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Spatial quantum search in a triangular network[†]

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The spatial search problem consists of minimising the number of steps required to find a given site in a network, with the restriction that only an oracle query or a translation to a neighbouring site is allowed at each step. We propose a quantum algorithm for the spatial search problem on a triangular lattice with N sites and torus-like boundary conditions. The proposed algorithm is a special case of the general framework for abstract search proposed by Ambainis, Kempe and Rivosh (AKR) in Ambainis *et al.* (2005) and Tulsi in Tulsi (2008) applied to a triangular network. The AKR–Tulsi formalism was employed to show that the time complexity of the quantum search on the triangular lattice is $O(\sqrt{N \log N})$.

1. Introduction

The spatial quantum search problem consists of using local unitary operations to search for one (or more) nodes within a set of N spatially arranged sites with an implicit notion of distance between them. The search nodes are identified in the usual way by the non-zero values of a binary function (the oracle). The spatial search problem (Ambainis and Aaronson 2003) incorporates the restriction that, in one step, one can either query the oracle at the current site or advance to a neighbouring site. Benioff (2002) pointed out that in a two-dimensional network under this restriction, Grover's search (Grover 1996; 1997) provides no advantage in terms of running time over a classical search due to the intrinsic non-locality of Grover's symmetrisation. Ambainis *et al.* (2005) proposed a generalised formalism for quantum walk (QW) based spatial search algorithms and worked out the specific case of a two-dimensional cartesian network, obtaining a $O(\sqrt{N} \log N)$ algorithm, which we shall refer to as AKR. Tulsi proposed an improvement to AKR, which requires an ancillary qubit and leads to an $O(\sqrt{N \log N})$ algorithm in two dimensions (Tulsi 2008).

It is not known whether the optimal solution is $O(\sqrt{N})$ for the two-dimensional spatial search problem. In contrast, it is known that the optimal solution is achieved in higher dimensions, such as a 3D-grid (Ambainis *et al.* 2005) and the SKW algorithm (Shenvi *et al.* 2003), which searches an item within $N = 2^n$ sites arranged in an n-dimensional ($n > 2$) hypercube. Magniez *et al.* (2009) and Krovi *et al.* (2010) showed that it is highly

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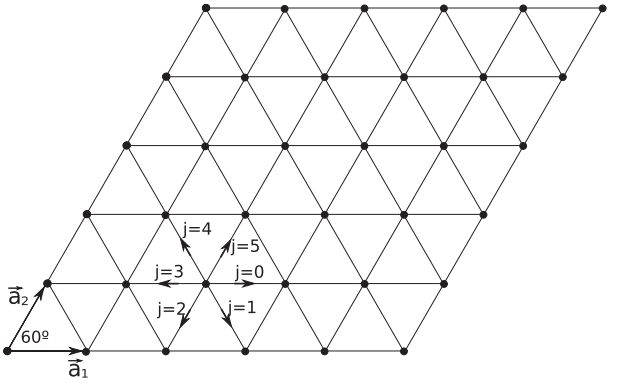


Figure 1. A $\sqrt{N} \times \sqrt{N}$ triangular network (here $N = 36$). The N sites form a Bravais lattice and the six directions of motion are labelled by an integer $j \in [0, 5]$.

unlikely that the complexity $O(\sqrt{N})$ will be achieved, since classical random walks on 2D-grids usually have complexity $\Theta(N \log N)$, and search techniques based on quantum walks usually provide at most a quadratic speed-up.

There are three ways to cover the plane with regular polygons: squares, hexagons and triangles. The resulting regular networks differ in their degree d (number of connections per node), which is 4, 3 and 6 respectively. As mentioned above, for the rectangular grid ($d = 4$), the AKR search algorithm finds a marked vertex in time $O(\sqrt{N} \log N)$. A QW-based spatial search algorithm of time complexity $O(\sqrt{N} \log N)$ has recently been implemented for a two-dimensional hexagonal network ($d = 3$) (Abal *et al.* 2010). Both of these algorithms can be improved to $O(\sqrt{N \log N})$ with Tulsi's modification. These results suggest that the degree of connectivity of a regular network does not affect the performance of a QW-based search algorithm.

