RAYLEIGH'S PROBLEM FOR A SEMI-INFINITE PLATE

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1. *Introduction.* In this paper we shall consider the solution of the following problem. With Cartesian axes $O(xyz)$ an infinitely thin semi-infinite plate coincident with the half plane $y = 0, x \geqslant 0$ is initially at rest surrounded by viscous incompressible fluid also at rest; starting at time $t = 0$ the plate is made to move uniformly with velocity W parallel to Oz, and we are required to determine the subsequent motion of the fluid.

In 1911 Rayleigh (1) considered a similar problem, the only difference being that in his case the plane was infinite in both directions, occupying the whole plane $y = 0$; the fluid velocity distribution is then given by $W[1 - \operatorname{erf} \frac{1}{2}y(\nu t)^{-\frac{1}{2}}]$. Of particular interest was Rayleigh's application of the results to form a qualitative picture of the standard two-dimensional 'flat-plate problem' of the *steady* flow past a semi-infinite plate $y = 0, z \geqslant 0$ whose leading edge is normal to the direction of flow. This he obtained by replacing t by z/W in his unsteady-flow solution. The parameter $y(\nu z/W)^{-\frac{1}{2}}$ brought to light by this analysis is the one which appears in the boundary-layer solution of the steady problem, though the deduced skin friction coefficient is some 70 % in excess of the exact value.

Similarly, now that interest is developing in edge effects in three-dimensional boundary-layer theory, one may consider the problem enunciated above as throwing some light on the flow past a flat plate occupying the quarter plane $y = 0, x \geqslant 0, z \geqslant 0$ and fixed in a uniform stream moving parallel to Oz. Apart from this application, however, the problem in itself is not without interest in that a solution of the exact equations of flow is possible.

It is convenient to develop the solution in two forms:

(i) A solution in series of the equations in polar coordinates.

(ii) A solution by operational methods of the equations in parabolic coordinates.

Both forms of solution have their places in the computation of results. The series converges rapidly for distances from the edge up to $(\nu t)^{\frac{1}{2}}$, and it is in this region where, in fact, most of the substantial changes due to the edge occur. For larger distances the convergence is slower. The operational solution, on the other hand, provides expressions in the form of single integrals which are valid everywhere and ideal for computation for distances from the edge greater than $(\nu t)^{\frac{1}{2}}$; at smaller distances computation is still possible but is made awkward by the proximity of two singularities in the integrand.

2. *The series solution in polar coordinates.* The equations of motion and continuity possess a solution in which the velocity components parallel to Ox and Oy vanish everywhere provided w, the velocity component parallel to Oz, can be found as a function of x, y, t satisfying the equation

$$\frac{\partial w}{\partial t} = \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right), \tag{1}$$

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where ν is the kinematic viscosity, and the boundary conditions

$$w = 0 \quad \text{for} \quad t = 0, \tag{2}$$

and, when $t > 0$,

$$\left.\begin{aligned} w &= W \quad \text{when} \quad y = 0,\ x > 0, \\ w &\to 0 \ \text{as} \ |y| \to 0 \quad \text{when} \quad x > 0, \\ w &\to 0 \ \text{as} \ |x| \ \text{or} \ |y| \to 0 \quad \text{when} \quad x < 0. \end{aligned}\right\} \tag{3}$$

If we introduce polar coordinates in the x, y plane and write

$$w' = W - w \tag{4}$$

we have

$$\frac{\partial w'}{\partial t} = \nu\left(\frac{\partial^2 w'}{\partial r^2} + \frac{1}{r}\frac{\partial w'}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w'}{\partial \theta^2}\right), \tag{5}$$

where $w' = W$ at $t = 0$, $w' = 0$ when $\theta = 0, 2\pi$, $w' \to W$ as $r \to \infty$ when $\theta \neq 0, 2\pi$.

A solution in which r and t appear in the combination $r/(\nu t)^{\frac{1}{2}}$ is clearly to be expected, and we shall look for a solution of the form

$$w' = w'(\eta, \theta), \quad \text{where} \quad \eta = \frac{r^2}{4\nu t} = \frac{R^2}{4}, \tag{6}$$

say. Then

$$\frac{\partial}{\partial t} = -\frac{\eta}{t}\frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial r} = \frac{r}{2\nu t}\frac{\partial}{\partial \eta}, \quad \frac{\partial^2}{\partial r^2} = \frac{1}{2\nu t}\frac{\partial}{\partial \eta} + \frac{\eta}{\nu t}\frac{\partial^2}{\partial \eta^2},$$

and equation (5) becomes

$$\frac{\partial^2 w'}{\partial \eta^2} + \left(1 + \frac{1}{\eta}\right)\frac{\partial w'}{\partial \eta} + \frac{1}{4\eta^2}\frac{\partial^2 w'}{\partial \theta^2} = 0, \tag{7}$$

with the boundary conditions $w' = 0$ when $\theta = 0, 2\pi$; $w' \to W$ as $\eta \to \infty$ when $\theta \neq 0, 2\pi$, the two conditions from $t = 0$ and $r \to \infty$ coalescing into this single condition. We must, of course, in addition, have w' finite for $\eta = 0$.

Equation (7) possesses solutions of the type

$$v_k(\eta)\begin{matrix}\cos k\theta \\ \sin k\theta\end{matrix}, \tag{8}$$

where $k > 0$ and v_k is given by

$$\frac{d^2 v_k}{d\eta^2} + \left(1 + \frac{1}{\eta}\right)\frac{dv_k}{d\eta} - \frac{k^2}{4\eta^2} v_k = 0. \tag{9}$$

Two independent solutions of (9) are

$$\left.\begin{aligned} f(\eta) &= \eta^{\frac{1}{2}k} e^{-\eta}\, {}_1F_1(1 + \tfrac{1}{2}k, 1 + k, \eta), \\ g(\eta) &= \eta^{-\frac{1}{2}k} e^{-\eta}\, {}_1F_1(1 - \tfrac{1}{2}k, 1 - k, \eta), \end{aligned}\right\} \tag{10}$$

when k is not an integer; $g(\eta)$ is singular at $\eta = 0$. Furthermore, for large η,

$${}_1F_1(1 + \tfrac{1}{2}k, 1 + k, \eta) \sim \frac{(k)!}{(\tfrac{1}{2}k)!}\eta^{-\frac{1}{2}k} e^{\eta},$$

so that

$$f(\eta) \sim \frac{(k)!}{(\tfrac{1}{2}k)!}\text{*}. \tag{11}$$

* Where the factorial notation is used instead of the gamma function.

