Ergod. Th. & Dynam. Sys. (2018), **38**, 2729–2747 © *Cambridge University Press, 2017*
doi:10.1017/etds.2017.15

On dynamical systems preserving weights

LAVY KOILPITCHAI and KUNAL MUKHERJEE

Indian Institute of Technology Madras, Chennai 600 036, India
(e-mail: lavykoilpitchai@gmail.com, kunal@iitm.ac.in)

(*Received* 29 *August* 2016 *and accepted in revised form* 10 *January* 2017)

Abstract. The canonical unitary representation of a locally compact separable group arising from an ergodic action of the group on a von Neumann algebra with separable predual preserving a faithful normal semifinite (infinite) weight is weak mixing. On the contrary, there exists a non-ergodic automorphism of a von Neumann algebra preserving a faithful normal semifinite trace such that the spectral measure and the spectral multiplicity of the induced action are respectively the Haar measure (on the unit circle) and ∞. Despite not even being ergodic, this automorphism has the same spectral data as that of a Bernoulli shift.

1. *Introduction*

Classical ergodic theory of probability measure preserving (p.m.p.) transformations and its non-commutative counterpart dealing with group actions on von Neumann algebras preserving faithful normal states are very well-developed areas of research. Developing the analogous theory in which p.m.p. transformations are replaced by transformations that preserve an infinite measure is quite hard, even when it comes to proving ergodic theorems. For an excellent account of the subject, see [**Aa**]. In the context of von Neumann algebras not much is known about group actions preserving faithful normal semifinite (f.n.s.) weights.

Connes proved that there exist separably acting III_1 factors that have no almost periodic faithful normal state or f.n.s. weight [**Co**]. Thus, the dynamics implemented by the modular automorphism groups in Connes' examples are always weak mixing on 'large regions'. For the same factors any ergodic action of any separable locally compact group preserving any faithful normal state will be weak mixing on 'large regions' as well ([**BCM**, §6]; see also [**HLS**]). From this point of view, it is interesting to investigate properties in ergodic hierarchy of f.n.s. weight preserving actions. Our interest in such dynamics arises from its relationship to the non-commutative generalization of relatively independent joinings of W^*-dynamical systems over a common subsystem.

To contrast with f.n.s. weight preserving systems, in the classical ergodic theory of p.m.p. (or faithful normal state preserving) dynamical systems, there are five levels

https://doi.org/10.1017/etds.2017.15 Published online by Cambridge University Press

of hierarchy, namely, Bernoulli $\Rightarrow$ Kolmogorov $\Rightarrow$ strong mixing $\Rightarrow$ weak mixing $\Rightarrow$ ergodicity. It is remarkable that there are instances of systems for which some if not all of the arrows in the aforesaid hierarchy can be reversed (in large regions, if not entirely on the phase space) (cf. [**Hal**, **Ka**, **BCM**, **JaPi**, **MSW**, **MuPa**]). The first occurrence of such rigidity in the ergodic hierarchy appeared in a work of Halmos where he proved that a (continuous) automorphism of a separable compact abelian group automatically preserves the normalized Haar measure, and the maximal spectral type of the induced $\mathbb{Z}$-action is Lebesgue and the spectral multiplicity is $\aleph_0$ [**Hal**]. This result was extended further by Kaplansky to the context of arbitrary separable compact groups by considering coordinates of irreducible representations in place of characters [**Ka**]. Kaplansky generalized Halmos's theorem except for the statement related to the spectral multiplicity. As far as is known today, for a compact metric abelian group (those for which the Pontryagin dual is countable), ergodic group automorphisms are abstractly modelled by independently and identically distributed stochastic processes popularly known as Bernoulli shifts (see [**MSW**] for related results) and this was the handiwork of several mathematicians.