In the current paper, we propose and analyse a new search algorithm for the case of a triangular network. The proposed algorithm is a special case of the general framework for abstract search (Ambainis *et al.* 2005; Tulsi 2008), so we can employ the AKR–Tulsi formalism to show that the time complexity of the quantum search on the triangular lattice is $O(\sqrt{N \log N})$. This confirms that the degree of the underlying regular network does not affect the performance of the quantum algorithm (Krovi *et al.* 2010).

The paper is organised as follows. In Section 2 we discuss the implementation of a QW on the triangular network. In Section 3 we analyse the time complexity of this search algorithm, with a numerical analysis in Section 3.2. Finally, we present our conclusions in Section 4.

2. QW on a triangular network

We consider N sites arranged in a triangular network covering a two-dimensional region, as shown in Figure 1. The network is $\sqrt{N} \times \sqrt{N}$, and we assume periodic boundary conditions. A site on the lattice is identified by two integers (n_1, n_2) through the equation

$$\mathbf{r} = n_1 \mathbf{a}_1 + n_2 \mathbf{a}_2, \tag{1}$$

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where $\mathbf{a}_1$ and $\mathbf{a}_2$ are unit vectors forming a 60^o angle, as indicated in Figure 1. These integers are such that $n_i \in [0, \sqrt{N} - 1]$ for $i = 1, 2$, so each of them takes $\sqrt{N}$ different values.

These sites define an orthonormal set of quantum state vectors $\{|n_1, n_2\rangle\}$, which span an N-dimensional Hilbert space $\mathscr{H}_P$. At a given site, there are six possible directions of motion, which we label with an integer $j \in [0, 5]$, as indicated in Figure 1. The orthonormal states $\{|j\rangle\}$ span a six-dimensional Hilbert space $\mathscr{H}_C$, which we shall refer to as the 'coin' subspace. The Hilbert space for this problem, $\mathscr{H} = \mathscr{H}_{\mathscr{C}} \otimes \mathscr{H}_{\mathscr{P}}$, is $6N$-dimensional. A generic state vector can be expressed as

$$|\Psi\rangle = \sum_{j=0}^{5} \sum_{n_1,n_2=0}^{\sqrt{N}-1} a_{j,\hat{n}} \, |j, \hat{n}\rangle, \tag{2}$$

where the $a_{j,\hat{n}}$ are complex amplitudes satisfying the normalisation constraint, and we have introduced the shorthand notation $\hat{n} \equiv (n_1, n_2)$.

The standard QW on this network is implemented using a unitary evolution operator of the form

$$U = S \cdot (C \otimes I_P) \tag{3}$$

where S is a shift operator in $\mathscr{H}$ (to be specified below), I_P is the identity operation in $\mathscr{H}_P$ and C is a unitary coin operation in $\mathscr{H}_C$. A useful coin operation for spatial search problems (Ambainis *et al.* 2005; Shenvi *et al.* 2003) is Grover's coin, whose matrix elements for a d-dimensional space are $G^{(d)}_{ij} = \frac{2}{d} - \delta_{ij}$. For the particular case $d = 6$, it is given by

$$G^{(6)} = \frac{1}{3} \begin{pmatrix} -2 & 1 & 1 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 & 1 & 1 \\ 1 & 1 & -2 & 1 & 1 & 1 \\ 1 & 1 & 1 & -2 & 1 & 1 \\ 1 & 1 & 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & 1 & 1 & -2 \end{pmatrix}. \tag{4}$$

We shall use $C = G^{(6)}$ for the coin operation in the current paper. The shift operator implements single-step displacements acting on the kets $|j; n_1, n_2\rangle$ as follows:

$$\begin{aligned} S|0; n_1, n_2\rangle &= |3; n_1 + 1, n_2\rangle \\ S|1; n_1, n_2\rangle &= |4; n_1 + 1, n_2 - 1\rangle \\ S|2; n_1, n_2\rangle &= |5; n_1, n_2 - 1\rangle \\ S|3; n_1, n_2\rangle &= |0; n_1 - 1, n_2\rangle \\ S|4; n_1, n_2\rangle &= |1; n_1 - 1, n_2 + 1\rangle \\ S|5; n_1, n_2\rangle &= |2; n_1, n_2 + 1\rangle. \end{aligned} \tag{5}$$

Note that S inverts the coin state. This inversion is crucial for the efficiency of the search algorithm described in the next section. Finally, the dynamics of the QW is obtained by applying U repeatedly $|\Psi(m)\rangle = U^m|\Psi(0)\rangle$ for some integer m.