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The boundary condition $w' = 0$ at $\theta = 0$ and 2π and the symmetry implied by the relation $w'(\theta) = w'(2\pi - \theta)$ suggest that we should look for a Fourier sine-series representation of w' in terms of odd multiples of $\frac{1}{2}\theta$, namely,

$$\frac{w'}{W} = \sum_{n=1}^{\infty} A_n(\eta) \sin(n - \tfrac{1}{2})\theta. \tag{12}$$

If we assume for the moment that the expression thereby defined is a continuous function of θ for any finite η, and that

$$\lim_{\eta \to \infty} \frac{w'}{W} = \sum_{n=1}^{\infty} \lim_{\eta \to \infty} A_n(\eta) \sin(n - \tfrac{1}{2})\theta, \tag{13}$$

then this expression satisfies all the boundary conditions provided that

$$\lim_{\eta \to \infty} A_n(\eta) = \frac{4}{(2n-1)\pi}.$$

Then, by (8), (10) and (11),

$$\frac{w'}{W} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\left(\frac{2n-1}{4}\right)! \, \eta^{(2n-1)/4}}{\left(\frac{2n-1}{2}\right)! \, (2n-1)} e^{-\eta} \, {}_1F_1\left(\frac{2n+3}{4}, \frac{2n+1}{2}, \eta\right) \sin(n - \tfrac{1}{2})\theta \tag{14}$$

$$= \frac{4}{\pi} \Sigma B_n(\eta, \theta), \tag{15}$$

say, satisfies the differential equation term by term and, provided that the two assumptions just made are valid, it satisfies all the boundary conditions, including the condition that w' remains finite at $\eta = 0$.

We shall provide justification for the assumptions and also for the term-by-term differentiation by proving

(i) that the required term-by-term derivatives are uniformly convergent in η and in θ for $0 \leqslant \eta \leqslant D < \infty$ and $0 \leqslant \theta \leqslant 2\pi$, thus justifying term-by-term differentiation and showing that w' is a continuous function of θ;

(ii) that $\Sigma B_n(\eta, \theta)$ does, in fact, converge uniformly for η in the interval $\eta \geqslant 0$ (for any θ, $0 \leqslant \theta \leqslant 2\pi$), thereby showing that

$$\lim_{\eta \to \infty} \Sigma B_n(\eta, \theta) = \Sigma \lim_{\eta \to \infty} B_n(\eta, \theta).$$

For (i) we notice that, when $0 < \alpha < \gamma$ and $\eta > 0$

$${}_1F_1(\alpha, \gamma, \eta) < e^{\eta},$$

so that

$$|B_n(\eta, \theta)| < \frac{\left(\frac{2n-1}{4}\right)! \, \eta^{(2n-1)/4}}{\left(\frac{2n-1}{2}\right)! \, (2n-1)} = C_n(\eta), \tag{16}$$

say. Now $\sum_1^{\infty} C_n(\eta)$ converges uniformly for η in the interval $0 \leqslant \eta \leqslant D < \infty$ and hence so does $\sum_1^{\infty} B_n(\eta, \theta)$. By similar reasoning one can show that, omitting the initial terms which are singular at $\eta = 0$, $\sum_3^{\infty} \partial B_n(\eta, \theta)/\partial\eta$ and $\sum_5^{\infty} \partial^2 B_n(\eta, \theta)/\partial\eta^2$ converge uniformly in the same interval.

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Furthermore, the comparison series $\Sigma C_n(\eta)$ which we have used is independent of θ and hence $\Sigma B_n(\eta, \theta)$ is uniformly convergent in θ for $0 \leqslant \theta \leqslant 2\pi$ and any η, $0 \leqslant \eta \leqslant D < \infty$; by similar reasoning so are the term-by-term derivatives $\Sigma \frac{\partial}{\partial \theta} B_n(\eta, \theta)$ and $\Sigma \frac{\partial^2}{\partial \theta^2} B_n(\eta, \theta)$.

For (ii) let us write

$$u_n(\theta) = \frac{\sin(n-\frac{1}{2})\theta}{(2n-1)} \tag{17}$$

and

$$E_n(\eta) = \frac{\left(\frac{2n-1}{4}\right)!}{\left(\frac{2n-1}{2}\right)!} e^{-\eta} \eta^{(2n-1)/4} {}_1F_1\left(\frac{2n+3}{4}, \frac{2n+1}{2}, \eta\right), \tag{18}$$

so that

$$B_n(\eta, \theta) = E_n(\eta) u_n(\theta).$$

Now it can readily be seen that for given η in the interval $\eta \geqslant 0$ $E_n(\eta)$ is positive, bounded, and such that

$$E_{n+2} \leqslant E_n{}^*, \quad E_n = 0 \quad \text{when} \quad \eta = 0.$$

Then, since $\sum_{n=1}^{\infty} u_{2n-1}(\theta)$ and $\sum_{n=1}^{\infty} u_{2n}(\theta)$ both converge, a slight modification of the proof of Abel's theorem is sufficient to establish the uniform convergence of $\Sigma B_n(\eta, \theta)$. Therefore, we have finally, from (4) and (14), that

$$\frac{w}{W} = 1 - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\left(\frac{2n-1}{4}\right)!}{\left(\frac{2n-1}{2}\right)!(2n-1)} \left(\frac{R}{2}\right)^{n-\frac{1}{2}} e^{-\frac{1}{4}R^2} {}_1F_1\left(\frac{2n+3}{4}, \frac{2n+1}{2}, \frac{R^2}{4}\right) \sin(n-\tfrac{1}{2})\theta \tag{19}$$

represents the solution of our problem. It provides a very convenient expression for small values of R where the convergence is quite rapid†.

3. *The operational solution in parabolic coordinates.* Introducing Heaviside's operator p for $\partial/\partial t$ in equation (1) and writing q^2 for p/ν, we require to solve the equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - q^2 w = 0 \tag{20}$$

with the boundary conditions (3).

Gunn (2), following Lamb (3) and Sommerfeld (4), has considered solutions of equation (20) of the type we require by putting $w = e^{-qy} u$ or $w = e^{-qy} v$ and introducing parabolic coordinates ξ, η, where

$$x + iy = (\xi + i\eta)^2, \quad x = \xi^2 - \eta^2, \quad y = 2\xi\eta, \tag{21}$$

* In fact

$$1 - \frac{E_{n+2}}{E_n} = \frac{{}_1F_1\left(\frac{2n+3}{4}, \frac{2n+3}{2}, \eta\right)}{{}_1F_1\left(\frac{2n+3}{4}, \frac{2n+1}{2}, \eta\right)}.$$

† This solution can also be deduced from Carslaw ((5), § 62), since our equation (1) is the same as the equation of heat conduction, by putting the wedge angle $\theta_0 = 2\pi$ and following out the limiting process mentioned therein and developed in detail for application to a different problem in § 69. For our particular problem the method of solution adopted above, however, appears to be simpler.

