered the spherical counterpart of the problems discussed previously: the n random points are chosen from a (closed) half-sphere of S^d, and a random spherical polytope is obtained by taking their spherical convex hull, i.e. the intersection of all (closed) half-spheres containing them. It was shown by the authors that the expected number of facets tends to a constant (dependent on the dimension d) as $n \to \infty$, and this constant coincides with the limit obtained in the corresponding-dimensional Euclidean spindle convex case.

Complementing this result, Besau, Ludwig, and Werner [2] investigated a spherical model in which the points are chosen uniformly from a spherical convex body that is contained in an open half-sphere of S^d. By using weighted floating bodies in Euclidean space, they were able to obtain asymptotic results for the expected volume and vertex number.

In [14], the authors introduced the notion of *spherical spindle convexity* (see also [4]) as an extension of these models in dimension two. Namely, for some set $X \subset S^2$ contained in a spherical circular disc of radius $0 < r \leq \pi/2$, its (closed) *spherical r-spindle convex hull*, denoted by $\mathrm{conv}_r(X)$, is the intersection of all closed spherical circular discs of radius r containing X, and X is spherically r-spindle convex if it coincides with $\mathrm{conv}_r(X)$. Similarly to the planar case, for a finite point set X, $\mathrm{conv}_r(X)$ is called a spherical disc–polygon, which is the intersection of finitely many spherical circular discs. We note that as a special case, $r = \pi/2$ gives us half-spheres and the usual notion of spherical convexity. The authors in [14] then determined the following result. Let K be a spherically convex disc with C^2 boundary and with the property that $\kappa_g(x) > \cot r$ for every $x \in \partial K$, where $\kappa_g(x)$ denotes the geodesic curvature of ∂K at boundary point x, and let K_n be a uniform spherical disc–polygon in K, i.e. the spherical spindle convex hull of n uniform independent and identically distributed points in K. Then

$$\lim_{n\to\infty} \mathbb{E}(\mathrm{SArea}(K \setminus K_n))n^{2/3} = \sqrt[3]{\frac{2A^2}{3}}\Gamma\left(\frac{5}{3}\right)\int_{\partial K} (\kappa_g - \cot r)^{1/3}\,\mathrm{d}s, \tag{1}$$

where A denotes the surface area $\mathrm{SArea}(K)$, $\Gamma(\cdot)$ is the Gamma function, the integration is taken with respect to arc length, and $\kappa_g = \kappa_g(x(s))$ if $x(s)$ is an arc-length parametrisation of ∂K. We note that $\cot r$ is the geodesic curvature of a circle of radius r on the sphere, and also that the

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curvature condition is equivalent to saying that K is r_0-spindle convex for some $0 < r_0 < r$. It was also determined in [14] that in the case when K is a spherical circular disc of radius r, the expectation of the missed surface area has an order of magnitude n^{-1}, and the expected number of vertices tends to the constant $\pi^2/2$ without normalisation.

As a comparison, the following is known about the analogous model in Euclidean space (see [6]). When the points are chosen from a circular disc of radius r, the expected number of vertices tends to $\pi^2/2$, coinciding with the spherical results. In the C^2 case, under the boundary condition that the curvature $\kappa = \kappa(x)$ satisfies $\kappa(x) > 1/r$ for all $x \in \partial K$, it holds that

$$\lim_{n\to\infty} \mathbb{E}(\mathrm{Area}(K \setminus K_n))n^{2/3} = \sqrt[3]{\frac{2A^2}{3}}\Gamma\left(\frac{5}{3}\right)\int_{\partial K}\left(\kappa - \frac{1}{r}\right)^{1/3} \mathrm{d}s,$$

where again $\kappa = \kappa(x(s))$ if $x(s)$ is an arc-length parametrisation of ∂K. Compared to the formula on the sphere, instead of the geodesic curvature deviation of ∂K and the boundary of a circular disc, the curvature deviation occurs. Due to the embedding, it seems natural that in the spherical case, the geodesic curvature appears instead of the curvature itself, as the geodesic curvature is essentially the curvature of the curve viewed from the embedding surface S^2.

In this work, we first determine a similar result concerning the perimeter deviation in the uniform spherical spindle convex model.

Theorem 1. *Let $K \subset S^2$ be a spherical r-convex disc with C^5 boundary and with the property that $\kappa_g(x) > \cot r$ for every $x \in \partial K$, where $\kappa_g(x)$ denotes the geodesic curvature of ∂K at the point x. Then*

$$\lim_{n\to\infty} \mathbb{E}(\mathrm{Per}(K) - \mathrm{Per}(K_n))n^{2/3} = \sqrt[3]{\frac{2A^2}{3}}\Gamma\left(\frac{5}{3}\right)\int_{\partial K}(\kappa_g - \cot r)^{1/3}\left(\frac{3\kappa_g + \cot r}{4}\right) \mathrm{d}s,$$

where A denotes $\mathrm{SArea}(K)$, *$\Gamma(\cdot)$ is the Gamma function, and the integration is taken with respect to the arc length, with κ_g being a function of s, namely $\kappa_g = \kappa_g(x(s))$ if x is the arc-length parametrisation of the boundary.*

Similarly to the case of the surface area, if the points are chosen from a spherical circular disc of radius r, we obtain a different order of magnitude.

Theorem 2. *Let $K \subset S^2$ be a spherical circular disc of radius r. Then*

$$\mathbb{E}(\mathrm{Per}(K) - \mathrm{Per}(K_n)) = n^{-1} \cdot \pi^3(1 - \cos r)\cot r + n^{-2} \cdot 2\pi^3\frac{(1 - \cos r)^2}{\sin r}(1 - \cot^2 r) + o(n^{-2}) \quad \text{as } n \to \infty.$$

Note that this means that for $0 < r < \pi/2$, the expectation of the perimeter deviation is of order of magnitude n^{-1}. However, when $r = \pi/2$, i.e. we are in the case of standard spherical convexity, the coefficient of n^{-1} is 0, and hence the order of magnitude is actually n^{-2}. These results again complement the corresponding results in Euclidean spindle convexity (see [6]) well; furthermore, the case $r = \pi/2$ gives back the results obtained in the usual notion of spherical convexity [1].

In addition to the results on perimeter deviation, we also introduce a notion of duality, which is the spherical counterpart of the planar spindle convex duality defined in [7] (see also [3]).

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For a set $H \subset S^2$, we define its *spherical r-convex dual* as

$$H^r = \{y \in S^2 \mid H \subseteq B(y, r)\} = \bigcap_{x \in H} B(x, r),$$

where $B(x, r)$ is the spherical circular disc with centre x and radius r. This notion will be discussed further in Section 4; we only note here that the transformation is inclusion-reversing, and that $(H^r)^r$ is the spherical spindle convex hull of H. The former is clear from the definition, and the latter holds since $(H^r)^r = \bigcap_{y \in H^r} B(y, r) = \bigcap_{y: H \subseteq B(y,r)} B(y, r)$. Using these properties, we have a natural way to define a model for circumscribed disc–polygons. For a spherically spindle convex disc K, let $X_n = \{P_1, \dots, P_n\}$ be an independently chosen set of points from K^r according to the uniform probability distribution. Then a random circumscribed disc–polygon $K_{(n)}$ can be defined as the dual of $\mathrm{conv}_r(X_n)$; this does in fact contain $K = (K^r)^r$, by $\mathrm{conv}_r(X_n) \subseteq K^r$ and the inclusion-reversing property. This is the spherical analogue of the model defined in [9, Section 5]. In this model of random circumscribed spherical disc–polygons, we obtain the following results.

Theorem 3. *Let $K \subset S^2$ be a spherical r-convex disc with C^5 boundary and with the property that $\kappa_g(x) > \cot r$ for every $x \in \partial K$. With the notation above,*

$$\lim_{n\to\infty} \mathbb{E}(\mathrm{SArea}(K_{(n)} \setminus K))n^{2/3}$$
$$= \Gamma\left(\frac{5}{3}\right)(\sin r)^{-2/3} \sqrt[3]{\frac{2((1-\cos r)\cdot 2\pi - \sin r \cdot L + \cos r \cdot A)^2}{3}} \cdot \frac{3}{4} \int_{\partial K} \Delta\kappa_g^{-1/3}\, \mathrm{d}s,$$

$$\lim_{n\to\infty} \mathbb{E}(\mathrm{Per}(K_{(n)}) - \mathrm{Per}(K))n^{2/3}$$
$$= \Gamma\left(\frac{5}{3}\right)(\sin r)^{-2/3} \sqrt[3]{\frac{2((1-\cos r)\cdot 2\pi - \sin r \cdot L + \cos r \cdot A)^2}{3}} \int_{\partial K} \Delta\kappa_g^{-1/3}\left(\kappa_g - \frac{\cot r}{4}\right) \mathrm{d}s,$$

where L and A denote $\mathrm{Per}(K)$ *and* $\mathrm{SArea}(K)$*, respectively, $\Gamma(\cdot)$ is the Gamma function, $\Delta\kappa_g = \kappa_g - \cot r$, and the integration is taken with respect to the arc length, with κ_g being a function of s, namely $\kappa_g = \kappa_g(x(s))$ if x is the arc-length parametrisation of the boundary.*

The paper is structured as follows. In Section 2, we state and prove some geometric lemmas necessary for the proof of Theorems 1 and 2, which are discussed in Section 3. In Section 4, we define the spherical spindle convex dual of a set, and consider the relationship between a set and its dual. Finally, in Section 5, we show the assertions of Theorem 3.