Likewise, in the context of W^*-dynamics, an ergodic action of $\mathbb{R}$ on a von Neumann algebra preserving a KMS state is weakly mixing, that is, zero is the only simple eigenvalue of the associated Liouvillean (see [**JaPi**, Theorem 1.3]). Recently, in [**BCM**, §6], it was proved that if a locally compact separable group acts ergodically on a separably acting von Neumann algebra M preserving a faithful normal state φ, then any finite-dimensional invariant subspace of the induced unitary representation of the group in the associated GNS space lies within the image inside the GNS space of the centralizer M^{φ}, thereby strengthening a well-known result of Høegh-Krohn, Landstad, and Størmer [**HLS**]. Consequently, if $M^{\varphi} = \mathbb{C}1$ (which of course forces M to be a III_1 factor), then every φ-preserving ergodic action is weakly mixing. In the same vein, if a countable discrete group acts on a separable compact quantum group by quantum automorphisms, then ergodicity of the action is equivalent to topological transitivity and weak mixing of the induced action on the GNS space associated to the Haar state (see [**MuPa**, §3]). Kaplansky's result has also been generalized to the context of compact quantum groups in [**MuPa**, §5].

However, similar results are not known in the context of actions of groups on von Neumann algebras preserving f.n.s. weights. In the classical case, Hajian proved that if T is an ergodic measure preserving transformation (automorphism) of a σ-finite measure space (X, μ) such that $\mu(X) = \infty$, then, for any two measurable subsets $A, B \subseteq X$ of finite measure and any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $\mu(T^n A \cap B) < \epsilon$; see [**Ha**, Theorem 2]. Under the same set-up, Hajian and Kakutani proved that T always admits a weakly wandering† set; see [**HaKa**, Theorem 2]. The last statement is the fundamental difference between the two kinds of ergodic measure preserving transformations (finite and infinite measure preserving), as a p.m.p. ergodic transformation can never admit such a set. Again, Hajian also proved that in the context of infinite ergodic theory, an ergodic measure preserving transformation always admits a weakly wandering set of infinite measure [**Ha**, Theorem 3]. For many more illuminating facts the interested reader is referred to [**HaKa**].

† A weakly wandering set W is a measurable subset of X with $\mu(W) > 0$ such that there exists a subsequence $n_k < n_{k+1}$ with $\{T^{n_k}(W)\}$ being mutually disjoint.

https://doi.org/10.1017/etds.2017.15 Published online by Cambridge University Press

It is to be noted that f.n.s. weight preserving actions of groups on von Neumann algebras arise naturally. The action of modular automorphism groups of f.n.s. weights is the first example. Since the unitaries implementing the modular automorphisms generate an abelian von Neumann algebra A on the standard Hilbert space associated to the underlying f.n.s. weight, there is an abundance of unitaries that commute with A and that can implement fruitful actions on the von Neumann algebra. In fact, any action of a group that preserves an f.n.s. weight arises this way; see §3 for a short discussion.

Let us quickly sketch one more natural example. Let β be an action of a group G on a von Neumann algebra M with separable predual by preserving a faithful normal state φ. Let $L^2(M, \varphi)$ and J_φ respectively denote the GNS space and the standard anti-linear involution. Let $1 \in B \subseteq M$ be an invariant subalgebra of the action β such that there exists a φ-preserving, faithful, normal conditional expectation $\mathbb{E}_B^\varphi$ from M onto B. Denote by $\mathcal{H}_B$ the associated Stinespring correspondence arising from the u.c.p. map $\mathbb{E}_B^\varphi$ and let e_B denote the Jones projection associated to B. Similar to the tracial setting, one can define the basic construction $\langle M, e_B\rangle$ to be the von Neumann subalgebra $J_\varphi B' J_\varphi$ of $\mathbf{B}(L^2(M, \varphi))$. It is also easily seen to be the w^*-closure of $Me_BM = \text{span}\{xe_By : x, y \in M\}$. Moreover, there is a unique f.n.s. weight $\widehat{\varphi}$ on $\langle M, e_B\rangle$ such that $\widehat{\varphi}(xe_By) = \varphi(xy)$, $x, y \in M$. Then $\mathcal{H}_B$ and $L^2(\langle M, e_B\rangle, \widehat{\varphi})$ are isomorphic as M–M correspondences via the map $x \otimes y \mapsto xe_By$, $x, y \in M$. The action β on (M, φ) extends further to an action $\widehat{\beta}$ on $(\langle M, e_B\rangle, \widehat{\varphi})$ via $\widehat{\beta}_g(xe_By) = \beta_g(x)e_B\beta_g(y)$, $x, y \in M$ and $g \in G$. Thus, there are plenty of examples and these examples lead to the investigation of f.n.s. weight preserving dynamical systems.