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in the resulting equations for u and v. Finally, if we introduce coordinates ζ, τ, where

$$\zeta = \xi + \eta, \quad \tau = \xi - \eta, \tag{22}$$

it becomes evident that a particular solution of (20) is

$$w = e^{-qy}\left[A_1\int^{\xi+\eta} e^{q\zeta^2}d\zeta + B_1\int^{\xi-\eta} e^{-q\zeta^2}d\zeta + C_1\right] + e^{qy}\left[A_2\int^{\xi+\eta} e^{-q\zeta^2}d\zeta + B_2\int^{\xi-\eta} e^{q\zeta^2}d\zeta + C_2\right], \tag{23}$$

where A_1, A_2, B_1, B_2, C_1, C_2 are constants.

Now, in the four quadrants of the x, y plane, y, $\xi+\eta$, $\xi-\eta$ have the following signs:

Quadrant	y	$\xi+\eta$	$\xi-\eta$
1	+	+	+
2	+	+	−
3	−	+	−
4	−	−	−

and it can readily be verified that the expression

$$\frac{w}{W} = \left(\frac{q}{\pi}\right)^{\frac{1}{2}}\left[e^{-qy}\int_{\eta-\xi}^{\infty} e^{-q\zeta^2}d\zeta + e^{qy}\int_{\eta+\xi}^{\infty} e^{-q\zeta^2}d\zeta\right], \tag{24}$$

which is of the form (23), satisfies the boundary conditions (3) and is therefore the required operational solution. We may write (24) in the alternative form

$$\frac{w}{W} = \frac{1}{\pi^{\frac{1}{2}}}\left[e^{-qy}\int_{q^{\frac{1}{2}}(\eta-\xi)}^{\infty} e^{-\tau^2}d\tau + e^{qy}\int_{q^{\frac{1}{2}}(\eta+\xi)}^{\infty} e^{-\tau^2}d\tau\right] \tag{25}$$

$$= \tfrac{1}{2}\{2\cosh qy - e^{-qy}\operatorname{erf} q^{\frac{1}{2}}(\eta-\xi) - e^{qy}\operatorname{erf} q^{\frac{1}{2}}(\xi+\eta)\}, \tag{26}$$

where

$$\operatorname{erf} x = \frac{2}{\pi^{\frac{1}{2}}}\int_0^x e^{-v^2}dv.$$

4. *Interpretation of the operational solution.* (*a*) *First form.* The interpretation of equation (26) is provided by the contour integral

$$\frac{w}{W} = \frac{1}{4\pi i}\int_M \frac{e^{zt}}{z}[2\cosh(yz^{\frac{1}{2}}\nu^{-\frac{1}{2}}) - \exp(-yz^{\frac{1}{2}}\nu^{-\frac{1}{2}})\operatorname{erf} z^{\frac{1}{2}}\nu^{-\frac{1}{2}}(\eta-\xi) - \exp(yz^{\frac{1}{2}}\nu^{-\frac{1}{2}})\operatorname{erf} z^{\frac{1}{2}}\nu^{-\frac{1}{2}}(\eta+\xi)]dz, \tag{27}$$

where M consists of

(i) the line $z = \lambda e^{-i\pi}$ from $\lambda = \infty$ to $\lambda = r$,

(ii) the small circle $z = re^{i\theta}$ from $\theta = -\pi$ to $\theta = \pi$,

(iii) the line $z = \lambda e^{i\pi}$ from $\lambda = r$ to $\lambda = \infty$,

as $r \to 0$.

Since, when x is small, $\operatorname{erf} x \sim x$, the only contribution to the integral from the small circle arises from the first term in the integrand which gives a contribution $4\pi i$. Hence

$$\frac{w}{W} - 1 = \frac{1}{4\pi i}\int_0^{\infty}\frac{e^{-\lambda t}}{\lambda}\Big\{[\operatorname{erf}\{e^{-\frac{1}{2}i\pi}\lambda^{\frac{1}{2}}\nu^{-\frac{1}{2}}(\eta+\xi)\} - \operatorname{erf}\{e^{\frac{1}{2}i\pi}\lambda^{\frac{1}{2}}\nu^{-\frac{1}{2}}(\eta-\xi)\}]\exp(-i\lambda^{\frac{1}{2}}\nu^{-\frac{1}{2}}y) + [\operatorname{erf}\{e^{-\frac{1}{2}i\pi}\lambda^{\frac{1}{2}}\nu^{-\frac{1}{2}}(\eta-\xi)\} - \operatorname{erf}\{e^{\frac{1}{2}i\pi}\lambda^{\frac{1}{2}}\nu^{-\frac{1}{2}}(\eta+\xi)\}]\exp i\lambda^{\frac{1}{2}}\nu^{-\frac{1}{2}}y\Big\}d\lambda. \tag{28}$$

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Now consider
$$\operatorname{erf} z = \frac{2}{\pi^{\frac{1}{2}}}\int_0^z e^{-w^2}\,dw, \tag{29}$$

where, since e^{-w^2} has no singularities in the finite w plane any finite path from 0 to z is permissible. Let us choose the straight line and put $z = re^{\frac{1}{4}i\pi}$; then

$$\operatorname{erf} re^{\frac{1}{4}i\pi} = \frac{2}{\pi^{\frac{1}{2}}}\int_0^{re^{\frac{1}{4}i\pi}} e^{-w^2}\,dw = \frac{2}{\pi^{\frac{1}{2}}}e^{\frac{1}{4}i\pi}\int_0^r e^{-i\chi^2}\,d\chi \tag{30}$$

on putting $w = \chi e^{\frac{1}{4}i\pi}$.

Similarly
$$\operatorname{erf} re^{-\frac{1}{4}i\pi} = \frac{2}{\pi^{\frac{1}{2}}}e^{-\frac{1}{4}i\pi}\int_0^r e^{i\chi^2}\,d\chi, \tag{31}$$

and therefore (28) can be written in the form

$$\frac{w}{W}-1 = \frac{1}{\pi^{\frac{1}{2}}}\int_0^\infty \frac{e^{-\lambda t}}{\lambda}\left\{\int_0^{\lambda^{\frac{1}{2}}\nu^{-\frac{1}{2}}(\xi+\eta)} \sin(\chi^2-\lambda^{\frac{1}{2}}\nu^{-\frac{1}{2}}y-\tfrac{1}{4}\pi)\,d\chi + \int_0^{\lambda^{\frac{1}{2}}\nu^{-\frac{1}{2}}(\eta-\xi)} \sin(\chi^2+\lambda^{\frac{1}{2}}\nu^{-\frac{1}{2}}y-\tfrac{1}{4}\pi)\,d\chi\right\}d\lambda. \tag{32}$$

