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MANY CUBIC SURFACES CONTAIN RATIONAL POINTS

T. D. BROWNING

For Klaus Roth, in memoriam

Abstract. Building on recent work of Bhargava, Elkies and Schnidman and of Kriz and Li, we produce infinitely many smooth cubic surfaces defined over the field of rational numbers that contain rational points.

§1. *Introduction.* This paper is concerned with the Hasse principle for the family of projective cubic surfaces

$$f(x_0, x_1) = g(x_2, x_3), \tag{1.1}$$

where $f, g \in \mathbb{Q}[u, v]$ are binary cubic forms. Our main result shows that a positive proportion of these surfaces, when ordered by height, possess a $\mathbb{Q}$-rational point. First, we discuss the question of local solubility. The following result will be addressed in §2.2.

THEOREM 1.1. *Approximately 99% of the cubic surfaces (1.1), when ordered by height, are everywhere locally soluble.*

It has long been known that the Hasse principle does not always hold for cubic surfaces and so local solubility is not enough to ensure that the surface (1.1) has a $\mathbb{Q}$-rational point. In the special case that f and g are diagonal there are many known counter-examples to the Hasse principle, the most famous being the surface

$$5x_0^3 + 9x_1^3 + 10x_2^3 + 12x_3^3 = 0,$$

that was discovered by Cassels and Guy [**6**]. As explained by Manin [**16**], this example is accounted for by the Brauer–Manin obstruction. It has been conjectured by Colliot-Thélène and Sansuc [**7**, §V] that this obstruction explains all failures of the Hasse principle for smooth cubic surfaces. This conjecture remains wide open in general. However, work of Swinnerton-Dyer [**21**] confirms it for a special family of diagonal cubic surfaces, conditionally under the assumption that the Tate–Shafarevich group of elliptic curves is finite.

A recent investigation of Bright [**4**] has focused on families of varieties over number fields that have no Brauer–Manin obstruction to the Hasse principle. In §2.1 we shall check that the conditions of his main result are satisfied for the family of cubic surfaces (1.1), thereby leading to the following conclusion.

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THEOREM 1.2 (Bright). *Assume that the Brauer–Manin obstruction is the only obstruction to the Hasse principle for smooth cubic surfaces. Then 100% of the cubic surfaces (1.1), when ordered by height, satisfy the Hasse principle.*

There are very few results which establish the existence of $\mathbb{Q}$-rational points on cubic surfaces unconditionally. Given coprime $a, b \in \mathbb{Z}$, Heath-Brown and Moroz [**12**, Corollary 1.2] have shown that any cubic surface

$$x_0^3 + 2x_1^3 + ax_2^3 + bx_3^3 = 0$$

has a $\mathbb{Q}$-rational point, provided that either $a \equiv \{\pm 2, \pm 3\} \bmod 9$, or $b \equiv \{\pm 2, \pm 3\} \bmod 9$, or $a \equiv \pm b \bmod 9$. This result combines sophisticated sieve methods, which allow primes to be represented by binary cubic polynomials, with a result of Satgé [**17**, Proposition 3.3], which demonstrates that the curve $u^3 + 2v^3 = pz^3$ has a $\mathbb{Q}$-rational point for any prime $p \equiv 2 \bmod 9$.

Our main result provides a much denser set of smooth cubic surfaces which are in possession of a rational point.

THEOREM 1.3. *A positive proportion of the cubic surfaces (1.1), when ordered by height, possess a $\mathbb{Q}$-rational point.*

Recall that the set of $\mathbb{Q}$-rational points is Zariski dense on the cubic surface (1.1) as soon as it is non-empty, as proved by Segre [**18**].

Theorem 1.3 will be established in §3. Our proof follows a strategy of Bhargava [**1**, §4], which was developed to show that a positive proportion of plane cubic curves have a $\mathbb{Q}$-rational point. The key input is recent work of Kriz and Li [**15**], concerning the family of Mordell curves

$$E_k : \quad y^2 = x^3 + k,$$

for $k \in \mathbb{Z}$. Using a Heegner point construction, Kriz and Li show that a positive proportion of these elliptic curves have rank one. The curves E_k have also been the focus of novel work by Bhargava *et al* [**3**] and we shall draw heavily on their investigation. It follows from [**3**, Theorem 8], in particular, that at least 41.1% of the curves E_k, for $k \in \mathbb{Z}$, have rank one, but this is only achieved under the assumption that the Tate–Shaferevich group is finite, whereas the work of Kriz and Li is unconditional.

Associated to E_k is an obvious 3-isogeny φ and we shall monopolize on the correspondence between elements of the φ-Selmer group and locally soluble binary cubic forms. Let $V(\mathbb{Z})$ be the lattice of (integer-matrix) binary cubic forms

$$f(u, v) = au^3 + 3bu^2v + 3cuv^2 + dv^3,$$

for $a, b, c, d \in \mathbb{Z}$. Associated to each $f \in V(\mathbb{Z})$ is the genus-one plane curve

$$C_f = \{z^3 = f(u, v)\} \subset \mathbb{P}^2. \tag{1.2}$$

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To establish Theorem 1.3, it will suffice to show that there is a positive proportion of irreducible $f \in V(\mathbb{Z})$, when ordered by height, such that $C_f(\mathbb{Q}) \neq \emptyset$. To see this we first note that, since f is irreducible, we must have $z \neq 0$ in any rational point $(u : v : z) \in C_f(\mathbb{Q})$. Now if the binary cubic forms $f, g \in V(\mathbb{Z})$ give rise to the rational points $(u : v : z) \in C_f(\mathbb{Q})$ and $(u' : v' : z') \in C_g(\mathbb{Q})$, with $zz' \neq 0$, then it follows that the rational point $(z'u : z'v : zu' : zv') \in \mathbb{P}^3(\mathbb{Q})$ lies on the cubic surface (1.1).

Remark 1.4. It is also possible to say something about appropriate K3 surfaces. Consider the family of Kummer surfaces with affine equation

$$z^2 = g_1(x)g_2(y), \tag{1.3}$$

where g_1, g_2 are irreducible quartic polynomials defined over $\mathbb{Q}$. Recent work of Harpaz and Skorobogatov [**11**] establishes the Hasse principle for these surfaces, provided that g_1, g_2 satisfy suitable genericity conditions, albeit under an assumption about the finiteness of certain associated Tate–Shaferevich groups. It has been shown by Bhargava [**1**, Theorem 11] that a positive proportion of the genus-one curves $y^2 = g(u, v)$ possess a $\mathbb{Q}$-rational point, when the binary quartic forms $g \in \mathbb{Q}[u, v]$ are ordered by height. An immediate consequence of this is the unconditional statement that a positive proportion of the Kummer surfaces (1.3), when ordered by height, possess a $\mathbb{Q}$-rational point.

§2. *The family of cubic surfaces and local solubility.* Let $X \subset \mathbb{P}^7 \times \mathbb{P}^3$ be the biprojective hypersurface defined by

$$ax_0^3 + bx_0^2x_1 + cx_0x_1^2 + dx_1^3 = ex_2^3 + fx_2^2x_3 + gx_2x_3^2 + hx_3^3.$$

We obtain a flat surjective morphism $\pi : X \to \mathbb{P}^7$ by projecting to the point $(a : b : \cdots : h)$, with X smooth, projective and geometrically integral. We denote by X_P the fibre above any point $P = (a : b : \cdots : h) \in \mathbb{P}^7(\mathbb{Q})$. Let $\eta : \mathrm{Spec}(K) \to \mathbb{P}^7$ denote the generic point and $\overline{\eta} : \mathrm{Spec}(\overline{K}) \to \mathbb{P}^7$ a geometric point above η, where K is the function field of $\mathbb{P}^7$ and $\overline{K}$ is an algebraic closure of K. The geometric generic fibre $X_{\overline{\eta}}$ is a smooth cubic surface and so it is connected and has torsion-free Picard group. Denote by $\overline{X}$ the base change of X to an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and let $\mathbf{A}_{\mathbb{Q}}$ denote the ring of adèles of $\mathbb{Q}$.

2.1. *Proof of Theorem 1.2.* The statement follows from [**4**, Theorem 1.4], provided that the following hypotheses are met:

(1) $X(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$;
(2) the fibre of π at each codimension-one point of $\mathbb{P}^7$ is geometrically integral;
(3) the fibre of π at each codimension-two point of $\mathbb{P}^7$ has a geometrically reduced component;
(4) $\mathrm{H}^1(\mathbb{Q}, \mathrm{Pic}\,\overline{X}) = 0$;
(5) $\mathrm{Br}\,\overline{X}$ is trivial;
(6) $\mathrm{H}^2(\mathbb{Q}, \mathrm{Pic}\,\mathbb{P}^7) \to \mathrm{H}^2(\mathbb{Q}, \mathrm{Pic}\,\overline{X})$ is injective;
(7) $\mathrm{Br}\,X_{\overline{\eta}} = 0$.

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Since X is $\mathbb{Q}$-rational the conditions (1), (4) and (5) are automatically met. The cubic surface X_P has equation $f(x_0, x_1) = g(x_2, x_3)$ for suitable binary cubic forms f and g. If X_P fails to be geometrically integral then it must be singular, which can only happen when $\operatorname{disc}(f) = \operatorname{disc}(g) = 0$. This is a codimension-two condition, which thereby establishes (2). The only way that (3) can fail is if f and g are proportional and both have repeated roots. But this is a codimension-three condition on the set of coefficients. Condition (6) follows from [**5**, Proposition 5.17]. Finally, $X_{\overline{\eta}}$ is a smooth cubic surface over an algebraically closed field of characteristic 0. Thus it is rational and it follows that $\operatorname{Br} X_{\overline{\eta}} = 0$, as required for condition (7). This completes the proof of Theorem 1.2.