2. Geometric tools

In the section, we introduce some terminology, and prove some lemmas used in the proofs of Theorems 1 and 2.

Let $0 < r \leq \pi/2$ denote the radius of spindle convexity. As r is fixed throughout the paper, we will generally omit it from the notation. By the distance of two points on S^2, we will generally mean their spherical distance. The set $B \subset S^2$ is a *spherical circular disc* of radius r if it is the set of points that has distance at most r from a fixed point of S^2, called the centre of the disc. The boundary of B is then the intersection of S^2 with a plane that has distance $\cos r$ from the origin, and is hence a Euclidean circle of radius $\sin r$. If a set $K \subset S^2$ is spherically

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r-spindle convex – that is, it coincides with its spherical r-spindle convex hull – we will refer to it as a *spherical r-convex disc* or *spherical spindle convex disc*; we note that we emphasize the word *circular* in the previous definition to make the distinction between the two clear.

A subset D of a spherical spindle convex disc K is called a *spherical disc-cap* if $D = \mathrm{cl}\,(K \cap B^{\mathrm{c}})$ for some spherical circular disc B of radius r, where $\mathrm{cl}(\cdot)$ and $\cdot^{\mathrm{c}}$ denote the closure and complement of a set, respectively. We note that the disc-cap D is bounded by a segment of ∂K, and a segment of ∂B. As discussed above, while the latter is obtained from a spherical circular disc, it is also a Euclidean circular arc, which has (Euclidean) radius $\sin r$, and some central angle measured in the plane in which the arc lies.

We use the usual o and O notation: $g = o(h)$ as $x \to x_0$ if $\lim_{x\to x_0} g(x)/h(x) = 0$; $g = O(h)$ if $\limsup_{x\to x_0} g(x)/h(x) < \infty$. We also write $g \sim h$ as $x \to x_0$ when $g = g(x)$ and $h = h(x)$ are asymptotically equal, i.e. $g(x)/h(x) \to 1$ as $x \to x_0$.

We first collect some definitions and results from differential geometry.

Lemma 1. *Let $\varrho = \varrho(s)$ be a positively oriented arc-length parametrisation of a twice continuously differentiable curve embedded in S^2. We denote by $\kappa(s)$, $\kappa_{\mathrm{g}}(s)$, and $\kappa_{\mathrm{n}}(s)$ the curvature, geodesic curvature, and normal curvature of the curve at $\varrho(s)$, respectively. Then, for all s,*

(i) $\kappa_{\mathrm{n}}(s) \equiv 1$ *and* $\kappa_{\mathrm{g}}^2(s) + 1 = \kappa^2(s)$;

(ii) $\ddot{\varrho}(s) = \kappa_{\mathrm{g}}(s) \cdot \varrho(s) \times \dot{\varrho}(s) - \varrho(s)$.

Proof. First, it is known that all curves on a surface passing through a given point and having the same tangent line at that point have the same normal curvature (a formulation of Meusnier's theorem; see, for example, [16]). This implies that, for any point $\varrho(s)$, the normal curvature is equal to the normal curvature of a great circle, implying that $\kappa_{\mathrm{n}} \equiv 1$. The second part of (i) follows, since $\kappa_{\mathrm{g}}(s)^2 + \kappa_{\mathrm{n}}(s)^2 = \kappa(s)^2$.

To prove (ii), we can use the so-called *Darboux frame* of the curve. We define the Darboux frame of an arc-length parametrised curve embedded in an orientable surface as follows. For an abritrary point on the curve, the frame consists of the three vectors T, V, and U, where T is the tangent vector of the curve, V is the unit normal vector of the embedding surface, and $U = T \times V$. Clearly, this triad forms an orthonormal basis. It is then known that $\dot{T} = -\kappa_{\mathrm{g}} \cdot U - \kappa_{\mathrm{n}} \cdot V$ (see [16]).

We note that changing the orientation of the parametrisation, or choosing the opposite normal vector, changes the signs in the formula. In our case, since we chose ϱ to be positively oriented, this version of the formula holds if V is the outer normal of the sphere. Thus we have that $V = \varrho$, $T = \dot{\varrho}$, and $U = \dot{\varrho} \times \varrho = -\varrho \times \dot{\varrho}$. Using $\kappa_{\mathrm{n}} \equiv 1$, this implies the formula for $\dot{T} = \ddot{\varrho}$. □

To state the second lemma, regarding disc-caps of spindle convex sets, we recall the following. For a differentiable curve $\gamma \subset S^2$, its *normal plane* at a point $x_0 \in \gamma$ is defined as the plane that contains x_0 and is perpendicular to the tangent vector of γ at x_0. Since we know that the vector pointing from the origin to x_0 is perpendicular to the tangent plane of the sphere at x_0, it is also perpendicular to the tangent vector of γ at x_0. This implies that the normal plane contains the origin, and hence intersects the sphere in a great circle.

Lemma 2. *([14], Lemma 2.) Let $K \subset S^2$ be a spherical r-convex disc with C_+^2 boundary, and let $D = \mathrm{cl}\,(K \cap B^{\mathrm{c}})$ be a nonempty spherical disc-cap. Then there is a unique point x_0 of ∂K and a nonnegative real number t such that the spherical centre of B is at a distance $r + t$ from x_0 along the great circle determined by the normal plane of ∂K at $x_0 \in \partial K$.*

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We refer to x_0 as the *vertex*, and to t as the *height* of the cap.

Lemma 3. *Let K be a spherical r-convex disc with C^5 boundary and with the property that $\kappa_g(x) > \cot r$ for every $x \in \partial K$, and let ϱ be an arc-length parametrisation of ∂K. We denote by $A(t, s_0)$ the area of the spherical disc-cap with vertex $\varrho(s_0)$ and height t, and by $\Delta\theta(t, s_0)$ the central angle of the circular arc that determines this disc-cap. Then*

$$\Delta\theta(t, s_0) = \vartheta_1 t^{1/2} + \vartheta_2 t^{3/2} + O(t^2), \quad A(t, s_0) = v_1 t^{3/2} + v_2 t^{5/2} + O(t^{7/2}),$$

where the coefficients are dependent on s_0, namely

$$\vartheta_1 = \frac{2}{\sin r}\sqrt{\frac{2}{\Delta\kappa_g}}, \qquad v_1 = \frac{4}{3}\sqrt{\frac{2}{\Delta\kappa_g}},$$

$$\vartheta_2 = \sqrt{\frac{2}{\Delta\kappa_g}}\,\frac{-\frac{1}{2}\Delta\kappa_g^4 - \frac{3}{2}\cot r\cdot(\Delta\kappa_g)^3 - \frac{5}{6}(1+\cot^2 r)(\Delta\kappa_g)^2 - \frac{1}{6}\kappa_g''\Delta\kappa_g + \frac{5}{18}(\kappa_g')^2}{\sin r\cdot(\Delta\kappa_g)^3},$$

$$v_2 = \sqrt{\frac{2}{\Delta\kappa_g}}\,\frac{-\frac{1}{5}\Delta\kappa_g^4 - \frac{3}{5}\cot r\cdot(\Delta\kappa_g)^3 - \frac{3}{5}(1+\cot^2 r)(\Delta\kappa_g)^2 - \frac{1}{15}\kappa_g''\Delta\kappa_g + \frac{1}{9}(\kappa_g')^2}{(\Delta\kappa_g)^3},$$

where $\Delta\kappa_g = \Delta\kappa_g(s_0) = \kappa_g(s_0) - \cot r$ and $\kappa_g' = \kappa_g'(s_0)$.

