Ergod. Th. & Dynam. Sys., page 1 of 25 © The Author(s), 2026. Published by Cambridge University Press.
doi:10.1017/etds.2026.10341

Hyperbolic foliated entropy of suspensions

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(*Received* 9 *January* 2026 *UTC and accepted in revised form* 17 *July* 2026 *UTC*)

Abstract. We study the hyperbolic entropies of foliations obtained by suspensions of a representation in the sense of Dinh, Nguyên and Sibony (topological and measure-theoretic). We establish a link between this type of entropy and an adapted version of an entropy defined by Ghys, Langevin and Walczak for pseudo-groups of homeomorphisms. Such a link has various consequences. Among them, it implies that the hyperbolic entropy of foliations is not invariant by diffeomorphisms, and that a minimal entropy suspension admits an invariant measure. Finally, this allows us to study thoroughly the simple case in which the image of the representation is isomorphic to $\mathbb{Z}$. In that case, we give the first exact estimate of the hyperbolic entropy, and prove a Brin–Katok-type theorem and a variational principle, relying strongly on the standard ones for the entropy of maps.

Key words: holomorphic foliations, topological hyperbolic entropy, metric hyperbolic entropy, suspensions
2020 Mathematics Subject Classification: 37F75, 37A15 (Primary); 37B40, 22D40 (Secondary)

1. *Introduction*

The dynamical theory of laminations by Riemann surfaces has received much attention in the last twenty years. In particular, much research has been focused on the case where all the leaves are hyperbolic Riemann surfaces. In the case of foliations on projective spaces, this is the most common setup. Indeed, every polynomial vector field on $\mathbb{C}^n$ induces a singular holomorphic foliation on $\mathbb{P}^n$. For generic foliations of degree at least 2, all the leaves are hyperbolic. This result is due to Jouanolou [23], Lins Neto and Soares [27], Lins Neto [26] and Glutsyuk [22]. In that case, considering Brownian motions, one can define Lyapunov exponents of a measure. When $n = 2$, Nguyên uses the integrability of the holonomy cocycle [29] to compute exactly the Lyapunov exponent [31] of the unique ergodic measure of a generic foliation obtained by Fornæss and Sibony [16]. It is worth noting that the unique ergodicity is extended later by Dinh, Nguyên and Sibony [12] to a generic foliation on every compact Kähler surface. We refer the reader to the survey articles [13, 15, 28, 30] for a more detailed exposition of this theory.

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Here, we are mostly interested in an ergodic theory for laminations by hyperbolic Riemann surfaces. Solving heat equations with respect to harmonic currents, Dinh, Nguyên and Sibony [9] obtain an effective geometric analog of the Birkhoff theorem. In this theorem, the quantity that plays the role of time is the Poincaré distance in the uniformization of the leaves by the hyperbolic disk, and the usual invariant measures are replaced by harmonic measures. Indeed, many foliations do not admit any invariant measure. Since Garnett [19] (respectively, Berndtsson and Sibony [4]) shows that any compact lamination (respectively, any compact holomorphic foliation with isolated singularities) does admit harmonic ones, we need to consider them. The hyperbolic interpretation of time then leads to notions of entropy, both measure-theoretic and topological, that the three authors define in a series of two articles [10, 11].

Let us introduce some notation and give a rough definition of the entropy (see §2.2 for a more precise treatment). Let $\mathscr{F} = (X, \mathscr{L})$ be a lamination by hyperbolic Riemann surfaces, with (X, d_X) a compact metric space. Then all the leaves are uniformized by the Poincaré disk and one can choose parametrizations $\phi_x : \mathbb{D} \to L_x$ such that $\phi_x(0) = x$, for $x \in X$, where L_x is the leaf passing through x. For $R > 0$, define a *Bowen distance*

$$d_R(x, y) = \inf_{\theta \in \mathbb{R}} \sup_{\xi \in \overline{\mathbb{D}}_R} d_X(\phi_x(\xi), \phi_y(e^{i\theta}\xi)), \tag{1.1}$$

where $\mathbb{D}_R \subset \mathbb{D}$ is the disk of hyperbolic radius R centered at 0. Note that the infimum over all possible parametrizations ensures that d_R is independent of the choice of the ϕ_x, for $x \in X$. Given Bowen balls, Dinh, Nguyên and Sibony can define a topological entropy $h(\mathscr{F})$ in the usual Bowen way. It measures the exponential rate with R, at which leaves go apart with respect to d_R. Also, given a harmonic measure μ, one can consider quantities introduced by Katok [24] and Brin and Katok [6] as definitions for a measure-theoretic entropy $h(\mu)$, and for local upper and lower entropies $h^{\pm}(\mu, x)$, respectively (see §5).

So far, little is known about these quantities. They are finite (with no upper bound) in some cases [3, 10, 11]. If μ is ergodic and with additional assumptions [2, 10], the $h^{\pm}(\mu, x)$ are known to be constant almost everywhere and we have

$$2 \leq h^{-}(\mu) \leq h(\mu) \leq h^{+}(\mu) \leq h(\mathscr{F}).$$

In particular, there is no clear interpretation of the fact that the entropy is minimal, that is, equal to 2, nor is there any known non-trivial example in which it is computable. We expect, as in the Brin–Katok theorem, that one could show $h^{-}(\mu) = h^{+}(\mu)$. We also hope to find a variational principle $h(\mathscr{F}) = \sup_{\mu} h(\mu)$, but no example can argue in favor of such a belief. As a matter of fact, there is not even a written example on which it is known to be greater than 2.

The aim of this note is to give first examples of explicit bounds and values for these entropies with suspensions. Consider a compact smooth Riemann surface Σ of genus $g \geq 2$ and a representation $\rho : \Gamma = \pi_1(\Sigma) \to \mathrm{Homeo}(T)$, where (T, d) is a compact metric space. Then Γ acts diagonally on $\mathbb{D} \times T$, and $X = (\mathbb{D} \times T)/\Gamma$ is foliated by the image of horizontal disks $\mathbb{D} \times \{t\}$, $t \in T$ (see §2.3). From the point of view of the entropies, such a lamination has the advantage of having completely explicit parametrizations ϕ_x, for $x \in X$. Hence, we can compare the entropy $h(\mathscr{F})$ with that given by the holonomy ρ. More

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precisely, given a symmetric set of generators $\mathcal{G}$ of Γ, Ghys, Langevin and Walczak [21] define a Bowen distance for $N \in \mathbb{N}$, and for $t, s \in T$,

$$d_N(t,s) = \sup\{d(\rho(g_1 \dots g_n)(t), \rho(g_1 \dots g_n)(s));\ n \le N,\ g_1, \dots, g_n \in \mathcal{G}\},$$

Here, the quantity that plays the role of time is the word distance in Γ with generators $\mathcal{G}$. As usual, a Bowen distance induces an entropy. Here, we use the same kind of distance with a weight $\omega\colon \mathcal{G} \to \mathbb{R}_+^*$, to define an entropy $h(\rho, \mathcal{G}, \omega)$, closely related to that of Ghys, Langevin and Walczak. Our main result is the following theorem (see Theorem 4.10 for a more precise statement).

THEOREM 1.1. *Consider a suspension as above. There exists $K = K(\Gamma) > 1$ such that*

$$2 + h(\rho, \mathcal{G}, \omega) \le h(\mathscr{F}) \le 2 + Kh(\rho, \mathcal{G}, \omega).$$

In particular, this has the following consequences. For the second one, we use a result of Ghys, Langevin and Walczak [21].

COROLLARY 1.2 (Proposition 4.11). *Entropy is not a smooth invariant of laminations.*

COROLLARY 1.3 (Proposition 5.4). *Let $\mathscr{F}$ be a suspension with entropy* 2. *Then $\mathscr{F}$ admits an invariant measure.*

Next, we consider the measure-theoretic entropies. For suspensions, Alvarez [1] shows that for a well-chosen probability measure m on Γ, there is a one-to-one correspondence between m-stationary measures on T and harmonic measures on $\mathscr{F}$, for which invariant measures correspond to invariant measures. So, we obtain the same kind of comparison of the measure-theoretic entropies $h^-(\mu)$, $h(\mu)$, $h^+(\mu)$, with entropies of the action of Γ on T. In particular, when the image of ρ in Homeo(T) is generated by a single map f, we obtain the following link with the usual topological entropy $h_{\text{top}}(f)$ and the usual measure-theoretic entropy $h_\nu(f)$ of f.

THEOREM 1.4 (Theorem 5.8). *In the above setup, let μ be an ergodic harmonic measure on $\mathscr{F}$ and ν be the corresponding m-stationary measure on T given by Alvarez* [1]. *Then μ and ν are in fact invariant and there exists a $K_0 > 0$ such that:*

- $h(\mathscr{F}) = 2 + 2K_0 h_{\text{top}}(f)$;
- $h^-(\mu) = h(\mu) = h^+(\mu) = 2 + 2K_0 h_\nu(f)$.

In particular, we have the variational principle $h(\mathscr{F}) = \sup_\mu h(\mu)$.

Let us briefly describe the proofs of these results. The main problem is to compare on the one hand word distance (with weight) in Γ, with generators $\mathcal{G}$; and on the other hand the Poincaré distance to 0 in the disk. More precisely, the weight $\omega\colon \mathcal{G} \to \mathbb{R}_+^*$ is defined so that, for $\alpha = g_1 \dots g_n \in \Gamma$, $d_P(0, \alpha(0))$ is comparable to $\sum_{i=1}^n \omega(g_i)$. This result is obtained in Proposition 3.5. The second problem is to show that we can restrict our attention to a transversal and to the holonomy ρ. This is done in §4.2, in which we show that we can decompose the entropy into a leafwise one equal to 2, and a transversal one given by the holonomy. Here, the arguments are very similar to [2, 10, 11]. A crucial property that we show, is that one can get rid of the reparametrization issue in the definition

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(1.1) of d_R, by choosing canonical parametrizations, and that one still computes the same entropy. Finally, similar gymnastics of a different Bowen distance for the same entropy give us that the transversal entropy is comparable to $h(\rho, \mathcal{G}, \omega)$, as in Theorem 1.1.

Our more precise results for the measure-theoretic entropy come from a subadditive argument. If f is a generator of $\rho(\Gamma)$, we show that at the limit, the maximal power $f^{n(R)}$ represented in $\mathbb{D}_R$ is essentially $n(R) = K_0 R$, for some constant K_0. Therefore, all that remains to do is to adapt the proof of the Brin–Katok theorem to an invertible case, that is, to powers of f between $-n(R)$ and $n(R)$, instead of from 0 to n.

The paper is organized as follows. In §2, we present the notions of lamination, topological entropy, and suspension. In §3, we recall well-known facts about Fuchsian groups and prove our comparison of word distance and hyperbolic distance. In §4, we start by constructing the weighted entropy $h(\rho, \mathcal{G}, \omega)$, then we show that we can restrict our attention to the transversal, and we prove the main theorem. Finally, in §5, we recall the basics about invariant, stationary and harmonic measures and Alvarez's theorem, before proving Theorem 1.4.

1.1. *Notation.* Throughout this paper, we denote by $\mathbb{D}$ the unit disk of $\mathbb{C}$. For $R \in \mathbb{R}_+^*$, we also denote by $\mathbb{D}_R = \{\xi \in \mathbb{D};\ d_P(0, \xi) < R\}$ the open disk of hyperbolic radius R in $\mathbb{D}$, where $d_P(0, \xi) = \ln[(1 + |\xi|)/(1 - |\xi|)]$ is the Poincaré distance between 0 and ξ.

We will also consider a distance $d = d_X$ on an ambient compact metric space X, coming from a Riemannian metric on a manifold M. For $\phi, \psi : K \to X$ or $\tau, \sigma : K \to \mathbb{D}$ (typically, $K = \overline{\mathbb{D}}_R$, and the maps are in fact defined on whole $\mathbb{D}$), we write

$$d_K(\phi, \psi) = \sup_{\xi \in K} d(\phi(\xi), \psi(\xi)), \quad d_{P,K}(\tau, \sigma) = \sup_{\xi \in K} d_P(\tau(\xi), \sigma(\xi)).$$

For integers $p \leq q \in \mathbb{Z}$, we denote by $[\![p, q]\!] = \{p, p + 1, \ldots, q - 1, q\}$ the integer interval between p and q. If x is a real number, we denote by $\lfloor x \rfloor$ and $\lceil x \rceil$ the floor and ceiling parts of x. That is, $\lfloor x \rfloor$ (respectively, $\lceil x \rceil$) is the greatest (respectively, smallest) integer $p \in \mathbb{Z}$ such that $p \leq x$ (respectively, $p \geq x$).