Finally, put
$$\lambda t = u^2, \quad \chi^2 = v, \quad y = (\nu t)^{\frac{1}{2}}Y, \quad r = (x^2+y^2)^{\frac{1}{2}} = (\nu t)^{\frac{1}{2}}R, \tag{33}$$

so that, since
$$(\xi+\eta)^2 = r+y, \quad (\xi-\eta)^2 = r-y,$$

$$\frac{w}{W}-1 = \frac{1}{\pi^{\frac{1}{2}}}\int_0^\infty \frac{e^{-u^2}}{u}\left\{\int_0^{u(R+Y)} \sin(v-uY-\tfrac{1}{4}\pi)\,v^{-\frac{1}{2}}\,dv + \int_0^{u(R-Y)} \sin(v+uY-\tfrac{1}{4}\pi)\,v^{-\frac{1}{2}}\,dv\right\}du, \tag{34}$$

when $\xi+\eta$ and $\eta-\xi$ are both positive, that is, in the second quadrant, whilst in the first quadrant, where $\xi+\eta$ and $\xi-\eta$ are both positive,

$$\frac{w}{W}-1 = \frac{1}{\pi^{\frac{1}{2}}}\int_0^\infty \frac{e^{-u^2}}{u}\left\{\int_0^{u(R+Y)} \sin(v-uY-\tfrac{1}{4}\pi)\,v^{-\frac{1}{2}}\,dv - \int_0^{u(R-Y)} \sin(v+uY-\tfrac{1}{4}\pi)\,v^{-\frac{1}{2}}\,dv\right\}du. \tag{35}$$

The integrals inside the curly brackets in equations (34) and (35) are, of course, expressible in terms of Fresnel's integrals but, even though these integrals are tabulated, the expressions (34) and (35) are not well suited to computation. We shall now consider a transformation of these expressions into forms more suited for calculation.

(*b*) *Second form.* Let us write

$$\phi_1 = 2^{\frac{1}{2}}\int_0^\infty \frac{e^{-u^2}}{u}\int_0^{u(R-Y)} \sin(v+uY-\tfrac{1}{4}\pi)\,v^{-\frac{1}{2}}\,dv\,du, \tag{36}$$

$$\phi_2 = -2^{\frac{1}{2}}\int_0^\infty \frac{e^{-u^2}}{u}\int_0^{u(R+Y)} \sin(v-uY-\tfrac{1}{4}\pi)\,v^{-\frac{1}{2}}\,dv\,du, \tag{37}$$

so that
$$1-\frac{w}{W} = \frac{1}{(2\pi^3)^{\frac{1}{2}}}(\phi_2 \pm \phi_1), \tag{38}$$

respectively, in the first and second quadrants. Now

$$\phi_1 = \int_0^\infty \frac{e^{-u^2}}{u}\int_0^{u(R-Y)} \{\sin(v+uY)-\cos(v+Yu)\}\,v^{-\frac{1}{2}}\,dv\,du \tag{39}$$

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and
$$\phi_2 = \int_0^\infty \frac{e^{-u^2}}{u} \int_0^{u(R+Y)} \{\cos(v - Yu) - \sin(v - Yu)\} v^{-\frac{1}{2}} dv\, du. \tag{40}$$

Let us consider ϕ_2 in detail, putting $v = ux$ in the inner integral; then

$$\phi_2 = \int_0^\infty \frac{e^{-u^2}}{u^{\frac{1}{2}}} \int_0^{R+Y} \{\cos(x - Y)u - \sin(x - Y)u\} x^{-\frac{1}{2}} dx\, du. \tag{41}$$

Now
$$\int_0^{R+Y} |\cos(x - Y)u - \sin(x - Y)u|\, x^{-\frac{1}{2}} dx \leqslant 4(R + Y)^{\frac{1}{2}},$$

and hence ϕ_2 converges absolutely since $\int_0^\infty u^{-\frac{1}{2}} e^{-u^2} du$ converges. Moreover,

$$\frac{e^{-u^2}}{u^{\frac{1}{2}}} \int_0^{R+Y} \{\cos(x - Y)u - \sin(x - Y)u\} x^{-\frac{1}{2}} dx$$

is a continuous function of u except at the isolated point $u = 0$. Therefore we may interchange the order of integration and write

$$\phi_2 = \int_0^{R+Y} \frac{dx}{x^{\frac{1}{2}}} \int_0^\infty \frac{e^{-u^2}}{u^{\frac{1}{2}}} \{\cos(x - Y)u - \sin(x - Y)u\} du, \tag{42}$$

provided that
$$\frac{1}{x^{\frac{1}{2}}} \int_0^\infty \frac{e^{-u^2}}{u^{\frac{1}{2}}} \{\cos(x - Y)u - \sin(x - Y)u\} du$$

is a continuous function of x except at isolated points. Let us write

$$\chi_2 = \int_0^\infty \frac{e^{-u^2}}{u^{\frac{1}{2}}} \{\cos(x - Y)u - \sin(x - Y)u\} du, \tag{43}$$

so that when $x > Y$ we may put

$$\chi_2 = \{\tfrac{1}{2}\pi(x - Y)\}^{\frac{1}{2}} \int_0^\infty e^{-u^2} \{J_{-\frac{1}{4}}(x - Y)u - J_{\frac{1}{4}}(x - Y)u\} du. \tag{44}$$

Now*
$$\int_0^\infty e^{-u^2} J_\nu(\beta u)\, du = \tfrac{1}{2}\sqrt{\pi}\, e^{-\frac{1}{8}\beta^2} I_\nu(\tfrac{1}{8}\beta^2). \tag{45}$$

Therefore, with $z = x - Y$,

$$\chi_2 = \frac{\pi}{2^{\frac{1}{2}}} z^{\frac{1}{2}} e^{-\frac{1}{8}z^2} [I_{-\frac{1}{4}}(\tfrac{1}{8}z^2) - I_{\frac{1}{4}}(\tfrac{1}{8}z^2)] \tag{46}$$

$$= \tfrac{1}{2} z^{\frac{1}{2}} e^{-\frac{1}{8}z^2} K_{\frac{1}{4}}(\tfrac{1}{8}z^2). \tag{47}$$

Similarly, when $x < Y$ we see from (43) that

$$\chi_2 = \{\tfrac{1}{2}\pi(Y - x)\}^{\frac{1}{2}} \int_0^\infty e^{-u^2} [J_{-\frac{1}{4}}(Y - x)u + J_{\frac{1}{4}}(Y - x)u]\, du, \tag{48}$$

and therefore, with $\zeta = Y - x$,

$$\chi_2 = \frac{\pi}{2^{\frac{1}{2}}} \zeta^{\frac{1}{2}} e^{-\frac{1}{8}\zeta^2} [I_{-\frac{1}{4}}(\tfrac{1}{8}\zeta^2) + I_{\frac{1}{4}}(\tfrac{1}{8}\zeta^2)]. \tag{49}$$

Therefore, since $x^{-\frac{1}{2}}\chi_2$ is a continuous function of x except for the isolated point $x = 0$, it follows from (42) that

$$\phi_2 = \frac{1}{2} \int_0^R \left(\frac{z}{z + Y}\right)^{\frac{1}{2}} e^{-\frac{1}{8}z^2} K_{\frac{1}{4}}(\tfrac{1}{8}z^2)\, dz + \frac{\pi}{2^{\frac{1}{2}}} \int_0^Y \left(\frac{\zeta}{Y - \zeta}\right)^{\frac{1}{2}} e^{-\frac{1}{8}\zeta^2} [I_{-\frac{1}{4}}(\tfrac{1}{8}\zeta^2) + I_{\frac{1}{4}}(\tfrac{1}{8}\zeta^2)]\, d\zeta. \tag{50}$$

* See, for example, Watson ((6), p. 394).