Remark 2.1. Although we shall not need it in our work, it is possible to show that $\operatorname{Br} X_\eta / \operatorname{Br} K = 0$ for the family of cubic surfaces considered here, using an argument suggested to us by Jörg Jahnel. According to Swinnerton-Dyer [**20**], there is no Brauer–Manin obstruction whenever the action of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the 27 lines is as large as possible. A binary cubic form splits into three linear forms, generically by an S_3-extension. Thus, generically, there is an $S_3 \times S_3$-extension $k/\mathbb{Q}$, such that the surface may be written in the classical Cayley–Salmon form $l_1 l_2 l_3 = l_4 l_5 l_6$, for binary linear forms $l_1, l_2, l_3 \in k[x_0, x_1]$ and $l_4, l_5, l_6 \in k[x_2, x_3]$. There are nine obvious k-rational lines, given by $l_i = l_j = 0$ for $i \in \{1, 2, 3\}$ and $j \in \{4, 5, 6\}$, and six obvious tritangent k-rational planes. The latter form a pair of Steiner trihedra, both of which are defined over $\mathbb{Q}$. The group $S_3 \times S_3 \times S_3$ is the maximal Galois group stabilizing two such Steiner trihedra. Consider now the cubic surface $S \subset \mathbb{P}^3$ with equation

$$3x_0^3 + 7x_0^2 x_1 + 11x_0 x_1^2 + 13x_1^3 = 17x_2^3 + 19x_2^2 x_3 + 23x_2 x_3^2 + 31x_3^3.$$

A calculation in `magma` shows that the Galois group operating on the 27 lines is precisely the maximal possible group $S_3 \times S_3 \times S_3$. Since the Galois group has order 216, we may consult the appendix in work of Jahnel [**13**] to conclude that $\operatorname{Br} S / \operatorname{Br} \mathbb{Q} = 0$. This suffices to show the claim.

2.2. *Proof of Theorem 1.1: initial steps.* Conditions (1) and (2) in §2.1 are sufficient to ensure that all of the hypotheses of [**5**, Theorem 1.3] are met. Thus it follows that

$$\lim_{H \to \infty} \frac{\#\{(a, b, \dots, h) \in (\mathbb{Z} \cap [-H, H])^8 : X_{(a:b:\cdots:h)}(\mathbf{A}_\mathbb{Q}) \neq \emptyset\}}{(2H)^8} = \prod_v \vartheta_v > 0,$$

where ϑ_v is the density of cubic surfaces X_P which have $\mathbb{Q}_v$-points, for each place v of $\mathbb{Q}$. Clearly $\vartheta_\infty = 1$. We shall show that $\vartheta_p = 1 - \chi_p(1/p)$ for a suitable rational function $\chi_p \in \mathbb{Q}(x)$. Let

$$Q(x) = 9(1 + x + x^2 + x^3 + x^4)^2 (1 + x + x^2)(1 - x + x^2)(1 - x^2).$$

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Then we shall show that

$$\chi_p(x) = \frac{(1+x^4)(1+x^2)x^4}{Q(x)}$$

when $p \equiv 1 \bmod 3$, while if $p \not\equiv 1 \bmod 3$ then

$$\chi_p(x) = \frac{P(x)x^4}{(3-2x^{10})(3-2x^5)(1-x+x^2-x^3+x^4)(1-x^2+x^4)(1+x^2)^3Q(x)},$$

where

$$\begin{aligned} P(x) = {} & 4x^{35} - 4x^{34} + 16x^{33} - 16x^{32} + 56x^{31} - 58x^{30} + 126x^{29} - 132x^{28} \\ & + 228x^{27} - 236x^{26} + 292x^{25} - 314x^{24} + 342x^{23} - 378x^{22} + 342x^{21} \\ & - 361x^{20} + 197x^{19} - 184x^{18} - 68x^{17} + 110x^{16} - 293x^{15} + 336x^{14} \\ & - 489x^{13} + 543x^{12} - 669x^{11} + 702x^{10} - 639x^9 + 639x^8 - 477x^7 \\ & + 450x^6 - 315x^5 + 306x^4 - 180x^3 + 180x^2 - 45x + 45. \end{aligned}$$

We note that $1 - \chi_3(1/3) = 0.9965901\ldots$. Inserting our expressions for $\chi_p(1/p)$ into a computer and taking a product over primes $\leqslant 10\,000$ we find that

$$\prod_{\substack{p \leqslant 10^4 \\ p \equiv 1 \bmod 3}} \vartheta_p = 0.9973776\ldots \quad \text{and} \quad \prod_{\substack{p \leqslant 10^4 \\ p \equiv 2 \bmod 3}} \vartheta_p = 0.9998049\ldots.$$

Combining these, one verifies that $\prod_p \vartheta_p$ has numerical value $0.993782\ldots$, which is satisfactory for Theorem 1.1.

Our calculation of ϑ_p, for a given prime p, is based on work of Bhargava *et al* [2], who perform the same sort of calculation for the family of all ternary cubic forms. We shall say that a binary cubic form over $\mathbb{F}_p$ is diagonal if it takes the shape $au^3 + bv^3$ for $a, b \in \mathbb{F}_p^*$. We begin by recording the following result.

LEMMA 2.2.

(1) *The probability that a random monic cubic over $\mathbb{F}_p$ has a simple root in $\mathbb{F}_p$ is $\sigma_1 = \frac{2}{3}(p^2-1)/p^2$, and the probability that it has a triple root is $\tau_1 = 1/p^2$.*

(2) *The probability that a random primitive binary cubic form over $\mathbb{F}_p$ has a simple root in $\mathbb{P}^1(\mathbb{F}_p)$ is $\sigma_* = \frac{1}{3}p(2p+1)/(p^2+1)$, and the probability that it has a triple root is $\tau_* = 1/(p^2+1)$.*

(3) *The probability that a random diagonal binary cubic form over $\mathbb{F}_p$ has a simple root in $\mathbb{P}^1(\mathbb{F}_p)$ is*

$$\sigma_2 = \begin{cases} \frac{1}{3} & \text{if } p \not\equiv 2 \bmod 3, \\ 1 & \text{if } p \equiv 2 \bmod 3. \end{cases}$$

Proof. Part (1) is proved in [2, Corollary 4]. According to [2, Corollary 6], the probability that a random binary cubic form over $\mathbb{F}_p$ has a simple root in

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$\mathbb{P}^1(\mathbb{F}_p)$ is $\sigma = \frac{1}{3}(p^2-1)(2p+1)/p^3$, and the probability that it has a triple root is $\tau = (p^2-1)/p^4$. Part (2) therefore follows on noting that $\sigma_* = \sigma p^4/(p^4-1)$ and $\tau_* = \tau p^4/(p^4-1)$. Consider now the special case of diagonal binary cubic forms over $\mathbb{F}_p$. The lemma follows on noting that a random element of $\mathbb{Z}_p^*$ is a cube in $\mathbb{Q}_p^*$ with probability σ_2. □

We now proceed with our calculation of the density ϑ_p for a given prime p. Over $\mathbb{Q}_p$, cubic surfaces of the shape (1.1) are determined by pairs (f, g) of binary cubic forms $f \in \mathbb{Z}_p[u, v]$ and $g \in \mathbb{Z}_p[x, y]$, where

$$f(u, v) = c_0u^3 + c_1u^2v + c_2uv^2 + c_3v^3,$$
$$g(x, y) = d_0x^3 + d_1x^2y + d_2xy^2 + d_3y^3.$$

We say that the surface (f, g) is soluble over $\mathbb{Q}_p$ if there exist $u, v, x, y \in \mathbb{Z}_p$, with $\min\{v_p(u), v_p(v), v_p(x), v_p(y)\} = 0$, such that $f(u, v) = g(x, y)$. We say that a binary cubic form over $\mathbb{Z}_p$ is primitive if not all of its coefficients are divisible by p. We are only interested in surfaces (f, g) defined over $\mathbb{Z}_p$ for which f or g is primitive.

Mimicking [2], we shall say that the surface (f, g) has valuations

$$\geqslant \varphi_0 \geqslant \varphi_1 \geqslant \varphi_2 \geqslant \varphi_3$$
$$\geqslant \gamma_0 \geqslant \gamma_1 \geqslant \gamma_2 \geqslant \gamma_3,$$

if we happen to know that $v_p(c_i) \geqslant \varphi_i$ and $v_p(d_i) \geqslant \gamma_i$, for $0 \leqslant i \leqslant 3$. Likewise, we shall replace the inequality symbol with equality if and only if we know the exact p-adic valuation of the relevant coefficient. We may clearly assume without loss of generality that $\min_{0\leqslant i\leqslant 3} v_p(c_i) = 0$ (i.e. f is primitive). Through a change of variables, furthermore, we will also be able to restrict attention to the case where $\min_{0\leqslant i\leqslant 3} v_p(d_i) = \xi \in \{0, 1, 2\}$, in which case we write $g = p^\xi \tilde{g}$ for a primitive form $\tilde{g}$.

We shall examine the solubility of $(f, g) = (f, p^\xi \tilde{g})$ by considering its reduction modulo p. Let $\overline{f}$ (respectively, $\overline{g}$) be the reduction of f (respectively, $\tilde{g}$) modulo p. Whenever $\overline{f}$ has a simple root over $\mathbb{F}_p$, then this root can be lifted to a root over $\mathbb{Z}_p$ and it follows that the cubic surface (f, g) is soluble over $\mathbb{Q}_p$. We claim the same is true when $\overline{g}$ has a simple root. This is obvious when $\xi = 0$ and so we suppose that $\xi > 0$. In this case we merely substitute pu, pv for u, v and divide by p^ξ to obtain a new cubic surface $(p^{3-\xi} f, \tilde{g})$, which is plainly soluble over $\mathbb{Q}_p$.