Finally, we use C, C', C'', etc. to denote positive constants which may change from line to line.

2. *Laminations, suspensions and hyperbolic entropy*

2.1. *Basics about laminations.* In this section, we recall some concepts in the context needed for this paper. In particular, we will not define laminations in the broadest sense. The reader may consult [30] for a more general definition.

Definition 2.1. Let (X, d_X) be a compact metric space. A *lamination by Riemann surfaces* (henceforth, 'lamination') $\mathscr{F} = (X, \mathscr{L})$ on X is the data of an atlas $\mathscr{L}$ of X, given by homeomorphisms

$$\Phi_i : U_i \to \mathbb{D} \times \mathbb{T}_i,$$

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where $(U_i)_{i\in I}$ is an open covering of X, $\mathbb{T}_i$ are topological spaces (the *transversals*), and the transition maps $\Phi_{ij} = \Phi_i \circ \Phi_j^{-1}$ are of the form

$$\mathbb{D} \times \mathbb{T}_j \to \mathbb{D} \times \mathbb{T}_i, \quad (z, t) \mapsto (\psi_{ij}(z, t), \lambda_{ij}(t)), \tag{2.1}$$

with ψ_{ij} holomorphic in z and continuous in t and λ_{ij} continuous.

If the maps ψ_{ij} and λ_{ij} have more regularity, for example if they are $\mathscr{C}^r$, Lipschitz, etc., we will say that *the lamination $\mathscr{F}$ is $\mathscr{C}^r$*, *Lipschitz*, etc.

A coordinate chart $U \simeq \mathbb{D} \times \mathbb{T}$ as above is called a *flow box*. The level sets $\Phi_i^{-1}(\mathbb{D} \times \{t\})$, for $t \in \mathbb{T}_i$ are called *plaques*. The form (2.1) of the transition maps ensures that the plaques are compatible, in the sense that two different plaques either have empty intersection, or have an open subset of $\mathbb{D}$ in common. A non-empty connected subset of X which contains each plaque it intersects, and which is minimal for this property, is called a *leaf* of $\mathscr{F}$. The fact that ψ_{ij} is holomorphic in z in (2.1) implies that a leaf L of $\mathscr{F}$ inherits a Riemann surface structure.

Here, we will only consider laminations which are embedded in a real manifold M. That way, using a partition of unity, one can build a smooth Riemannian metric g_M, which restricts to a Hermitian metric on the plaques. We also suppose that the distance d_X on X is that coming from the distance induced by g_M on M. This is actually the only reason why we need to consider embedded laminations.

We are mainly interested in laminations, the leaves of which are all hyperbolic Riemann surfaces. If $\mathscr{F} = (X, \mathscr{L})$ is such a lamination, denote by g_P the leafwise Poincaré metric. Two Hermitian metrics on a Riemann surface being pointwise proportional, there exists a map

$$\eta \colon X \to \mathbb{R}_+^* \quad \text{such that} \quad \eta^2 g_P = g_M.$$

On the right-hand side of the above equation, we still denote by g_M its restriction to the tangent space to the leaves of $\mathscr{F}$. When X is compact, it is easy to see [**10**, pp. 572] (and in the context of suspensions as in §2.3, the arguments can be made even simpler) that there exists a constant $c_0 > 1$ such that

$$c_0^{-1} \leq \eta(x) \leq c_0, \quad x \in X. \tag{2.2}$$

2.2. *Hyperbolic entropy.* Consider a lamination $\mathscr{F} = (X, \mathscr{L})$, the leaves of which are hyperbolic, where (X, d_X) is a compact metric space. For each $x \in X$, denote by L_x the leaf of $\mathscr{F}$ through x and by $\phi_x \colon \mathbb{D} \to L_x$ a uniformization of L_x such that $\phi_x(0) = x$. Such a uniformization is unique up to precomposition by a rotation.

Definition 2.2. [**10**, §3] For $R > 0$, write $\mathbb{D}_R = \{\xi \in \mathbb{D};\ d_P(0, \xi) < R\}$, where d_P is the Poincaré distance on $\mathbb{D}$. Define the *Bowen distance*

$$d_R(x, y) = \inf_{\theta\in\mathbb{R}} \sup_{\xi\in\overline{\mathbb{D}}_R} d_X(\phi_x(\xi), \phi_y(e^{i\theta}\xi)), \quad x, y \in X. \tag{2.3}$$

Such a quantity measures the distance between the orbits of x and y until hyperbolic time R, up to reparametrization by a rotation. Clearly, it is independent of the choice of

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uniformizations ϕ_x, ϕ_y. The ball of center $x \in X$ and radius $\varepsilon \in \mathbb{R}_+^*$ for the distance d_R will be denoted by $B_R(x, \varepsilon)$ and called the *R-Bowen ball of center x and radius* ε.

For $R, \varepsilon > 0$ and $Y \subset X$, denote by $N(Y, R, \varepsilon)$ (respectively, $N'(Y, R, \varepsilon)$) the minimal $N \in \mathbb{N}$ such that there exist $x_1, \ldots, x_N \in X$ (respectively, $x_1, \ldots, x_N \in Y$) with

$$Y \subset \bigcup_{i=1}^{N} B_R(x_i, \varepsilon).$$

The subset $\{x_1, \ldots, x_N\}$ is called the (R, ε)-*covering of* Y (respectively, (R, ε)-*dense in* Y). Also, denote by $M(Y, R, \varepsilon)$ the maximal $M \in \mathbb{N}$ such that there exist $x_1, \ldots, x_M \in Y$ with the $B_R(x_i, \varepsilon)$, $i \in \{1, \ldots, M\}$, pairwise disjoint. The subset $\{x_1, \ldots, x_M\}$ is called (R, ε)-*separated in* Y.

Such numbers are in fact closely related to each other, as can be seen in the following lemma.

LEMMA 2.3. (Dinh–Nguyên–Sibony [10, Proposition 3.1]) *In the above setup, we have*

$$N(Y, R, \varepsilon) \leq N'(Y, R, \varepsilon) \leq M(Y, R, \varepsilon) \leq N(Y, R, \varepsilon/2).$$

Definition 2.4. [10, §3] For $Y \subset X$, the *hyperbolic entropy of* Y is defined as the quantity

$$h(Y) = \sup_{\varepsilon > 0} \limsup_{R \to +\infty} \frac{1}{R} \log N(Y, R, \varepsilon). \tag{2.4}$$

For $Y = X$, we denote it by $h(\mathscr{F})$. By Lemma 2.3, the same equivalent definition could have been made with the numbers $N'(Y, R, \varepsilon)$ or $M(Y, R, \varepsilon)$.

Remark 2.5. The above entropy will sometimes be referred as a *topological* entropy, as opposed to a *measure-theoretic* entropy (see §5). As we will see in Proposition 4.11, it is not necessarily invariant under homeomorphisms, so this may not be the best choice of terminology.

On the other hand, for compact laminations, it is easy to check that it is independent of the choice of the Riemannian metric g_M. Also, it is invariant under applications which are transversally continuous and leafwise biholomorphic.

Remark 2.6. This setup can be made much more general by considering a metric space X and a family of distances $(d_R)_{R \geq 0}$ on X. In practice, we suppose that $d_0 = d_X$, that d_R increases with R, and most of the time the distances d_R are continuous with respect to d_X. Then there is a notion of d_R-balls of radius ε, of covering, dense and separated subsets. Lemma 2.3 still holds and one can define an entropy for this family of distances by formula (2.4). See [10, §3] for various dynamically meaningful setups. This is something we will constantly do, for families of distances that will induce the same entropy (or a comparable one) as that of d_R. See, for example, §4.1 or Remark 4.8.

2.3. *Suspensions.* Consider a compact smooth Riemann surface Σ of genus $g \geq 2$ and denote by $\Gamma = \pi_1(\Sigma)$ its fundamental group. More details about the structure of Γ will

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be recalled in the next subsection. For now, consider a representation $\rho\colon \Gamma \to \mathrm{Homeo}(T)$, where T is a compact metric space. Then Γ acts diagonally on $\mathbb{D} \times T$ by

$$\Gamma \times (\mathbb{D} \times T) \to \mathbb{D} \times T, \quad \gamma \cdot (z, t) = (\gamma \cdot z, \rho(\gamma)(t)),$$

where $\gamma \cdot z$ is the action by deck transformations of $\gamma \in \pi_1(\Sigma)$ on a fixed uniformization $\mathbb{D} \to \Sigma$. Since the action of Γ on $\mathbb{D}$ is free, properly discontinuous and cocompact, it is also such on $\mathbb{D} \times T$. Therefore, if we define

$$X = (\mathbb{D} \times T)/\Gamma,$$

then X is compact. We will suppose that it is embedded in a real manifold M (this is the case if, for example, T is itself a real manifold), to inherit a well-chosen Riemannian structure. For $A \subset \mathbb{D} \times T$, denote by $[A]$ its image under the quotient map $\mathbb{D} \times T \to X$. Then X is laminated by the hyperbolic Riemann surfaces $[\mathbb{D} \times \{t\}]$, for $t \in T$. All these leaves are coverings of Σ. Note $\mathscr{F} = (X, \mathscr{L})$ in this lamination. It is called the *suspension of the representation* ρ.

Above, we presented suspensions in the context that is necessary to define our entropy, that is, that of hyperbolic leaves. The reader can consult [7, Ch. V, §4] for a broader definition. In our setup, Ghys [20, pp. 51–52] (see also [30, Example 2.31]) shows that when $T = \mathbb{P}^1$ and $\rho(\Gamma) \subset \mathrm{Aut}(\mathbb{P}^1)$, X is an algebraic surface that can be embedded in $\mathbb{P}^N$, for some $N \geq 3$.

From the point of view of entropy, suspensions constructed as above have the important and simple property that the covering maps of the leaves are completely explicit. This will allow us to have good estimates of entropy, in terms of the representation ρ.

More precisely, denote $\mathbb{T}_0 = [\{0\} \times T]$. This is a global transversal to the lamination $\mathscr{F}$, which is isomorphic to T and will be called the *distinguished transversal*. We will sometimes blur the distinction between a point $t \in T$ and the point $[0, t] \in \mathbb{T}_0$. For $t \in T$, denote by ϕ_t the uniformization of the leaf L_t passing through t given by

$$\phi_t\colon \mathbb{D} \to L_t, \quad z \mapsto [z, t]. \tag{2.5}$$

It is of course such that $\phi_t(0) = t$ (with $t \simeq [0, t]$), as in the notation of §2.2.

3. *Some hyperbolic geometry*

For this section, our main references are the books by Katok [25] and by Farb and Margalit [14]. We will only focus on the hyperbolic geometry needed for further estimates of entropy. Also, as far as Fuchsian groups are concerned, we will only consider those which are fundamental groups of smooth compact Riemann surfaces. The reader is invited to consult the aforementioned books for a more complete presentation.

3.1. *Cocompact Fuchsian groups.*

In this subsection, we recall the definition of a Fuchsian group, with a focus on those which are fundamental groups of compact smooth hyperbolic Riemann surfaces.

Definition 3.1. A *Fuchsian group* is a subgroup Γ of $\mathrm{Aut}(\mathbb{D})$ that acts properly and discontinuously on $\mathbb{D}$, that is, for every compact K of $\mathbb{D}$, $\{\alpha \in \Gamma;\ \alpha(K) \cap K \neq \emptyset\}$ is finite.

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If Γ is a Fuchsian group, the quotient $\Sigma = \mathbb{D}/\Gamma$ is a hyperbolic Riemann surface (with possibly orbifold singularities). We say that Γ is *cocompact* if Σ is compact.