Proof. We may assume without loss of generality that ϱ is a parametrisation with positive orientation, $s_0 = 0$, $\varrho(0) = (0, 0, 1)$, and $\dot\varrho(0) = (1, 0, 0)$; hence, $\varrho(0) \times \dot\varrho(0) = (0, 1, 0)$. Using Lemma 1, we can express the higher derivatives of $\varrho(s)$ in terms of the orthonormal basis $\varrho(s)$, $\dot\varrho(s)$, and $\varrho(s) \times \dot\varrho(s)$. By the smoothness condition on the boundary, this leads us to obtain the fourth-order series expression for $\varrho(s)$ around $\varrho(0)$:

$$\begin{aligned}\varrho(s) = \Big(& s - \frac{s^3}{6}\big(\kappa_g^2 + 1\big) - \frac{s^4}{8}\kappa_g\kappa_g' + O(s^5),\\ & \frac{s^2}{2}\kappa_g + \frac{s^3}{6}\kappa_g' + \frac{s^4}{24}\big(\kappa_g'' - \kappa_g - \kappa_g^3\big) + O(s^5),\\ & 1 - \frac{s^2}{2} + \frac{s^4}{24}\big(\kappa_g^2 + 1\big) + O(s^5)\Big).\end{aligned} \tag{2}$$

Now, let $C = C(t)$ be the spherical circular disc that determines the spherical disc-cap with vertex $\varrho(0)$ and height t. Then, by Lemma 2, the centre of C is at distance $r + t$ from the origin along the great circle $S^2 \cap \{x = 0\}$, implying that C has centre $(0, \sin(r+t), \cos(r+t))$. This vector forms an orthonormal basis together with the vectors $(1, 0, 0)$ and $(0, -\cos(r+t), \sin(r+t))$, thus ∂C can be parametrised as

$$\begin{aligned}\big(& \sin r\cos\varphi,\\ & \cos r\sin(r+t) + \sin r\cos(r+t)\sin\varphi,\\ & \cos r\cos(r+t) - \sin r\sin(r+t)\sin\varphi\big), \quad \varphi \in [0, 2\pi],\end{aligned} \tag{3}$$

since the Euclidean radius of ∂C is $\sin r$, and its centre has distance $\cos r$ from the origin. Note that the closest point of ∂C to $\varrho(0)$ is at the parameter value $\varphi = 3\pi/2$; hence, for small enough t, the arc of ∂C that is contained in K is given by parameter values in $[\pi, 2\pi]$.

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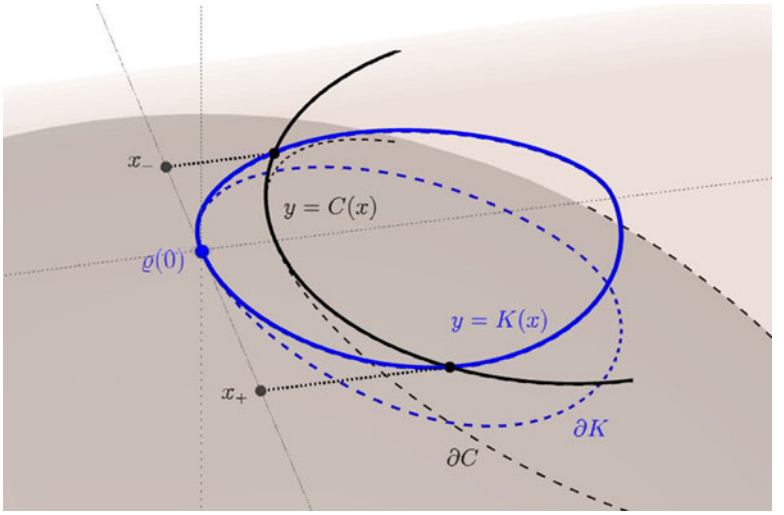


FIGURE 1. Projection of a disc-cap to the tangent plane.

Our first goal is to express the intersection points of ∂K and ∂C in terms of t. Let x_- and x_+ denote the x-coordinates of the points of intersection.

Consider the orthogonal projection of the construction to the tangent plane $z = 1$ to the sphere at $\varrho(0)$, or equivalently, the xy plane, see Figure 1. The projection of ∂K is the graph of the function

$$K(x) := \frac{x^2}{2}\kappa_g + \frac{x^3}{6}\kappa_g' + \frac{x^4}{24}\left(\kappa_g'' + 3\kappa_g + 3\kappa_g^3\right) + f(x)$$

in the xy plane near the origin, where $f(x) = O(x^5)$ as $x \to 0$. This can be obtained from the series expansion (2) of $\varrho(s)$ as follows. Let x denote the first coordinate of the parametrisation. Then the powers of x, and thus any polynomial expression of x, can be expressed in terms of s (up to some $O(s^m)$ term), and matching the coefficients to the second coordinate of the expansion yields the expression for $K(x)$. The necessary arc of the projection of ∂C is described for $x \le \sin r$ by the function

$$C(x) := \cos r \sin(r + t) - \cos(r + t)\sqrt{\sin^2 r - x^2}$$

for arbitrary t when $r = \pi/2$, and for $t < \pi/2 - r$ when $r < \pi/2$. This can be assumed since we are interested in the behaviour as $t \to 0$.

The equality $K(x) = C(x)$ gives us an implicit relationship between t and x at the points of intersection. We use implicit differentiation (see [12]) to derive the fourth-order Taylor expansion of t around 0. This can be done as follows. Introduce the notation $F(x, t) := K(x) - C(x)$,

$$F_1(x, t) := -\frac{\partial F(x, t)/\partial x}{\partial F(x, t)/\partial t},$$

and, recursively,

$$F_k(x, t) := \frac{\partial F_{k-1}(x, t)}{\partial x} + \frac{\partial F_{k-1}(x, t)}{\partial t} \cdot F_1(x, t)$$

for $k \ge 2$. Then, applying the two-dimensional chain rule to $F(x,t(x))$, where we now consider $t = t(x)$ as a function of x, we have $t'(x) := \mathrm{d}t(x)/\mathrm{d}x = F_1(x, t(x))$; further, by differentiating

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$F_{k-1}(x, t(x))$, we obtain

$$F_k(x, t(x)) = \frac{\partial F_{k-1}(x, t(x))}{\partial x}.$$

Thus, the kth derivative of $t = t(x)$ (with respect to x) coincides with $F_k(x, t(x))$. This gives us a method to derive the fourth-order Taylor expansion of t around 0. Since F is polynomial in x and trigonometric in t, this is a lengthy but straightforward computation. As such, we chose not to include the steps here explicitly; however, to ensure the correctness of the calculations, and make the subsequent formula verifiable, we refer to the supplementary computational script [15]. Applying this method yields that at the points of intersection,

$$t = \frac{x^2}{2}(\kappa_g - \cot r) + \frac{x^3}{6}\kappa_g' + \frac{x^4}{24}\left(\kappa_g'' + 3\kappa_g^3 + 9\kappa_g - 3\cot^3 r - 9\cot r\right) + O(x^5)$$

as $x \to 0$. Note that since ∂K is of class C^5, the function $K(x)$, and consequently the remainder term $f(x)$, is four times differentiable. If a function is $O(x^n)$ and is differentiable at 0, its derivative is $O(x^{n-1})$ around zero, which in our case implies that the first four derivatives of $f(x)$ equal 0 at $x = 0$. Since the implicit differentiation used above requires at most the fourth derivative of $K(x)$ at point $x = 0$, the term $f(x)$ does not affect the results.

By introducing the notation $\Delta\kappa_g = \kappa_g - \cot r$, the above formula can be reformulated as

$$t = \frac{x^2}{2}\Delta\kappa_g + \frac{x^3}{6}\kappa_g' + \frac{x^4}{24}(\kappa_g'' + 3(\Delta\kappa_g)^3 + 9\cot r(\Delta\kappa_g)^2 + 9(1+\cot^2 r)\Delta\kappa_g) + O(x^5).$$

We aim now to obtain x as a function of t. There are two solutions around $x = 0$, $x_- < 0 < x_+$, obtained by inverting the function on the negative and positive domains, respectively. To do this, we use [8, Lemma 2] (see also [11, Lemma 2] in a more general form). By the assumptions, $\Delta\kappa_g > 0$, hence the function $t(x)$ is strictly increasing in a small enough neighbourhood $(0, \varepsilon)$, and the above lemma can be used. The inverse on the negative domain can be obtained by inverting the function $t(-x)$ on $x > 0$, which by the same argument is strictly increasing in an appropriate interval there, and taking its negative. With this approach, we get

$$x_+ = C_1 t^{1/2} + C_2 t + C_3 t^{3/2} + O(t^2), \qquad x_- = -C_1 t^{1/2} + C_2 t - C_3 t^{3/2} + O(t^2),$$

where

$$C_1 = \sqrt{\frac{2}{\Delta}\kappa_g}, \qquad C_2 = \frac{-\kappa_g'}{3(\Delta\kappa_g)^2},$$
$$C_3 = \sqrt{\frac{2}{\Delta}\kappa_g}\,\frac{-\frac{1}{4}\Delta\kappa_g^4 - \frac{3}{4}\cot r\cdot(\Delta\kappa_g)^3 - \frac{3}{4}(1+\cot^2 r)(\Delta\kappa_g)^2 - \frac{1}{12}\kappa_g''\Delta\kappa_g + \frac{5}{36}(\kappa_g')^2}{(\Delta\kappa_g)^3}.$$