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The standard QW is best analysed in the Fourier-transformed space, so we consider the reciprocal lattice vectors $\{\mathbf{g}_1, \mathbf{g}_2\}$, which satisfy the usual requirements from condensed matter physics (Kittel 1995):

$$\mathbf{g}_i \cdot \mathbf{a}_j = \frac{2\pi}{\sqrt{N}}\delta_{ij}. \tag{6}$$

A site in this *reciprocal lattice* is located by

$$\mathbf{k} = k_1\mathbf{g}_1 + k_2\mathbf{g}_2, \tag{7}$$

with k_1, k_2 integers in $[0, \sqrt{N} - 1]$. We use the notation $\hat{k} = (k_1, k_2)$ and write a generic state vector in the Fourier representation as

$$|\Psi\rangle = \sum_{j,\hat{k}} f_{j,\hat{k}} |j, \hat{k}\rangle. \tag{8}$$

The kets $|j, \hat{k}\rangle$ and $|j, \hat{n}\rangle$ are related by the discrete Fourier transform

$$|j, \hat{k}\rangle = \frac{1}{\sqrt{N}} \sum_{\hat{n}} \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}} |j, \hat{n}\rangle \tag{9}$$

$$|j, \hat{n}\rangle = \frac{1}{\sqrt{N}} \sum_{\hat{k}} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} |j, \hat{k}\rangle, \tag{10}$$

and one can check that $\langle j, \hat{k} | j, \hat{n}\rangle = \omega^{\hat{n}\cdot\hat{k}}/\sqrt{N}$ where $\omega \equiv \mathrm{e}^{2\pi i/\sqrt{N}}$ and $\hat{n} \cdot \hat{k} = n_1k_1 + n_2k_2$. The action of S on the kets $|j, \hat{k}\rangle$ of the Fourier representation can be obtained from Equations (5) as

$$\begin{aligned}
S|0, \hat{k}\rangle &= \omega^{k_1} |3, \hat{k}\rangle \\
S|1, \hat{k}\rangle &= \omega^{k_1 - k_2} |4, \hat{k}\rangle \\
S|2, \hat{k}\rangle &= \omega^{-k_2} |5, \hat{k}\rangle \\
S|3, \hat{k}\rangle &= \omega^{-k_1} |0, \hat{k}\rangle \\
S|4, \hat{k}\rangle &= \omega^{-k_1 + k_2} |1, \hat{k}\rangle \\
S|5, \hat{k}\rangle &= \omega^{k_2} |2, \hat{k}\rangle.
\end{aligned} \tag{11}$$

Thus, in the $\hat{k}$-representation, S acts diagonally, that is,

$$S = \sum_{\hat{k}} S_{\hat{k}} |\hat{k}\rangle\langle\hat{k}|,$$

where $S_{\hat{k}} = \langle\hat{k}|S|\hat{k}\rangle$ is the reduction of S to $\mathscr{H}_C$. Therefore, the evolution operator, Equation (3), is also diagonal in the Fourier representation and can be expressed as

$$U = \sum_{\hat{k}} U_{\hat{k}} |\hat{k}\rangle\langle\hat{k}|,$$

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where $U_{\hat{k}} = \langle k|U|k\rangle$ acts in $\mathscr{H}_C$. The matrix elements of the reduced operator can be calculated from Equation (11), with the result

$$U_{\hat{k}} = \frac{1}{3}\begin{pmatrix} \omega^{-k_1} & \omega^{-k_1} & \omega^{-k_1} & -2\omega^{-k_1} & \omega^{-k_1} & \omega^{-k_1} \\ \omega^{-k_1+k_2} & \omega^{-k_1+k_2} & \omega^{-k_1+k_2} & \omega^{-k_1+k_2} & -2\omega^{-k_1+k_2} & \omega^{-k_1+k_2} \\ \omega^{k_2} & \omega^{k_2} & \omega^{k_2} & \omega^{k_2} & \omega^{k_2} & -2\omega^{k_2} \\ -2\omega^{k_1} & \omega^{k_1} & \omega^{k_1} & \omega^{k_1} & \omega^{k_1} & \omega^{k_1} \\ \omega^{k_1-k_2} & -2\omega^{k_1-k_2} & \omega^{k_1-k_2} & \omega^{k_1-k_2} & \omega^{k_1-k_2} & \omega^{k_1-k_2} \\ \omega^{-k_2} & \omega^{-k_2} & -2\omega^{-k_2} & \omega^{-k_2} & \omega^{-k_2} & \omega^{-k_2} \end{pmatrix}. \tag{12}$$