Moreover, Σ is smooth if and only if every element of Γ besides the identity has no fixed point in $\mathbb{D}$, in which case $\pi_1(\Sigma) = \Gamma$ (see [25, §3.6]). Conversely, let Σ be a compact smooth hyperbolic Riemann surface and let $\phi\colon \mathbb{D} \to \Sigma$ be a uniformization of Σ. Then one can write $\Sigma = \mathbb{D}/\Gamma$, with $\Gamma = \pi_1(\Sigma)$ being a Fuchsian group acting by deck transformations on $\mathbb{D}$, and every element of Γ besides the identity having no fixed point.

The next three results gather together some very well-known facts we will use about Fuchsian groups.

PROPOSITION 3.2. [25, Corollary 4.2.3] *With the notation above, Σ is compact if and only if Γ has a compact fundamental domain D.*

PROPOSITION 3.3. [14, 10.4.1] *Consider two smooth compact Riemann surfaces $\Sigma_i = \mathbb{D}/\Gamma_i$ of genus $g_i \geq 2$, for $i \in \{1, 2\}$. Then Σ_1 and Σ_2 are homeomorphic if and only if they are diffeomorphic, if and only if Γ_1 and Γ_2 are group-isomorphic, if and only if $g = g_1 = g_2$. In that case, the Γ_i have the representation*

$$\Gamma_i = \langle \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g \mid \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \dots \alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1} = \mathrm{id}\rangle.$$

We now prove a simple result that will be useful in §4.4 to provide examples of diffeomorphic suspensions having different entropies.

PROPOSITION 3.4. *Let $g \in \mathbb{N}$ with $g \geq 2$ and $\varepsilon > 0$. Then there exists a Fuchsian group Γ_ε, with $\mathbb{D}/\Gamma_\varepsilon$ a compact smooth Riemann surface of genus g, a set of generators $\mathcal{G}_\varepsilon$ of Γ_ε with $\mathcal{G}_\varepsilon = \mathcal{G}_\varepsilon^{-1}$, $\mathrm{id} \notin \mathcal{G}_\varepsilon$, and an element $\alpha_\varepsilon \in \mathcal{G}_\varepsilon$ such that $d_P(0, \alpha_\varepsilon(0)) \leq \varepsilon$.*

Proof. The proof goes as for Poincaré's theorem [25, Theorem 3.4.2]. Consider a convex $4g$-gon, bounded by geodesics, the vertices of which are $re^{i\theta_j}$, $j \in [\![1, 4g]\!]$, with $\theta_1 = \mu/2, \theta_2 = \mu/2 + \nu, \theta_{j+1} = \theta_j + \mu$, $j \in [\![2, 4g-2]\!]$, $\theta_{4g} = \theta_{4g-1} + \nu = 2\pi - \mu/2$ (see Figure 1). In other words, we choose one big angle ν and one small angle μ, with $2\nu + (4g-2)\mu = 2\pi$, put $4g$ points on a circle with two pairs of them at angle ν and the others at angle μ, so that the two big geodesics are separated by only one geodesic. Write $A_j = re^{i\theta_j}$ and denote by D the closed $4g$-gon bounded by the geodesics A_jA_{j+1}, $j \in [\![1, 4g]\!]$, with $A_{4g+1} = A_1$. Now, as for Poincaré's theorem, the sum of the internal angles of D goes to $(4g-2)\pi$ when r goes to 0, and to 0 when r goes to 1. So, there is an $r \in (0, 1)$ such that this sum is 2π, and the usual identification of the sides of D defines a Fuchsian group (see [14, 10.4.2]).

We wish to show that it satisfies our condition. Denote by α_0 the generator sending the geodesic $(A_{4g-1}A_{4g})$ to the geodesic (A_1A_2). It is enough to prove that for well-chosen μ and ν, $d_P(0, \alpha_0(0)) < \varepsilon$. First, let us estimate this distance in terms of these angles. Note B_1 is the middle of the geodesic $[A_{4g-1}A_{4g}]$ and B_2 the middle of $[A_1A_2]$. Then $\alpha_0(B_1) = B_2$ and since B_1 and B_2 are complex conjugate,

$$d_P(0, \alpha_0(0)) \leq d_P(0, B_2) + d_P(\alpha_0(B_1), \alpha_0(0)) = 2d_P(0, B_2). \tag{3.1}$$

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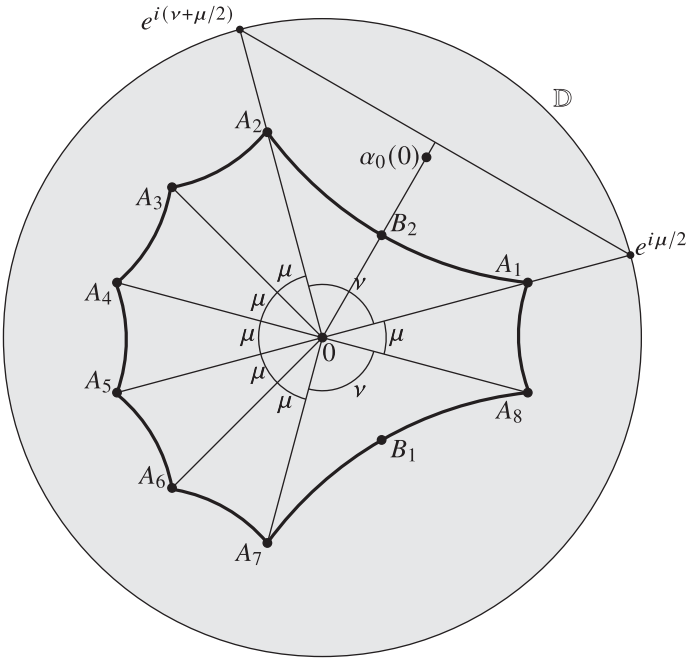


FIGURE 1. The polygon D, for some $r \in (0, 1)$ and $g = 2$, in the proof of Proposition 3.4.

In fact, we even have the equality but we do not need it. Now, B_2 is a point $r'e^{i(\mu+\nu)/2}$ and is in the Euclidean triangle $0A_1A_2$, which is itself contained in the Euclidean triangle whose vertices are 0, $e^{i\mu/2}$, $e^{i(\mu/2+\nu)}$. By elementary trigonometry, $r' \leq \cos(\nu/2)$ and if ν is sufficiently close to π, this implies that $d_P(0, B_2) < \varepsilon/2$. Equation (3.1) then gives the result and concludes the proof. □

3.2. *Covering estimates.* In this subsection, we give estimates that allow us to compare hyperbolic distance with word distance in a Fuchsian group with generators. Our main result is the following proposition.

PROPOSITION 3.5. *Let Γ be a Fuchsian group with $\mathbb{D}/\Gamma$ being a smooth compact Riemann surface of genus $g \geq 2$. Denote by D a compact fundamental domain of Γ, with $0 \in D$, and by $\mathcal{G}$ a symmetric set of generators of Γ such that $\mathcal{G} = \mathcal{G}^{-1}$. Define the radius $R_0 = \max\{d_P(0, \alpha(0)); \alpha \in \mathcal{G}\}$. Then there exists $K = K(\Gamma) > 1$ such that, for any $\delta \in (0, K^{-1})$ and for $N \in \mathbb{N}$ sufficiently large,*

$$\mathbb{D}_{(K^{-1}-\delta)N} \subset \bigcup_{\substack{\alpha_1,\dots,\alpha_n \in \mathcal{G} \\ n \leq N}} \alpha_1 \dots \alpha_n(D) \subset \mathbb{D}_{N(R_0+\delta)}.$$

Beginning of the proof. Let us start with the right inclusion. Let δ_0 be the diameter of D and $N \in \mathbb{N}$ be sufficiently large so that $N\delta > \delta_0$. Take $\alpha_1, \dots, \alpha_n \in \mathcal{G}$ for $n \leq N$, $\zeta \in D$,

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and denote $\xi = \alpha_1 \dots \alpha_n(\zeta)$. We need to show that $d_P(0, \xi) < N(R_0 + \delta)$. Denote $\xi_k = \alpha_1 \dots \alpha_k(0)$, for $k \in [\![0, n]\!]$, with $\xi_0 = 0$. Using the triangle inequality,

$$d_P(0, \xi) \leq \sum_{k=0}^{n-1} d_P(\xi_k, \xi_{k+1}) + d_P(\xi_n, \xi) = \sum_{k=0}^{n-1} d_P(0, \alpha_{k+1}(0)) + d_P(0, \zeta).$$

Above, we used that all the α_j, $j \in [\![1, n]\!]$, are isometries of the Poincaré distance. Now, by definition, all the terms of the sum are at most R_0 and $d_P(0, \zeta) \leq \delta_0$. Hence,

$$d_P(0, \xi) \leq nR_0 + \delta_0 < N(R_0 + \delta),$$

thanks to the above choice of N. This concludes the proof of the right inclusion. □

Remark 3.6. In §4 (see Proposition 4.11), we will encounter situations in which some generators $\alpha \in \mathcal{G}$, on the one hand have very small distance $d_P(0, \alpha(0))$, thanks to Proposition 3.4; and on the other hand are 'the only ones to count'. Indeed, in the representation ρ giving a suspension, the other generators will be sent to the identity, so that we will be able to disregard them. In such a situation, we will need a refinement of the inclusion we just proved. Namely, consider $\alpha_1, \dots, \alpha_n \in \mathcal{G}$, $\omega_j = d_P(0, \alpha_j(0))$, for $j \in [\![1, n]\!]$. Then, with the same proof as above,

$$\alpha_1 \dots \alpha_n(D) \subset \mathbb{D}_{\sum_{j=1}^n \omega_j + \delta_0}. \tag{3.2}$$

For the other inclusion, and for finding $K(\Gamma)$, we need the following lemma.

LEMMA 3.7. *Recall that δ_0 was defined as the diameter of D, in the notation of Proposition 3.5. There exist a finite number $\beta_1, \dots, \beta_p$ of elements of Γ such that, for any $\xi \in \mathbb{D}$ with $d_P(0, \xi) > \delta_0 + 1$,*

$$d_P(\beta_k(0), \xi) \leq d_P(0, \xi) - 1 \quad \textit{for some } k \in [\![1, p]\!].$$

Proof. Let us first construct some β_k, $k \in [\![1, p]\!]$, that satisfy

$$\partial \mathbb{D}_{\delta_0+1} \subset \bigcup_{k=1}^{p} \beta_k(D). \tag{3.3}$$

Since Γ acts properly discontinuously on $\mathbb{D}$, the set $B = \{\beta \in \Gamma;\ d_P(0, \beta(0)) \leq 2\delta_0 + 1\}$ is finite. Now, for $\xi \in \partial \mathbb{D}_{\delta_0+1}$, there exists $\beta \in \Gamma$ such that $\xi \in \beta(D)$. But

$$d_P(0, \beta(0)) \leq d_P(0, \xi) + d_P(\xi, \beta(0)) \leq 2\delta_0 + 1,$$

since $\xi = \beta(\zeta)$, $\zeta \in D$, β is an isometry, and by definition of δ_0. Hence, $\beta \in B$, and this completes the proof of (3.3), with $B = \{\beta_1, \dots, \beta_p\}$.

With the $\beta_1, \dots, \beta_p$ constructed, let us turn to the proof of the lemma. Let $\xi \in \mathbb{D}$ with $d_P(0, \xi) > \delta_0 + 1$ and let ξ_1 be the point in the geodesic joining 0 to ξ, lying between 0 and ξ and belonging to $\partial \mathbb{D}_{\delta_0+1}$. By (3.3), there exist $k \in [\![1, p]\!]$ and $\zeta \in D$ such that $\xi_1 = \beta_k(\zeta)$. Now, by the triangle inequality,

$$\begin{aligned} d_P(\beta_k(0), \xi) &\leq d_P(\beta_k(0), \beta_k(\zeta)) + d_P(\xi_1, \xi) \leq \delta_0 + d_P(0, \xi) - (\delta_0 + 1) \\ &= d_P(0, \xi) - 1. \end{aligned}$$

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Above, we used that β_k is an isometry of the Poincaré distance, that $d_P(0,\zeta) \leq \delta_0$ by definition of δ_0, and that $d_P(0,\xi) = d_P(0,\xi_1) + d_P(\xi_1,\xi)$, thanks to their configuration on the same geodesic. □

End of proof of Proposition 3.5. Consider the β_k, $k \in [\![1,p]\!]$, given by Lemma 3.7 and decompose each of the $\beta_k = \alpha_{1,k} \dots \alpha_{\ell_k,k}$ as a product of generators $\alpha_{j,k} \in \mathcal{G}$, for $k \in [\![1,p]\!]$ and $j \in [\![1,\ell_k]\!]$. Denote $K = \max\{\ell_k;\ k \in [\![1,p]\!]\}$.