Now, since the intersection points lie on ∂C, by (3) we have $\sin r \cos\theta_+ = x_+$ for some parameter value $\theta_+ \in [0, 2\pi]$. As noted before, we know that the segment of ∂C contained in K is obtained for parameter values in $[\pi, 2\pi]$ for small enough t. Hence, we obtain

$$\theta_+ = \frac{3\pi}{2} + \arccos\left(\frac{x_+}{\sin r}\right) = \frac{3\pi}{2} + \frac{x_+}{\sin r} + \frac{x_+^3}{6\sin^3 r} + O(x_+^5)$$
$$= \frac{3\pi}{2} + \frac{C_1}{\sin r}t^{1/2} + \frac{C_2}{\sin r}t + \left(\frac{C_3}{\sin r} + \frac{C_1^3}{6\sin^3 r}\right)t^{3/2} + O(t^2),$$

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where we used the expansion of the function $\arccos(x)$ around $x = 0$. Using a similar argument for the solution to $\sin r \cos\theta_- = x_-$, we obtain

$$\Delta\theta := \theta_+ - \theta_- = \frac{2C_1}{\sin r} t^{1/2} + \frac{2}{\sin r}\left(C_3 + \frac{C_1^3}{6\sin^2 r}\right) t^{3/2} + O(t^2),$$

which yields the statement of the lemma after some algebraic manipulations.

The surface area can be computed via the integral

$$\int_{x_-}^{x_+}\int_{K(x)}^{C(x)} \frac{1}{\sqrt{1-x^2-y^2}}\,\mathrm{d}y\,\mathrm{d}x = \int_{x_-}^{x_+}\int_{K(x)}^{C(x)} \left(1 + \frac{x^2}{2} + \frac{3x^4}{8} + h(x) + \frac{y^2}{2} + g(y)\right)\mathrm{d}y\,\mathrm{d}x,$$

where $h(x) = O(x^6)$ and $g(y) = O(y^4)$. We will also use an expansion of $C(x)$, namely

$$C(x) = t + O(t^3) + \frac{x^2}{2}(\cot r - t + O(t^2)) + \frac{x^4}{8}(\cot r + O(t)) + O(x^6).$$

Clearly, $K(x)$ and $C(x)$ are positive for all values of $x \in [x_-, x_+]$ by construction. For small enough t, and some $M > 0$, it also holds that

$$\begin{aligned} x_+ &< Mt^{1/2}, & C(x) &< Mt, & |h(x)| &< Mx^6, \\ x_- &> -Mt^{1/2}, & K(x) &< Mx^2, & |g(y)| &\le My^4 \end{aligned}$$

for the values of x and y that are in the examined interval for that value of t. Then,

$$\left|\int_{x_-}^{x_+}\int_{K(x)}^{C(x)} h(x)\,\mathrm{d}y\,\mathrm{d}x\right| < \int_{x_-}^{x_+}\int_{0}^{C(x)} |h(x)|\,\mathrm{d}y\,\mathrm{d}x < \int_{x_-}^{x_+} Mt\cdot Mx^6\,\mathrm{d}x = \frac{M^2}{7}\cdot t\left(x_+^7 - x_-^7\right) < \frac{2M^9}{7}t^{9/2}.$$

Similarly,

$$\left|\int_{x_-}^{x_+}\int_{K(x)}^{C(x)} g(y)\,\mathrm{d}y\,\mathrm{d}x\right| < \int_{x_-}^{x_+}\int_{0}^{C(x)} |g(y)|\,\mathrm{d}y\,\mathrm{d}x < \frac{M}{5}\int_{x_-}^{x_+} C^5(x)\,\mathrm{d}x < \frac{M^6}{5}t^5(x_+ - x_-) < \frac{2M^7}{5}\cdot t^{11/2}.$$

Hence, the integrals of $h(x)$ and $g(y)$ carry no significant terms.

After integrating with respect to y, we obtain that the surface area is equal to

$$\begin{aligned} &\int_{x_-}^{x_+} t + O(t^3) - \frac{x^2}{2}(\Delta\kappa_g + O(t^2)) - \frac{x^3}{6}\kappa_g' \\ &\quad + \frac{x^4}{24}\left(\kappa_g'' + 3(\Delta\kappa_g)^3 + 9\cot r(\Delta\kappa_g)^2 + 9\Delta\kappa_g(1+\cot^2 r) + O(t^2)\right)\mathrm{d}x + O(t^{7/2}). \end{aligned}$$

Using a similar argument as before, the integral of $O(t^3)$ and $x^2O(t^2)$ is $O(t^{7/2})$, and the integral of $x^4O(t^2)$ is $O(t^{9/2})$. Thus, the terms not explicitly determined are not significant, and the remaining are.

As a consequence, it is enough to compute the polynomial integral

$$\int_{x_-}^{x_+} t - \frac{x^2}{2}\Delta\kappa_g - \frac{x^3}{6}\kappa_g' + \frac{x^4}{24}\left(\kappa_g'' + 3(\Delta\kappa_g)^3 + 9\cot r(\Delta\kappa_g)^2 + 9\Delta\kappa_g(1+\cot^2 r)\right)\mathrm{d}x.$$

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This yields the desired result after a long but straightforward calculation, which can be verified in the supplementary file [15]. □

For the proof of Theorem 2, we need a similar geometric result in the case when K is a spherical circular disc of radius r. In that case, a disc-cap of height t is the complement of the intersection of two circular discs of radius r whose spherical centres are at distance t, or equivalently their set-theoretical difference. Clearly, the surface area and central angle do not depend on the vertex of the disc-cap, hence we reduce the above notation to $A(t)$ for the surface area and $\Delta\theta(t)$ for the central angle. By [14, Lemma 6], the corresponding central angle is

$$\Delta\theta(t) = 2\arccos\left(\frac{\sin t}{1+\cos t}\cdot\cot r\right).$$

For the area, we need an improvement on [14, Lemma 5].

Lemma 4. *Let $A(t)$ denote the surface area of the set-theoretical difference of two spherical circular discs of radius r whose centres are at distance t. Then*

$$A(t) = 2\left(2\arcsin\left(\frac{\sin(t/2)}{\sin r}\right) + 2\cos r\arccos\left(\frac{\tan(t/2)}{\tan r}\right) - \pi\cos r\right).$$

Moreover, the third-order expansion of $A(t)$ around $t=0$ is

$$A(t) = 2\sin r\cdot t - \frac{1}{12}\cos r\cot r\cdot t^3 + O(t^5).$$

Proof. We determine the surface area of the intersection of two such discs, which yields the statement when subtracted from $\mathrm{SArea}(K) = 2\pi(1-\cos r)$.

Let A be the centre of one of the circular discs, B and C the two intersection points of their boundaries, and D the midpoint of the segment connecting B and C, which coincides with the midpoint of the segment connecting the two centres; see Figure 2. By construction, $d(A,B) = r$, $d(A,D) = t/2$, and the angle ADB is $\pi/2$. Let β and α denote the angles DBA and BAC, respectively, and let $a = d(B,D)$. We use spherical trigonometry in the triangle ABD. By the spherical sine theorem, $\sin\beta = \sin(t/2)/\sin r$. By the spherical cosine theorem used for the side AB, $\cos r = \cos a\cos(t/2)$, and for the side BD,

$$\cos(\alpha/2) = \frac{\cos a - \cos r\cos(t/2)}{\sin r\sin(t/2)} = \frac{\tan(t/2)}{\tan r}.$$

The surface area of half of the intersection is then obtained by subtracting the area of the triangle from the area of the circular sector, $\alpha\cdot(1-\cos r) - (\alpha + 2\beta - \pi)$, which yields the assertions after substituting in the above formulas obtained for α and β. □

3. Proofs of Theorems 1 and 2

In the proof of Theorem 1, we closely follow the proof of the corresponding planar result in [6, Theorem 1.2], whose argument is based on the technique used in [17].