Note that Hajian's theorem [**Ha**, Theorem 2] requires Birkhoff's pointwise ergodic theorem which is not available in the context of von Neumann algebras beyond semifinite traces; Birkhoff's theorem extends to a von Neumann algebra setting with respect to a given normal state by a result of Lance, and with respect to a semifinite trace as proved by Yeadon. However, we will justify that at the core of generalizing Hajian's theorem (or Hajian's theorem itself) is polar decomposition. In this paper, using Hilbert algebra techniques, we extend this classical theorem to the context of f.n.s. weights. We prove that if a separable locally compact group acts on a separably acting von Neumann algebra M preserving an f.n.s. weight φ, then ergodicity of the action forces weak mixing of the induced unitary representation of the group in the GNS space. Our proof relies heavily on the techniques developed in [**BCM**]. We also show that weak mixing of the aforesaid unitary representation does not entail ergodicity of the action. In fact, we provide an example of an f.n.s. trace preserving dynamical system with underlying group $\mathbb{Z}$ for which the induced unitary representation on the GNS space has countable Lebesgue spectrum but the action fails to be ergodic.

The layout of the paper is as follows. In §2 we lay out the preliminary background on Hilbert algebras that is needed to address the problem. In §3 we discuss the invariance of objects associated with the standard forms of von Neumann algebras under f.n.s. weight preserving automorphisms. In §4 we investigate polar decompositions of operators affiliated to (full) left (respectively, right) Hilbert algebras and their connections with automorphisms. In §5 we address the amplification of dynamical systems arising from tensoring von Neumann algebras with matrix algebras. Finally, spectral properties of f.n.s.

https://doi.org/10.1017/etds.2017.15 Published online by Cambridge University Press

weight preserving dynamical systems, the non-commutative version of Hajian's theorem (the main result of this paper) and the striking example as depicted above appear in §6. Readers familiar with Hilbert algebras can skip §2 and §3.

2. *Preliminaries on Hilbert algebras*

In this section we collect all the necessary facts about left (respectively, right) Hilbert algebras that will be needed in the sequel. Much of this material can be found in [**StZs**]. We are mainly interested in automorphisms of a von Neumann algebra that preserve a fixed f.n.s. weight. Thus, it is important to know to what extent such automorphisms preserve and transform varieties of objects associated to a von Neumann algebra. In this paper all Hilbert spaces are separable, all von Neumann algebras have separable preduals and inner products are linear in the left variable.

Let $\mathfrak{A}$ be a complex left Hilbert algebra with involution given by $\mathfrak{A} \ni \xi \mapsto \xi^{\#} \in \mathfrak{A}$ and a (definite) scalar product $\langle \cdot\,, \cdot \rangle$. Let $\mathcal{H}$ denote the Hilbert space obtained by completing $(\mathfrak{A}, \langle \cdot\,, \cdot \rangle)$ with respect to the induced norm. Thus, for $\xi \in \mathfrak{A}$, the densely defined operator L_ξ^0 defined by $L_\xi^0(\eta) = \xi\eta$, $\eta \in \mathfrak{A}$, has a bounded extension denoted by L_ξ to $\mathcal{H}$ and further $(L_\xi)^* = L_{\xi^\#}$. Consequently, $\mathfrak{A} \ni \xi \mapsto L_\xi \in \mathbf{B}(\mathcal{H})$ is a $*$-representation of $\mathfrak{A}$ in $\mathcal{H}$, and $\mathfrak{L}(\mathfrak{A}) = \{L_\xi : \xi \in \mathfrak{A}\}''$ is said to be the left von Neumann algebra of $\mathfrak{A}$. The aforesaid representation is actually injective. Moreover, $\mathcal{H} \supseteq \mathfrak{A} \ni \xi \mapsto \xi^\# \in \mathfrak{A} \subseteq \mathcal{H}$ is closable and anti-linear.