Now, arguing as for (3.3), note that $\overline{\mathbb{D}}_{\delta_0+1}$ is covered by a finite number of copies of the fundamental domain D. Hence, there is $N_0 \in \mathbb{N}$ such that for $\xi \in \overline{\mathbb{D}}_{\delta_0+1}$, one can find $\alpha_1, \dots, \alpha_n \in \mathcal{G}$, with $n \leq N_0$ and $\xi \in \alpha_1 \dots \alpha_n(D)$. Let us show by induction on $L \in \mathbb{N}$ that if $\xi \in \mathbb{D}$ satisfies $d_P(0,\xi) \leq \delta_0 + 1 + L$, then there are $\alpha_1, \dots, \alpha_n \in \mathcal{G}$, with $n \leq KL + N_0$ and $\xi \in \alpha_1 \dots \alpha_n(D)$. We have just done the initialization for $L = 0$. Let $L \in \mathbb{N}$, and suppose that the induction hypothesis is satisfied up to rank L and let $\xi \in \mathbb{D}$ be such that $d_P(0,\xi) \in (\delta_0 + L + 1, \delta_0 + L + 2]$. By Lemma 3.7, one finds β_k, for some $k \in [\![1,p]\!]$, with $d_P(\beta_k(0),\xi) \leq d_P(0,\xi) - 1$. Denote $\xi' = \beta_k^{-1}(\xi)$. Then $d_P(0,\xi') \leq \delta_0 + L + 1$. By the induction hypothesis, there exist $\alpha_1, \dots, \alpha_n \in \mathcal{G}$, with $n \leq KL + N_0$ and $\xi' \in \alpha_1 \dots \alpha_n(D)$. Hence,

$$\xi = \beta_k(\xi') \in \beta_k \alpha_1 \dots \alpha_n(D) = \alpha_{1,k} \dots \alpha_{\ell_k,k} \alpha_1 \dots \alpha_n(D),$$

with $n + \ell_k \leq K(L+1) + N_0$, since $\ell_k \leq K$ by definition of K. The induction is complete.

Finally, consider $\delta \in (0, K^{-1})$ and $N \in \mathbb{N}$ sufficiently large. Let $\xi \in \mathbb{D}_{(K^{-1}-\delta)N}$, and define $L = \lceil N(K^{-1} - \delta) - (\delta_0 + 1) \rceil$. Recall that $\lceil x \rceil$ denotes the smallest integer $n \in \mathbb{Z}$ such that $n \geq x$, for $x \in \mathbb{R}$. Applying the result of the previous paragraph, we obtain $\alpha_1, \dots, \alpha_n \in \mathcal{G}, n \leq KL + N_0$, such that $\xi \in \alpha_1 \dots \alpha_n(D)$. So, it is enough to show that if N is sufficiently large, $KL + N_0 \leq N$. By minimality of L, we have

$$KL + N_0 \leq K(N(K^{-1} - \delta) - \delta_0) + N_0 \leq N - KN\delta + N_0 \leq N,$$

if $N \geq N_0 K^{-1} \delta^{-1}$. This completes the proof. □

4. *Comparing entropies*

4.1. *Entropy of a homeomorphisms representation with weighted generators.*

Here, we consider the abstract setup of a compact metric space (T, d_T), a group $\mathcal{H}$, with a (typically finite) set of generators $\mathcal{G}$ such that $\mathcal{G} = \mathcal{G}^{-1}$, and a morphism $\rho\colon \mathcal{H} \to \mathrm{Homeo}(T)$. We will construct some quantity that generalizes slightly an entropy defined by Ghys, Langevin and Walczak [**21**, §2] in this setup.

Consider also $\omega\colon \mathcal{G} \to \mathbb{R}_+^*$ a *weight function*. Define the *Bowen distance* for $R > 0$ and $t, s \in T$ to be

$$d_R^{\rho,\mathcal{G},\omega}(t,s) = \sup\{d_T(\rho(g_1 \dots g_k)(t), \rho(g_1 \dots g_k)(s))\},$$

where the sup is taken over $g_i \in \mathcal{G}$, $i \in [\![1,k]\!]$, such that $\sum_{i=1}^k \omega(g_i) \leq R$, for any $k \in \mathbb{N}$.

Given such a Bowen distance, we can mimic the vocabulary of §2.2 (see Remark 2.6) for (R,ε)-covering, dense and separated subsets for $(\rho,\mathcal{G},\omega)$. Denote by $B_R^{\rho,\mathcal{G},\omega}(t,\varepsilon)$ the Bowen ball $\{s \in T;\ d_R^{\rho,\mathcal{G},\omega}(t,s) < \varepsilon\}$, for $t \in X$ and $R, \varepsilon > 0$. Let $R, \varepsilon > 0$ and

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$Y \subset T$. Define $N(Y, R, \varepsilon, \rho, \mathcal{G}, \omega)$ (respectively, $N'(Y, R, \varepsilon, \rho, \mathcal{G}, \omega)$) to be the minimal cardinality of an (R, ε)-covering subset of Y (respectively, dense subset in Y), and $M(Y, R, \varepsilon, \rho, \mathcal{G}, \omega)$ the maximal cardinality of an (R, ε)-separated subset in Y. With exactly the same proof, we have the same property as Lemma 2.3. Hence, define the *entropy of ρ with respect to the set of generators $\mathcal{G}$ and weight ω* by

$$h(Y, \rho, \mathcal{G}, \omega) = \sup_{\varepsilon>0} \limsup_{R\to+\infty} \frac{1}{R} \log N(Y, R, \varepsilon, \rho, \mathcal{G}, \omega).$$

Of course, such a definition would yield the same result if we took $N'(Y, R, \varepsilon, \rho, \mathcal{G}, \omega)$ or $M(Y, R, \varepsilon, \rho, \mathcal{G}, \omega)$ instead. If $Y = T$, denote it simply by $h(\rho, \mathcal{G}, \omega)$.

Remark 4.1. Let us make a comparison with the similar definition of Ghys, Langevin and Walczak [**21**, §2]. They define the same kind of entropy (without making the Bowen distance explicit) for pseudo-groups of homeomorphisms. Here, we do not need pseudo-groups, since the holonomies of suspensions are global. However, we prefer to define it at the level of representations and not at the level of groups, since we will consider contexts in which two generators have the same image but not the same weight.

Let us denote by $h_{\mathrm{GLW}}(\rho(\mathcal{H}), \rho(\mathcal{G}))$ the entropy defined by Ghys. Langevin and Walczak. For reasons of clarity, they need to include the identity in $\mathcal{G}$ but we ignore this issue (or we add it to our setup with $\omega(\mathrm{id}) = 0$). Our entropy generalizes theirs in the sense that if 1 denotes the constant weight equal to 1, then $h_{\mathrm{GLW}}(\rho(\mathcal{H}), \rho(\mathcal{G})) = h(\rho, \mathcal{G}, 1)$. More generally, we have the following lemma.

LEMMA 4.2. *Retain the notation above. If $c_1 \leq \omega \leq c_2$, for $c_1, c_2 \in \mathbb{R}_+^*$ (this always happens for some $c_1, c_2 \in \mathbb{R}_+^*$ if $\mathcal{G}$ is finite), then*

$$c_2^{-1} h_{\mathrm{GLW}}(\rho(\mathcal{H}), \rho(\mathcal{G})) \leq h(\rho, \mathcal{G}, \omega) \leq c_1^{-1} h_{\mathrm{GLW}}(\rho(\mathcal{H}), \rho(\mathcal{G})).$$

Proof. It follows from the definition of $d_R^{\rho,\mathcal{G},\omega}$ that $d_{c_2^{-1}R}^{\rho,\mathcal{G},1} \leq d_R^{\rho,\mathcal{G},\omega} \leq d_{c_1^{-1}R}^{\rho,\mathcal{G},1}$. Then separated (respectively, covering) sets for $d_R^{\rho,\mathcal{G},\omega}$ being separated (respectively, covering) for $d_{c_1^{-1}R}^{\rho,\mathcal{G},\omega}$ (respectively, $d_{c_2^{-1}R}^{\rho,\mathcal{G},\omega}$), we obtain

$$N(Y, c_2^{-1}R, \varepsilon, \rho, \mathcal{G}, 1) \leq N(Y, R, \varepsilon, \rho, \mathcal{G}, \omega),$$
$$M(Y, R, \varepsilon, \rho, \mathcal{G}, \omega) \leq M(Y, c_1^{-1}R, \varepsilon, \rho, \mathcal{G}, 1).$$

Taking logarithms, dividing by R and taking the relevant limits, we obtain

$$c_2^{-1} h(Y, \rho, \mathcal{G}, 1) \leq h(Y, \rho, \mathcal{G}, \omega) \leq c_1^{-1} h(Y, \rho, \mathcal{G}, 1).$$

We finish the proof by applying Remark 4.1. □

Having proven that our entropy is linked to that of the three authors, we refer the reader to their article [**21**] for examples, estimates, and links to other notions of entropy.

4.2. *Entropy of the distinguished transversal.* Henceforth, fix $\mathscr{F} = (X, \mathscr{L})$ a lamination given by the suspension process of §2.3. Retain the notation introduced there. Also,

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denote by D a compact fundamental domain of Γ, with $0 \in D$, as was done in §3.2 and denote by δ_0 the diameter of D. Moreover, consider a smooth Riemannian metric g_M on X, which restricts to a Hermitian metric on the leaves, as explained in §2.1. Denote by d the distance on X induced by g_M. We want to reduce the problem of computing the entropy to the distinguished transversal $\mathbb{T}_0$. More precisely, we first want to obtain the following result.

THEOREM 4.3. *With the notation above, $h(\mathscr{F}) = h(\mathbb{T}_0) + 2$.*

To prove the latter, we need several lemmas. The first two concern how fast two automorphisms of the disk diverge from each other if they are close in 0. The first one comes from our previous work. The formulation is slightly modified for our purposes, but the statement is exactly the one that is proven (with the notation below, because $e^{-R} \leq 1 - |\xi|^2 \leq 2e^{-R}$, if R is sufficiently large). The second one comes from Dinh, Nguyên and Sibony's work on the entropy of foliations. A more precise version of it can be found in [2, Lemma 3.14], but the one here is sufficient for our needs.

LEMMA 4.4. [2, Lemma 3.11] *Let $\varepsilon > 0$ be sufficiently small, $\theta \in [-\pi, \pi]$ and $\xi \in \mathbb{D}$ be such that $R = d_P(0, \xi)$ is sufficiently large.*

(1) *If $e^{-R} \geq 8\varepsilon^{-1}|\sin(\theta/2)|$, then $d_P(\xi, e^{i\theta}\xi) \leq \varepsilon$.*
(2) *If $e^{-R} \leq \frac{1}{4}\varepsilon^{-1}|\sin(\theta/2)|$, then $d_P(\xi, e^{i\theta}\xi) \geq \varepsilon$.*

LEMMA 4.5. [10, Lemma 3.8] *Let $\varepsilon > 0$ be sufficiently small. There exists $A > 1$ such that for all $a, b \in \mathbb{D}$, the following assertions hold.*

(1) *If $d_P(a, b) \leq A^{-1}e^{-R}$, then there exist $\tau_a, \tau_b \in \mathrm{Aut}(\mathbb{D})$ such that $\tau_a(0) = a$, $\tau_b(0) = b$ and $d_{P,\overline{\mathbb{D}}_R}(\tau_a, \tau_b) \leq \varepsilon$.*
(2) *If $d_P(a, b) \geq Ae^{-R}$, then for all $\tau_a, \tau_b \in \mathrm{Aut}(\mathbb{D})$ such that $\tau_a(0) = a$ and $\tau_b(0) = b$, $d_{P,\overline{\mathbb{D}}_R}(\tau_a, \tau_b) \geq \varepsilon$.*

Recall the notation

$$d_{P,K}(\tau, \sigma) = \sup_{\xi \in K} d_P(\tau(\xi), \sigma(\xi)),$$

for $K \subset \mathbb{D}$ and $\tau, \sigma \in \mathrm{Aut}(\mathbb{D})$.