Proof of Theorem 1. Let P_1 and P_2 be two distinct points in K. There are exactly two spherical circular discs of radius r that contain both points in their boundaries. Let $D^-(P_1,P_2)$ and $D^+(P_1,P_2)$ denote the spherical disc-caps determined by these circular discs, i.e. the closure

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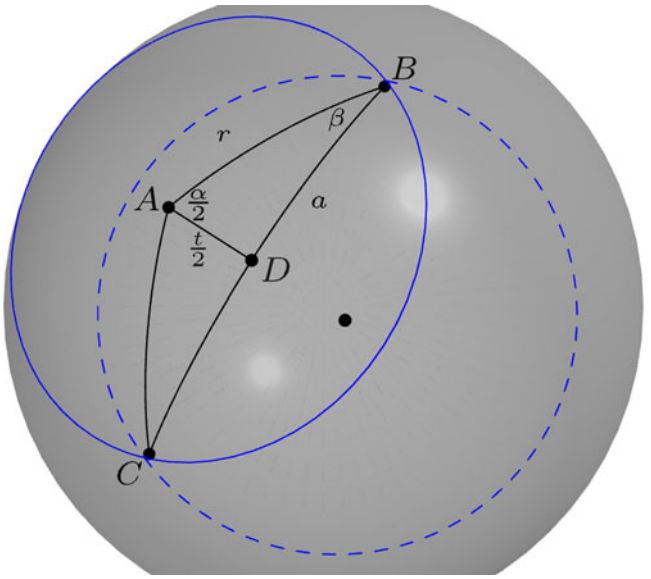


FIGURE 2. The surface area of the intersection of two spherical circular discs.

of the subset of K that is not covered by the corresponding disc. We also introduce the notation $A^-(P_1, P_2) = A(D^-(P_1, P_2))$ and $A^+(P_1, P_2) = A(D^+(P_1, P_2))$. Furthermore, let $i(P_1, P_2)$ denote the length of the shorter r-arc connecting P_1 and P_2.

The random pair of points $P_1, P_2 \in K$ determines an edge of K_n if and only if $D^-(P_1, P_2)$ or $D^+(P_1, P_2)$ don't contain more of the chosen n points. Hence, our goal is to determine $\mathbb{E}(\operatorname{Per}(K) - \operatorname{Per}(K_n)) = \operatorname{Per}(K) - \binom{n}{2} \cdot I_n$, where

$$I_n = \frac{1}{A^2} \int_K \int_K \left(\left(1 - \frac{A^-(P_1, P_2)}{A} \right)^{n-2} + \left(1 - \frac{A^+(P_1, P_2)}{A} \right)^{n-2} \right) i(P_1, P_2) \, \mathrm{d}P_1 \, \mathrm{d}P_2.$$

To compute this integral, we carry out the integral transformation in [14, Section 3], which can be described as follows. Let $\varrho = \varrho(s)$ be an arc-length parametrisation of ∂K with positive orientation, and let $C = C(t, s) \in S^2$ be the point at distance $r + t$ from $\varrho(s)$ in the direction of $\varrho(s) \times \dot{\varrho}(s)$ along the great circle determined by the normal plane of ∂K at $\varrho(s)$. The normal plane is then the plane spanned by $\varrho(s)$ and $\varrho(s) \times \dot{\varrho}(s)$, implying that

$$C = C(t, s) = \cos(r + t) \cdot \varrho(s) + \sin(r + t) \cdot \varrho(s) \times \dot{\varrho}(s).$$

Let $K_{t,s}$ denote the spherical circular disc with centre $C(t, s)$ and radius r.

An orthonormal basis in the orthogonal complement of C is given by

$$w_1(t, s) := \dot{\varrho}(s), \qquad w_2(t, s) := -\sin(r + t) \cdot \varrho(s) + \cos(r + t) \cdot \varrho(s) \times \dot{\varrho}(s).$$

Any pair of points (P_1, P_2) in K determines exactly two disc-caps, and by Lemma 2 each has a unique vertex $\varrho(s)$ and height t. Hence, the points can be reparametrised as

$$P_i = \cos r \cdot C(t, s) + \sin r \cdot [\cos(\varphi_i) \cdot w_1(t, s) + \sin(\varphi_i) \cdot w_2(t, s)]$$

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for some parameters $\varphi_1, \varphi_2 \in [0, 2\pi]$. In general, this transformation is twofold, but because of the properties of the specific integrand here, every cap gets counted exactly twice.

Note that in this parametrisation, the area of the disc-cap is only dependent on t and s, and will be denoted by $A(t, s)$, which is exactly the quantity defined in Lemma 3. Moreover, the circular arc connecting P_1 and P_2 has central angle $|\varphi_1 - \varphi_2|$ in the plane in which it lies, and hence $i(P_1, P_2) = \sin r \cdot |\varphi_1 - \varphi_2|$.

Let $t(s)$ denote the greatest value of t for which $K \cap K_{t,s}$ is nonempty, and let $[\theta_1, \theta_2]$ be the interval (depending on t and s) parametrising $\partial K_{t,s} \cap K$. Then, by [14, Lemma 3], carrying out the above transformation yields

$$I_n = \frac{1}{A^2} \int_{\partial K} \int_0^{t(s)} \int_{\theta_1}^{\theta_2} \int_{\theta_1}^{\theta_2} \left(1 - \frac{A(t,s)}{A}\right)^{n-2} \sin r |\varphi_1 - \varphi_2| \\ \times \sin^2 r (\sin(r+t)\kappa_g - \cos(r+t)) |\sin(\varphi_1 - \varphi_2)| \, \mathrm{d}\varphi_1 \, \mathrm{d}\varphi_2 \, \mathrm{d}t \, \mathrm{d}s.$$

Note that $\kappa_g = \kappa_g(\varrho(s))$ is a function of s; however, we omit this from the notation to be as concise as possible. After integrating with respect to φ_1 and φ_2, we obtain

$$I_n = \frac{\sin^3 r}{A^2} \int_{\partial K} \int_0^{t(s)} \left(1 - \frac{A(t,s)}{A}\right)^{n-2} (\sin(r+t)\kappa_g - \cos(r+t)) \\ \times 2(2 - \Delta\theta \sin \Delta\theta - 2\cos \Delta\theta) \, \mathrm{d}t \, \mathrm{d}s,$$

where $\Delta\theta := \theta_2 - \theta_1$. This corresponds to the central angle $\Delta\theta = \Delta\theta(t, s)$ defined in Lemma 3.

We first show that I_n is negligible on the interval $t \geq t'$ for arbitrary fixed $t' > 0$. It is clear that, for fixed s, the function $A(t, s)$ is increasing in t; furthermore, it follows from a compactness argument that $\alpha := \min_s A(t', s) > 0$, implying that $A(t, s) \geq \alpha$ for all s and $t \geq t'$. Hence, the exponential term of the integrand can be upper bounded by $(1 - \alpha/A)^{n-2}$. The other terms can be bounded by a universal constant, again by a compactness argument, thus

$$n^{2/3} \binom{n}{2} \frac{\sin^3 r}{A^2} \int_{\partial K} \int_{t'}^{t(s)} \left(1 - \frac{A(t,s)}{A}\right)^{n-2} (\sin(r+t)\kappa_g(s) - \cos(r+t)) \\ \times 2(2 - \Delta\theta \sin \Delta\theta - 2\cos \Delta\theta) \, \mathrm{d}t \, \mathrm{d}s \\ \leq C n^{2/3} \binom{n}{2} \left(1 - \frac{\alpha}{A}\right)^{n-2} \to 0 \quad \text{as } n \to \infty,$$

where C is some absolute constant. As a consequence, we aim to determine

$$\lim_{n\to\infty} \mathbb{E}(\operatorname{Per}(K) - \operatorname{Per}(K_n)) n^{2/3} = \lim_{n\to\infty} \left(\operatorname{Per}(K) - \binom{n}{2} \int_{\partial K} \hat{I}_n(s) \, \mathrm{d}s\right) n^{2/3},$$

where

$$\hat{I}_n(s) := \frac{\sin^3 r}{A^2} \int_0^{t'} \left(1 - \frac{A(t,s)}{A}\right)^{n-2} (\sin(r+t)\kappa_g - \cos(r+t)) \\ \times 2(2 - \Delta\theta \sin \Delta\theta - 2\cos \Delta\theta) \, \mathrm{d}t.$$

Fix $\varepsilon > 0$. We can choose $t' > 0$ such that the following conditions hold for all $t \in [0, t']$ and all parameters s:

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(i) $|\sin(r+t)\kappa_g - \cos(r+t) - \sin r(\Delta\kappa_g + t(1+\cot^2 r + \cot r\Delta\kappa_g))| < \varepsilon t$ by the first-order series expansion of the left-hand side;

(ii) $|A(t,s) - (v_1 t^{3/2} + v_2 t^{5/2})| < \varepsilon t^{5/2}$ and

(iii) $1 \le \left(1 - \dfrac{A(t,s)}{A}\right)^{-2} < 1 + \dfrac{3v_1}{A} t^{3/2}$ from the expansion of $A(t,s)$ in Lemma 3;

(iv) $\left|2(2 - \Delta\theta \sin\Delta\theta - 2\cos\Delta\theta) - \dfrac{1}{6}\left(\vartheta_1^4 t^2 + \left(4\vartheta_1^3\vartheta_2 - \dfrac{\vartheta_1^6}{15}\right)t^3\right)\right| < \dfrac{\varepsilon}{6} t^3$ using the expansion of $\Delta\theta$ in Lemma 3, as well as the sixth-order Taylor polynomial $x^4/6 - x^6/90 + o(x^6)$ of $2(2 - x\sin x - 2\cos x)$;