The map $\mathcal{H} \supseteq \mathfrak{A}^2 \ni \sum_i \xi_i\eta_i \mapsto (\sum_i \xi_i\eta_i)^\# \in \mathcal{H}$, defines a closed anti-linear operator S. The adjoint S^* of S obeys $\langle S^*\eta, \xi\rangle = \langle \eta, S\xi\rangle$, for all $\xi \in \mathfrak{D}(S)$ and $\eta \in \mathfrak{D}(S^*)$. Moreover, $S^2 \subseteq 1$, $(S^*)^2 \subseteq 1$, $S = S^{-1}$ and $S^* = (S^*)^{-1}$. Let $\Delta = S^*S$ denote the modular operator of the Hilbert algebra $\mathfrak{A}$. Then Δ is positive, self-adjoint, non-singular and $\Delta^{-1} = SS^*$. Let $S = J\Delta^{1/2}$ be the polar decomposition of S; then J is an anti-unitary with $J = J^* = J^{-1}$ and $J^2 = 1$. Further, $J\Delta J = \Delta^{-1}$, $S = J\Delta^{1/2} = \Delta^{-1/2}J$ and $S^* = J\Delta^{-1/2} = \Delta^{1/2}J$. Furthermore, $\Delta^{it}J = J\Delta^{it}$ for all $t \in \mathbb{R}$ and in general $Jf(\Delta)J = \overline{f}(\Delta^{-1})$ for all complex-valued Borel functions on $[0, \infty)$. Note that $t \mapsto \Delta^{it}$ is a strong operator topology continuous group of unitaries and is called the modular group associated to $\mathfrak{A}$.

There are a few more natural operators associated to the left Hilbert algebra $\mathfrak{A}$ that will be useful for our purpose. For $\eta \in \mathfrak{D}(S^*)$, the linear operator $R_\eta^0 : \mathfrak{A} \subseteq \mathcal{H} \to \mathcal{H}$ defined by $R_\eta^0(\xi) = L_\xi(\eta)$, $\xi \in \mathfrak{A}$, is closable, with closure denoted by R_η, and R_η is affiliated to $\mathfrak{L}(\mathfrak{A})'$. Again, $\mathfrak{A}' = \{\eta \in \mathfrak{D}(S^*) : R_\eta \in \mathbf{B}(\mathcal{H})\}$ is a right Hilbert algebra equipped with product $\eta_1\eta_2 = R_{\eta_2}(\eta_1)$, $\eta_1, \eta_2 \in \mathfrak{A}'$, involution given by $\eta^\flat = S^*\eta$, $\eta \in \mathfrak{A}'$, and scalar product of $\mathcal{H}$. Then $\mathfrak{A}' \ni \eta \mapsto R_\eta \in \mathbf{B}(\mathcal{H})$ is a $*$-antirepresentation of $\mathfrak{A}'$ and $\mathfrak{R}(\mathfrak{A}') = \{R_\eta : \eta \in \mathfrak{A}'\}'' = \mathfrak{L}(\mathfrak{A})'$ is the right von Neumann algebra associated to $\mathfrak{A}$.

Dualzing the above discussion, note that $\mathcal{H} \supseteq (\mathfrak{A}')^2 \ni \sum_i \eta_i\xi_i \mapsto (\sum_i \eta_i\xi_i)^\flat \in \mathcal{H}$, with $\eta_i, \xi_i \in \mathfrak{A}'$, extends to a closed anti-linear operator F such that $F = S^*$. For $\xi \in \mathfrak{D}(S)$, the map $L_\xi^0 : \mathfrak{A}' \subseteq \mathcal{H} \to \mathcal{H}$ by $L_\xi^0(\eta) = R_\eta(\xi)$, $\eta \in \mathfrak{A}'$, is closable and its closure L_ξ is affiliated to $\mathfrak{L}(\mathfrak{A})$. Further, $\mathfrak{A}'' = \{\xi \in \mathfrak{D}(S) : L_\xi \in \mathbf{B}(\mathcal{H})\}$ is a left Hilbert algebra equipped with multiplication $\xi_1\xi_2 = L_{\xi_1}(\xi_2)$, $\xi_1, \xi_2 \in \mathfrak{A}''$, and obvious involution given by S, and $\mathfrak{A}$ is a left Hilbert subalgebra of $\mathfrak{A}''$. The left (respectively, right) Hilbert algebra $\mathfrak{A}''$ (respectively, $\mathfrak{A}'$) is full.