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Similarly,
$$\phi_1 = \int_0^\infty \frac{e^{-u^2}}{u^{\frac{1}{2}}} \int_0^{R-Y} [\sin(x+Y)u - \cos(x+Y)u] x^{-\frac{1}{2}}\, dx\, du \tag{51}$$
$$= -\frac{1}{2}\int_Y^R \left(\frac{v}{v-Y}\right)^{\frac{1}{2}} e^{-\frac{1}{8}v^2} K_{\frac{1}{4}}(\tfrac{1}{8}v^2)\, dv, \tag{52}$$

since $x+Y \geqslant 0$ throughout the range of integration.

(c) *Third form.* These expressions for ϕ_1 and ϕ_2 are still not very convenient for computation, particularly when R is large. We can achieve some further simplification by considering the limits of ϕ_1 and ϕ_2 as R tends to infinity. Consider ϕ_1; we may put, with $\alpha = R-Y$,

$$\phi_1 = \psi_1 - \psi_2, \tag{53}$$

where
$$\psi_1 = \int_0^\infty \frac{e^{-u^2}}{u} \sin Yu \int_0^{\alpha u} (\cos v + \sin v)\, v^{-\frac{1}{2}}\, dv\, du, \tag{54}$$
$$\psi_2 = \int_0^\infty \frac{e^{-u^2}}{u} \cos Yu \int_0^{\alpha u} (\cos v - \sin v)\, v^{-\frac{1}{2}}\, dv\, du. \tag{55}$$

Now
$$\int_0^\infty \frac{\cos v}{v^{\frac{1}{2}}} dv = \int_0^\infty \frac{\sin v}{v^{\frac{1}{2}}} dv = (\tfrac{1}{2}\pi)^{\frac{1}{2}}.$$

Therefore,
$$\psi_1 = (2\pi)^{\frac{1}{2}} \int_0^\infty \frac{e^{-u^2}}{u} \sin Yu\, du - \chi_1, \tag{56}$$

where
$$\chi_1 = \int_0^\infty \frac{e^{-u^2}}{u} \sin Yu \int_{\alpha u}^\infty (\cos v + \sin v)\, v^{-\frac{1}{2}}\, dv\, du. \tag{57}$$

Now integration by parts suffices to show that
$$\left| \int_{\alpha u}^\infty \frac{\cos v}{v^{\frac{1}{2}}} dv \right| \leqslant \frac{2}{(\alpha u)^{\frac{1}{2}}} \quad \text{and} \quad \left| \int_{\alpha u}^\infty \frac{\sin v}{v^{\frac{1}{2}}} dv \right| \leqslant \frac{2}{(\alpha u)^{\frac{1}{2}}}.$$

Hence, since $|\sin Yu| < Yu$,
$$|\chi_1| \leqslant \frac{4Y}{\alpha^{\frac{1}{2}}} \int_0^\infty \frac{e^{-u^2}}{u^{\frac{1}{2}}} du,$$

so that $\chi_1 \to 0$ **as** $\alpha \to \infty$. **Therefore**
$$\lim_{R\to\infty} \psi_1 = (2\pi)^{\frac{1}{2}} \int_0^\infty \frac{e^{-u^2}}{u} \sin Yu\, du = 2^{\frac{1}{2}}\pi \int_0^{\frac{1}{2}Y} e^{-u^2} du = (\tfrac{1}{2}\pi^3)^{\frac{1}{2}} \operatorname{erf} \tfrac{1}{2}Y. \tag{58}$$*

Next consider ψ_2, **putting** $x = u/\beta$, **where** $\beta = 1/\alpha = 1/(R-Y)$, **so that**
$$\psi_2 = \int_0^\infty e^{-\beta^2 x^2} \cos \beta Yx \int_0^x (\cos v - \sin v)\, v^{-\frac{1}{2}}\, dv \frac{dx}{x}, \tag{59}$$

and let us introduce χ_3 **defined by**
$$\chi_3 = \int_0^\infty e^{-\beta^2 x^2} \int_0^x (\cos v - \sin v)\, v^{-\frac{1}{2}}\, dv \frac{dx}{x}. \tag{60}$$

Now χ_3 **converges uniformly in** β **at both termini for** $\beta \geqslant 0$ **by Abel's test since, as will be shown below,**
$$\int_0^\infty \frac{1}{x} \int_0^x (\cos v - \sin v)\, v^{-\frac{1}{2}}\, dv\, dx$$

* See, for example, Carslaw and Jaeger ((7), p. 94).

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converges. Furthermore,

$$\left|\int_0^x (\cos v - \sin v)\, v^{-\frac{1}{2}}\, dv\right| = \left|\int_x^\infty (\cos v - \sin v)\, v^{-\frac{1}{2}}\, dv\right| \leqslant \frac{4}{x^{\frac{1}{2}}}$$

as before, so that

$$|\chi_3 - \psi_2| \leqslant 4\int_0^\infty e^{-\beta^2 x^2}(1-\cos\beta Yx)\, x^{-\frac{1}{2}}\, dx = 4(\beta Y)^{\frac{1}{2}}\int_0^\infty e^{-z^2/Y^2}(1-\cos z)\, z^{-\frac{1}{2}}\, dz$$

on putting $z = \beta Yx$. Therefore since this last integral converges

$$\begin{aligned}\lim_{R\to\infty}\psi_2 = \lim_{\beta\to 0}\chi_3 &= \int_0^\infty \frac{1}{x}\int_0^x (\cos v - \sin v)\, v^{-\frac{1}{2}}\, dv\, dx \\ &= -\int_0^\infty (\cos x - \sin x)\, x^{-\frac{1}{2}} \log x\, dx \\ &= -(\tfrac{1}{2}\pi)^{\frac{1}{2}}\int_0^\infty [J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x)] \log x\, dx \\ &= (\tfrac{1}{2}\pi^3)^{\frac{1}{2}}. \qquad (61)^*\end{aligned}$$

Therefore from equations (53), (58) and (61)

$$\lim_{R\to\infty}\phi_1 = (\tfrac{1}{2}\pi^3)^{\frac{1}{2}}(\operatorname{erf}\tfrac{1}{2}Y - 1). \qquad (62)$$