The characteristic polynomial factors as follows:

$$P(\lambda) = (\lambda-1)^2(\lambda+1)^2(\lambda^2 - 2\cos(\theta_k)\lambda + 1), \tag{13}$$

where θ_k is defined by

$$\cos(\theta_k) \equiv \frac{1}{3}\left(\cos(\tilde{k_1}) + \cos(\tilde{k_2}) + \cos(\tilde{k_1} - \tilde{k_2})\right), \tag{14}$$

and $\tilde{k}_i \equiv 2\pi k_i/\sqrt{N}$ for $i = 1, 2$. The eigenvalues are ± 1, each with multiplicity 2, and $\mathrm{e}^{\pm i\theta_k}$. We use $|v_{\pm 1}\rangle$ and $|v_{\pm k}\rangle$ to denote the normalised eigenvectors of $U_{\hat{k}}$ associated with the eigenvalues ± 1 and $\mathrm{e}^{\pm i\theta_k}$, respectively. Then $|v_{\pm 1}, \hat{k}\rangle$ and $|v_{\pm k}, \hat{k}\rangle$ are eigenvectors of U associated with the same eigenvalues, where $|\hat{k}\rangle$ is defined in Equation (9).

3. Time complexity of the search algorithm

We shall use Tulsi's version (Tulsi 2008) of the framework of the *abstract search algorithm* (Ambainis *et al.* 2005) to analyse the time complexity of a search algorithm on the triangular network. We assume that there is a single marked vertex $|\hat{t}\rangle$, which we want to find. The search algorithm uses a conditional coin operation, which acts as $-I_C$ on the searched site $|\hat{t}\rangle$ and as $G^{(6)}$ otherwise. Thus, the modified evolution operator is $U' = S \cdot C'$, where C' acts in $\mathscr{H}$ in the way just described, that is,

$$C' = -I_C \otimes |\hat{t}\rangle\langle\hat{t}| + \sum_{\hat{n}\neq\hat{t}} G^{(6)} \otimes |\hat{n}\rangle\langle\hat{n}|. \tag{15}$$

Using the above equation, one can show that $U' = U \cdot R_{\hat{t}}$, where $R_{\hat{t}} = I_{6N} - 2|u_C, \hat{t}\rangle\langle u_C, \hat{t}|$, U is given by Equation (3) and $|u_C\rangle$ is the uniform superposition of the computational basis of the coin space. Ambainis *et al.* (2005) showed that the evolution of the modified quantum walk U' may be analysed using the eigenspectrum of the standard QW evolution operator U. This fact reduces the analysis of the search algorithm to a tractable eigenproblem for the unitary operator U.

Using the expression $U' = U \cdot R_{\hat{t}}$ as a starting point, Tulsi proposed a new version of the search algorithm, which requires an extra register (an ancillary qubit) used as a control for the operators $R_{\hat{t}}$ and U. The operators acting on the ancillary register are described

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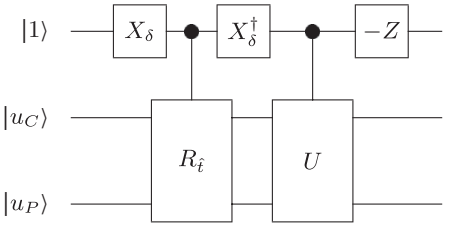


Figure 2. Tulsi's circuit diagram for the one-step evolution operator of the quantum walk search algorithm.

in Figure 2, where $-Z$ is the negative of Pauli's Z operator and

$$X_\delta = \begin{pmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{pmatrix} \tag{16}$$

where $\cos\delta \propto 1/\sqrt{\log N}$.

The new evolution operator is

$$U'' = (-Z \otimes I) \cdot C(U) \cdot (X_\delta^\dagger \otimes I) \cdot C(R_{\hat{t}}) \cdot (X_\delta \otimes I), \tag{17}$$

where $C(U)$ and $C(R_{\hat{t}})$ are the controlled operations shown in Figure 2 and I is the identity operator in $\mathscr{H}$. We will show that U'' must be iterated $O(\sqrt{N \log N})$ times, taking $|1\rangle|u_C\rangle|u_P\rangle$ as the initial condition, in order to maximise the overlap with the search element.