Suppose now that $\overline{f}$ does not have any $\mathbb{F}_p$-roots and that $\overline{g}$ does not have a simple root. Then, either $\xi = 0$, in which case $(\overline{f}, \overline{g})$ is absolutely irreducible and it follows that the surface has a smooth $\mathbb{F}_p$-point which can be lifted, or else $\xi > 0$. But in the latter case, substituting pu, pv for u, v and dividing by p^ξ, we obtain a new cubic surface $(p^{3-\xi} f, \tilde{g})$. If $\overline{g}$ does not have any $\mathbb{F}_p$-roots then certainly (f, g) is not soluble over $\mathbb{Q}_p$. Next, suppose that $\overline{f}$ has a triple root and $\overline{g}$ has no $\mathbb{F}_p$-roots. If $\xi = 0$ then once again the surface $(\overline{f}, \overline{g})$ has a smooth $\mathbb{F}_p$-point which can be lifted to ensure solubility over $\mathbb{Q}_p$.

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For each $\xi \in \{0, 1, 2\}$, we define the following probabilities:

- $\alpha_1(\xi)$ is the probability of solubility for a pair $(f, p^\xi \tilde{g})$ such that both $\overline{f}$ and $\overline{g}$ have a triple root;
- $\alpha_2(\xi)$ is the probability of solubility for a pair $(f, p^\xi \tilde{g})$ such that $\overline{f}$ has a triple root and $\overline{g}$ is insoluble;
- $\alpha_3(\xi)$ is the probability of solubility for a pair $(f, p^\xi \tilde{g})$ such that $\overline{f}$ is insoluble and $\overline{g}$ has a triple root;
- $\alpha_4(\xi)$ is the probability of solubility for a pair $(f, p^\xi \tilde{g})$ such that $\overline{f}$ and $\overline{g}$ are both insoluble.

Our work above already implies that

$$\alpha_2(0) = \alpha_3(0) = \alpha_4(0) = 1 \quad \text{and} \quad \alpha_4(1) = \alpha_4(2) = 0.$$

Next we claim that $\alpha_3(\xi) = \alpha_2(3 - \xi)$ for $\xi \in \{1, 2\}$. Indeed, over $\mathbb{F}_p$, the only solutions have $u = v = 0$. Substituting pu for u and pv for v and dividing by p^2 we are left with the cubic surface $(p^{3-\xi} f, \tilde{g})$. This establishes the claim, implying that

$$\alpha_2(\xi) + \alpha_3(\xi) = \alpha_2(\xi) + \alpha_2(3 - \xi) = \alpha_2(1) + \alpha_2(2),$$

for any $\xi \in \{1, 2\}$.

We now return to our calculation of ϑ_p. For each $1 \leqslant i \leqslant 4$, we let $\beta_i(\xi)$ be the probability that our pair of cubic forms $(f, g) = (f, p^\xi \tilde{g})$ has the reduction type considered in the definition of $\alpha_i(\xi)$. Let $S \subset \mathbb{Z}_p^4$ be a set of coefficients producing primitive binary cubic forms. Then for any $\xi \in \{0, 1, 2\}$, the probability that a binary cubic form takes the shape $p^k \tilde{g}$ for $k \equiv \xi \bmod 3$, where $\tilde{g}$ has coefficient vector in S, is

$$\sum_{k \equiv \xi \bmod 3} \frac{1}{p^{4k}} \mu(S) = \frac{1}{p^{4\xi}} \left(1 - \frac{1}{p^{12}}\right)^{-1} \mu(S),$$

where $\mu(S)$ is the probability that a primitive binary cubic form has coefficient vector in S. It now follows from Lemma 2.2 that $\beta_i(\xi) = p^{-4\xi} \beta_i / (1 - 1/p^{12})$, for $1 \leqslant i \leqslant 4$, where

$$\beta_1 = \tau_*^2, \qquad \beta_2 = \beta_3 = \tau_*(1 - \sigma_* - \tau_*), \qquad \beta_4 = (1 - \sigma_* - \tau_*)^2.$$

Hence we deduce that the probability of insolubility of a cubic surface (1.1) with coefficients in $\mathbb{Q}_p$ is equal to

$$\begin{aligned} 1 - \vartheta_p &= \sum_{\xi \in \{0,1,2\}} \sum_{1 \leqslant i \leqslant 4} \beta_i(\xi)(1 - \alpha_i(\xi)) \\ &= \frac{\tau_*^2}{1 - 1/p^{12}} \left(1 - \alpha_1(0) + \frac{1 - \alpha_1(1)}{p^4} + \frac{1 - \alpha_1(2)}{p^8}\right) \\ &\quad + (2 - \alpha_2(1) - \alpha_2(2)) \frac{(1 + 1/p^4)\tau_*(1 - \sigma_* - \tau_*)}{p^4(1 - 1/p^{12})} \\ &\quad + \frac{(1 - \sigma_* - \tau_*)^2(1 + 1/p^4)}{p^4(1 - 1/p^{12})}. \end{aligned} \tag{2.1}$$

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It remains to analyse $\alpha_1(\xi)$ for $\xi \in \{0, 1, 2\}$ and $\alpha_2(1), \alpha_2(2)$. In doing so, when the reduction of the form has a triple point, the density will not depend on what this triple point is and so we shall always take it to be $(1 : 0)$.

In the special case $\alpha_1(1)$, which corresponds to surfaces $(f, p\tilde{g})$, where $\overline{f}$ and $\overline{g}$ both have triple roots, we may assume that $\overline{f}$ and $\overline{g}$ both have the triple root $(1 : 0)$. Substituting pv for v and dividing by p, it is easily seen that $\alpha_1(1)$ is equal to the probability of solubility over $\mathbb{Q}_p$ for a cubic surface with valuations

$$\begin{aligned} &\geqslant 0 \geqslant 1 \geqslant 2 = 2 \\ &\geqslant 1 \geqslant 1 \geqslant 1 = 0, \end{aligned}$$

in which one is only interested in solutions with $\min\{v_p(u), v_p(x)\} = 0$.

As in [2], we will develop recursions in order to solve for the densities of soluble cubics among those whose reductions produce the bad configurations outlined above.

2.3. *Technical lemmas.* The purpose of this section is to collect together definitions and calculations of particular probabilities that will feature in our analysis.

LEMMA 2.3. *Let λ be the probability that a cubic surface (f, g) with valuations*

$$\begin{aligned} &= 2 \geqslant 2 \geqslant 2 = 1 \\ &\geqslant 0 \geqslant 0 \geqslant 1 = 1 \end{aligned}$$

is soluble over $\mathbb{Z}_p$ with $x \in \mathbb{Z}_p^*$. Then

$$\lambda = \left(1 - \frac{1-\sigma_2}{p^5}\right)^{-1} \left(1 - \frac{1}{p} + \frac{1}{p^2} - \frac{(1-\sigma_2)^2}{p^4} - \frac{(1-\sigma_2)\sigma_2}{p^5}\right).$$

Proof. If $v_p(d_1) = 0$ (which happens with probability $1 - 1/p$) there is a simple root which can be lifted to a $\mathbb{Q}_p$-point on the cubic surface with $x \in \mathbb{Z}_p^*$. If $v_p(d_0) = 0$ and $v_p(d_1) \geqslant 1$ (probability $(p-1)/p^2$) then the equation is insoluble over $\mathbb{Q}_p$ with $x \in \mathbb{Z}_p^*$. If $v_p(d_0), v_p(d_1) \geqslant 1$ (probability $1/p^2$) then we divide by p to get the valuations

$$\begin{aligned} &= 1 \geqslant 1 \geqslant 1 = 0 \\ &\geqslant 0 \geqslant 0 \geqslant 0 = 0. \end{aligned}$$

With probability σ_1 we obtain a simple root of g over $\mathbb{F}_p$ which can be lifted. Note that if $\overline{g}$ is insoluble (probability $1 - \sigma_1 - \tau_1$) then the equation $\overline{f} = \overline{g}$ is absolutely irreducible and so has a smooth $\mathbb{F}_p$-point which can be lifted. Finally,

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with probability τ_1 there is a triple root over $\mathbb{F}_p$, which we may take to be $(1:0)$. Now with probability σ_2 the surface is soluble over $\mathbb{Z}_p$ with $x \in \mathbb{Z}_p^*$. With probability $1-\sigma_2$ the diagonal cubic form is insoluble over $\mathbb{F}_p$ and we replace v by pv and y by pv. Dividing by p leads us to the valuations

$$\begin{aligned} &= 0 \geqslant 1 \geqslant 2 = 2 \\ &\geqslant 0 \geqslant 1 \geqslant 2 = 2. \end{aligned}$$

If $v_p(d_0) = 0$ (probability $1 - 1/p$) then we get solubility over $\mathbb{Q}_p$ with probability σ_2, since when the diagonal cubic form is insoluble over $\mathbb{F}_p$ we will not obtain solutions with $x \in \mathbb{Z}_p^*$. Alternatively, if $v_p(d_0) \geqslant 1$ (probability $1/p$) then we replace u by pu and divide once more by p to get the valuations

$$\begin{aligned} &= 2 \geqslant 2 \geqslant 2 = 1 \\ &\geqslant 0 \geqslant 0 \geqslant 1 = 1, \end{aligned}$$

for which the probability of solubility over $\mathbb{Q}_p$ with $x \in \mathbb{Z}_p^*$ is λ. All in all it follows that

$$\lambda = 1 - \frac{1}{p} + \frac{1}{p^2}\left\{1 - \tau_1 + \tau_1\left(\sigma_2 + (1-\sigma_2)\left(\frac{(p-1)\sigma_2}{p} + \frac{\lambda}{p}\right)\right)\right\}.$$