https://doi.org/10.1017/etds.2017.15 Published online by Cambridge University Press

The (full) left Hilbert algebra $\mathfrak{A}''$ and the (full) right Hilbert algebra $\mathfrak{A}'$ crucially interact and this interaction is known as Tomita's fundamental theorem, which states:

(i) $J\mathfrak{A}''=\mathfrak{A}'$ and $S^*J\xi=JS\xi$, $R_{J\xi}=JL_\xi J$,

(i)$'$ $\Delta^{it}\mathfrak{A}''=\mathfrak{A}''$ and $S\Delta^{it}\xi=\Delta^{it}S\xi$, $L_{\Delta^{it}\xi}=\Delta^{it}L_\xi\Delta^{-it}$, $t\in\mathbb{R}$, for all $\xi\in\mathfrak{A}''$;

(ii) $J\mathfrak{A}'=\mathfrak{A}''$ and $SJ\eta=JS^*\eta$, $L_{J\eta}=JR_\eta J$,

(ii)$'$ $\Delta^{it}\mathfrak{A}'=\mathfrak{A}'$ and $S^*\Delta^{it}\eta=\Delta^{it}S^*\eta$, $R_{\Delta^{it}\eta}=\Delta^{it}R_\eta\Delta^{-it}$, $t\in\mathbb{R}$, for all $\eta\in\mathfrak{A}'$;

and further $\mathfrak{L}(\mathfrak{A})\ni x\mapsto Jx^*J\in\mathfrak{L}(\mathfrak{A})'$ is a $*$-anti-isomorphism. Consequently, $\mathfrak{L}(\mathfrak{A})$ acting on $\mathcal{H}$ is in standard form. Moreover, $\sigma_t(x)=\Delta^{it}x\Delta^{-it}$, $x\in\mathfrak{L}(\mathfrak{A})$ and $t\in\mathbb{R}$, implements a one-parameter strong*-continuous group of automorphisms of $\mathfrak{L}(\mathfrak{A})$ and is called the modular automorphism group associated to $\mathfrak{A}$.

Note that the association

$$\varphi_{\mathfrak{A}}(x)=\begin{cases}\|\xi\|^2, & \text{if there exists } \xi\in\mathfrak{A}'' \text{ such that } L_\xi=x^{1/2},\ x\in\mathfrak{L}(\mathfrak{A})_+,\\ +\infty, & \text{otherwise,}\end{cases}\tag{1}$$

defines an f.n.s. weight on $\mathfrak{L}(\mathfrak{A})$ which is invariant with respect to (σ_t) and $\varphi_{\mathfrak{A}}$ is said to be the weight associated to the left Hilbert algebra $\mathfrak{A}$. Similarly, one can define the weight $\varphi_{\mathfrak{A}'}$ associated to the right Hilbert algebra $\mathfrak{A}'$. Moreover, $\varphi_{\mathfrak{A}}$ satisfies the KMS condition for any pair of elements in $\mathfrak{N}_{\varphi_{\mathfrak{A}}}\cap\mathfrak{N}_{\varphi_{\mathfrak{A}}}^*$, where $\mathfrak{N}_{\varphi_{\mathfrak{A}}}=\{x\in\mathfrak{L}(\mathfrak{A}):\varphi_{\mathfrak{A}}(x^*x)<\infty\}$ and the map $\mathfrak{A}''\ni\xi\mapsto L_\xi\in\mathfrak{N}_{\varphi_{\mathfrak{A}}}\cap\mathfrak{N}_{\varphi_{\mathfrak{A}}}^*$ is a $*$-isomorphism such that $\varphi_{\mathfrak{A}}(L_\zeta^*L_\xi)=\langle\xi,\zeta\rangle$, $\xi,\zeta\in\mathfrak{A}''$.