The expression for the controlled $R_{\hat{t}}$ is

$$C(R_{\hat{t}}) = I_{12N} - 2|1, u_C, \hat{t}\rangle\langle 1, u_C, \hat{t}|.$$

We first define

$$|\delta_1\rangle \equiv X_\delta^\dagger|1\rangle = -\sin\delta\,|0\rangle + \cos\delta\,|1\rangle, \tag{18}$$

and then a new reflection operator

$$\bar{R}_{\hat{t}} \equiv (X_\delta^\dagger \otimes I) \cdot C(R_{\hat{t}}) \cdot (X_\delta \otimes I) = I_{12N} - 2|\delta_1, u_C, \hat{t}\rangle\langle\delta_1, u_C, \hat{t}|. \tag{19}$$

The effective target state is

$$|\bar{t}\rangle = |\delta_1, u_C, \hat{t}\rangle. \tag{20}$$

We now define

$$\bar{U} = (-Z \otimes I) \cdot C(U). \tag{21}$$

Note that the eigenspectrum of $\bar{U}$ is determined by the eigenspectrum of U. In fact, for $\hat{k} \neq 0$, we have

$$\bar{U}|0\rangle|v_{\pm k}, \hat{k}\rangle = -|0\rangle|v_{\pm k}, \hat{k}\rangle \tag{22}$$

$$\bar{U}|1\rangle|v_{\pm k}, \hat{k}\rangle = \mathrm{e}^{\pm i\theta_k}|1\rangle|v_{\pm k}, \hat{k}\rangle. \tag{23}$$

The eigenvectors $|l\rangle|v_{\pm 1}, \hat{k}\rangle$, $l = 0, 1$ will not be used since $\langle u_C|v_{\pm 1}\rangle = 0$, and they are thus orthogonal to both the initial condition and the target. For $\hat{k} = 0$, the initial condition is the only eigenvector with eigenvalue 1 that will be used.

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The search algorithm consists of applying U'' repeatedly, taking the initial condition as

$$|\Psi_0\rangle \equiv |1, u_C, u_P\rangle = \frac{1}{\sqrt{6\,N}} \sum_{j,\hat{n}} |1, j, \hat{n}\rangle, \tag{24}$$

where $|u_C\rangle$ and $|u_P\rangle$ are the uniform superposition in the coin and position spaces, respectively. The initial condition can be prepared in $O(\sqrt{N})$ time steps. The number of iterations of U'' is given by $\pi/4\alpha$, where α is defined as follows. The eigenvalues of U'' that are different from 1 have the form $e^{\pm i\alpha_j}$, where $0 < \alpha_j \leqslant \pi/2$. Then $\alpha = \min\{\alpha_1, \alpha_2, \cdots\}$. Ambainis *et al.* (2005) showed that it is possible to estimate α knowing the eigenspectrum of the evolution operator $\bar{U}$.

3.1. *Theoretical analysis*

The first step in the analysis of the algorithm is to decompose the target state in the eigenbasis of $\bar{U}$, that is, we have to calculate coefficients a_0, a_k and b_k such that

$$\begin{aligned} |\delta_1, u_C, \hat{t}\rangle = \frac{a_0}{\sqrt{N}}|\Psi_0\rangle + \frac{1}{\sqrt{N}} \sum_{\hat{k}\neq 0} a_k \left(|1, v_{+k}, \hat{k}\rangle + |1, v_{-k}, \hat{k}\rangle\right) \\ + \frac{1}{\sqrt{N}} \sum_{\hat{k}\neq 0} b_k \left(|0, v_{+k}, \hat{k}\rangle + |0, v_{-k}, \hat{k}\rangle\right). \end{aligned} \tag{25}$$

The coefficients b_k are related to the (-1)-eigenspace. The coefficients a_k and b_k are real. In order to satisfy these conditions, eigenvectors $|l, v_{+k}, \hat{k}\rangle$ for $l = 0, 1$ must be chosen appropriately. The procedure is as follows. If $\langle u_C|v_{+k}\rangle$ has a non-zero phase $e^{i\lambda}$, then $|l, v_{+k}, \hat{k}\rangle$ must be redefined to $e^{-i\lambda}|l, v_{+k}, \hat{k}\rangle$. The same procedure must also be applied to $|l, v_{-k}, \hat{k}\rangle$. After those redefinitions, Equation (25) is valid. For the triangular network, a straightforward calculation yields

$$a_k = \langle 1|\delta_1\rangle\langle v_{\pm k}|u_C\rangle = \frac{\cos\delta}{\sqrt{2}} \tag{26}$$

$$b_k = \langle 0|\delta_1\rangle\langle v_{\pm k}|u_C\rangle = -\frac{\sin\delta}{\sqrt{2}} \tag{27}$$

This result is remarkably similar to the result obtained for the square grid. Note that the corresponding expressions for the honeycomb lattice (Abal *et al.* 2010) are more complex because they depend on k.