The lemma follows on rearranging and recalling that $\tau_1 = 1/p^2$. □

LEMMA 2.4. *Let ϱ be the probability that a cubic surface (f, g) with valuations*

$$\begin{aligned} &= 0 \geqslant 1 \geqslant 1 = 1 \\ &\geqslant 0 \geqslant 1 \geqslant 2 = 2 \end{aligned}$$

is soluble over $\mathbb{Z}_p$ with $x \in \mathbb{Z}_p^*$. Then

$$\varrho = \left(1 - \frac{1}{p^5}\right)^{-1}\left(\left(1 - \frac{1}{p} + \frac{1}{p^2} - \frac{1}{p^3}\right)\sigma_2 + \frac{1}{p} - \frac{1}{p^2} + \frac{\sigma_1}{p^3}\right).$$

Proof. The first case to consider is when $v_p(d_0) = 0$ (which happens with probability $(p-1)/p$). In this case the cubic surface is soluble over $\mathbb{Q}_p$ with $x \in \mathbb{Z}_p^*$ with probability σ_2, else it is insoluble over $\mathbb{Q}_p$. Suppose next that $v_p(d_0) \geqslant 1$ (probability $1/p$). Then we must have $v_p(u) \geqslant 1$. Replacing u by pu and simplifying, we get a cubic surface with valuations

$$\begin{aligned} &= 2 \geqslant 2 \geqslant 1 = 0 \\ &\geqslant 0 \geqslant 0 \geqslant 1 = 1. \end{aligned}$$

When $v_p(d_1) = 0$ (probability $(p-1)/p$) this is soluble over $\mathbb{Q}_p$. When $v_p(d_0) = 0$ and $v_p(d_1) \geqslant 1$ (probability $(p-1)/p^2$) this is soluble over $\mathbb{Q}_p$ with probability σ_2. Finally, we suppose that $v_p(d_0), v_p(d_1) \geqslant 1$ (probability $1/p^2$).

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In this case we must have $v_p(v) \geqslant 1$. Replacing v by pv and simplifying we get the valuations

$$\begin{aligned} &= 1 \geqslant 2 \geqslant 2 = 2 \\ &\geqslant 0 \geqslant 0 \geqslant 0 = 0. \end{aligned}$$

With probability σ_1 the second cubic form has a simple root which can be lifted to get solubility over $\mathbb{Q}_p$. With probability τ_1 we get a triple root, which on moving to $(1 : 0)$ leads us to replace y by py. On simplification, this gives a cubic surface with valuations whose probability of solubility is ϱ.

All in all, it follows that

$$\varrho = \frac{(p-1)\sigma_2}{p} + \frac{1}{p}\left\{\frac{p-1}{p} + \frac{(p-1)\sigma_2}{p^2} + \frac{\sigma_1 + \tau_1\varrho}{p^2}\right\}.$$

Rearranging this and recalling that $\tau_1 = 1/p^2$, the lemma follows. □

LEMMA 2.5. *Let ω be the probability that a cubic surface (f, g) with valuations*

$$\begin{aligned} &= 0 \geqslant 1 \geqslant 2 = 2 \\ &\geqslant 1 \geqslant 1 \geqslant 1 = 0 \end{aligned}$$

is soluble over $\mathbb{Z}_p$ with $\min\{v_p(u), v_p(x)\} = 0$. Then

$$\omega = \left(1 - \frac{1-\sigma_2}{p^5}\right)^{-1}\left(\sigma_2 + \frac{1-\sigma_2}{p}\left(1 - \frac{1-\sigma_2}{p} + \frac{1-\sigma_2}{p^2} - \frac{1}{p^4}\right)\right).$$

Proof. With probability σ_2 there is a solution over $\mathbb{F}_p$ which can be lifted to ensure solubility over $\mathbb{Q}_p$. Otherwise, with probability $1 - \sigma_2$, the only solution has $v_p(u), v_p(y) \geqslant 1$. Replacing u by pu, y by py and simplifying, we arrive at the valuations

$$\begin{aligned} &= 2 \geqslant 2 \geqslant 2 = 1 \\ &\geqslant 0 \geqslant 1 \geqslant 2 = 2, \end{aligned}$$

in which we are interested in solubility over $\mathbb{Q}_p$ with $x \in \mathbb{Z}_p^*$. If $v_p(d_0) = 0$ (probability $(p-1)/p$) then there are no solutions. If, on the other hand, $v_p(d_0) \geqslant 1$ then we simplify to get the valuations

$$\begin{aligned} &= 1 \geqslant 1 \geqslant 1 = 0 \\ &\geqslant 0 \geqslant 0 \geqslant 1 = 1. \end{aligned}$$

Now if $v_p(d_1) = 0$ (probability $(p-1)/p$) then there is a simple $\mathbb{F}_p$-root of g which can be lifted to get solubility over $\mathbb{Q}_p$. If $v_p(d_1) \geqslant 1$ and $v_p(d_0) = 0$ (probability $(p-1)/p^2$) then the cubic surface is soluble over $\mathbb{Q}_p$ with probability σ_2. If $v_p(d_0), v_p(d_1) \geqslant 1$ (probability $1/p^2$) then we replace v by pv

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and simplify, in order to get the valuations

$$\begin{aligned} &= 0 \geqslant 1 \geqslant 2 = 2 \\ &\geqslant 0 \geqslant 0 \geqslant 0 = 0. \end{aligned}$$

With probability σ_1 the second polynomial has a simple root over $\mathbb{F}_p$ which can be lifted to get solubility over $\mathbb{Q}_p$. With probability $1 - \sigma_1 - \tau_1$ the form $\overline{g}$ is irreducible over $\mathbb{F}_p$ and we also get solubility over $\mathbb{Q}_p$. Finally, with probability τ_1, the form $\overline{g}$ has a triple root. Without loss of generality we may assume that the root is $(1 : 0)$ and the probability of solubility over $\mathbb{Q}_p$ for a cubic surface with these valuations is ω.

Putting everything together, we obtain

$$\omega = \sigma_2 + \frac{1-\sigma_2}{p}\left(1 - \frac{1}{p} + \left(1 - \frac{1}{p}\right)\frac{\sigma_2}{p} + \frac{1 - \tau_1 + \tau_1\omega}{p^2}\right).$$

The lemma follows on rearranging and recalling that $\tau_1 = 1/p^2$. □

LEMMA 2.6. *For $i \in \{0, 1\}$ let γ_i be the probability that a cubic surface (f, g) is soluble over $\mathbb{Z}_p$ with $x \in \mathbb{Z}_p^*$, assuming that $\overline{f}$ is irreducible over $\mathbb{F}_p$ and the surface has valuations*

$$\begin{aligned} &\geqslant 0 \geqslant 0 \geqslant 0 = 0 \\ &\geqslant 0 \geqslant i \geqslant i + 1 = i + 1. \end{aligned}$$

Then

$$\gamma_i = \left(1 - \frac{1}{p^5}\right)^{-1}\left(1 - \frac{1}{p^2} + \frac{\sigma_1}{p^{2+i}}\right).$$

Proof. If $v_p(d_0) = 0$ (probability $1 - 1/p$) then the cubic surface is soluble over $\mathbb{Q}_p$. If $v_p(d_0) \geqslant 1$ and $v_p(d_1) = 0$ (probability $(1 - i)(1 - 1/p)/p$) then we also get solubility over $\mathbb{Q}_p$ with $x \in \mathbb{Z}_p^*$. If $v_p(d_0), v_p(d_1) \geqslant 1$ (probability $(1 + i(p - 1))/p^2$) then we replace u by pu, v by pv and simplify to get the valuations

$$\begin{aligned} &\geqslant 2 \geqslant 2 \geqslant 2 = 2 \\ &\geqslant 0 \geqslant 0 \geqslant i = i. \end{aligned}$$

We seek solubility over $\mathbb{Q}_p$ with $x \in \mathbb{Z}_p^*$.

Suppose first that $i = 0$. Then with probability σ_1 the second cubic form g has a simple $\mathbb{F}_p$-root which can be lifted. With probability $1 - \sigma_1 - \tau_1$ it is not soluble over $\mathbb{Q}_p$ and with probability τ_1 the form g has a triple root, which we can move to $(1 : 0)$. Replacing y by py we are led to the new valuations

$$\begin{aligned} &\geqslant 1 \geqslant 1 \geqslant 1 = 1 \\ &\geqslant 0 \geqslant 1 \geqslant 2 = 2. \end{aligned}$$

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If $v_p(d_0) = 0$ then there are no solutions. If $v_p(d_0) \geqslant 1$ (probability $1/p$) then we simplify and observe that the probability of solubility is now γ_0.

Suppose next that $i = 1$. If $v_p(d_1) = 0$ (probability $1 - 1/p$) then we get solubility over $\mathbb{Q}_p$. If $v_p(d_1) \geqslant 1$ and $v_p(d_0) = 0$ then the surface is not soluble over $\mathbb{Q}_p$ with $x \in \mathbb{Z}_p^*$. If $v_p(d_0), v_p(d_1) \geqslant 1$ (probability $1/p^2$) then we simplify to arrive at the valuations

$$\begin{aligned} &\geqslant 1 \geqslant 1 \geqslant 1 = 1 \\ &\geqslant 0 \geqslant 0 \geqslant 0 = 0. \end{aligned}$$

With probability σ_1 the second cubic form g has a simple $\mathbb{F}_p$-root which can be lifted to ensure the desired solubility over $\mathbb{Q}_p$. With probability $1 - \sigma_1 - \tau_1$ it is not soluble over $\mathbb{Q}_p$ and with probability τ_1 the form g has a triple root, which we can move to $(1 : 0)$. Replacing y by py we are led to the a cubic surface for which the probability of solubility with $x \in \mathbb{Z}_p^*$ is precisely γ_1.