We now also need a simple result which compares the Poincaré metric on the disk and on close points in S. Recall the notation ϕ_t, for $t \in \mathbb{T}_0$, introduced in (2.5).

LEMMA 4.6. *There exist $h > 0$ and $c > 1$ such that:*

(1) *if $t \in \mathbb{T}_0$ and $\zeta, \xi \in \mathbb{D}$, then $d(\phi_t(\zeta), \phi_t(\xi)) \leq cd_P(\zeta, \xi)$;*
(2) *if $t, s \in \mathbb{T}_0$ and $\zeta, \xi \in \mathbb{D}$ are such that $d_P(\zeta, \xi) \leq h$, then we have $d(\phi_t(\zeta), \phi_s(\xi)) \geq c^{-1}d_P(\zeta, \xi)$.*

Proof. Note that $\phi_t(\zeta) = \phi_{t'}(\zeta')$, for some $\zeta' \in D$ and $t' \in \mathbb{T}_0$. Then statement (1) follows immediately from the equivalence of Hermitian metrics (2.2) and a compactness argument. For statement (2), let us choose $h > 0$ such that

$$d_P(\xi, \alpha(\xi)) > 2h, \quad \xi \in \mathbb{D}, \alpha \in \Gamma \setminus \{\mathrm{id}\}. \tag{4.1}$$

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It is classical that such an h exists, since Γ is cocompact and since each $\alpha \in \Gamma \setminus \{\mathrm{id}\}$ has no fixed point. Reducing h if necessary, it is sufficient to prove the statement only for $\phi_t(\zeta)$ and $\phi_s(\xi)$ belonging to a common chart $U \simeq \mathbb{D} \times \mathbb{B}^p$ of the ambient Riemannian manifold M, where $\mathbb{B}^p$ is the unit ball of some $\mathbb{R}^p$. Take a Riemannian geodesic joining $\phi_t(\zeta)$ and $\phi_s(\xi)$. If h is sufficiently small, it stays inside U and its projection in the leafwise direction joins ζ to some $\alpha(\xi)$, $\alpha \in \Gamma$. So that it is of Poincaré length at least $d_P(\zeta, \xi)$, thanks to condition (4.1). We conclude, by compactness of $D \times \mathbb{T}_0$, the equivalence of Riemannian metrics and (2.2). □

The following result will be the last in our preparation for Theorem 4.3, and will still be very useful in the next subsection (see Remark 4.8).

LEMMA 4.7. *Let $h > 0$ be given by Lemma 4.6 and $\delta > 0$. For all $\varepsilon > 0$ sufficiently small, and all $R > 0$ sufficiently large, we have the following inequality. Let $\zeta \in \mathbb{D}_h$ and $t, s \in \mathbb{T}_0$ be such that $d_R(\phi_t(\zeta), \phi_s(\zeta)) < \varepsilon$. Then*

$$d_{\overline{\mathbb{D}}_{(1-\delta)R}}(\phi_t, \phi_s) < 2\varepsilon.$$

Here again and in the proof below, we use the notation

$$d_K(\phi, \psi) = \sup_{\xi \in K} d(\phi(\xi), \psi(\xi)),$$

for $K \subset \mathbb{D}$ and $\phi, \psi : \mathbb{D} \to X$.

Proof. Let $\tau \in \mathrm{Aut}(\mathbb{D})$ be such that $\tau(0) = \zeta$ and define $\psi_t = \phi_t \circ \tau$, $\psi_s = \phi_s \circ \tau$. Since $d_R(\psi_t(0), \psi_s(0)) < \varepsilon$, one finds $\theta \in [-\pi, \pi]$ with $d_{\overline{\mathbb{D}}_R}(\psi_t, \psi_s \circ r_\theta) < \varepsilon$, where r_θ is the rotation of angle θ in $\mathbb{D}$. We first show that $|\theta| \leq e^{-R}$. Note that if there is some $\xi \in \overline{\mathbb{D}}_R$ with $d_P(\xi, e^{i\theta}\xi) \geq c\varepsilon$, then by continuity, there is a $\xi \in \overline{\mathbb{D}}_R$ such that $h \geq d_P(\xi, e^{i\theta}\xi) \geq c\varepsilon$. Applying statement (2) of Lemma 4.6 to $\tau(\xi)$ and $\tau(e^{i\theta}\xi)$, it follows that $d(\psi_t(\xi), \psi_s(e^{i\theta}\xi)) \geq \varepsilon$, a contradiction. Hence, for all $\xi \in \overline{\mathbb{D}}_R$, $d_P(\xi, e^{i\theta}\xi) \leq c\varepsilon$. By statement (2) of Lemma 4.4, one obtains $|\sin(\theta/2)| \leq 4\varepsilon e^{-R}$, so that $|\theta| \leq e^{-R}$, for ε sufficiently small.

Now, since $\overline{\mathbb{D}}_{(1-\delta)R} \subset \tau(\overline{\mathbb{D}}_{(1-\delta/2)R})$, for R sufficiently large,

$$d_{\overline{\mathbb{D}}_{(1-\delta)R}}(\phi_t, \phi_s) \leq d_{\overline{\mathbb{D}}_{(1-\delta/2)R}}(\psi_t, \psi_s) \leq d_{\overline{\mathbb{D}}_R}(\psi_t, \psi_s \circ r_\theta) + d_{\overline{\mathbb{D}}_{(1-\delta/2)R}}(\psi_s, \psi_s \circ r_\theta).$$

Above, the first term of the right-hand side is at most ε, and the second also by statement (1) of Lemma 4.4. This concludes the proof. □

Proof of Theorem 4.3. We prove the two inequalities separately. Let us first show that $h(\mathscr{F}) \leq 2 + h(\mathbb{T}_0)$. Take F an $(R + \delta_0, \varepsilon/2)$-dense subset in $\mathbb{T}_0$ of minimal cardinality and G an $A^{-1}e^{-R}$-dense subset of $\mathbb{D}_{\delta_0}$ for the Poincaré distance of minimal cardinality. Here, A is the constant of Lemma 4.5. Define $H = \{\phi_t(\zeta);\ t \in F,\ \zeta \in G\}$. We claim that H is (R, ε)-dense in X. Let $x = \phi_s(\xi) \in X$, for $s \in \mathbb{T}_0$ and $\xi \in D$. Take $t \in F$ with $d_{R+\delta_0}(t, s) < \varepsilon/2$. In particular, one finds $\theta \in \mathbb{R}$ with $d_{\overline{\mathbb{D}}_{R+\delta_0}}(\phi_t, \phi_s \circ r_\theta) < \varepsilon/2$. Let τ be an automorphism of the disk satisfying $\tau(0) = e^{-i\theta}\xi$. Since $d_P(0, \xi) \leq \delta_0$, $d_{\overline{\mathbb{D}}_R}(\phi_t \circ \tau, \phi_s \circ r_\theta \circ \tau) < \varepsilon/2$. Moreover, one can find some $\zeta \in G$ such that we have

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$d_P(\zeta, \tau(0)) < A^{-1}e^{-R}$, and an automorphism σ of $\mathbb{D}$ such that we have $\sigma(0) = \zeta$ and $d_{P,\overline{\mathbb{D}}_R}(\sigma, \tau) \leq c^{-1}(\varepsilon/2)$, by statement (1) of Lemma 4.5. Putting everything together, we obtain

$$\begin{aligned} d_R(x, \phi_t(\zeta)) &\leq d_{\overline{\mathbb{D}}_R}(\phi_s \circ r_\theta \circ \tau, \phi_t \circ \sigma) \\ &\leq d_{\overline{\mathbb{D}}_R}(\phi_s \circ r_\theta \circ \tau, \phi_t \circ \tau) + d_{\overline{\mathbb{D}}_R}(\phi_t \circ \tau, \phi_t \circ \sigma). \end{aligned}$$

On the right-hand side, the first term is smaller than $\varepsilon/2$, and the second too, by statement (1) of Lemma 4.6. It follows that H is (R, ε)-dense in X. Hence,

$$N'(X, R, \varepsilon) \leq \operatorname{card}(H) = \operatorname{card}(F)\operatorname{card}(G) \leq Ce^{2R}N'\left(\mathbb{T}_0, R + \delta_0, \frac{\varepsilon}{2}\right).$$

Taking $(1/R)$ log and relevant limits, we obtain $h(\mathscr{F}) \leq 2 + h(\mathbb{T}_0)$.

For the other inequality, fix $\delta > 0$, F a $((1-\delta)R, 2\varepsilon)$-separated subset in $\mathbb{T}_0$ of maximal cardinality and G an Ae^{-R}-separated subset for the Poincaré distance in $\mathbb{D}_{h/2}$ of maximal cardinality. Let us show that $H = \{\phi_t(\zeta);\ t \in F,\ \zeta \in G\}$ is (R, ε)-separated in X. Take $t, s \in F$ and $\zeta, \xi \in G$ such that $d_R(\phi_t(\zeta), \phi_s(\xi)) < \varepsilon$. We need to prove that $t = s$, $\zeta = \xi$. Since $d_R(\phi_t(\zeta), \phi_s(\xi)) < \varepsilon$, one finds two automorphisms $\tau, \sigma \in \operatorname{Aut}(\mathbb{D})$ such that $\tau(0) = \zeta$, $\sigma(0) = \xi$ and $d_{\overline{\mathbb{D}}_R}(\phi_t \circ \tau, \phi_s \circ \sigma) < \varepsilon$. By the same argument as in Lemma 4.7, since $d_P(\zeta, \xi) \leq h$, it follows that $d_{P,\overline{\mathbb{D}}_R}(\tau, \sigma) \leq c\varepsilon$. Choosing A above to apply Lemma 4.5(2), one obtains that $d_P(\zeta, \xi) \leq Ae^{-R}$ and that $\zeta = \xi$, since they both belong to G. Now, apply Lemma 4.7 to obtain that $d_{\overline{\mathbb{D}}_{(1-\delta)R}}(\phi_t, \phi_s) < 2\varepsilon$. Since $t, s \in F$, one obtains that $t = s$. Hence, H is (R, ε)-separated in X and

$$M(X, R, \varepsilon) \geq \operatorname{card}(H) = \operatorname{card}(F)\operatorname{card}(G) \geq C^{-1}e^{2R}M(\mathbb{T}_0, (1-\delta)R, 2\varepsilon).$$

Taking $(1/R)$ log and limits, one obtains $h(\mathscr{F}) \geq 2 + (1-\delta)h(\mathbb{T}_0)$, and since δ is arbitrary, this concludes the proof. □

Remark 4.8. Lemma 4.7 will be very useful in the next subsection, since it will help us to get rid of reparametrization in (2.3) to compute $h(\mathbb{T}_0)$. Indeed, define another Bowen distance to be, for $R > 0$,

$$d'_R(t, s) = \sup_{\xi \in \overline{\mathbb{D}}_R} d(\phi_t(\xi), \phi_s(\xi)), \quad t, s \in \mathbb{T}_0. \tag{4.2}$$

Compared to d_R, instead of being the distance between the two closest parametrizations of L_t, L_s, this is the distance between the two canonical parametrizations ϕ_t of L_t and ϕ_s of L_s. Given the Bowen distance d'_R, we define completely analogous concepts of being $(R, \varepsilon)'$-covering, dense or separated, for subsets of $\mathbb{T}_0$. Also, adding primes in the definition, we obtain an analogous notion of entropy denoted by $h'(Y)$, for $Y \subset \mathbb{T}_0$.

In fact, $h(Y) = h'(Y)$. Indeed, it is clear that $d'_R \geq d_R$. Therefore, $h'(Y) \geq h(Y)$. On the other hand, Lemma 4.7 gives that, for $\delta > 0$, there is a $C > 0$ such that $d'_{(1-\delta)R} \leq Cd_R$, for R sufficiently large. It follows that $h(Y) \geq (1-\delta)h'(Y)$ and, since δ is arbitrary, $h(Y) = h'(Y)$. It will be much more convenient to work with d'_R in the following to avoid reparametrization issues. Here, we showed that we are still computing the same entropy.