Similarly,

$$\lim_{R\to\infty}\phi_2 = (\tfrac{1}{2}\pi^3)^{\frac{1}{2}}(\operatorname{erf}\tfrac{1}{2}Y + 1). \qquad (63)$$

Finally, from equations (52) and (50) we may make use of these two results to write

$$\phi_1 = (\tfrac{1}{2}\pi^3)^{\frac{1}{2}}(\operatorname{erf}\tfrac{1}{2}Y - 1) + \frac{1}{2}\int_R^\infty \left(\frac{v}{v-Y}\right)^{\frac{1}{2}} e^{-\frac{1}{8}v^2} K_{\frac{1}{4}}(\tfrac{1}{8}v^2)\, dv \qquad (64)$$

and

$$\phi_2 = (\tfrac{1}{2}\pi^3)^{\frac{1}{2}}(\operatorname{erf}\tfrac{1}{2}Y + 1) - \frac{1}{2}\int_R^\infty \left(\frac{v}{v+Y}\right)^{\frac{1}{2}} e^{-\frac{1}{8}v^2} K_{\frac{1}{4}}(\tfrac{1}{8}v^2)\, dv. \qquad (65)$$

Therefore from equation (38)

$$1 - \frac{w}{W} = \operatorname{erf}\tfrac{1}{2}Y + \frac{1}{(2\pi)^{\frac{1}{2}}}\int_R^\infty \left[\left(\frac{v}{v-Y}\right)^{\frac{1}{2}} - \left(\frac{v}{v+Y}\right)^{\frac{1}{2}}\right] e^{-\frac{1}{8}v^2} K_{\frac{1}{4}}(\tfrac{1}{8}v^2)\, dv \qquad (66)$$

in the first quadrant, and

$$\frac{w}{W} = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_R^\infty \left[\left(\frac{v}{v-Y}\right)^{\frac{1}{2}} + \left(\frac{v}{v+Y}\right)^{\frac{1}{2}}\right] e^{-\frac{1}{8}v^2} K_{\frac{1}{4}}(\tfrac{1}{8}v^2)\, dv \qquad (67)$$

in the second quadrant.

5. *Skin friction.* Since at the wall

$$\left(\frac{\partial w}{\partial y}\right)_{\substack{r=\text{const.}\\ y=0}} = \left(\frac{\partial w}{\partial y}\right)_{\substack{x=\text{const.}\\ y=0}}, \qquad (68)$$

the skin friction

$$\tau = -\mu\left(\frac{\partial w}{\partial y}\right)_{\substack{x=\text{const.}\\ y=0}} \qquad (69)$$

* This integral may be evaluated from Weber's infinite integral

$$\int_0^\infty \frac{J_\nu(t)}{t^{\nu-\mu+1}}\, dt = (\tfrac{1}{2}\mu - 1)!/2^{\mu-\nu+1}(\nu - \tfrac{1}{2}\mu)!$$

(with the factorial notation) by differentiating with respect to μ (see Watson (6), pp. 391, 329 and equation (4), p. 393).

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is, from equation (66), given by

$$\tau = \frac{\mu W}{(\nu t)^{\frac{1}{2}}}\left\{\frac{1}{\pi^{\frac{1}{2}}} + \frac{1}{(2\pi)^{\frac{1}{2}}}\left[\frac{\partial}{\partial Y}\int_R^\infty e^{-\frac{1}{4}v^2}\left[\left(\frac{v}{v-Y}\right)^{\frac{1}{2}} - \left(\frac{v}{v+Y}\right)^{\frac{1}{2}}\right] K_{\frac{1}{4}}(\tfrac{1}{8}v^2)\,dv\right]_{Y=0}\right\}. \tag{70}$$

We may differentiate under the integral sign, since the resulting integral is uniformly convergent for Y in the interval $0 \leqslant Y \leqslant \frac{1}{2}R$, say, and has an integrand continuous in v and Y. Hence, for $R > 0$,

$$\tau = \frac{\mu W}{(\nu t)^{\frac{1}{2}}}\left[\frac{1}{\pi^{\frac{1}{2}}} + \frac{1}{(2\pi)^{\frac{1}{2}}}\int_R^\infty e^{-\frac{1}{4}v^2} K_{\frac{1}{4}}(\tfrac{1}{8}v^2)\frac{dv}{v}\right]. \tag{71}$$

As might be expected the integral diverges at $R = 0$ and behaves like $R^{-\frac{1}{2}}$ for small values of R. However, the integral

$$E = \frac{\mu W}{(2\pi)^{\frac{1}{2}}}\int_0^\infty dR\int_R^\infty e^{-\frac{1}{4}v^2} K_{\frac{1}{4}}(\tfrac{1}{8}v^2)\frac{dv}{v}, \tag{72}$$

which represents the integrated excess of the skin friction over Rayleigh's value of $\mu W(\pi\nu t)^{-\frac{1}{2}}$, converges. Moreover, it can readily be shown that the order of the integrations in E can be interchanged, and so we have

$$E = \frac{\mu W}{(2\pi)^{\frac{1}{2}}}\int_0^\infty \frac{e^{-\frac{1}{4}v^2}}{v} K_{\frac{1}{4}}(\tfrac{1}{8}v^2)\int_0^v dR\,dv = \frac{\mu W}{(2\pi)^{\frac{1}{2}}}\int_0^\infty e^{-\frac{1}{4}v^2} K_{\frac{1}{4}}(\tfrac{1}{8}v^2)\,dv. \tag{73}$$

The value of the last integral in equation (73) may be deduced by making $Y \to 0$ and $R \to \infty$ in equation (52) and then using (62). Thus

$$\frac{1}{(2\pi^3)^{\frac{1}{2}}}\int_0^\infty e^{-\frac{1}{4}v^2} K_{\frac{1}{4}}(\tfrac{1}{8}v^2)\,dv = 1. \tag{74}$$

Hence

$$E = \tfrac{1}{2}\mu W, \tag{75}$$

and is independent of the time.