The Tomita algebra

$$\begin{aligned}\mathfrak{T}=\Big\{&\xi\in\bigcap_{z\in\mathbb{C}}\mathfrak{D}(\Delta^z):\Delta^z\xi\in\mathfrak{A}''\cap\mathfrak{A}',\\ &\mathfrak{D}(\Delta^zL_\xi\Delta^{-z})=\mathfrak{D}(\Delta^{-z}) \text{ and } \Delta^zL_\xi\Delta^{-z}\subset L_{\Delta^z\xi},\\ &\mathfrak{D}(\Delta^zR_\xi\Delta^{-z})=\mathfrak{D}(\Delta^{-z}) \text{ and } \Delta^zR_\xi\Delta^{-z}\subset R_{\Delta^z\xi} \text{ for all } z\in\mathbb{C}\Big\}\end{aligned}\tag{2}$$

is a left Hilbert subalgebra of $\mathfrak{A}''$ such that $\mathfrak{T}''=\mathfrak{A}''$ and $\mathfrak{T}'=\mathfrak{A}'$. Note that $\mathfrak{T}$ is a core for Δ^z and $\Delta^z\mathfrak{T}=\mathfrak{T}$, $\Delta^z(\xi\eta)=(\Delta^z\xi)(\Delta^z\eta)$ for all $z\in\mathbb{C}$ and $\xi,\eta\in\mathfrak{T}$. Also $J\mathfrak{T}=\mathfrak{T}$, $J(\xi\eta)=(J\xi)(J\eta)$ and $\Delta^zJ\xi=J\Delta^{-\bar z}\xi$ for all $\xi,\eta\in\mathfrak{T}$. Clearly, $\mathfrak{T}$ is dense in $\mathcal{H}$.

Note that $\mathfrak{P}_S=\{\xi\in\mathfrak{D}(S):S\xi=\xi \text{ and } L_\xi\geq 0\}$ and $\mathfrak{P}_{S^*}=\{\eta\in\mathfrak{D}(S^*):S^*\eta=\eta \text{ and } R_\eta\geq 0\}$ are closed convex cones which are polar to each other, $\Delta^{1/2}\mathfrak{P}_S=J\mathfrak{P}_S=\mathfrak{P}_{S^*}$ and hence $\mathfrak{P}=\overline{\Delta^{1/4}\mathfrak{P}_S}=\overline{\Delta^{-1/4}\mathfrak{P}_{S^*}}$ is a self-polar (dual) closed convex cone in $\mathcal{H}$ invariant under Δ^{it}, $t\in\mathbb{R}$. Further, $J\xi=\xi$ for $\xi\in\mathfrak{P}$, $\mathcal{H}=\mathfrak{P}-\mathfrak{P}+i(\mathfrak{P}-\mathfrak{P})$ and $\mathfrak{P}=\overline{\{\xi(J\xi):\xi\in\mathfrak{T}\}}=\overline{\{\xi(J\xi):\xi\in\mathfrak{A}''\}}=\overline{\{(J\eta)\eta:\eta\in\mathfrak{A}'\}}$.

3. *Invariance under automorphisms*

In this section we record the invariance of various objects associated to a von Neumann algebra under an automorphism preserving an f.n.s. weight and certain translational properties under automorphisms.

Let M be a von Neumann algebra and let φ be an f.n.s. weight on M. Let $\mathfrak{F}_\varphi=\{x\in M_+:\varphi(x)<\infty\}$, $\mathfrak{N}_\varphi=\{x\in M:\varphi(x^*x)<\infty\}$ and $\mathfrak{M}_\varphi=\mathfrak{N}_\varphi^*\mathfrak{N}_\varphi\subseteq\mathfrak{N}_\varphi\cap\mathfrak{N}_\varphi^*$. Then $\mathfrak{N}_\varphi$,

https://doi.org/10.1017/etds.2017.15 Published online by Cambridge University Press