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4.3. *Comparing with the weighted entropy.* The results of the previous subsection are very general to non-singular laminations (see also [11, Proposition 4.1] for the singular case) and do not really depend on the suspension structure, with minor changes. What should be changed is that we would no longer consider the global transversal $\mathbb{T}_0$ but the union of local transversals in flow boxes. Also, there would be nothing like Lemma 4.7 since there would be no *canonical* parametrization. In this subsection, we enter more deeply into the structure of a suspension, and into the underlying Fuchsian group.

For $R > 0$, denote $\Gamma_R = \{\alpha \in \Gamma;\ d_P(0, \alpha(0)) \leq R\}$, and for $t, s \in \mathbb{T}_0$, consider yet another Bowen-like distance

$$d_{\Gamma_R}(t, s) = \max\{d(\phi_t(\alpha(0)), \phi_s(\alpha(0)));\ \alpha \in \Gamma_R\}. \tag{4.3}$$

This distance is somehow equivalent to the distance d'_R defined in (4.2), in the sense of the following lemma.

LEMMA 4.9. *For $t, s \in \mathbb{T}_0$ and $R > 0$, we have $d_{\Gamma_R}(t, s) \leq d'_R(t, s)$. Moreover, for any $\varepsilon > 0$, there exists $\lambda > 0$, independent of R, t and s, such that $d_{\Gamma_{R+\delta_0}}(t, s) \leq \lambda$ implies $d'_R(t, s) \leq \varepsilon$.*

Proof. The first inequality is immediate. For the second, define $\Delta_R = \bigcup_{\alpha\in\Gamma_R} \alpha(\overline{\mathbb{D}}_{\delta_0})$. Then $\overline{\mathbb{D}}_R \subset \Delta_{R+\delta_0}$. Hence, for $t, s \in \mathbb{T}_0$,

$$\begin{aligned} d'_R(t, s) = d_{\overline{\mathbb{D}}_R}(\phi_t, \phi_s) \leq d_{\Delta_{R+\delta_0}}(\phi_t, \phi_s) &\leq \max_{\alpha\in\Gamma_{R+\delta_0}} d_{\overline{\mathbb{D}}_{\delta_0}}(\phi_t \circ \alpha, \phi_s \circ \alpha), \\ &\leq \max_{\alpha\in\Gamma_{R+\delta_0}} d'_{\delta_0}(\rho(\alpha)(t), \rho(\alpha)(s)). \end{aligned} \tag{4.4}$$

The function d'_{δ_0} is uniformly continuous on the compact $\mathbb{T}_0 \times \mathbb{T}_0$. So, for $\varepsilon > 0$, we can find $\lambda > 0$ such that $d(t', s') \leq \lambda$ implies $d'_{\delta_0}(t', s') \leq \varepsilon$, $t', s' \in \mathbb{T}_0$. Suppose now that $d_{\Gamma_{R+\delta_0}}(t, s) \leq \lambda$. Then, for all $\alpha \in \Gamma_{R+\delta_0}$, $d(\rho(\alpha)(t), \rho(\alpha)(s)) \leq \lambda$ and thus $d'_{\delta_0}(\rho(\alpha)(t), \rho(\alpha)(s)) \leq \varepsilon$. It follows from (4.4) that $d'_R(t, s) \leq \varepsilon$, as expected. □

The previous lemma will enable us to compare the entropy of $\mathscr{F}$ to some weighted entropy for Γ and ρ, in the sense of §4.1. Consider $\mathcal{G}$ a set of generators such that $\mathcal{G} = \mathcal{G}^{-1}$, and for $\alpha \in \mathcal{G}$, define $\omega(\alpha) = d_P(0, \alpha(0))$. Note that the weight ω depends only on the Fuchsian group, and not on the representation ρ.

THEOREM 4.10. *With the notation above, there exists a constant $K' = K'(\Gamma)$, depending only on the Fuchsian group Γ and not on the representation ρ, such that*

$$h(\rho, \mathcal{G}, \omega) \leq h(\mathbb{T}_0) \leq K' h(\rho, \mathcal{G}, \omega).$$

Proof. Here, we use the results of §§3.2 and 4.1. By (3.2) and Lemma 4.9, we have $d'_R \geq d_{\Gamma_R} \geq d_R^{\rho,\mathcal{G},\omega}$. Hence, considering an $(R, \varepsilon)'$-dense subset in $\mathbb{T}_0$, it is ε-dense for $d_R^{\rho,\mathcal{G},\omega}$ and $h(\mathbb{T}_0) = h'(\mathbb{T}_0) \geq h(\rho, \mathcal{G}, \omega)$, by Remark 4.8.

For the other inequality, by Proposition 3.5, $d_{\Gamma_{(K^{-1}-\delta)N}} \leq d_N^{\rho,\mathcal{G},1} \leq d_{c_2 N}^{\rho,\mathcal{G},\omega}$, for a constant K depending only on Γ, any $\delta \in (0, K^{-1})$ and N sufficiently large. Here, we used the notation of Lemma 4.2. In particular, $c_2 = \max\{\omega(\alpha);\ \alpha \in \mathcal{G}\}$ only depends on Γ.

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So, using Lemma 4.9, if $d_{c_2N}^{\rho,\mathcal{G},\omega}(t,s) \leq \lambda$, then $d'_{(K^{-1}-\delta)N-\delta_0}(t,s) \leq \varepsilon$. Considering a λ-dense subset for $d_{c_2N}^{\rho,\mathcal{G},\omega}$, it is $((K^{-1}-\delta)N-\delta_0,\varepsilon)'$-dense. Taking $(1/N)$ log and limits, we obtain $c_2 h(\rho,\mathcal{G},\omega) \geq (K^{-1}-\delta)h'(\mathbb{T}_0)$. By Remark 4.8 and since δ is arbitrary, we obtain $h(\mathbb{T}_0) \leq c_2 K h(\rho,\mathcal{G},\omega)$, with $K' = c_2 K$ only depending on Γ. □

4.4. *Deriving some consequences.* We begin by constructing an example showing that entropy is not a smooth invariant. Consider a complex manifold T and $f\colon T \to T$ to be an automorphism of positive entropy $h(f)$. Take $\rho\colon \Gamma \to \mathrm{Aut}(T)$ a representation such that $\rho(\alpha_1) = f$, and $\rho(\alpha_k) = \mathrm{id}$, $k \in [\![2, g]\!]$, $\rho(\beta_k) = \mathrm{id}$, for $k \in [\![1, g]\!]$. Here, we used the notation of the representation of Proposition 3.3. For such a representation, Ghys, Langevin and Walczak [**21**, Example 2.2] showed that $h_{\mathrm{GLW}}(\rho(\Gamma),\rho(\mathcal{G})) = 2h(f)$. Moreover, we also obtain that $h(\rho,\mathcal{G},\omega) = (1/(\omega(\alpha_1)))h_{\mathrm{GLW}}(\rho(\Gamma),\rho(\mathcal{G}))$ with the arguments of Lemma 4.2. This has the following consequence.

PROPOSITION 4.11. *There exist two smoothly diffeomorphic foliations $\mathscr{F}_1$, $\mathscr{F}_2$ such that $h(\mathscr{F}_1) \neq h(\mathscr{F}_2)$. In other words, entropy is not a $\mathscr{C}^\infty$-invariant.*

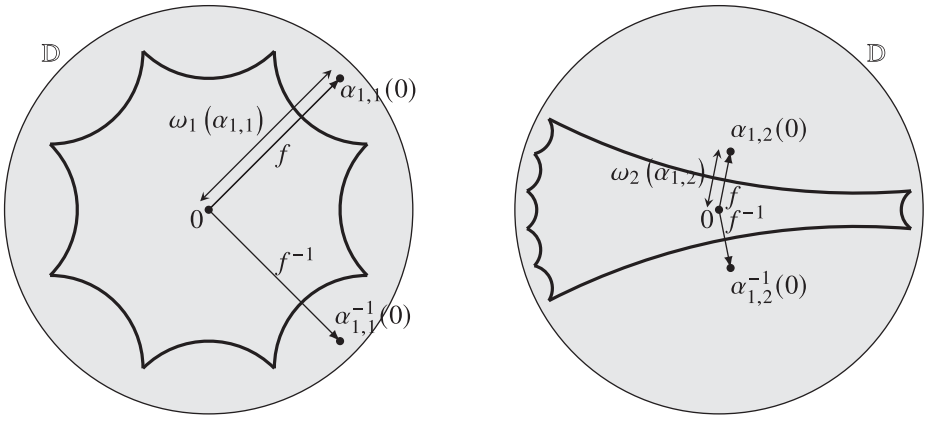
Proof. Denote by $\mathscr{F}_1$, $\mathscr{F}_2$ two suspensions obtained by the process above, with different but group-isomorphic Fuchsian groups Γ_1, Γ_2, respective generators $\mathcal{G}_1$, $\mathcal{G}_2$, and with the same T and f. Denote by $\alpha_{1,j} \in \mathcal{G}_j$, $j \in \{1,2\}$, the generator of Γ_j sent to f. Since the Riemann surfaces $\Sigma_j = \mathbb{D}/\Gamma_j$ are diffeomorphic (being compact and of the same genus) by a diffeomorphism sending $\alpha_{1,1}$ to $\alpha_{1,2}$, the foliations $\mathscr{F}_1$ and $\mathscr{F}_2$ are diffeomorphic. Moreover, by the paragraph above and by Theorems 4.3 and 4.10,

$$h(\mathscr{F}_1) \leq 2 + K'(\Gamma_1)h(\rho_1,\mathcal{G}_1,\omega_1) = 2 + 2h(f)\frac{K'(\Gamma_1)}{\omega_1(\alpha_{1,1})},$$
$$h(\mathscr{F}_2) \geq 2 + h(\rho_2,\mathcal{G}_2,\omega_2) = 2 + \frac{2h(f)}{\omega_2(\alpha_{1,2})}.$$

So, fixing Γ_1 and choosing Γ_2 in consequence, it is enough to find it, along with a generator $\alpha_{1,2}$ such that $\omega_2(\alpha_{1,2}) = d_P(0,\alpha_{1,2}(0)) < (\omega_1(\alpha_{1,1})/(K'(\Gamma_1)))$, which is a fixed constant. This can be achieved by Proposition 3.4 (see Figure 2). The whole point is that one can find a fixed generator $\alpha_{1,1} \in \mathcal{G}_1$, which will be sent to the α_ε of Proposition 3.4 by the diffeomorphism. This is guaranteed by the construction of α_ε and by [**14**, 10.4.2]. □

Remark 4.12. We believe that such a construction can show that there are suspensions for which all values in $(2,\infty)$ can be achieved in their class of diffeomorphism. Indeed, the proof above shows that arbitrarily large entropy can be achieved. Moreover, Proposition 3.4 can be adapted to have a generator of arbitrary large weight. In §5.3, we will even get an exact value of the entropy, of the form $2K_0h(f)$, with a constant K_0 depending only on Γ. However, this is not sufficient, since the constant K_0 is not easily seen to depend continuously on the Fuchsian group, and we are not able to control it enough to show that arbitrarily small entropy greater than 2 can be achieved.

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$$h(\mathscr{F}_1) \leq 2 + \frac{2K'(\Gamma_1)}{\omega_{1,1}(\alpha_1)} h(f); \qquad h(\mathscr{F}_2) \geq 2 + \frac{2}{\omega_2(\alpha_{1,2})} h(f)$$

FIGURE 2. Fundamental domains in the proof of Proposition 3.4.

So, the entropy is not a smooth invariant. However, in the case of suspensions and of homeomorphisms like that in the proof of Proposition 4.11, we have the following proposition.

PROPOSITION 4.13. *Let* $\Phi\colon \Sigma_1 \to \Sigma_2$ *be a homeomorphism between two hyperbolic Riemann surfaces inducing an isomorphism* $\varphi\colon \Gamma_1 = \pi_1(\Sigma_1) \to \Gamma_2 = \pi_1(\Sigma_2)$, *let* $\Psi\colon T_1 \to T_2$ *a homeomorphism between compact metric spaces, and for* $i \in \{1, 2\}$, *let* $\rho_i\colon \Gamma_i \to \mathrm{Homeo}(T_i)$, $i \in \{1, 2\}$, *be representations satisfying the conjugacy* $\rho_2(\gamma_2) = \Psi \circ \rho_1(\varphi^{-1}(\gamma_2)) \circ \Psi^{-1}$, *for* $\gamma_2 \in \Gamma_1$.