We may compute the actual skin friction at any value of R from (71) or from the series obtained by differentiating (19), namely,

$$\tau = \frac{2\mu W e^{-\frac{1}{4}R^2}}{\pi(\nu t)^{\frac{1}{2}}}\left[\sum_{n=1}^{\infty}\frac{\left(\frac{2n-1}{4}\right)!}{\left(\frac{2n-1}{2}\right)!}\frac{R^{n-\frac{1}{2}}}{2^{n-\frac{1}{2}}}\,{}_1F_1(\tfrac{1}{2}n+\tfrac{3}{4}, n+\tfrac{1}{2}, \tfrac{1}{4}R^2)\right]. \tag{76}$$

Equation (76) leads to the series

$$\frac{(\nu t)^{\frac{1}{2}}}{\mu W}\tau = R^{-\frac{1}{2}}[0{\cdot}46040 + 0{\cdot}15549R + 0{\cdot}01918R^2 - 0{\cdot}00388R^3 - 0{\cdot}00086R^4 + \ldots], \tag{77}$$

which is very convenient for computation for $R \leqslant 1$. We can also obtain the value of $\int_0^\tau \tau\,dr$ in the form

$$\frac{1}{\mu W}\int_0^\tau \tau\,dr = R^{\frac{1}{2}}[0{\cdot}9208 + 0{\cdot}1037R + 0{\cdot}0077R^2 - 0{\cdot}0011R^3 - 0{\cdot}0002R^4 + \ldots]. \tag{78}$$

For $R \geqslant 1$ it is convenient to make use of equation (71) where, for ease in computation, we replace v as the variable of integration by $\chi = \frac{1}{8}v^2$ so that

$$\frac{(\nu t)^{\frac{1}{2}}}{\mu W}\tau = \left[\frac{1}{\pi^{\frac{1}{2}}} + \frac{2^{\frac{1}{2}}}{8\pi^{\frac{1}{2}}}\int_{\frac{1}{8}R^2}^\infty e^{-\chi} K_{\frac{1}{4}}(\chi)\frac{d\chi}{\chi}\right]. \tag{79}$$

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The following asymptotic expansion of the integral in (79) can readily be obtained, namely,

$$\int_\chi^\infty e^{-\chi} K_{\frac{1}{4}}(\chi)\frac{d\chi}{\chi} = \frac{(2\pi)^{\frac{1}{2}}}{4} e^{-2\chi}\left[\frac{1}{\chi^2} - \frac{35}{32}\frac{1}{\chi^3} + \frac{945}{512}\frac{1}{\chi^4} + \ldots\right] \tag{80}$$

for large values of χ.

Results for τ obtained from the series in (77) for $R \leqslant 1$ and for $R \geqslant 1$ by computation from (79) and (80) are shown in Table 1. Also included in the table are values of $\frac{1}{\mu W}\int_0^\tau (\tau - \tau_R)\,dr$, where τ_R is Rayleigh's value of the skin friction for the whole plane, namely,

$$\tau_R = \frac{\mu W}{(\pi\nu t)^{\frac{1}{2}}}. \tag{81}$$

Table 1

$R = \frac{r}{(\nu t)^{\frac{1}{2}}}$	$\left(\frac{W^2 t}{\nu}\right)^{\frac{1}{2}} \frac{\tau}{\rho W^2}$	$\frac{1}{\mu W}\int_0^\tau (\tau - \tau_R)\,dr$
0·01	4·620	0·087
0·05	2·094	0·179
0·10	1·506	0·238
0·20	1·101	0·308
0·40	0·831	0·384
0·60	0·723	0·424
0·80	0·665	0·450
1·0	0·630	0·466
1·2	0·608	0·477
1·4	0·584	0·483
1·6	0·577	0·486
1·8	0·573	0·489
2·0	0·570	0·491
2·5	0·567	0·493
3·0	0·565	0·494
3·5	0·565	0·494
4·0	0·564	0·495
$\lim R \to \infty$	0·564	0·500

6. *Velocity distribution.* The velocity distribution may similarly be obtained from the series in equation (19) for $R \leqslant 1$ and from the integrals in (66) and (67) when $R \geqslant 1$. The following expressions for $1 - w/W$ have been obtained from (19) for $R = 0{\cdot}2$, $0{\cdot}4$, $0{\cdot}6$, $0{\cdot}8$, $1{\cdot}0$ respectively:

$0{\cdot}4111 \sin\tfrac{1}{2}\theta + 0{\cdot}0092 \sin\tfrac{3}{2}\theta + 0{\cdot}0003 \sin\tfrac{5}{2}\theta + \ldots,$

$0{\cdot}5785 \sin\tfrac{1}{2}\theta + 0{\cdot}0259 \sin\tfrac{3}{2}\theta + 0{\cdot}0015 \sin\tfrac{5}{2}\theta + 0{\cdot}0001 \sin\tfrac{7}{2}\theta + \ldots,$

$0{\cdot}7028 \sin\tfrac{1}{2}\theta + 0{\cdot}0469 \sin\tfrac{3}{2}\theta + 0{\cdot}0041 \sin\tfrac{5}{2}\theta + 0{\cdot}0004 \sin\tfrac{7}{2}\theta + \ldots,$

$0{\cdot}8025 \sin\tfrac{1}{2}\theta + 0{\cdot}0710 \sin\tfrac{3}{2}\theta + 0{\cdot}0083 \sin\tfrac{5}{2}\theta + 0{\cdot}0010 \sin\tfrac{7}{2}\theta + 0{\cdot}0001 \sin\tfrac{9}{2}\theta + \ldots,$

$0{\cdot}8847 \sin\tfrac{1}{2}\theta + 0{\cdot}0981 \sin\tfrac{3}{2}\theta + 0{\cdot}0141 \sin\tfrac{5}{2}\theta + 0{\cdot}0020 \sin\tfrac{7}{2}\theta + 0{\cdot}0003 \sin\tfrac{9}{2}\theta + \ldots.$

For $R < 0{\cdot}2$ the two leading terms in the expansion will be sufficient for most practical purposes. Thus we have for R sufficiently small

$$1 - \frac{w}{W} = 0{\cdot}9208 R^{\frac{1}{2}} \sin\tfrac{1}{2}\theta + 0{\cdot}1037 R^{\frac{1}{2}} \sin\tfrac{3}{2}\theta.$$

The results are shown graphically in Fig. 1, where w/W is plotted as a function of $X = x(\nu t)^{-\frac{1}{2}}$ for discrete values of $Y = y(\nu t)^{-\frac{1}{2}}$.

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7. *Discussion of results.* From Table 1 we see that the edge effect on the skin friction is less than 1 % when $X = R > 2$, and this fact is reflected in the velocity distribution near the plate, where X being positive in this region, the changes in velocity when $Y = \frac{1}{2}$ and $Y = 1$ are almost entirely confined to the region $X < 1$. As one might expect the effect of the edge on the velocity tends to become more diffuse as Y increases.

When X is negative it is evident from (67) that when R is large and Y is finite the leading term in the asymptotic expansion is independent of Y, and as will be seen the curves in Fig. 1 all run together; the material changes in velocity in this region are confined to $|X| < 2$.