Thus, for $i \in \{0, 1\}$, we have shown that

$$\gamma_i = 1 - \frac{1}{p^2} - \frac{i(p-1)}{p^2} + \frac{1 + i(p-1)}{p^2} \times \begin{cases} \sigma_1 + \dfrac{\tau_1 \gamma_0}{p} & \text{if } i = 0, \\ 1 - \dfrac{1}{p} + \dfrac{\sigma_1 + \tau_1 \gamma_1}{p^2} & \text{if } i = 1. \end{cases}$$

The lemma follows on rearranging and recalling that $\tau_1 = 1/p^2$. □

LEMMA 2.7. *Let ϵ be the probability that a cubic surface (f, g) with valuations*

$$\begin{aligned} &\geqslant 0 \geqslant 0 \geqslant 1 = 1 \\ &\geqslant 1 \geqslant 1 \geqslant 1 = 0 \end{aligned}$$

is soluble over $\mathbb{Z}_p$ with $\min\{v_p(u), v_p(x)\} = 0$. Then $\epsilon = A + \alpha_1(1)/p^4$, where

$$A = 1 - \frac{1}{p} + \frac{p-1}{p^2}\left(\sigma_2 + (1-\sigma_2)\left(\left(1 - \frac{1}{p}\right)\sigma_2 + \frac{\lambda}{p}\right)\right) + \frac{\sigma_1 + (1-\sigma_1)\gamma_1}{p^2} - \frac{\gamma_1}{p^4}.$$

Proof. If $v_p(c_1) = 0$ (probability $(p-1)/p$) then the surface is soluble over $\mathbb{Q}_p$.

If $v_p(c_1) \geqslant 1$ and $v_p(c_0) = 0$ (probability $(p-1)/p^2$) then with probability σ_2 the surface is soluble over $\mathbb{Q}_p$. Alternatively (probability $1-\sigma_2$) we must have $v_p(u), v_p(y) \geqslant 1$ in any solution. Replacing u by pu, v by pv and simplifying, we are led to the valuations

$$\begin{aligned} &= 2 \geqslant 2 \geqslant 1 = 0 \\ &\geqslant 0 \geqslant 1 \geqslant 2 = 2, \end{aligned}$$

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in which we are interested in solubility over $\mathbb{Q}_p$ with $x \in \mathbb{Z}_p^*$. If $v_p(d_0) = 0$ (probability $(p-1)/p$) then this is soluble over $\mathbb{Q}_p$ with probability σ_2. If $v_p(d_0) \geqslant 1$ (probability $1/p$) then we replace v by pv and simplify to get valuations for which the probability of solubility for such cubic surfaces is precisely λ, in the notation of Lemma 2.3.

We now turn to the case $v_p(c_1), v_p(c_0) \geqslant 1$ (probability $1/p^2$) in the original equation. Replacing y by py and simplifying we arrive at the valuations

$$\begin{aligned} &\geqslant 0 \geqslant 0 \geqslant 0 = 0 \\ &\geqslant 0 \geqslant 1 \geqslant 2 = 2, \end{aligned}$$

in which we are interested in solubility over $\mathbb{Q}_p$ with $\min\{v_p(u), v_p(x)\} = 0$. With probability σ_1 the first cubic form has a simple root which can be lifted. With probability $1 - \sigma_1 - \tau_1$ the first cubic form is irreducible $\mathbb{F}_p$ and the probability of solubility is equal to γ_1, in the notation of Lemma 2.6. Finally, with probability τ_1 it has a triple root, which we may move $(1:0)$. In this final case, in view of our earlier discussions, the probability of solubility is equal to $\alpha_1(1)$. The lemma easily follows on recalling that $\tau_1 = 1/p^2$. □

2.4. *Calculation of* $\alpha_2(1)$ *and* $\alpha_2(2)$. For $\xi \in \{1, 2\}$ we proceed to calculate $\alpha_2(\xi)$. The density does not depend on the triple point that $\overline{f}$ has and so we may take it to be $(1:0)$. Substituting pv for v and dividing by p we arrive at the cubic surface $(f', p^{\xi-1}\tilde{g})$ where the reduction $\overline{g}$ of $\tilde{g}$ modulo p is insoluble and the coefficients of f' have valuations

$$\geqslant 0 \geqslant 1 \geqslant 2 = 2.$$

If $\xi = 1$ then clearly $\alpha_2(1) = \gamma_1$, in the notation of Lemma 2.6. Suppose next that $\xi = 2$. If $v_p(c_0) = 0$ then we must have $v_p(u) \geqslant 1$ and a change of variables from u to pu leads us to the conclusion that the surface is insoluble over $\mathbb{Q}_p$ since $\overline{g}$ is insoluble over $\mathbb{F}_p$. If $v_p(c_0) \geqslant 1$ (probability $1/p$) then we divide by p to get the cubic surface $(f'', \tilde{g})$, in which the coefficients of f'' have valuations

$$\geqslant 0 \geqslant 0 \geqslant 1 = 1.$$

The probability of solubility in this case is precisely γ_0, in the notation of Lemma 2.6. Thus we have shown that

$$\alpha_2(1) = \gamma_1 \quad \text{and} \quad \alpha_2(2) = \frac{\gamma_0}{p}.$$

2.5. *Calculation of* $\alpha_1(0)$. We may assume that the coefficients of (f, g) have valuations

$$\begin{aligned} &\geqslant 1 \geqslant 1 \geqslant 1 = 0 \\ &\geqslant 1 \geqslant 1 \geqslant 1 = 0. \end{aligned}$$

With probability σ_2 the equation $c_3 v^3 = d_3 y^3$ has a simple root over $\mathbb{F}_p$ which can lifted, thereby ensuring solubility over $\mathbb{Q}_p$. With probability $1 - \sigma_2$ it is insoluble over $\mathbb{F}_p$. In the latter case we replace v by pv, y by py and simplify.

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Suppose first that $\min\{v_p(c_0), v_p(d_0)\} \geqslant 2$ (which happens with probability $1/p^2$). Then we get a new pair of cubic forms with valuations

$$\begin{aligned} &\geqslant 0 \geqslant 0 \geqslant 1 = 1 \\ &\geqslant 0 \geqslant 0 \geqslant 1 = 1. \end{aligned}$$

If $v_p(c_1) = 0$ or $v_p(d_1) = 0$ (probability $1-1/p^2$) then there is a simple root over $\mathbb{F}_p$ which can be lifted. If, on the other hand, $v_p(c_1), v_p(d_1) \geqslant 1$ (probability $1/p^2$) then the probability of solubility is κ, where κ denotes the probability that a pair of cubics with valuations

$$\begin{aligned} &\geqslant 0 \geqslant 1 \geqslant 1 = 1 \\ &\geqslant 0 \geqslant 1 \geqslant 1 = 1 \end{aligned}$$

is soluble over $\mathbb{Z}_p$ with $\min\{v_p(u), v_p(x)\} = 0$. We claim that

$$\kappa = \left(1 - \frac{1}{p}\right)^2 \sigma_2 + \frac{1}{p^2} - \frac{1}{p^6} + \frac{\alpha_1(0)}{p^6} + \frac{2\kappa_1}{p}\left(1 - \frac{1}{p}\right),$$

where κ_1 is the probability of solubility for a cubic surface with valuations

$$\begin{aligned} &= 0 \geqslant 1 \geqslant 1 = 1 \\ &\geqslant 1 \geqslant 1 \geqslant 1 = 1, \end{aligned}$$

with $x \in \mathbb{Z}_p^*$. To see this, if $v_p(c_0) = v_p(d_0) = 0$ (probability $(1 - 1/p)^2$), then with probability σ_2 we get a simple root over $\mathbb{F}_p$ which can be lifted (and it is insoluble over $\mathbb{Q}_p$ otherwise). If $v_p(c_0), v_p(d_0) \geqslant 1$ (probability $1/p^2$) then we simplify and obtain the valuations

$$\begin{aligned} &\geqslant 0 \geqslant 0 \geqslant 0 = 0 \\ &\geqslant 0 \geqslant 0 \geqslant 0 = 0. \end{aligned}$$

With probability σ_1 the cubic form f has a simple $\mathbb{F}_p$-root which can be lifted to get solubility over $\mathbb{Q}_p$. With probability $1 - \sigma_1 - \tau_1$ the form f is irreducible over $\mathbb{F}_p$ and we also get solubility over $\mathbb{Q}_p$. Finally, with probability τ_1 the form f has a triple root. In this case, with probability $1 - \tau_1$ we get solubility over $\mathbb{Q}_p$, since either g has a simple root over $\mathbb{F}_p$ or it is irreducible over $\mathbb{F}_p$. Alternatively, with probability τ_1 the form g also has a triple root over $\mathbb{F}_p$ and the probability of solubility is precisely $\alpha_1(0)$. The claim follows on recalling that $\tau_1 = 1/p^2$ and noting that the probability of solubility is κ_1 when $v_p(c_0) = 0$, $v_p(d_0) \geqslant 1$ or $v_p(c_0) \geqslant 1$, $v_p(d_0) = 0$.