Let $\mathscr{F}_i$ *be the usual suspension of the representation* ρ_i, $i \in \{1, 2\}$. *Then* $h(\mathscr{F}_1) = 2$ *if and only if* $h(\mathscr{F}_2) = 2$, *and* $h(\mathscr{F}_1) = +\infty$ *if and only if* $h(\mathscr{F}_2) = +\infty$.

Proof. By Theorems 4.3 and 4.10, we have $h(\mathscr{F}_i) = 2$ (respectively, $h(\mathscr{F}_i) = +\infty$) if and only if $h_{\mathrm{GLW}}(\rho_i(\Gamma_i), \rho_i(\mathcal{G}_i)) = 0$ (respectively, $= +\infty$), for $i \in \{1, 2\}$. Since Ψ is uniformly continuous, it is easy to obtain the statement of the proposition from this observation. □

It is worth noting that having infinite entropy is quite a wild phenomenon. Indeed, for a general compact lamination, it was shown by Dinh, Nguyên and Sibony (see [**10**]) that if it is of class $\mathscr{C}^{2+\alpha}$, for $\alpha > 0$, then it has finite entropy. In the case of suspensions, thanks to a result of Ghys, Langevin and Walczak, one can do better.

PROPOSITION 4.14. *Let* $\mathscr{F}$ *be a Lipschitz suspension. Then* $h(\mathscr{F}) < +\infty$.

Proof. We use the notation of the previous subsection. If $\rho(\mathcal{G})$ contains only Lipschitz maps, it follows from [**21**, Proposition 2.7] that $h_{\mathrm{GLW}}(\rho(\Gamma), \rho(\mathcal{G}))$ is finite. We apply Theorem 4.10 and Lemma 4.2 to conclude the proof. □

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On the other hand, still by previous work of Ghys, Langevin and Walczak, we have the following necessary condition for having minimal entropy 2. Note that *a priori*, having an invariant measure for the action of a non-amenable group Γ is a rare phenomenon (see, for example, [8, 18, 33] and references therein).

PROPOSITION 4.15. *Let $\mathscr{F}$ be a suspension with entropy 2. Then there exists a Borel probability measure ν on T which is invariant by Γ (see Definition 5.2).*

Proof. It is shown in [21, §5] that if Γ is a group of homeomorphisms of a compact metric space T and $\mathcal{G}$ is a set of generators of Γ such that $\mathcal{G} = \mathcal{G}^{-1}$ and $h_{\mathrm{GLW}}(\Gamma, \mathcal{G}) = 0$, then there exists a probability measure ν on T which is invariant by Γ. Since by Theorem 4.10 and Lemma 4.2, $h(\mathscr{F}) \geq 2 + c_2 h_{\mathrm{GLW}}(\rho(\Gamma), \rho(\mathcal{G}))$, for $c_2 = \max(\omega)$, this result applies. □

5. *Measure-theoretic entropy*

In this section, we consider a measure-theoretic entropy for hyperbolic laminations, defined by Dinh, Nguyên and Sibony [10].

5.1. *Harmonic, invariant and stationary measures.*

Definition 5.1. Roughly speaking, harmonic measures are quasi-invariant measures by the lamination $\mathscr{F}$. More precisely, let $\mathscr{F} = (X, \mathscr{L})$ be a hyperbolic lamination and μ be a Borel probability measure on X. The measure μ is said to be *harmonic* if, for every flow box $U \simeq \mathbb{D} \times \mathbb{T}$, there exist a Radon measure ν on $\mathbb{T}$ and harmonic functions $h_t : \mathbb{D} \to \mathbb{R}_+$, for $t \in \mathbb{T}$, such that for every integrable function $f : U \to \mathbb{R}$,

$$\int f d\mu = \int_{\mathbb{T}} \int_{\mathbb{D}} h_t(z) f(t, z) g_P(z)\, d\nu(t). \tag{5.1}$$

If, moreover, the harmonic functions h_t, $t \in \mathbb{T}$, are all constant, μ is called an invariant measure.

There are analogous concepts for group actions.

Definition 5.2. Let Γ be a group acting on a topological space T and m be a probability measure on Γ. A Borel measure ν on T is called *invariant* by Γ if, for every $\gamma \in \Gamma$, $\gamma_* \nu = \nu$. It is called *m-stationary* if it is invariant on average, that is, if

$$\int_{\Gamma} \gamma_* \nu \, dm(\gamma) = \nu.$$

In fact, in the case of suspensions, there is a bridge between these two pairs of notions with the following result.

THEOREM 5.3. (Alvarez [1]) *Let $\Sigma = \mathbb{D}/\Gamma$ be a smooth compact Riemann surface of genus $g \geq 2$. There exists a probability measure m on Γ such that $m(\gamma) > 0$ for every $\gamma \in \Gamma$ and satisfying the following assertion. Let $\mathscr{F}$ be a suspension obtained as in §2.3. Then the map*

$$\mu \mapsto h_t(0)\, d\nu(t),$$

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with the notation of (5.1) *on the flow box* $U \times \mathbb{T}_0$*, for* U *a small neighborhood of* 0 *in* $\mathbb{D}$*, sends bijectively harmonic measures for* $\mathscr{F}$ *onto* m*-stationary measures for* Γ*. Moreover, it maps bijectively invariant measures for* $\mathscr{F}$ *onto invariant measures for* Γ*, and bijectively ergodic measures for* $\mathscr{F}$ *onto ergodic measures for* Γ*.*

The theorem above is valid in a broader context, but we state it in the one that is relevant here. Note that Proposition 4.15 now gives the following proposition.

PROPOSITION 5.4. *Let* $\mathscr{F}$ *be a suspension with entropy* 2*. Then there exists an invariant measure for* $\mathscr{F}$*.*

5.2. *Entropy of a harmonic measure.* For the dynamics of a single map, the usual definition of the entropy of a measure relies on measurable partitions. In our context, such a definition does not work, but thanks to the Katok [24] and Brin–Katok [6] theorems, one can still have analogies with only a notion of Bowen balls.

Definition 5.5. [10, §4] Let $\mathscr{F} = (X, \mathscr{L})$ be a hyperbolic lamination and μ a harmonic measure for $\mathscr{F}$. For $R, \varepsilon, \delta > 0$, denote by $N(R, \varepsilon, \delta)$ the minimal amount of (R, ε)-Bowen balls required to cover a subset of μ-measure at least $1 - \delta$. Define the *entropy of* μ to be

$$h(\mu) = \sup_{\delta>0} \sup_{\varepsilon>0} \limsup_{R\to+\infty} \frac{1}{R} \log N(R, \varepsilon, \delta).$$

Given $x \in X$, one also defines *local upper and lower entropies of* μ *at* x:

$$h^+(\mu, x) = \sup_{\varepsilon>0} \limsup_{R\to+\infty} -\frac{1}{R} \log \mu(B_R(x, \varepsilon)),$$
$$h^-(\mu, x) = \sup_{\varepsilon>0} \liminf_{R\to+\infty} -\frac{1}{R} \log \mu(B_R(x, \varepsilon)).$$

Finally, consider a local flow box $U \simeq \mathbb{D} \times \mathbb{T}$ and a local disintegration of μ into a leafwise harmonic density and a transversal Radon measure ν as in (5.1). Rescale the h_t and ν if necessary to have that $h_t(0) = 1$, for all $t \in \mathbb{T}$. Let $\pi_{\mathbb{T}} \colon U \to \mathbb{T}$ be the projection on the second coordinate. Define the *local transversal upper and lower entropies of* μ *at* x to be

$$\widetilde{h}^+(\mu, x) = \sup_{\varepsilon>0} \limsup_{R\to+\infty} -\frac{1}{R} \log \nu(\pi_{\mathbb{T}}(B_R(x, \varepsilon))),$$
$$\widetilde{h}^-(\mu, x) = \sup_{\varepsilon>0} \liminf_{R\to+\infty} -\frac{1}{R} \log \nu(\pi_{\mathbb{T}}(B_R(x, \varepsilon))).$$

In our context, such quantities were studied by Dinh, Nguyên and Sibony [10] (see also [2] for a singular context). In particular, they show the following theorem.

THEOREM 5.6. (Dinh–Nguyên–Sibony [10, Theorem 4.2 and Proposition 4.5]) *Consider* $\mathscr{F} = (X, \mathscr{L})$ *a compact hyperbolic lamination and* μ *a harmonic measure. Then the quantities* $x \mapsto \widetilde{h}^{\pm}(\mu, x)$ *and* $x \mapsto h^{\pm}(\mu, x)$ *are leafwise constant with*

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$h^{\pm}(\mu, x) = 2 + \widetilde{h}^{\pm}(\mu, x)$. *In particular, if* μ *is ergodic, then all these quantities are constant almost everywhere. Denote them by* $\widetilde{h}^{\pm}(\mu)$, $h^{\pm}(\mu)$. *We have*

$$2 \leq h^{-}(\mu) \leq h(\mu) \leq h^{+}(\mu) \leq h(\mathscr{F}).$$

In [10, Problems 5 and 6], the three authors ask whether $h^{+}(\mu) = h^{-}(\mu)$ holds (this would be a Brin–Katok-type theorem) and whether one has the variational principle $h(\mathscr{F}) = \sup_{\mu} h(\mu)$. Except when the entropy equals 2, for which it is trivial, there are no examples yet in which one can show whether or not it holds. Below, in Theorem 5.8, we will study a toy example in which one can use the classical theorems for maps to show them on foliations.

Before ending this subsection, let us give a more adapted way to compute the transversal entropies in the case of suspensions.

LEMMA 5.7. *Let* $\mathscr{F}$ *be a suspension with the usual notation,* μ *be a harmonic measure for* $\mathscr{F}$*, and recall the Bowen distances* d'_R *defined in (4.2). For* $t \in T$*, define*

$$B_R^{\mathbb{T}}(t, \varepsilon) = \{s \in T;\ d'_R(s, t) < \varepsilon\}, \quad B_R^{\mathbb{T},\Gamma}(t, \varepsilon) = \{s \in T;\ d_{\Gamma_R}(t, s) < \varepsilon\},$$

where d_{Γ_R} *is defined in* (4.3)*. Then, if* ν *denotes the measure on* T *given by Theorem 5.3,*

$$\widetilde{h}^{+}(\mu, t) = \sup_{\varepsilon>0} \limsup_{R\to+\infty} -\frac{1}{R} \log \nu(B_R^{\mathbb{T}}(t, \varepsilon)) = \sup_{\varepsilon>0} \limsup_{R\to+\infty} -\frac{1}{R} \log \nu(B_R^{\mathbb{T},\Gamma}(t, \varepsilon)),$$
$$\widetilde{h}^{-}(\mu, t) = \sup_{\varepsilon>0} \liminf_{R\to+\infty} -\frac{1}{R} \log \nu(B_R^{\mathbb{T}}(t, \varepsilon)) = \sup_{\varepsilon>0} \liminf_{R\to+\infty} -\frac{1}{R} \log \nu(B_R^{\mathbb{T},\Gamma}(t, \varepsilon)).$$

Proof. The right equalities are consequences of Lemma 4.9. For the left ones, with the notation of Definition 5.5, for some small $h > 0$ and if ε is sufficiently small,

$$\pi_{\mathbb{T}}(B_R(t, \varepsilon)) = \{s \in T;\ \text{there exists } \zeta \in \mathbb{D}_h \text{ with } d_R(\phi_s(\zeta), t) < \varepsilon\}.$$

Since $d'_R > d_R$, it is clear that we have the inequality '$\leq$' in the two equalities to be shown. Now consider $s \in \pi_{\mathbb{T}}(B_R(t, \varepsilon))$ and take $\zeta \in \mathbb{D}_h$ with $d_R(\phi_s(\zeta), t) < \varepsilon$. By the same arguments as in Lemma 4.7, we have $d_P(0, \zeta) < Ae^{-R}$. It follows by statement (1) of Lemma 4.5 that $d_{R(1-\delta)}(\phi_s(\zeta), \phi_s(0)) < \varepsilon$, for any $\delta > 0$ and R sufficiently large. Hence, $d_{R(1-\delta)}(t, s) < 2\varepsilon$ and $d'_{R(1-\delta)^2}(s, t) < 4\varepsilon$ by Lemma 4.7. We conclude by the same arguments as in Remark 4.8. □

5.3. *The case of a single map.* Here, we consider a suspension $\mathscr{F}$, for which the image of the representation $\rho(\Gamma) = \langle f \rangle \simeq \mathbb{Z}$ is generated by a single map f, and with the usual notation. More precisely, for $\alpha \in \mathcal{G}$, denote by $n(\alpha) \in \mathbb{Z}$ the integer such that $\rho(\alpha) = f^{n(\alpha)}$. Our main result is the following theorem.