Tulsi has shown that α can be determined using the expression

$$\frac{1}{\alpha} = O\left(\frac{1}{a_0}\sqrt{\sum_{\hat{k}\neq 0} \frac{a_k^2}{1-\cos\theta_k} + \sum_{\hat{k}\neq 0} \frac{b_k^2}{4}}\right) \tag{28}$$

when $N \gg 1$. Equations (26) and (27) show that a_k and b_k do not depend on $\hat{k}$. The non-trivial sum inside the square root may be calculated using

$$\frac{1}{2}\sum_{k\neq 0} \frac{1}{1-\cos\theta_k} \approx \frac{N}{16}\frac{1}{(\pi-\varepsilon)^2} \iint_{\varepsilon}^{2\pi-\varepsilon} d\tilde{k}_2 d\tilde{k}_1 \frac{1}{1-\cos\theta_k}, \tag{29}$$

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where we have used

$$\sum_{k\neq 0} \to \frac{N}{8}\frac{1}{(\pi-\varepsilon)^2}\iint_{\varepsilon}^{2\pi-\varepsilon} d\tilde{k}_1 d\tilde{k}_2$$

with $\varepsilon = 2\pi\sqrt{2/N} \ll 1$. Using Equation (14) and $\varepsilon \ll 1$, Equation (29) can be approximated by

$$\frac{3N}{16}\frac{1}{\pi^2}\int_{\varepsilon}^{2\pi-\varepsilon} d\tilde{k}_2 \int_{\varepsilon}^{2\pi-\varepsilon} \frac{d\tilde{k}_1}{\tilde{k}_1^2+\tilde{k}_2^2-\tilde{k}_1\tilde{k}_2} \sim N\log\left(\frac{2\pi}{\varepsilon}\right) \sim N\log N. \tag{30}$$

Substituting this result into Equation (28) and using the fact that $a_0 = \cos\delta$ and $\cos\delta = \Theta\left(1/\sqrt{\log N}\right)$, together with Equations (26) and (27), we obtain

$$\frac{1}{\alpha} = O\left(\sqrt{N\log N}\right). \tag{31}$$

Since the number of iterations of U'' is $\pi/4\alpha$, we conclude that the time complexity of the search algorithm is $O(\sqrt{N\log N})$. We still need to show that the probability of finding the marked vertex is constant. This is where Tulsi's method comes into its own.

The overlap of the final state after $t = \pi/4\alpha$ iterations of U'' with the target state is

$$|\langle\delta_1, u_C, \hat{t}|(U'')^t|\Psi_0\rangle| = \Theta\left(\min\left\{\frac{1}{\sqrt{\sum_{\hat{k}\neq 0} a_k^2 \cot^2\frac{\theta_k}{4}}}, 1\right\}\right). \tag{32}$$

As Tulsi showed, the coefficients b_k play no role in this overlap. The calculation at this point for the triangular network is similar to one performed by Ambainis *et al.* (2005) for the 2D grid, except that a_k has the constant factor of $\cos\delta$. Using the fact that:

(1) $\cot\theta_k \leqslant 1/\theta_k^2$
(2) $1/\theta_k^2$ is bounded from below and above by $const/(1-\cos\theta_k)$ and
(3) the result of Equation (30),

we obtain

$$|\langle\delta_1, u_C, \hat{t}|(U'')^t|\Psi_0\rangle| = \Theta\left(\min\left\{\sqrt{\cos^2(\delta)\log N}, 1\right\}\right). \tag{33}$$

Using the fact that $\cos\delta = \Theta\left(1/\sqrt{\log N}\right)$, we obtain

$$|\langle\delta_1, u_C, \hat{t}|(U'')^t|\Psi_0\rangle| = \Theta(1). \tag{34}$$

So the probability of finding the marked vertex is constant – we have confirmed this result using a numerical simulation, as shown in Figure 4.