Next we calculate κ_1, noting that we must have $v_p(u) \geqslant 1$ in any solution. Replacing u by pu and simplifying we arrive at the valuations

$$\begin{aligned} &= 2 \geqslant 2 \geqslant 1 = 0 \\ &\geqslant 0 \geqslant 0 \geqslant 0 = 0. \end{aligned}$$

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With probability $1-\tau_1$ this is soluble over $\mathbb{Q}_p$. With probability τ_1 the second form g has a triple root over $\mathbb{F}_p$ which we can move to $(1:0)$. But then with probability σ_2 the cubic surface is soluble over $\mathbb{Q}_p^*$ and with probability $1-\sigma_2$ the only solution has $v_p(v), v_p(y) \geqslant 1$. Replacing v by pv, y by py and simplifying, we therefore obtain the valuations

$$\begin{aligned} &= 1 \geqslant 2 \geqslant 2 = 2 \\ &\geqslant 0 \geqslant 1 \geqslant 1 = 1. \end{aligned}$$

If $v_p(d_0) = 0$ there are no solutions over $\mathbb{Q}_p^*$, while if $v_p(d_0) \geqslant 1$ we may remove a factor of p from all of the coefficients. In the new pair of forms we get solubility if $v_p(d_1) = 0$ (probability $1-1/p$), while we get solubility with probability σ_2 if $v_p(d_1) \geqslant 1$ and $v_p(d_0) = 0$ (probability $(p-1)/p^2$). Finally, the probability of solubility is κ_1 if $v_p(d_1), v_p(d_0) \geqslant 1$ (probability $1/p^2$). Recalling that $\tau_1 = 1/p^2$ and rearranging, we easily conclude that

$$\kappa_1 = \left(1 - \frac{1-\sigma_2}{p^5}\right)^{-1}\left(1 - \frac{1-\sigma_2}{p^2} + \frac{1-\sigma_2}{p^3}\left(1 - \frac{1-\sigma_2}{p} - \frac{\sigma_2}{p^2}\right)\right).$$

Next, suppose that $v_p(c_0) = 1$ and $v_p(d_0) \geqslant 2$ (probability $(p-1)/p^2$). Then our cubic surface has valuations

$$\begin{aligned} &= 0 \geqslant 1 \geqslant 2 = 2 \\ &\geqslant 1 \geqslant 1 \geqslant 2 = 2. \end{aligned}$$

Hence we replace u by pu and divide by p to get the new valuations

$$\begin{aligned} &= 2 \geqslant 2 \geqslant 2 = 1 \\ &\geqslant 0 \geqslant 0 \geqslant 1 = 1. \end{aligned}$$

But the probability that a pair of cubics with these valuations is soluble over $\mathbb{Z}_p$ with $x \in \mathbb{Z}_p^*$ is precisely equal to λ in the notation of Lemma 2.3. The case in which $v_p(c_0) \geqslant 2$ and $v_p(d_0) = 1$ yields the same conclusion.

Finally, we suppose that $v_p(c_0) = v_p(d_0) = 1$ (probability $(p-1)^2/p^2$). This leads to the valuations

$$\begin{aligned} &= 0 \geqslant 1 \geqslant 2 = 2 \\ &= 0 \geqslant 1 \geqslant 2 = 2. \end{aligned}$$

With probability σ_2 the equation $c_0u^3 = d_0x^3$ has a simple root over $\mathbb{F}_p$ which can be lifted. The alternative case leads to insolubility over $\mathbb{Q}_p$.

Summarizing our argument so far, we have shown that

$$\alpha_1(0) = \sigma_2 + (1-\sigma_2)\left\{\frac{1}{p^2}\left(1 + \frac{\kappa - 1}{p^2}\right) + \frac{2(p-1)\lambda}{p^2} + \left(1 - \frac{1}{p}\right)^2\sigma_2\right\}.$$

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Our calculations show that $\kappa = B + \alpha_1(0)/p^6$, where

$$B = \left(1 - \frac{1}{p}\right)^2 \sigma_2 + \frac{1}{p^2} - \frac{1}{p^6} + \frac{2(p-1)}{p^2(1-(1-\sigma_2/p^5))}\left(1 - \frac{1-\sigma_2}{p^2} + \frac{1-\sigma_2}{p^3}\left(1 - \frac{1-\sigma_2}{p} - \frac{\sigma_2}{p^2}\right)\right).$$

Substituting κ and rearranging, we finally conclude that

$$\alpha_1(0) = \left(1 - \frac{1-\sigma_2}{p^{10}}\right)^{-1} \times \left(\sigma_2 + (1-\sigma_2)\left\{\frac{1}{p^2} + \frac{B-1}{p^4} + \frac{2\lambda}{p}\left(1 - \frac{1}{p}\right) + \left(1 - \frac{1}{p}\right)^2 \sigma_2\right\}\right).$$

2.6. *Calculation of* $\alpha_1(1)$. In this section we calculate the value of $\alpha_1(1)$ recorded at the close of §2.2. If $v_p(c_0) = 0$ (probability $1 - 1/p$) then the probability of solubility is precisely ω, in the notation of Lemma 2.5. Alternatively, with probability $1/p$, we have $v_p(c_0) \geqslant 1$. Replacing y by py and simplifying, we deduce that

$$\alpha_1(1) = \left(1 - \frac{1}{p}\right)\omega + \frac{\omega_1}{p},$$

where ω_1 is the probability of solubility over $\mathbb{Q}_p$ for a cubic surface with valuations

$$\begin{aligned} &\geqslant 0 \geqslant 0 \geqslant 1 = 1 \\ &\geqslant 0 \geqslant 1 \geqslant 2 = 2, \end{aligned}$$

with $\min\{v_p(u), v_p(x)\} = 0$.

It remains to calculate ω_1. If $v_p(c_1) = 0$ (probability $(p-1)/p$) then there is a simple root over $\mathbb{F}_p$ which can lifted. If $v_p(c_1) \geqslant 1$ and $v_p(c_0) = 0$ (probability $(p-1)/p^2$) then the probability of solubility over $\mathbb{Q}_p$ is precisely ϱ, in the notation of Lemma 2.4. Suppose now that $v_p(c_1), v_p(c_0) \geqslant 1$ (probability $1/p^2$). If $v_p(d_0) = 0$ (probability $(p-1)/p$) then we must replace x by px in order to arrive at the valuations

$$\begin{aligned} &\geqslant 0 \geqslant 0 \geqslant 0 = 0 \\ &= 2 \geqslant 2 \geqslant 2 = 1. \end{aligned}$$

Arguing as in the close of the proof of Lemma 2.4, the probability of solubility over $\mathbb{Z}_p$ with $u \in \mathbb{Z}_p^*$ is found to be $\sigma_1 + \tau_1\varrho$.

If $v_p(d_0) \geqslant 1$ (probability $1/p$), we simplify to get the valuations

$$\begin{aligned} &\geqslant 0 \geqslant 0 \geqslant 0 = 0 \\ &\geqslant 0 \geqslant 0 \geqslant 1 = 1. \end{aligned}$$

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We are interested in solubility over $\mathbb{Q}_p$ with $\min\{v_p(u), v_p(x)\} = 0$. With probability σ_1 the first form has a simple $\mathbb{F}_p$-root which can be lifted. With probability $1 - \sigma_1 - \tau_1$ the first form is irreducible over $\mathbb{F}_p$ and the overall probability of solubility is equal to γ_0, in the notation of Lemma 2.6. Finally, with probability τ_1 the first form has a triple root which can be moved to $(1 : 0)$. In this case the probability of solubility is equal to $\epsilon = A + \alpha_1(1)/p^4$, in the notation of Lemma 2.7.

Recalling that $\tau_1 = 1/p^2$, we have shown that $\omega_1 = C + \alpha_1(1)/p^9$, where

$$C = 1 - \frac{1}{p} + \frac{(p-1)\varrho}{p^2} + \frac{(p-1)(\sigma_1 + \varrho/p^2)}{p^3} + \frac{\sigma_1 + (1 - \sigma_1 - 1/p^2)\gamma_0}{p^3} + \frac{A}{p^5}.$$

We now substitute this into our earlier expression and rearrange things to finally conclude that

$$\alpha_1(1) = \left(1 - \frac{1}{p^{10}}\right)^{-1} \left\{ \left(1 - \frac{1}{p}\right)\omega + \frac{C}{p} \right\}.$$

2.7. *Calculation of* $\alpha_1(2)$. It remains to study the probability of solubility over $\mathbb{Q}_p$ for cubic surfaces with valuations

$$\geqslant 1 \geqslant 1 \geqslant 1 = 0$$
$$\geqslant 3 \geqslant 3 \geqslant 3 = 2.$$

We claim that

$$\alpha_1(2) = \left(1 - \frac{1}{p}\right)\varrho + \frac{A}{p} + \frac{\alpha_1(1)}{p^5},$$

in the notation of Lemmas 2.4 and 2.7. To begin with, if $v_p(c_0) = 1$ (with probability $(p-1)/p$) then we must have $v_p(u), v_p(v), v_p(y) \geqslant 1$ in any solution. Making the obvious transformations and simplifying, the probability of solubility over $\mathbb{Q}_p$ with $x \in \mathbb{Z}_p^*$ is precisely ϱ. If, on the other hand, $v_p(c_0) \geqslant 2$ (probability $1/p$) then we replace v by pv to arrive at valuations for which the probability of solubility over $\mathbb{Q}_p$ is ϵ, in the notation of Lemma 2.7. The claim follows.