THEOREM 5.8. *In the setup above, there exists a constant* K_0 *that does not depend on* f*, such that the following assertions hold.*

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(1) *We have* $h(\mathscr{F}) = 2 + 2K_0 h_{\text{top}}(f)$.
(2) *Every harmonic measure* μ *on* $\mathscr{F}$ *is actually invariant. Denote by* ν *its trace on* T *given by Theorem 5.3. Then, if* μ *is ergodic,*

$$h^-(\mu) = h(\mu) = h^+(\mu) = 2 + 2K_0 h_\nu(f).$$

(3) *We have* $h(\mathscr{F}) = \sup_\mu h(\mu) = \sup_{\mu \text{ ergodic}} h(\mu)$.

Here, the fact that K_0 does not depend on f means more precisely that it depends only on the Fuchsian group Γ, and on the powers $n(\alpha)$, for $\alpha \in \mathcal{G}$, but not on other choices. Moreover, we denoted by $h_{\text{top}}(f)$ the topological entropy of f in the usual sense, and by $h_\nu(f)$ its measure-theoretic entropy for ν. Note that item (3) is a consequence of the first two, and of the classical variational principle for maps (see [**34**, Theorem 8.6 and Corollary 8.6.1]). Let us emphasize the 2 in the theorem above, which comes from the fact that powers of f from $-n$ to n are considered, instead of classically from 0 to n.

Remark 5.9. Theorem 5.8 shows that there is no converse to Proposition 4.15. Even a weak version stating that μ invariant implies $h(\mu) = 0$ does not hold. Here, even if all harmonic measures are invariant, we do not necessarily have that $h(\mathscr{F}) = 2$. Other examples of random dynamical systems for which all stationary measures are invariant (so-called *stiffness*) can be found in [**8**, **33**] and references therein.

Let us start by finding the constant K_0. For $R > 0$, denote

$$n(R) = \max\{n \in \mathbb{N};\ \text{there exists } \alpha \in \Gamma,\ d_P(0, \alpha(0)) \le R \text{ and } \rho(\alpha) = f^n\}.$$

LEMMA 5.10. *The limit* $K_0 = \lim_{R \to +\infty}((n(R))/R)$ *exists in* $\mathbb{R}_+^*$.

Proof. It is easy to see that $R \mapsto -n(R)$ is subadditive. Hence, the limit of $n(R)/R$ exists in $[0, \infty]$ by [**34**, Theorem 4.9]. Moreover, there exists $K_1 > 1$ with $K_1^{-1} R \le n(R) \le K_1 R$, for sufficiently large R. Indeed, on the one hand $n(R) \ge \lfloor R/(d_P(0, \alpha(0)))\rfloor |n(\alpha)|$, for $\alpha \in \mathcal{G}$. On the other hand, using the notation of Proposition 3.5 and applying it to $\delta = K^{-1}/2$, one obtains $n(R) \le \lceil 2KR \rceil \max_{\alpha \in \mathcal{G}} n(\alpha)$. Therefore, K_0 is neither 0 nor ∞. Recall that for $x \in \mathbb{R}$, we denote by $\lfloor x \rfloor$ (respectively, $\lceil x \rceil$) the highest (respectively, lowest) integer $n \in \mathbb{Z}$ such that $n \le x$ (respectively, $n \ge x$). □

Proof of item (1) of Theorem 5.8. Note that the constant K_0 in the lemma above depends only on the Fuchsian group and on the powers $n(\alpha)$, $\alpha \in \mathcal{G}$. Now, recall the Bowen distance d_{Γ_R} defined in (4.3) and denote $d_n^{f^{\pm 1}}(t, s) = \max_{i \in [\![-n,n]\!]} d(f^i(s), f^i(s))$, for $s, t \in T$ and $n \in \mathbb{N}$. Finally, denote by $d_n^f(s, t) = \max_{i \in [\![0,n]\!]} d(f^i(s), f^i(t))$ the classical Bowen distance. Write $B_{\Gamma_R}(t, \varepsilon)$, $B_n^{f^{\pm 1}}(t, \varepsilon)$, $B_n^f(t, \varepsilon)$ for the corresponding Bowen balls and $N_\Gamma(T, R, \varepsilon)$, $N_f(T, n, \varepsilon)$, $N_{f^{\pm 1}}(T, n, \varepsilon)$ for the corresponding minimal number of Bowen balls required to cover T, as in Definition 2.2. Then the Bowen definition of the topological entropy [**34**, Definitions 7.9 and 7.10] is

$$h_{\text{top}}(f) = \sup_{\varepsilon > 0} \limsup_{n \to +\infty} \frac{1}{n} \log N_f(T, n, \varepsilon).$$

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On the other hand, for $t \in T$ and $n, \varepsilon > 0$, $B_n^{f^{\pm 1}}(t, \varepsilon) = f^n(B_{2n}^f(f^{-n}(t), \varepsilon))$, so that $N_{f^{\pm 1}}(T, n, \varepsilon) = N_f(T, 2n, \varepsilon)$. Now if we define

$$I(R) = \{i \in \mathbb{Z};\ \text{there exists } \alpha \in \Gamma,\ d_P(0, \alpha(0)) \leq R,\ \rho(\alpha) = f^i\},$$

by definition $I(R) \subset [\![-n(R), n(R)]\!]$. Moreover, the maximal step between two consecutive points of $I(R)$ is at most $P = \max_{\alpha \in \mathcal{G}}(n(\alpha))$. Therefore, by a uniform continuity argument on f^p, for $p \in [\![1, P]\!]$, as in Lemma 4.9, we have that for all $\varepsilon > 0$, there exists $\lambda > 0$ such that $B_{\Gamma_R}(t, \lambda) \subset B_{n(R)}^{f^{\pm 1}}(t, \varepsilon) \subset B_{\Gamma_R}(t, \varepsilon)$, for any $t \in T$. Hence,

$$N_\Gamma(\mathbb{T}_0, R, \lambda) \geq N_{f^{\pm 1}}(T, n(R), \varepsilon) = N_f(T, 2n(R), \varepsilon) \geq N_\Gamma(\mathbb{T}_0, R, \varepsilon).$$

We conclude with Lemma 5.10, the usual arguments that d_{Γ_R} and d_R define the same entropy, and Theorem 4.3. □

Note that by the above inclusions, and by Lemma 5.7, we have the following lemma.

LEMMA 5.11. *With the setup of Theorem 5.8, for $t \in T$,*

$$\widetilde{h}^+(\mu, t) = K_0 \sup_{\varepsilon > 0} \limsup_{n \to +\infty} \left(-\frac{1}{n} \log \nu \left(B_n^{f^{\pm 1}}(t, \varepsilon) \right) \right),$$
$$\widetilde{h}^-(\mu, t) = K_0 \sup_{\varepsilon > 0} \liminf_{n \to +\infty} \left(-\frac{1}{n} \log \nu \left(B_n^{f^{\pm 1}}(t, \varepsilon) \right) \right).$$

By the martingale theorem and the Hewitt–Savage 0–1 law [5], one can show that every bounded $\rho_* m$-harmonic function on $\mathbb{Z}$ is constant. This is actually a very simple case of the Choquet–Deny property of virtually nilpotent groups [17]. In particular, it implies that any m-stationary measure ν is actually invariant by f. Hence, by Theorems 5.3 and 5.6, the proof of Theorem 5.8 is reduced to the following proposition.

PROPOSITION 5.12. *Let $f : T \to T$ be a homeomorphism of a compact metric space and ν be an ergodic invariant measure. Then, for ν-almost every $t \in T$,*

$$\sup_{\varepsilon > 0} \liminf_{n \to +\infty} \left(-\frac{1}{n} \log \nu \left(B_n^{f^{\pm 1}}(t, \varepsilon) \right) \right) = \sup_{\varepsilon > 0} \limsup_{n \to +\infty} \left(-\frac{1}{n} \log \nu \left(B_n^{f^{\pm 1}}(t, \varepsilon) \right) \right) = 2h_\nu(f).$$

This is almost exactly like the Brin–Katok theorem [6], except that two-sided Bowen balls $B_n^{f^{\pm 1}}$ are involved, instead of usual Bowen balls B_n^f. Of course, our arguments will use the Brin–Katok theorem and its proof.

Proof. We only do the proof for $h_\nu(f) < +\infty$, since it is easy to adapt it for infinite entropy. Let us start by showing that the lim inf is greater than $2h_\nu(f)$. This is the most complicated part of Brin and Katok's proof, but they show [6, pp. 37] that for every $\delta > 0$, there exists $\varepsilon > 0$ with

$$\left(\nu \left\{ t \in T;\ \nu \left(B_n^f(t, \varepsilon) \right) > \exp(-(h_\nu(f) - \delta)n) \right\} \right)_{n \in \mathbb{N}}$$

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summable. Since $\nu(B_n^{f^{\pm 1}}(t,\varepsilon)) = \nu(B_{2n}^{f}(f^{-n}(t),\varepsilon))$ because ν is invariant,

$$\left(\nu\left\{t \in T;\ \nu\left(B_n^{f^{\pm 1}}(t,\varepsilon)\right) > \exp(-2(h_\nu(f)-\delta)n)\right\}\right)_{n\in\mathbb{N}}$$

is also summable. Here, we used again the ν-invariance by f. Just like Brin and Katok, we conclude by the Borel–Cantelli theorem that for ν-almost every $t \in T$,

$$\sup_{\varepsilon>0} \liminf_{n\to+\infty} \left(-\frac{1}{n}\log \nu\left(B_n^{f^{\pm 1}}(t,\varepsilon)\right)\right) \geq 2h_\nu(f). \tag{5.2}$$

Now let us proceed with showing that the lim sup is lower than $2h_\nu(f)$. Just like Brin and Katok, we use the Shannon–McMillan–Breiman theorem, except that an adapted version is needed. Consider ξ a finite measurable partition of T. For $x \in T$, denote by $c_n^{\pm}(x)$ the element of the partition $\bigvee_{i=-n}^{n} f^{-i}(\xi)$ containing x. Using the Birkhoff theorem for both f and f^{-1}, one can adapt the proof of [32, Theorem 2.5] to show that

$$-\frac{1}{2n}\log \nu\left(c_n^{\pm}(t)\right) \underset{n\to+\infty}{\longrightarrow} h_\nu(f,\xi) \quad \text{almost everywhere,}$$

where $h_\nu(f,\xi) \leq h_\nu(f)$ is the entropy of f, with respect to the partition ξ. Consider a finite measurable partition ξ, the elements of which all have diameter at most ε. Then $c_n^{\pm}(t) \subset B_n^{f^{\pm 1}}(t,\varepsilon)$. Hence,

$$\limsup_{n\to+\infty}\left(-\frac{1}{n}\log \nu\left(B_n^{f^{\pm 1}}(t,\varepsilon)\right)\right) \leq 2\limsup_{n\to+\infty}\left(-\frac{1}{2n}\log \nu\left(c_n^{\pm}(x)\right)\right) \leq 2h_\nu(f)$$

almost everywhere. Taking the sup with $\varepsilon > 0$, and together with (5.2), this finishes the proof. □

Acknowledgments. This work has been supported by the EIPHI Graduate school (contract 'ANR-17-EURE-0002'), by the Région 'Bourgogne Franche-Comté', and by the French National Research Agency under the project DynAtrois (contract 'ANR-24-CE40-1163'). The author would like to thank all those who asked 'naive' questions about the hyperbolic entropy of foliations. The aim of this paper is to answer most of them in a more precise way than by the usual waving of hands.

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