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An example of a flow on a non-compact surface without minimal set

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Abstract. An example is given of a one-dimensional foliation (or a flow) on an open surface which admits no minimal set.

Every flow on a compact manifold has a minimal set (i.e. a closed, non-empty invariant subset which does not contain any proper subset with these properties). In this paper we show, by producing a concrete example, that this does not hold in general when the manifold is not compact.

THEOREM. *There exists a flow on an open surface which admits no minimal set.*

This is an answer to problem 34-1 in the publication [**2**].

Remark. In the case of foliations with leaf dimension greater than one, Hector [**1**] has already constructed examples without minimal sets. Here, we want to note that in his examples there are a series of leaves having infinitely many ends and non-trivial holonomies, and that the existence of such leaves seems to be necessary for his construction to realize the required complicated dynamics. In the case of leaf dimension one (i.e. the flow case), similar methods do not work, because every non-compact leaf (trajectory) is diffeomorphic to the real line and hence has only two ends and trivial holonomy.

Proof of the theorem. Take sequences $\{a_k\}$, $\{b_k\}$, $\{c_k\}$, $\{d_k\}$, $\{e_k\}$ $(k = 1, 2, 3, \dots)$ such that: $a_1 = 9/10$, $a_k > b_k > c_k > d_k > e_k > a_{k+1}$ $(k = 1, 2, 3, \dots)$ and $\lim_{k\to\infty} a_k = 0$. Put $P = \{(x, y) \in \mathbb{R}^2 \mid -1 < x < 0,\ x - 2 < y < \min\{x + 1, 9/10\}\}$, $Q_k = \{(x, y) \in \mathbb{R}^2 \mid 9/10 < x < 21/10,\ c_k < y < a_k\}$ and $R_k = \{(x, y) \in \mathbb{R}^2 \mid 29/10 < x < 41/10, b_{k+1} < y < d_k\}$ $(k = 1, 2, 3, \dots)$, and consider on each of them a foliation by the horizontal lines $y =$ constant.

Let N be the quotient space obtained from the disjoint union $P \cup \bigcup_{k=1}^{\infty} Q_k \cup \bigcup_{k=1}^{\infty} R_k$ by the following identifications: $(x, y) \in Q_k$ $(9/10 < x < 1)$ is identified with $(x - 1, y - 2) \in P$, $(x, y) \in Q_k$ $(2 < x < 21/10)$ with $(x + y - 3, y) \in P$,

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$(x, y) \in R_k$ $(29/10 < x < 3)$ with $(x-3, y) \in P$ and $(x, y) \in R_k$ $(4 < x < 41/10)$ with $(4-x, y-1) \in P$. Denote by $\mathcal{F}_N$ the foliation on N induced from the horizontal foliations on the pieces.

Let $(N_n, \mathcal{F}_n)$ $(n = 1, 2, 3, \dots)$ be countably many copies of $(N, \mathcal{F}_N)$. Put $R^{(0)} = \{(x, y) \in \mathbb{R}^2 \mid 49/10 < x < 61/10,\ 0 < y < 9/10\}$ and let $(N_0, \mathcal{F}_0)$ be the quotient space obtained from the union $P \cup R^{(0)}$ by identifying $(x, y) \in R^{(0)}$ $(49/10 < x < 5)$ with $(x-5, y) \in P$ and $(x, y) \in R^{(0)}$ $(6 < x < 61/10)$ with $(6-x, y-1) \in P$.

For each $k = 1, 2, 3, \dots$, put

$$S_k^{(n)} = \{(x, y) \in P \subset N_n \mid a_{k+1} - 1 < x < e_k - 1\} \quad (n = 0, 1, 2, \dots)$$

and

$$T_k^{(n)} = \left\{(x, y) \in P \subset N_n \;\middle|\; \frac{d_k - e_k}{1 - e_k}x + d_k < y < \frac{b_k - c_k}{1 - c_k}x + b_k\right\} \quad (n = 1, 2, 3, \dots).$$

For each $n \geq 0$ and $k \geq 1$, choose a foliation-preserving diffeomorphism $\varphi_k^{(n)}$ from $S_k^{(n)}$ onto $T_k^{(n+1)}$ so that $\varphi_k^{(n)}((1-t)a_{k+1} + te_k - 1, a_{k+1}) = ((1-t)(c_k - 1), c_k)$ and that $\varphi_k^{(n)}((1-t)a_{k+1} + te_k - 1, e_k - 3) = ((1-t)(d_k - 1), d_k)$ for $0 < t < 1$.

Now, let M be the quotient space obtained from the disjoint union $\bigcup_{n=0}^{\infty} N_n$ by identifying $S_k^{(n)}$ with $T_k^{(n+1)}$ via $\varphi_k^{(n)}$ for all $n \geq 0$ and $k \geq 1$, and let $\mathcal{F}_M$ be the foliation on M induced from $(N_n, \mathcal{F}_n)$. We will show that $(M, \mathcal{F}_M)$ has the desired property.