(v) $$\left|\left(1 + \frac{3v_1}{A} t^{3/2}\right)(1 + \cot^2 r + \cot r\kappa_g \pm \varepsilon)\left(4\vartheta_1^3\vartheta_2 - \frac{\vartheta_1^6}{15} \pm \varepsilon\right)t^4 + \frac{3v_1}{A} t^{3/2}\left[\Delta\kappa_g \vartheta_1^4 t^2 + \Delta\kappa_g\left(4\vartheta_1^3\vartheta_2 - \frac{\vartheta_1^6}{15} \pm \varepsilon\right)t^3 + \vartheta_1^4(1 + \cot^2 r + \cot r\Delta\kappa_g \pm \varepsilon)t^3\right]\right| < \varepsilon t^3;$$

(vi) $e^{-y} e^{-(c+\varepsilon)y^{5/3} n^{-2/3}} \le \left[1 - \dfrac{y}{n} - c\left(\dfrac{y}{n}\right)^{5/3}\right]^n \le e^{-y} e^{-cy^{5/3} n^{-2/3}}$ and

(vii) $e^{-(1+\varepsilon)y} \le \left[1 - \dfrac{y}{n} - c\left(\dfrac{y}{n}\right)^{5/3}\right]^n \le e^{-(1-\varepsilon)y}$ for every $0 < y \le n \min_s v_1(s)(t')^{2/3}/A$ and $c \in [-a, a]$, with $a = A^{2/3} \max_s |v_2(s)/v_1^{5/3}(s)|$.

We also choose $\delta = \delta(\varepsilon)$ small enough that, for all $|y| \le \delta$, $e^{-y} \le 1 - (1-\varepsilon)y$. Lastly, let n_0 be large enough that

$$\max_s \frac{|v_2(s)|A^{2/3}}{v_1^{5/3}(s) n_0^{1/3}} \le \delta, \qquad \max_s \frac{(1-\varepsilon)(v_2(s) - \varepsilon)A^{2/3} D_2^\varepsilon}{v_1^{7/3}(s) n_0^{3/5}} \le \varepsilon,$$

with D_2^ε defined by (4). By assumptions (i)–(v),

$$\begin{aligned}\hat{I}_n(s) &\le \frac{\sin^4 r}{6A^2} \int_0^{t'} \left(1 - \frac{A(t,s)}{A}\right)^n \left(\Delta\kappa_g + (1 + \cot^2 r + \cot r\Delta\kappa_g + \varepsilon)t\right) \\ &\qquad \times \left(\vartheta_1^4 t^2 + \left(4\vartheta_1^3\vartheta_2 - \frac{\vartheta_1^6}{15} + \varepsilon\right)t^3\right)\left(1 + \frac{3v_1}{A} t^{3/2}\right) \mathrm{d}t \\ &\le \frac{\sin^4 r}{6A^2} \int_0^{t'} \left(1 - \frac{v_1}{A} t^{3/2} - \frac{v_2 - \varepsilon}{A} t^{5/2}\right)^n \left(D_1 t^2 + D_1 D_2^\varepsilon t^3\right) \mathrm{d}t,\end{aligned}$$

where

$$D_1 = \Delta\kappa_g \vartheta_1^4, \ D_1 D_2^\varepsilon = \vartheta_1^4(1 + \cot^2 r + \cot r\Delta\kappa_g + \varepsilon) + \Delta\kappa_g\left(4\vartheta_1^3\vartheta_2 - \frac{\vartheta_1^6}{15} + \varepsilon\right) + \varepsilon. \quad (4)$$

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Using the substitution $t^{3/2}v_1/A = y/n$, we have

$$\hat{I}_n(s) \le \frac{\sin^4 rD_1}{9n^2v_1^2}\int_0^{nt''}\left[1-\frac{y}{n}-\frac{(v_2-\varepsilon)A^{2/3}}{v_1^{5/3}}\left(\frac{y}{n}\right)^{5/3}\right]^n\left[1+D_2^{\varepsilon}\left(\frac{Ay}{nv_1}\right)^{2/3}\right]y\,\mathrm{d}y,$$

where $t'' := (t')^{3/2}v_1/A$.

We consider the integral on the interval $[n^{1/5}, nt'']$ first. As D_2^{ε} has a uniform upper bound, and v_1 a uniform (nonzero) lower bound, for large enough n the second factor is at most 2. Using assumption (vii), the expression is upper bounded by

$$\frac{2\sin^4 rD_1}{9n^2v_1^2}\int_{n^{1/5}}^{nt''}\mathrm{e}^{-(1-\varepsilon)y}nt''\,\mathrm{d}y \le \frac{2\sin^4 rD_1}{9nv_1^2}\mathrm{e}^{-(1-\varepsilon)n^{1/5}},$$

which tends to zero faster than any polynomial of n, and hence tends to zero when multiplied by $\binom{n}{2}\cdot n^{2/3}$.

For the integral on $[0, n^{1/5}]$ we use assumption (v), and the choice of δ and n_0, to obtain the following upper bound for $n \ge n_0$:

$$\frac{\sin^4 rD_1}{9n^2v_1^2}\int_0^{n^{1/5}}\mathrm{e}^{-y}\exp\left\{-\frac{(v_2-\varepsilon)A^{2/3}}{v_1^{5/3}}\frac{y^{5/3}}{n^{2/3}}\right\}\left[1+D_2^{\varepsilon}\left(\frac{Ay}{nv_1}\right)^{2/3}\right]y\,\mathrm{d}y$$

$$\le \frac{\sin^4 rD_1}{9n^2v_1^2}\int_0^{n^{1/5}}\mathrm{e}^{-y}\left(1-(1-\varepsilon)\frac{(v_2-\varepsilon)A^{2/3}}{v_1^{5/3}}\frac{y^{5/3}}{n^{2/3}}\right)\left[1+D_2^{\varepsilon}\left(\frac{Ay}{nv_1}\right)^{2/3}\right]y\,\mathrm{d}y.$$

Multiplying out the product, and again using the choice of n_0, we obtain the upper bound

$$\frac{\sin^4 rD_1}{9n^2v_1^2}\int_0^{n^{1/5}}\mathrm{e}^{-y}\left[1+\frac{A^{2/3}}{n^{2/3}}\left(\frac{D_2^{\varepsilon}}{v_1^{2/3}}y^{2/3}-(1-\varepsilon)\frac{(v_2-\varepsilon)}{v_1^{5/3}}y^{5/3}+\varepsilon\right)\right]y\,\mathrm{d}y.$$

Expanding the domain of the integration, and using the definition of the Gamma function, we get the bound

$$\frac{\sin^4 rD_1}{9n^2v_1^2}\left[1+\frac{A^{2/3}}{n^{2/3}}\left(\frac{D_2^{\varepsilon}}{v_1^{2/3}}\Gamma\left(\frac{8}{3}\right)-(1-\varepsilon)\frac{v_2-\varepsilon}{v_1^{5/3}}\Gamma\left(\frac{11}{3}\right)+2\varepsilon\right)\right].$$

We can obtain a lower bound in a similar way, and as ε was arbitrary, $\hat{I}_n(s)$ is asymptotically equal to this formula with $\varepsilon = 0$.

If we write in the formulas for D_1 and v_1, we see that $\sin^4 rD_1/(18v_1^2) = 1$, and since $\int_{\partial K} 1\,\mathrm{d}s = \mathrm{Per}\,(K)$, in the limit the perimeter cancels out, hence

$$\lim_{n\to\infty}\mathbb{E}(\mathrm{Per}\,(K)-\mathrm{Per}\,(K_n))n^{2/3} = \lim_{n\to\infty}\left(\mathrm{Per}\,(K)-\binom{n}{2}\int_{\partial K}\hat{I}_n(s)\,\mathrm{d}s\right)n^{2/3}$$

$$= \int_{\partial K}\frac{\sin^4 rD_1A^{2/3}}{18v_1^2}\left(\frac{v_2}{v_1^{5/3}}\Gamma\left(\frac{11}{3}\right)-\frac{D_2}{v_1^{2/3}}\Gamma\left(\frac{8}{3}\right)\right)\mathrm{d}s.$$

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After expanding the formulas (see also [15]), we obtain that the limit of the expectation is finally equal to

$$\sqrt[3]{\frac{2A^2}{3}}\Gamma\left(\frac{5}{3}\right)\frac{1}{4}\int_{\partial K}\Delta\kappa_{\mathrm{g}}^{1/3}(3\Delta\kappa_{\mathrm{g}}+4\cot r)-\frac{5(\kappa_{\mathrm{g}}')^2-3\kappa_{\mathrm{g}}''\Delta\kappa_{\mathrm{g}}}{3\Delta\kappa_{\mathrm{g}}^{8/3}}\,\mathrm{d}s$$
$$=\sqrt[3]{\frac{2A^2}{3}}\Gamma\left(\frac{5}{3}\right)\frac{1}{4}\int_{\partial K}\Delta\kappa_{\mathrm{g}}^{1/3}(3\kappa_{\mathrm{g}}+\cot r)-\frac{5(\kappa_{\mathrm{g}}')^2-3\kappa_{\mathrm{g}}''\Delta\kappa_{\mathrm{g}}}{3\Delta\kappa_{\mathrm{g}}^{8/3}}\,\mathrm{d}s.$$