2.8. *Conclusion.* Returning to (2.1), we substitute the values of σ_*, τ_* from part (2) of Lemma 2.2. This leads to the expression

$$\chi_p(1/p) = \frac{E(p)}{p^4(1 + 1/p^2)^2(1 - 1/p^{12})},$$

where

$$E(p) = 1 - \alpha_1(0) + \frac{1 - \alpha_1(1)}{p^4} + \frac{1 - \alpha_1(2)}{p^8} + \frac{(1 - 1/p)(1 + 1/p^4)}{3p^2}(2 - \alpha_2(1) - \alpha_2(2)) + \frac{(1 - 1/p)^2(1 + 1/p^4)}{9}.$$

Using a computer algebra package, it remains to substitute our various expressions for $\alpha_1(0), \alpha_1(1), \alpha_1(2)$ and $\alpha_2(1), \alpha_2(2)$ into this, before simplifying the expression. This ultimately leads to the expressions for $\chi_p(x)$ recorded in §2.2, with $x = 1/p$.

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§3. *Global solubility.* We have seen that in order to prove Theorem 1.3 it suffices to prove that there is a positive proportion of irreducible $f \in V(\mathbb{Z})$, when ordered by height, such that $C_f(\mathbb{Q}) \neq \emptyset$, where C_f is the associated genus-one curve (1.2). Our template for proving this is the argument developed by Bhargava [1, §4] to show that a positive proportion of plane cubic curves have a $\mathbb{Q}$-point.

3.1. *Selmer groups of Mordell curves.* Our work relies on the arithmetic of Mordell curves

$$E_k : \quad y^2 = x^3 + k,$$

for $k \in \mathbb{Z}$. These elliptic curves have j-invariant 0 and (potential) complex multiplication by the ring of integers of $\mathbb{Q}(\sqrt{-3})$. There is a 3-isogeny $\varphi = \varphi_k : E_k \to E_{-27k}$ and, by duality, a 3-isogeny $\widehat{\varphi} = \varphi_{-27k} : E_{-27k} \to E_k$. As k varies over the integers, Bhargava *et al* [3] have undertaken an extensive investigation into the average size of the φ-Selmer groups $\mathrm{Sel}_\varphi(E_k) \subset \mathrm{H}^1(G_\mathbb{Q}, E_k[\varphi])$ and $\mathrm{Sel}_{\widehat{\varphi}}(E_{-27k}) \subset \mathrm{H}^1(G_\mathbb{Q}, E_{-27k}[\widehat{\varphi}])$, where $G_\mathbb{Q} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, consisting of "locally soluble" cohomology classes. Let

$$\begin{aligned}\varrho &= \frac{103 \cdot 229}{2 \cdot 3^2 \cdot 7^2 \cdot 13} \\ &\quad \times \prod_{p \equiv 5 \bmod 6} \frac{(1 - 1/p)(1 + 1/p + 5/3p^2 + 1/p^3 + 5/3p^4 + 1/p^5)}{1 - 1/p^6} \\ &= 2.1265\ldots.\end{aligned}$$

While it does not play a role in our work, when the elliptic curves E_k are ordered by height, it follows from [3, Theorem 1] that the average of $\#\,\mathrm{Sel}_\varphi(E_k)$ is $1 + \varrho$ (respectively $1 + \varrho/3$) if $k < 0$ (respectively $k > 0$).

We shall be interested in soluble elements of $\mathrm{Sel}_\varphi(E_k)$ and $\mathrm{Sel}_{\widehat{\varphi}}(E_{-27k})$ According to the fundamental short exact sequence

$$0 \to E_{-27k}(\mathbb{Q})/\varphi(E_k(\mathbb{Q})) \to \mathrm{Sel}_\varphi(E_k) \to Ш(E_k)[\varphi] \to 0,$$

the soluble elements of $\mathrm{Sel}_\varphi(E_k)$ are precisely those that come from the group $E_{-27k}(\mathbb{Q})/\varphi(E_k(\mathbb{Q}))$. Similarly, soluble elements of $\mathrm{Sel}_{\widehat{\varphi}}(E_{-27k})$ are precisely those that come from $E_k(\mathbb{Q})/\widehat{\varphi}(E_{-27}(\mathbb{Q}))$. Noting that $\widehat{\varphi} \circ \varphi = [3]$, we may now turn to the exact sequence

$$0 \to \frac{E_{-27k}(\mathbb{Q})[\widehat{\varphi}]}{\varphi(E_k(\mathbb{Q})[3])} \to \frac{E_{-27k}(\mathbb{Q})}{\varphi(E_k(\mathbb{Q}))} \xrightarrow{\widehat{\varphi}} \frac{E_k(\mathbb{Q})}{3E_k(\mathbb{Q})} \to \frac{E_k(\mathbb{Q})}{\widehat{\varphi}(E_{-27k}(\mathbb{Q}))} \to 0,$$

which is recorded in [19, Remark X.4.7]. The groups appearing in this sequence are all $\mathbb{F}_3$-vector spaces and it follows that

$$\begin{aligned}\dim_{\mathbb{F}_3} \frac{E_{-27k}(\mathbb{Q})}{\varphi(E_k(\mathbb{Q}))} + \dim_{\mathbb{F}_3} \frac{E_k(\mathbb{Q})}{\widehat{\varphi}(E_{-27k}(\mathbb{Q}))} &\geqslant \dim_{\mathbb{F}_3} \frac{E_k(\mathbb{Q})}{3E_k(\mathbb{Q})} \\ &\geqslant \mathrm{rank}\, E_k(\mathbb{Q}). \qquad (3.1)\end{aligned}$$

(Since the two groups on the left-hand side sit inside the relevant Selmer groups, this automatically implies the inequality recorded in [3, Proposition 42(i)].)

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3.2. *Binary cubic forms.* For any Dedekind domain D, we write $V(D)$ for the set of binary cubic forms $f(u, v) = [a, 3b, 3c, d]$, with $a, b, c, d \in D$. We put

$$\operatorname{disc}(f) = -3b^2c^2 + 4ac^3 + 4b^3d + a^2d^2 - 6abcd,$$

for the (reduced) discriminant of f. We put $V(D)_A$ for the set of $f \in V(D)$ such that $\operatorname{disc}(f) = A$.

Taking $D = \mathbb{Z}$, we shall adopt the notation in [3] and introduce the sets

$$V(\mathbb{Z})^{\mathrm{sol}} = \{f \in V(\mathbb{Z}) : C_f(\mathbb{Q}) \neq \emptyset\},$$
$$V(\mathbb{Z})^{\mathrm{loc\,sol}} = \{f \in V(\mathbb{Z}) : C_f(\mathbb{Q}_p) \neq \emptyset \ \forall \text{ primes } p\},$$

where C_f is given by (1.2). Put $V(\mathbb{Z})_A^{\mathrm{sol}}$ and $V(\mathbb{Z})_A^{\mathrm{loc\,sol}}$ for the corresponding subsets restricted to have discriminant A. The group $\mathrm{GL}_2(\mathbb{Z})$ acts naturally on $V(\mathbb{R})$ by linear change of variables and the discriminant is $\mathrm{SL}_2(\mathbb{Z})$-invariant. We let $\mathcal{F}$ be a fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $V(\mathbb{R})$. We set

$$\mathcal{F}_X = \mathcal{F} \cap \{f \in V(\mathbb{R}) : 0 < |\operatorname{disc}(f)| \leqslant X\}.$$

This set has finite volume. The following result (cf. [3, Theorem 37]) is due to Davenport [8] and Davenport and Heilbronn [9].

LEMMA 3.1 (Davenport–Heilbronn). *We have*

$$\#\{f \in \mathcal{F}_X \cap V(\mathbb{Z}) : f \text{ irreducible}\} = c_1 X + o(X),$$

as $X \to \infty$, *where* $c_1 = \operatorname{meas}(\mathcal{F}_1) = \pi^2/3$.

The following result allows us to associate a φ-Selmer element of an elliptic curve E_k over $\mathbb{Q}$, with $k \in \mathbb{Z}$, to an element of $V(\mathbb{Z})$.

LEMMA 3.2. *Let* $k \in 3\mathbb{Z}$. *There exists an injective map from the set of non-identity elements of* $\mathrm{Sel}_{\widehat{\varphi}}(E_{-27k})$ *to the set of irreducible* $f \in \mathcal{F} \cap V(\mathbb{Z})_{4k}^{\mathrm{loc\,sol}}$.

Proof. Assume throughout the proof that $k \in 3\mathbb{Z}$. According to [3, Theorem 33] there is a bijection between the $\mathrm{SL}_2(\mathbb{Q})$-equivalence classes of $V(\mathbb{Q})_{4k}^{\mathrm{loc\,sol}}$ and elements of $\mathrm{Sel}_{\widehat{\varphi}}(E_{-27k})$. Under this bijection, the identity element of $\mathrm{Sel}_{\widehat{\varphi}}(E_{-27k})$ corresponds to the unique $\mathrm{SL}_2(\mathbb{Q})$-equivalence class of reducible forms in the space $V(\mathbb{Q})_{4k}$ (i.e. the orbit of $f(u, v) = ku^3 + 3uv^2$ under $\mathrm{SL}_2(\mathbb{Q})$). Next, it follows from [3, Theorem 34] that any $\mathrm{SL}_2(\mathbb{Q})$-equivalence class of $V(\mathbb{Q})_{4k}^{\mathrm{loc\,sol}}$ contains an element of $V(\mathbb{Z})_{4k}$. This therefore produces the required map from non-identity elements of $\mathrm{Sel}_{\widehat{\varphi}}(E_{-27k})$ to $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of irreducible elements of $V(\mathbb{Z})_{4k}^{\mathrm{loc\,sol}}$. Injectivity follows from injectivity of the map in [3, Theorem 33]. □

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3.3. *A positive proportion of curves with positive rank.* We now turn to the work of Kriz and Li [15]. Let $\mathcal{D} \subset \mathbb{Z}$ be the subset of fundamental discriminants. Then $d \in \mathcal{D}$ if and only if either $d \equiv 1 \bmod 4$ is square-free, or $d = 4d'$ for some square-free integer $d' \equiv \{2, 3\} \bmod 4$. We have the following result.