First let us check that M is a Hausdorff (i.e. ordinary) open surface. Let $q : \bigcup_{n=0}^{\infty} N_n \to M$ be the quotient map and let P_n $(\subset N_n)$ be the copy of P in N. Then one can easily see that $q(P_1 \cup P_0)$ is nothing but the open surface which is obtained from the open disk $q(P_1 \cup \{(x, y) \in P_0 \mid e_1 - 1 \leq x\})$ by attaching infinitely many open handles $\{(x, y) \in P_0 \mid e_i - 1 \leq x \leq a_i - 1\}$ $(i = 2, 3, \dots)$. Similarly, for every $p > 0$, $q(\bigcup_{n=0}^{p} P_n)$ is obtained from P_p by successively attaching infinitely many open handles, and is hence Hausdorff. Since $q(\bigcup_{n=0}^{\infty} P_n)$ is the increasing union of the open subsets $q(\bigcup_{n=0}^{p} P_n)$, it follows that $q(\bigcup_{n=0}^{\infty} P_n)$ is also Hausdorff. Finally, M is Hausdorff because M is obtained from $q(\bigcup_{n=0}^{\infty} P_n)$ by attaching infinitely many further handles $\{(x, y) \in R^{(0)} \mid 5 \leq x \leq 6\}$, $\{(x, y) \in Q_k^{(n)} \mid 1 \leq x \leq 2\}$ and $\{(x, y) \in R_k^{(n)} \mid 3 \leq x \leq 4\}$ $(k, n = 1, 2, 3, \dots)$, where $Q_k^{(n)}$ and $R_k^{(n)}$ $(\subset N_n)$ denote the copies of Q_k and R_k in N respectively. (Here we should note that all the attaching intervals of our open handles are 'apparently disjoint' to each other. Therefore no branching occurs in M.)

Next we will prove the non-existence of minimal sets in $\mathcal{F}_M$. It suffices to show that the closure $\overline{F}$ is not a minimal set for any leaf F of $\mathcal{F}_M$. Let L_n $(n = 0, 1, 2, \dots)$ be a leaf of $\mathcal{F}_M$ defined by $L_n = (-1, 0) \times \{0\} \subset P \subset N_n \subset M$. Then, by the construction, one can observe that $\overline{L_n} = \bigcup_{i \geq n} L_i$ for all $n \geq 0$. Thus, $\overline{L_n} \supsetneq \overline{L_{n+1}}$, and hence $\overline{L_n}$ is not a minimal set. Next let us consider a leaf F of $\mathcal{F}_M$ such that $F \neq L_n$ for any n. Then, by the construction, either (maybe both) of the two ends of F, say e, is of the form $(-1, s] \times \{y\} \subset P \subset N_m$ for some $m \geq 0$ and some s, $y < 0$. In N_{m+1} the end e behaves similarly to L_m and is asymptotic to L_{m+1}, that is, $\overline{F} \supset L_{m+1}$. This implies that $\overline{F}$ is not a minimal set because $\overline{F} \supsetneq \overline{L_{m+1}}$. This proves the non-existence.

The foliation $\mathcal{F}_M$ is transversely orientable, but not tangentially orientable. If one wants to get an example of a tangentially orientable foliation (or a flow), one has only to take a

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suitable double covering of M. This completes the proof of the theorem. □

The surface M constructed in the above proof has infinite genus. The infiniteness is inevitable since in the case of finite genus the following holds.

PROPOSITION. *Every one-dimensional foliation on an open surface of finite genus has a minimal set.*

Proof. Let $\mathcal{F}$ be a foliation on an open surface M. We suppose that $\mathcal{F}$ has no minimal set and will show that M has infinite genus. First of all, by passing to a suitable 4-fold covering if necessary, we may assume that $\mathcal{F}$ is both transversely and tangentially orientable.

First Step. Since $\mathcal{F}$ has no minimal set, one can find three leaves L_1, L_2 and L_3 in $\mathcal{F}$ such that $\overline{L_1} \supsetneqq \overline{L_2} \supsetneqq \overline{L_3}$. Take a point x in L_3 and a small curve T through x transverse to $\mathcal{F}$. Since $L_2 \cap T$ accumulates to x, one can take points $y, z \in L_2 \cap T$ so that the composite curve $\eta = \tau * \sigma$ is an embedded closed curve in M, where τ is the curve which joins y to z in T and σ is the curve which joins z to y in L_2. Then, η does not separate M (hence the genus of M is ≥ 1), because L_1 is asymptotic to L_2 and hence intersects τ infinitely many times. The saturation $S(\eta)$ of η does not coincide with M, because if $S(\eta) = M$ then the saturation of a holonomy invariant minimal set in τ (the compactness of τ assures the existence of such a minimal set) becomes a minimal set of $\mathcal{F}$. Since $S(\eta)$ is open in M, we can take a small compact neighborhood $N(\eta)$ of η so that $N(\eta)$ is contained in $S(\eta)$. This finishes the first step.

Second Step. We take three leaves L_1', L_2' and L_3' in $M - S(\eta)$ such that $\overline{L_1'} \supsetneqq \overline{L_2'} \supsetneqq \overline{L_3'}$. Again, this can be done since $\mathcal{F}$ has no minimal set. Then, since $N(\eta)$ is compact, by similar arguments as in the first step, we can find an embedded closed curve η' which is disjoint from $N(\eta)$ and does not separate $M - N(\eta)$. This shows that the genus of M is ≥ 2. This finishes the second step.

Since we are assuming that $\mathcal{F}$ has no minimal set, we can repeat similar procedures endlessly and can conclude that the genus of M is infinite. This proves the proposition. □

It is easy to see that the example constructed in this paper is not conjugate to a foliation defined by a closed 1-form (although it is without holonomy). So, the following question, posed to the author by H. Minakawa, might be interesting.

Question. Does a one-dimensional foliation on an open surface defined by a non-singular closed 1-form always admit a minimal set?

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