By an elementary calculation, the last term in the integral is exactly the derivative of the expression $\kappa_{\mathrm{g}}'/\Delta\kappa_{\mathrm{g}}^{5/3}$, and hence its integral is zero. The remaining expression is exactly the statement of the theorem. □

Proof of Theorem 2. In the case when K is the circular disc of radius r, we follow similar steps. The integral transformation used is geometrically similar to the previous one, and its Jacobian is given by [14, Lemma 1], hence we obtain

$$\lim_{n\to\infty}\mathbb{E}(\operatorname{Per}(K_n))=\lim_{n\to\infty}\binom{n}{2}\frac{2\sin^3 r}{A^2}\int_0^{t'}\left(1-\frac{A(t)}{A}\right)^n$$
$$\times\sin t\cdot(2-\Delta\theta\sin\Delta\theta-2\cos\Delta\theta)\,\mathrm{d}t,$$

where $A:=\operatorname{SArea}(K)=2\pi(1-\cos r)$, and the upper bound of the integral t' is some sufficiently small fixed number. Using the appropriate series expansions, we obtain that the integral is of the form

$$\int_0^{t'}\left(1-\frac{2\sin r}{A}t+\frac{\cot r\cos r}{12A}t^3+O(t^5)\right)^n\left(t-\frac{t^3}{6}+O(t^5)\right)(4-\pi\cot r\cdot t+O(t^3))\,\mathrm{d}t.$$

Using similar analytical tools as above, we obtain the statement of the theorem. □

4. Duality

We define the spherical r-convex dual of the set

$$H\text{ as }H^r=\{y\in S^2\mid H\subseteq B(y,r)\}=\bigcap_{x\in H}B(x,r).$$

As discussed in the introduction, this operation reverses inclusion, and if $H=K$ is an r-spindle convex disc, $K=(K^r)^r$, justifying the name 'duality'. If K has a C^1 boundary, then for any $x=\varrho(s)\in\partial K$, the smoothness guarantees that there is a unique point, denoted by $x^r:=x^r(s)$, such that the spherical circular disc with centre x^r and radius r *supports* K at x, i.e. $B(x^r,r)$ contains K, and their supporting great circles at x coincide (see Figure 3). By the definition of the dual set, $x^r\in K^r$, and more specifically it is a boundary point. It is also straightforward to check that $(x^r)^r=x$.

We note that in the special case of $r=\pi/2$, it is easy to see that $K^{\pi/2}=-K^\circ$, where $K^\circ=\{y\in S^2\mid\langle y,x\rangle\le 0\text{ for all }x\in K\}$ is the spherical polar body.

Lemma 5. *Let $K\subseteq S^2$ be a spherically convex disc. Then, for the r-spindle convex dual K^r of K, we have*

(i) $\operatorname{Per}(K^r)=\sin r\cdot 2\pi-\cos r\cdot\operatorname{Per}(K)-\sin r\cdot\operatorname{SArea}(K)$, *and*

(ii) $\operatorname{SArea}(K^r)=(1-\cos r)\cdot 2\pi-\sin r\cdot\operatorname{Per}(K)+\cos r\cdot\operatorname{SArea}(K)$.

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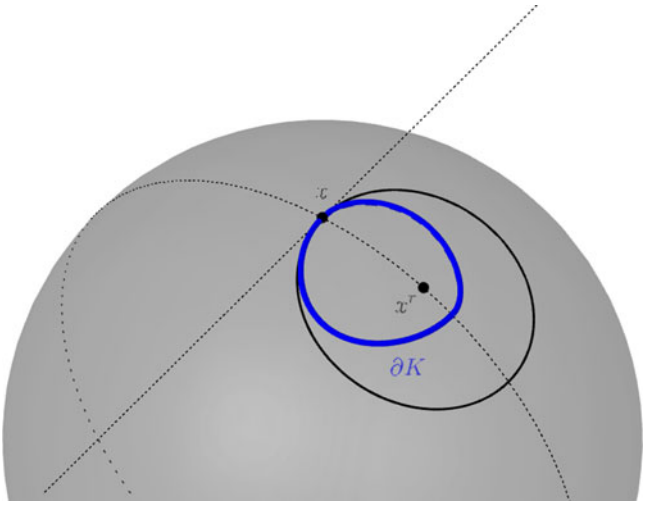


FIGURE 3. The spherical spindle convex dual.

Furthermore, if K has a C^2 boundary with the property that $\kappa_g(x) > \cot r$ for every $x \in \partial K$, the equality

(iii) $\sin^2 r \cdot (\kappa_g^r(x^r) - \cot r)(\kappa_g(x) - \cot r) = 1$ *also holds, where $\kappa_g^r(y)$ denotes the geodesic curvature of ∂K^r at a point y.*

Proof. First, we assume K is of class C^2 and $\kappa_g > \cot r$, and let $\varrho = \varrho(s)$ be an arc-length parametrisation of ∂K with positive orientation. Then the centre x^r of the support r-disc at $x = \varrho(s) \in \partial K$ can be explicitly determined using the parametrisation of ∂K, namely

$$x^r = \cos r \cdot \varrho(s) + \sin r \cdot \varrho(s) \times \dot{\varrho}(s).$$

Hence, a parametrisation of ∂K^r is given by

$$\zeta(s) = \cos r \cdot \varrho(s) + \sin r \cdot \varrho(s) \times \dot{\varrho}(s), \quad s \in [0, \operatorname{Per}(K)]. \tag{5}$$

Differentiating and using the relationship $\ddot{\varrho} = \kappa_g \cdot \varrho \times \dot{\varrho} - \varrho$ from Lemma 1, we obtain that $\dot{\zeta}(s) = -\dot{\varrho}(s) \cdot \sin r \cdot (\kappa_g(\varrho(s)) - \cot r)$. Hence,

$$\begin{aligned}\operatorname{Per}(K^r) = \int_0^{\operatorname{Per}(K)} |\dot{\zeta}(s)| \, \mathrm{d}s &= \int_0^{\operatorname{Per}(K)} \sin r(\kappa_g(\varrho(s)) - \cot r) \, \mathrm{d}s \\ &= \sin r \int_0^{\operatorname{Per}(K)} \kappa_g(\varrho(s)) \, \mathrm{d}s - \cos r \cdot \operatorname{Per}(K),\end{aligned}$$

and as the Gaussian curvature of S^2 is constant 1, by the Gauss–Bonnet theorem we have

$$\int_0^{\operatorname{Per}(K)} \kappa_g(\varrho(s)) \, \mathrm{d}s = 2\pi - \int_K 1 \, \mathrm{d}A = 2\pi - \operatorname{SArea}(K),$$

which yields the first part of the assertion. Since $(K^r)^r = K$, using the corresponding formula for $K := K^r$ and some algebraic manipulations, we obtain the second expression.

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To obtain the formula for the geodesic curvature, we first find an expression for the curvature κ^r of ∂K^r. To do this, we use the standard differential geometry formula

$$\kappa^r(\zeta(s)) = \frac{|\dot\zeta(s) \times \ddot\zeta(s)|}{|\dot\zeta(s)|^3}.$$

From (5), we see that

$$\begin{aligned}\ddot\zeta(s) &= -\ddot\varrho(s) \cdot \sin r \cdot (\kappa_g(\varrho(s)) - \cot r) - \dot\varrho(s) \cdot \sin r \cdot \kappa_g'(\varrho(s)) \\ &= -(\kappa_g(\varrho(s)) \cdot \varrho \times \dot\varrho - \varrho) \cdot \sin r \cdot (\kappa_g(\varrho(s)) - \cot r) - \dot\varrho(s) \cdot \sin r \cdot \kappa_g'(\varrho(s)),\end{aligned}$$

and hence $\dot\zeta(s) \times \ddot\zeta(s) = \sin^2 r(\kappa_g(\varrho(s)) - \cot r)^2 \cdot (\kappa_g(\varrho(s)) \cdot \varrho(s) - \varrho(s) \times \dot\varrho(s))$, implying that its length is

$$\sin^2 r(\kappa_g(\varrho(s)) - \cot r)^2 \cdot \left(\kappa_g^2(\varrho(s)) + 1\right)^{1/2} = \sin^2 r(\kappa_g(\varrho(s)) - \cot r)^2 \kappa(\varrho(s)),$$

and thus

$$\kappa^r(\zeta(s)) = \frac{\kappa(\varrho(s))}{\sin r \cdot (\kappa_g(\varrho(s)) - \cot r)}.$$

Since $(x^r)^r = x$, we can express $\kappa(\varrho(s))$ in terms of $\kappa^r(\zeta(s))$ in a similar way, which yields the desired equality.