3.2. *Simulation*

A generic state in the extended space is

$$|\Psi\rangle = \sum_j \sum_{\hat{n}} a_{j,\hat{n}}|0, j, \hat{n}\rangle + b_{j,\hat{n}}|1, j, \hat{n}\rangle, \tag{35}$$

where a, b are the amplitudes of the state vector extended to include the 2-dimensional Hilbert space associated with the ancillary qubit.

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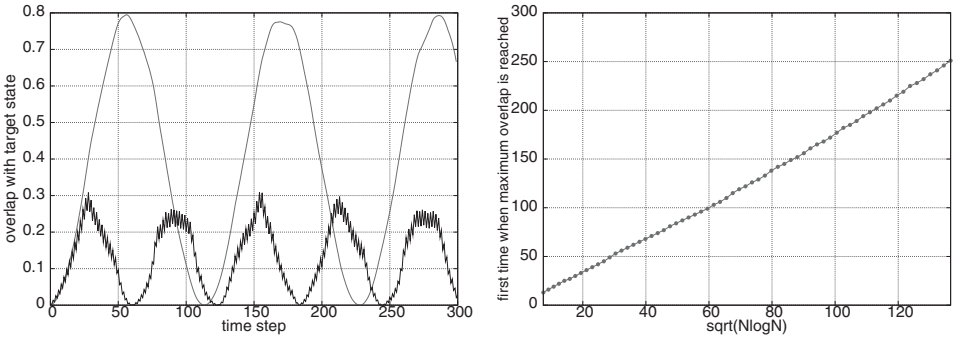


Figure 3. Left-hand panel: the probability of finding the searched state is plotted against the number of time steps for spatial search with (thin line) and without (thick line) Tulsi's modification for $N = 400$. Right-hand panel: t_{MAX} is plotted against $\sqrt{N \log N}$ using Tulsi's search – note the approximately linear dependence.

It is straightforward to show that the operator U'' in Equation (17) defines the following map:

$$\tilde{a}_{j,\hat{n}} = -a_{j,\hat{n}} + \frac{1}{3}\delta_{\hat{n},\hat{t}} \sum_{\ell} \left(a_{\ell,\hat{t}} \sin^2 \delta - b_{\ell,\hat{t}} \cos \delta \sin \delta \right) \tag{36}$$

$$\tilde{b}_{j,\hat{n}} = \sum_{k,l} G^{(6)}_{k,l} \sum_{\hat{n}'} S_{j,k,\hat{n},\hat{n}'} \left[b_{l,\hat{n}'} + \frac{1}{3}\delta_{\hat{n}',\hat{t}} \sum_{\ell} \left(a_{\ell,\hat{t}} \sin \delta \cos \delta - b_{\ell,\hat{t}} \cos^2 \delta \right) \right]$$

where $\delta_{\hat{n},\hat{t}}$ is a Kronecker-delta, which selects the searched state. This map allows the simulation of the abstract search algorithm using a digital computer. The initial condition is taken to be $|1\rangle|u_C\rangle|u_P\rangle$, and the effective target state is $|\bar{t}\rangle = |\delta_1\rangle|u_C\rangle|\hat{t}\rangle$, as explained in the previous section.

Fortran 90 was chosen as the programming language because it provides useful tools for large matrix computations and intrinsic functions for complex vector algebra. A parallel programming approach using OpenMP was implemented.

We present the results of the simulations in Figures 3 and 4. The left-hand panel in Figure 3 shows the time evolution of the probability of finding the searched state, $|\langle \bar{t}|\Psi\rangle|^2$, both for an algorithm using Tulsi's modification (thin curve), based on U'' defined in Equation (17), and without it (thick curve), based on $U' = S \cdot C'$ with the modified coin operator defined in Equation (15). Note that Tulsi's search yields a smoother curve, and although the first maximum (t_{max}) is reached later, the maximum probability of finding the searched element is higher than with the usual spatial search. A change in the sign of the local derivative was used to find the maximum point. In the right-hand panel of Figure 3, the time at which the maximum probability is reached for Tulsi's search is plotted against $\sqrt{N \log N}$. A straight-line fit to this data has a correlation coefficient given by $R^2 = 0.9988$.