LEMMA 3.3. *We have*

$$\#\{k \in \mathcal{D} \cap (0, X] : \operatorname{rank} E_{-432k}(\mathbb{Q}) = 1\} \geqslant \frac{1}{2\pi^2} X + o(X),$$

as $X \to \infty$.

Proof. It follows from [15, Theorem 1.8] that the analytic rank of E_{-432k} is one for at least $\frac{1}{6}$ of fundamental discriminants $k \in \mathcal{D}$. By appealing to the celebrated work of Gross and Zagier [10], and Kolyvagin [14], we deduce that the same is true for the arithmetic rank. Finally, the statement of the lemma follows on recalling that $\#\mathcal{D} \cap (0, X] = (3/\pi^2)X + o(X)$, as $X \to \infty$. □

3.4. *Proof of the main result.* Since $432k \in 3\mathbb{Z}$, we begin by deducing from Lemma 3.2 that there is an injective map from soluble non-identity elements of $\mathrm{Sel}_{\widehat{\varphi}}(E_{27\cdot 432k})$ to the set of irreducible elements in $\mathcal{F} \cap V(\mathbb{Z})^{\mathrm{sol}}_{-4\cdot 432k}$. Recalling that $\widehat{\varphi}_{-432k} = \varphi_{27\cdot 432k}$, we may make the change of variables $27k \to -k$, in order to deduce that there is an injective map from soluble non-identity elements of $\mathrm{Sel}_{\varphi}(E_{-432k})$ to the set of irreducible elements in $\mathcal{F} \cap V(\mathbb{Z})^{\mathrm{sol}}_{64k}$. In view of (3.1), if $k \in \mathbb{Z}$ is such that $\operatorname{rank} E_{-432k}(\mathbb{Q}) = 1$, then either $E_{27\cdot 432k}(\mathbb{Q})/\varphi(E_{-432k}(\mathbb{Q}))$ or $E_{-432k}(\mathbb{Q})/\widehat{\varphi}(E_{27\cdot 432k}(\mathbb{Q}))$ must have size at least 3. But then it follows that there are at least two soluble non-identity elements belonging to $\mathrm{Sel}_{\varphi}(E_{-432k})$ or $\mathrm{Sel}_{\widehat{\varphi}}(E_{27\cdot 432k})$. This implies that

$$\begin{aligned}\#\{f \in \mathcal{F}_X \cap V(\mathbb{Z})^{\mathrm{sol}} : f \text{ irreducible}\} &\geqslant \sum_{\substack{0<k\leqslant X/(4\cdot 432)\\ \operatorname{rank} E_{-432k}(\mathbb{Q})=1}} 2 \\ &\geqslant \frac{1}{2^6 \cdot 3^3 \cdot \pi^2} X + o(X), \qquad (3.2)\end{aligned}$$

by Lemma 3.3. We henceforth set $c_2 = 1/(2^6 \cdot 3^3 \cdot \pi^2)$ for the constant appearing in this bound.

We need a version of this which runs over irreducible elements of $V(\mathbb{Z})^{\mathrm{sol}}$ which are ordered by height, for which we ape an argument of Bhargava [1, §4]. Let $B = [-1, 1]^4$. We require a lower bound for the counting function

$$N(X) = \#\{f \in X^{1/4}B \cap V(\mathbb{Z})^{\mathrm{sol}} : f \text{ irreducible}\},$$

as $X \to \infty$. Since $\mathcal{F}_X$ has finite volume, it follows that for any given $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that

$$\operatorname{meas}(r_\varepsilon B \cap \mathcal{F}_X) \geqslant (1 - \varepsilon) \operatorname{meas}(\mathcal{F}_X),$$

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for all $X > 1$. Thus there exist constants $\delta > 0$ and $r > 0$ such that

$$\mathrm{meas}(rX^{1/4}B \cap \mathcal{F}_X) \geqslant \left(1 - \frac{c_2}{c_1} + \delta\right) \mathrm{meas}(\mathcal{F}_X) = (c_1 - c_2 + \delta c_1)X,$$

for any $X > 1$, where we recall from Lemma 3.1 that $c_1 = \mathrm{meas}(\mathcal{F}_1) = \pi^2/3$. Now it easily follows from Lemma 3.1 and Hilbert's irreducibility theorem that

$$\#\{f \in rX^{1/4}B \cap \mathcal{F}_X \cap V(\mathbb{Z}) : f \text{ irreducible}\} \geqslant (c_1 - c_2 + \delta c_1)X + o(X).$$

The number of binary cubic forms contributing to the left-hand side which do not belong $V(\mathbb{Z})^{\mathrm{sol}}$ is

$$\begin{aligned}&\leqslant \#\{f \in \mathcal{F}_X \cap V(\mathbb{Z}) : f \text{ irreducible}\} - \#\{f \in \mathcal{F}_X \cap V(\mathbb{Z})^{\mathrm{sol}} : f \text{ irreducible}\}\\ &\leqslant (c_1 - c_2)X + o(X),\end{aligned}$$

by Lemma 3.1 and (3.2). Putting $Y = r^{-4}X$, it therefore follows that

$$\begin{aligned}N(X) &= \#\{f \in rY^{1/4}B \cap V(\mathbb{Z})^{\mathrm{sol}} : f \text{ irreducible}\}\\ &\geqslant \#\{f \in rY^{1/4}B \cap \mathcal{F}_Y \cap V(\mathbb{Z})^{\mathrm{sol}} : f \text{ irreducible}\}\\ &\geqslant \delta c_1 Y + o(Y).\end{aligned}$$

This completes the proof of Theorem 1.3.

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References

1. M. Bhargava, A positive proportion of plane cubics fail the Hasse principle. *Preprint*, 2014, arXiv:1402.1131.
2. M. Bhargava, J. Cremona and T. Fisher, The proportion of plane cubic curves over $\mathbb{Q}$ that everywhere locally have a point. *Int. J. Number Theory* **12** (2016), 1077–1092.
3. M. Bhargava, N. Elkies and A. Shnidman, The average size of the 3-isogeny Selmer groups of elliptic curves $y^2 = x^3 + k$. *Preprint*, 2016, arXiv:1610.05759.
4. M. Bright, Obstructions to the Hasse principle in families. *Preprint*, 2016, arXiv:1607.01303.
5. M. Bright, T. D. Browning and D. Loughran, Failures of weak approximation in families. *Compos. Math.* **152** (2016), 1435–1475.
6. J. W. S. Cassels and M. J. T. Guy, On the Hasse principle for cubic surfaces. *Mathematika* **13** (1966), 111–120.
7. J.-L. Colliot-Thélène and J.-J. Sansuc, La descente sur les variétés rationnelles. In *Journées de géométrie algébrique d'Angers (Juillet 1979)*, Sijthoff and Noordhoff (Alphen aan den Rijn and Germantown, MD, 1980), 223–237.
8. H. Davenport, On the class-number of binary cubic forms I and II. *J. Lond. Math. Soc.* **26** (1951), 183–198.
9. H. Davenport and H. Heilbronn, On the density of discriminants of cubic fields II. *Proc. R. Soc. Lond. Ser.* A **322** (1971), 405–420.
10. B. H. Gross and D. Zagier, Heegner points and derivatives of L-series. *Invent. Math.* **84** (1986), 225–320.

https://doi.org/10.1112/S0025579317000195 Published online by Cambridge University Press

11. Y. Harpaz and A. Skorobogatov, Hasse principle for Kummer varieties. *Algebra Number Theory* **10** (2016), 813–841.
12. D. R. Heath-Brown and B. Z. Moroz, On the representation of primes by cubic polynomials in two variables. *Proc. Lond. Math. Soc.* (3) **88** (2004), 289–312.
13. J. Jahnel, *Brauer Groups, Tamagawa Measures, and Rational Points on Algebraic Varieties*, American Mathematical Society (Providence, RI, 2014).
14. V. A. Kolyvagin, Euler systems. In *The Grothendieck Festschrift,* Vol. II (*Progress in Mathematics* 87), Birkhäuser (Boston, MA, 1990), 435–483.
15. D. Kriz and C. Li, Goldfeld's conjecture and congruences between Heegner points. *Preprint*, 2016, arXiv:1609.06687.
16. Y. I. Manin, Le groupe de Brauer–Grothendieck en géométrie diophantienne. In *Actes du Congrès International des Mathématiciens (Nice, 1970),* Vol. 1, Gauthier-Villars (Paris, 1971), 401–411.
17. P. Satgé, Un analogue du calcul de Heegner. *Invent. Math.* **87** (1987), 425–439.
18. B. Segre, On arithmetical properties of singular cubic surfaces. *J. Lond. Math. Soc.* **19** (1944), 84–91.
19. J. H. Silverman, *The Arithmetic of Elliptic Curves*, Springer (1986).
20. P. Swinnerton-Dyer, The Brauer group of cubic surfaces. *Math. Proc. Cambridge Philos. Soc.* **113** (1993), 449–460.
21. P. Swinnerton-Dyer, The solubility of diagonal cubic surfaces. *Ann. Sci. Éc. Norm. Supér.* (4) **34** (2001), 891–912.

T. D. Browning,
School of Mathematics,
University of Bristol,
Bristol BS8 1TW,
U.K.
E-mail: t.d.browning@bristol.ac.uk

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