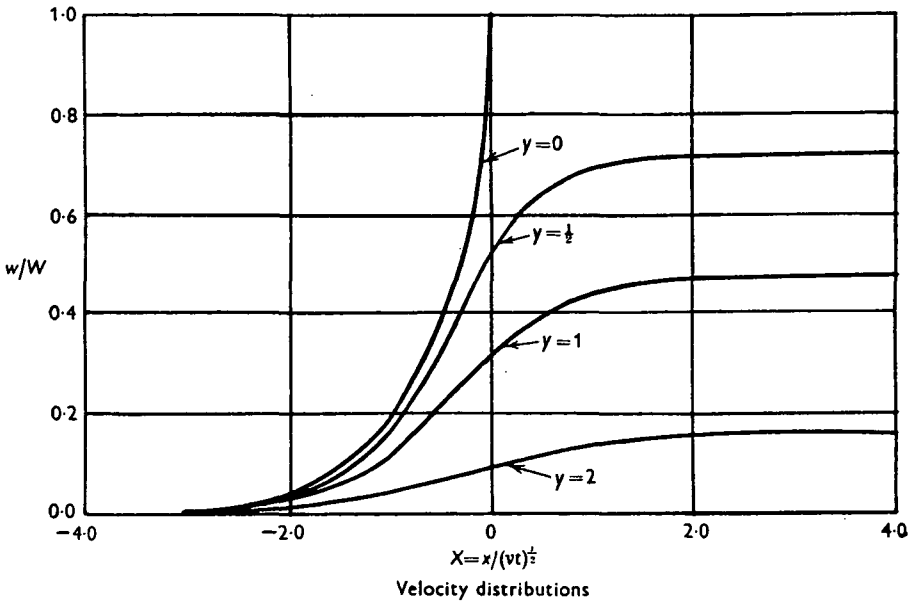


Fig. 1.

We may therefore expect our solution to be a reasonable approximation to conditions when a strip of breadth b (instead of a half-plane) is started to move in the same way provided that

$$4(\nu t)^{\frac{1}{2}} < b, \tag{82}$$

that is, provided that $t < b^2/(16\nu)$. Furthermore, the frictional force per unit length on such a plate would be expected to be, from (75),

$$\mu W\left[1 + \frac{b}{(\pi\nu t)^{\frac{1}{2}}}\right] \tag{83}$$

for one side of the plate when allowance is made for *both* edge effects.

Moreover, if in our original problem of the half-plane we follow Rayleigh and replace t by z/W, we should expect to form a qualitative picture of the flow with velocity W past the quarter plane $y = 0$, $x \geqslant 0$, $z \geqslant 0$, whose leading edge is normal to the incident flow. Thus we should expect velocity distributions $1 - w/W$ similar to those in Fig. 1

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for w/W when X and Y are interpreted as $x(W/\nu z)^{\frac{1}{2}}$ and $y(W/\nu z)^{\frac{1}{2}}$, the main edge effects being confined to the region

$$\frac{x}{z} < 2\left(\frac{\nu}{Wz}\right)^{\frac{1}{2}},$$

that is, to the interior of a half-parabola whose vertex is at the corner, whose axis is parallel to the incident stream, and whose latus rectum is of length $4\nu/W$.

We may state this result in a somewhat different way by introducing the idea of the boundary layer. When X is large and positive velocity changes are for all practical purposes confined to the region $Y < 4$; for smaller values of X the changes are confined to a region of similar order though somewhat smaller. Thus the effect of the edge on the boundary layer is to decrease the thickness a little in the neighbourhood of the edge. Furthermore, we may say that the edge effect, being confined to $X < 2$, affects a region whose breadth is of the order of about half the boundary thickness. That the edge effect should be confined to a region of the order of the boundary-layer thickness is not indeed surprising. We are also left in little doubt that the appropriate coordinates to introduce for an attack on this steady motion problem are $x(W/\nu z)^{\frac{1}{2}}$ and $y(W/\nu z)^{\frac{1}{2}}$, or some combination of them.

The skin friction would, on Rayleigh's hypothesis, be expected to be similar in form to that implied by Table 1 when t is replaced by z/W, the integrated excess frictional force due to edge being the same at all distances from the leading edge and of amount $\frac{1}{2}\mu W$ per unit length (in the direction Oz).

Finally, we may consider the implications of Rayleigh's hypothesis as far as it concerns a semi-infinite strip of breadth b whose leading edge is normal to the incident stream. From (82) and (83) we then expect that, provided that

$$b > 4\left(\frac{\nu z}{W}\right)^{\frac{1}{2}}, \tag{84}$$

we should have as the skin friction per unit length on a strip at distance z from the leading edge

$$\mu W\left[1 + b\left(\frac{W}{\pi\nu z}\right)^{\frac{1}{2}}\right], \tag{85}$$

corresponding to an effective increase in breadth of $(\pi\nu z/W)^{\frac{1}{2}}$.

Of course, unless the length of the strip is finite (84) will be violated sufficiently far downstream on the plate. Let us suppose we have a plate of finite length l and that $b > 4(\nu l/W)^{\frac{1}{2}}$; then the local skin friction coefficient c_f for a strip at distance z from the leading edge is given by

$$c_f = \frac{1}{\frac{1}{2}\rho W^2 b}\int_0^b \tau\, dx = 2\left(\frac{\nu}{\pi W z}\right)^{\frac{1}{2}}\left[1 + \frac{1}{b}\left(\frac{\pi\nu z}{W}\right)^{\frac{1}{2}}\right]. \tag{86}$$

Integrating to get the skin friction coefficient C_f for the whole of the plate of finite length l we have, for both sides,

$$C_f = \frac{2}{l}\int_0^l c_f\, dz = 8\left(\frac{\nu}{\pi W l}\right)^{\frac{1}{2}}\left[1 + \frac{l}{2b}\left(\frac{\pi\nu}{Wl}\right)^{\frac{1}{2}}\right]. \tag{87}$$

Thus the proportional correction in C_f for the edge effects is $\frac{1}{2}(l/b)(\pi\nu/Wl)^{\frac{1}{2}}$. The minimum breadth of plate for which this result could conceivably hold is $4(\nu l/W)^{\frac{1}{2}}$ when this value is $\frac{1}{8}\pi^{\frac{1}{2}}$, that is, about 20 %.

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SUMMARY

The problem considered in this paper is the fluid motion arising from a thin semi-infinite plate started to move impulsively from rest (in viscous incompressible fluid at rest) with a velocity, subsequently maintained uniform, parallel to the edge. Two solutions are given, one obtained in polar coordinates is in the form of an infinite series, whilst the other, derived operationally in parabolic coordinates, leads to a single integral for the velocity distribution. The former is convenient for computation in the vicinity of the edge, the latter being more convenient elsewhere.

A hypothesis originally introduced by Rayleigh after he had discussed the corresponding flow for an infinite plane has been used here to draw deductions about the steady flow past a quarter-plane whose leading edge is normal to the direction of flow and also to obtain approximate expressions for the effect of the edges on the skin friction of a sufficiently broad rectangle whose length is parallel to the incident stream.

After I had almost completed the analysis described in § 2 and the associated computations, I found, in conversation with Prof. Goldstein, that he, too, had been thinking about the same problem. I should like to thank him for drawing my attention to the merits of the approach I have used in § 3 and for leaving me free to complete the investigation.

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