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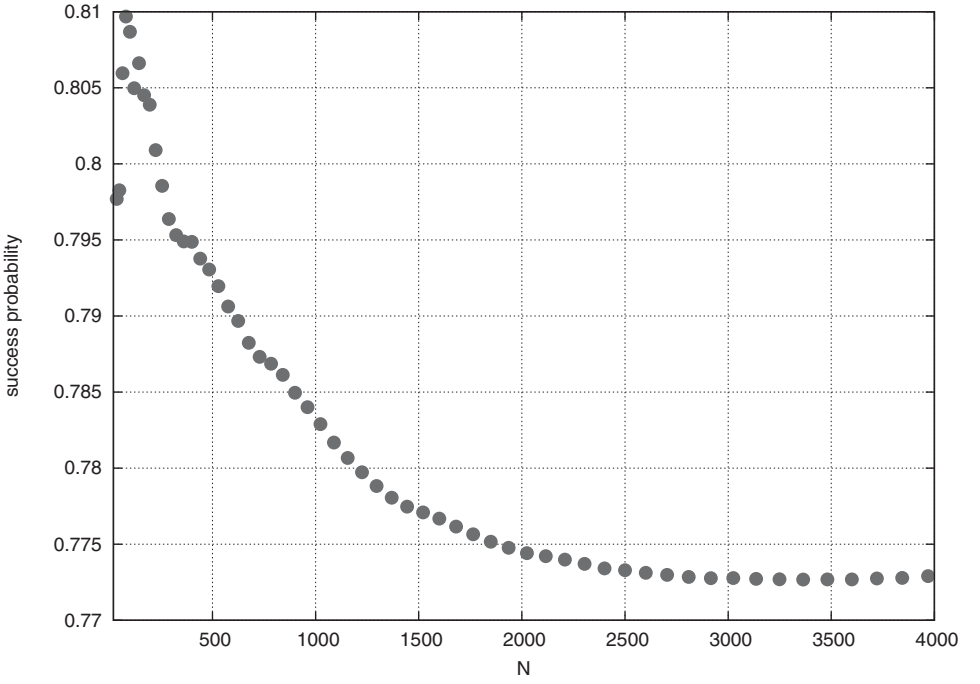


Figure 4. Success probability of the search algorithm plotted against N.

In order to confirm the complexity of Tulsi's search in the triangular network, Figure 4 shows a plot of the success probability against N. After an initial decay, the curve stabilises at approximately 0.773 for $N > 2000$.

4. Summary and conclusions

In this work, the problem of quantum spatial search on a periodic two-dimensional triangular network has been considered for the first time. In this problem, a searched item is to be located in a regular triangular network with N elements. We defined a quantum walk operator for this network and found the explicit Fourier representation for it. The eigenproblem was solved exactly and these results were then used to estimate the overlap and running times of the algorithm according to the generalised search formalism (Ambainis *et al.* 2005). This formalism gives a powerful insight into the running time of the algorithm for large values of N. In order to check our analytical results and to gain further knowledge of the detailed performance of the spatial search, we simulated the algorithms on a classical computer, using standard parallel techniques. This allowed us to obtain results for high values of N and to see how fast the convergence actually is to the theoretical expectations. The simulations were implemented both for a modified quantum walk search and for Tulsi's search, which uses an extra qubit as a control register.

Both spatial search algorithms are found to require $O(\sqrt{N \log N})$ steps to reach the point at which a measurement yields the searched state with constant probability. However, in

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the case of Tulsi's search, this result is obtained with a constant overlap with the searched state.

Previous work on the spatial search problem in a plane has considered the cases of a square grid (Ambainis *et al.* 2005) and a hexagonal grid (Abal *et al.* 2010). For both these quantum algorithms, the time complexity is $O(\sqrt{N \log N})$. In a sense, this work completes the program for the spatial search problem by providing the details of a search in a triangular grid. Since these regular graphs have different degree ($d = 4$ for rectangles, $d = 3$ for hexagons and $d = 6$ for triangles), this completes the proof that the degree of a regular graph does not affect the performance of a spatial search algorithm. The same conclusion is obtained in Krovi *et al.* (2010), but using a different model of quantum walk.

A search algorithm implemented on a real network will have to cope with a loss of symmetry due to imperfections. The issue of how robust these different spatial search protocols are when it comes to searching with some fraction of the links in the network missing is still an important open question for future work.

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