As the duality transformation is continuous, for general K we can obtain the results for surface area and perimeter by using standard approximating tools. □

Corollary 1 *For the spherical polar body $K^\circ = -K^{\pi/2}$ we have* $\operatorname{Per}(K^\circ) + \operatorname{SArea}(K) = 2\pi$.

We note that if we consider the perimeter and surface area as special intrinsic volumes, the case for the polar body follows from combining Theorem 4.3.1 and Corollary 4.3.3 of [10] in dimension 2.

Lemma 6. *Let K be a spherical spindle convex disc with C^2 boundary and with the property that $\kappa_g(x) > \cot r$ for every $x \in \partial K$. Then*

$$\int_{\partial K^r} F\left(\Delta\kappa_g^r\right) \mathrm{d}s = \int_{\partial K} F\left(\frac{1}{\sin^2 r \Delta\kappa_g}\right) \cdot \sin r \cdot \Delta\kappa_g \,\mathrm{d}s$$

for any integrable function F, where the integration is taken with respect to arc length on both sides, $\Delta\kappa_g := \kappa_g - \cot r$, $\Delta\kappa_g^r := \kappa_g^r - \cot r$, and κ_g (κ_g^r) is understood as $\kappa_g(\varrho(s))$ ($\kappa_g^r(\hat\varrho(s))$) if $\varrho(s)$ ($\hat\varrho(s)$) is an arc-length parametrisation of ∂K (∂K^r).

Proof. Let ϱ be an arc-length parametrisation of ∂K, and ζ the parametrisation of ∂K^r as described by (5). The integral transformation will be described by the mapping $\partial K \to \partial K^r$, $\varrho(s) \mapsto \zeta(s)$ ($s \in [0, \operatorname{Per} K]$). From Lemma 5, we know that $\Delta\kappa_g^r(s) = (\sin^2 r \Delta\kappa_g(s))^{-1}$, implying that the argument of the function F changes as stated.

We find the Jacobian as follows. The ratio of the arc lengths of ∂K^r and ∂K between the points $\zeta(s)$, $\zeta(s+h)$ and $\varrho(s)$, $\varrho(s+h)$, respectively, is

$$\frac{1}{h} \cdot \int_s^{s+h} |\dot\zeta(\tau)| \,\mathrm{d}\tau = \frac{1}{h} \cdot \int_s^{s+h} \sin r \cdot (\kappa_g(\varrho(\tau)) - \cot r) \,\mathrm{d}\tau.$$

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The limit of this expression as $h \to 0$ is, by definition,

$$\frac{\mathrm{d}}{\mathrm{d}h}\int_s^{s+h} \sin r \cdot (\kappa_{\mathrm{g}}(\varrho(\tau)) - \cot r)\,\mathrm{d}\tau = \sin r \cdot (\kappa_{\mathrm{g}}(\varrho(s)) - \cot r),$$

where we used the Leibniz integral rule to obtain the formal derivative. □

5. A circumscribed model

In this section, we prove the asymptotic formulae stated in Theorem 3, regarding the expectation of the area and perimeter deviations of a spherical spindle convex disc K and a circumscribed spherical disc–polygon $K_{(n)}$. Recall that $K_{(n)}$ is obtained by taking the spherical spindle convex dual of the spherical spindle convex hull of n uniformly distributed, independent points chosen from K^r.

By Lemma 5, we have

$$\begin{aligned} \mathrm{Per}\,(K_{(n)}) - \mathrm{Per}\,(K) &= \cos r \cdot \left(\mathrm{Per}\left(K^r\right) - \mathrm{Per}\left(K^r_{(n)}\right)\right) + \sin r \cdot \mathrm{SArea}\left(K^r \setminus K^r_{(n)}\right), \\ \mathrm{SArea}\left(K_{(n)} \setminus K\right) &= \sin r \cdot \left(\mathrm{Per}\,(K^r) - \mathrm{Per}\left(K^r_{(n)}\right)\right) - \cos r \cdot \mathrm{SArea}\left(K^r \setminus K^r_{(n)}\right), \end{aligned} \tag{6}$$

and, by definition, $K^r_{(n)}$ is a uniform disc–polygon in K^r.

To be concise, we use the notation $\Delta\kappa_{\mathrm{g}}(s) = \kappa_{\mathrm{g}}(s) - \cot r$ as previously, and $\Delta\kappa^r_{\mathrm{g}}(s) = \kappa^r_{\mathrm{g}}(s) - \cot r$ for the dual disc, as well as $A(\cdot)$ for $\mathrm{SArea}\,(\cdot)$. From (1), and using the integral transformation in Lemma 6, we obtain

$$\begin{aligned} \lim_{n\to\infty} \mathbb{E}\big(\mathrm{SArea}\left(K^r \setminus K^r_{(n)}\right)\big) \cdot n^{2/3} &= \sqrt[3]{\frac{2A(K^r)^2}{3}}\,\Gamma\left(\frac{5}{3}\right) \int_{\partial K^r} \left(\Delta\kappa^r_{\mathrm{g}}(s)\right)^{1/3} \mathrm{d}s \\ &= \sqrt[3]{\frac{2A(K^r)^2}{3}}\,\Gamma\left(\frac{5}{3}\right) \int_{\partial K} \left(\frac{1}{\sin^2 r\Delta\kappa_{\mathrm{g}}}\right)^{1/3} \sin r\Delta\kappa_{\mathrm{g}}\,\mathrm{d}s \\ &= \sqrt[3]{\frac{2A(K^r)^2}{3}}\,\Gamma\left(\frac{5}{3}\right) (\sin r)^{1/3} \int_{\partial K} \Delta\kappa_{\mathrm{g}}^{2/3}\,\mathrm{d}s. \end{aligned}$$

Similarly, from Theorem 1 we have

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{E}\big(\mathrm{Per}\,(K^r) - \mathrm{Per}\left(K^r_{(n)}\right)\big) \cdot n^{2/3} \\ &= \sqrt[3]{\frac{2A(K^r)^2}{3}}\,\Gamma\left(\frac{5}{3}\right) \int_{\partial K^r} \left(\Delta\kappa^r_{\mathrm{g}}\right)^{1/3} \cdot \left(\frac{3}{4}\Delta\kappa^r_{\mathrm{g}} + \cot r\right) \mathrm{d}s \\ &= \sqrt[3]{\frac{2A(K^r)^2}{3}}\,\Gamma\left(\frac{5}{3}\right) \int_{\partial K} \left(\frac{1}{\sin^2 r\Delta\kappa_{\mathrm{g}}}\right)^{1/3} \left(\frac{3}{4\sin^2 r\Delta\kappa_{\mathrm{g}}} + \cot r\right) \sin r\Delta\kappa_{\mathrm{g}}\,\mathrm{d}s \\ &= \sqrt[3]{\frac{2A(K^r)^2}{3}}\,\Gamma\left(\frac{5}{3}\right) (\sin r)^{1/3} \int_{\partial K} \Delta\kappa_{\mathrm{g}}^{2/3} \left(\frac{3}{4\sin^2 r\Delta\kappa_{\mathrm{g}}} + \cot r\right) \mathrm{d}s. \end{aligned}$$

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Combining these results with (6), we have

$$\lim_{n\to\infty} \mathbb{E}(\mathrm{SArea}\,(K_{(n)} \setminus K))n^{2/3}$$
$$= \sqrt[3]{\frac{2A(K^r)^2}{3}}\Gamma\left(\frac{5}{3}\right)(\sin r)^{1/3} \int_{\partial K} \Delta\kappa_{\mathrm{g}}^{2/3}\left(\sin r\left(\frac{3}{4\sin^2 r\Delta\kappa_{\mathrm{g}}} + \cot r\right) - \cos r\right) \mathrm{d}s,$$

and

$$\lim_{n\to\infty} \mathbb{E}(\mathrm{Per}\,(K_{(n)} - \mathrm{Per}\,(K))n^{2/3}$$
$$= \sqrt[3]{\frac{2A(K^r)^2}{3}}\Gamma\left(\frac{5}{3}\right)(\sin r)^{1/3} \int_{\partial K} \Delta\kappa_{\mathrm{g}}^{2/3}\left(\cos r\left(\frac{3}{4\sin^2 r\Delta\kappa_{\mathrm{g}}} + \cot r\right) + \sin r\right) \mathrm{d}s,$$

which yield the desired formulas after simplification of the integrand, and substituting $A(K^r)$ with $(1-\cos r)2\pi - \sin r\,\mathrm{Per}\,(K) + \cos r A(K)$ from Lemma 5.

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There were no competing interests to declare which arose during the preparation or publication process of this article.

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