$\mathfrak{M}_\varphi$ are respectively left ideal and $*$-subalgebra of M, $\mathfrak{M}_\varphi^+ = \mathfrak{F}_\varphi$ and $\mathfrak{M}_\varphi$ is the linear hull of $\mathfrak{F}_\varphi$. The weight φ extends uniquely to a linear functional on $\mathfrak{M}_\varphi$, and the extension is also denoted by φ, and φ is lower w-semicontinuous on M^+. Let M act faithfully on the GNS Hilbert space $\mathcal{H}_\varphi$ by left multiplication, the latter being obtained via separation and completion of the form $\langle x, y\rangle_\varphi = \varphi(y^*x)$, $x, y \in \mathfrak{N}_\varphi$. The GNS representation will be denoted by π_φ, while the norm and the inner product on $\mathcal{H}_\varphi$ will respectively be denoted by $\|\cdot\|_\varphi$ and $\langle\cdot\,,\cdot\rangle_\varphi$. For $x \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$, we will denote its image in $\mathcal{H}_\varphi$ by x_φ. Note that $\mathfrak{M}_\varphi^2$ is dense in $\mathfrak{M}_\varphi$ in $\|\cdot\|_\varphi$, which in turn is dense in $\mathcal{H}_\varphi$ in $\|\cdot\|_\varphi$. Then $\mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$ equipped with $*$-structure inherited from M and the inner product structure inherited from $\mathcal{H}_\varphi$ is a left Hilbert algebra $\mathfrak{A}_\varphi \subseteq \mathcal{H}_\varphi$ and $\pi_\varphi(M) = \mathfrak{L}(\mathfrak{A}_\varphi) = \{L_{x_\varphi} : x_\varphi \in \mathfrak{A}_\varphi\}''$. We will denote the canonical modular operator and the anti-linear unitary respectively by Δ_φ and J_φ, and S_φ will denote the closure of the involution in $\mathfrak{A}_\varphi$, that is, $S_\varphi = J_\varphi \Delta_\varphi^{1/2}$. Moreover, $\varphi_{\mathfrak{A}} = \varphi$. Further, in this case $\mathfrak{A}_\varphi'' = \{\xi \in \mathfrak{D}(S_\varphi) : L_\xi \in \mathbf{B}(\mathcal{H}_\varphi)\} = \mathfrak{A}_\varphi$, that is, $\mathfrak{A}_\varphi$ is a full left Hilbert algebra and $\{(\sigma^\varphi_{-i/2}(x))_\varphi : x \in \mathfrak{D}(\sigma^\varphi_{-i/2}) \cap \mathfrak{A}_\varphi\} \subseteq \mathfrak{A}_\varphi' = J_\varphi \mathfrak{A}_\varphi$.

Since the representation π_φ will be fixed throughout the paper, we will regard $\pi_\varphi(M) = M$ and write $\pi_\varphi(x) = x$ for $x \in M$ to reduce notation. Recall that $\sigma_t^\varphi(x) = \Delta_\varphi^{it} x \Delta_\varphi^{-it}$, for all $x \in M$, $t \in \mathbb{R}$, defines the modular automorphism group of M associated to φ and (σ_t^φ) is strong*-continuous (that is, (σ_t^φ) is the modular automorphism group associated to the left Hilbert algebra $\mathfrak{A}_\varphi$). Thus, (σ_t^φ) preserves each of $\mathfrak{F}_\varphi$, $\mathfrak{N}_\varphi$, $\mathfrak{M}_\varphi$. Denoting $\mathfrak{T}_\varphi$ and $\mathfrak{P}_\varphi$ respectively to be the Tomita algebra and the standard self-polar cone (of positives in $\mathcal{H}_\varphi$) associated to φ, it follows that Δ_φ^{it} preserves $\mathfrak{A}_\varphi$, $\mathfrak{A}_\varphi'$ (from (i)–(iv) in §2), $\mathfrak{T}_\varphi$ and $\mathfrak{P}_\varphi$ for all $t \in \mathbb{R}$. For more details, we refer the reader to [**St**, **Ta2**].

Let $\alpha \in \mathrm{Aut}(M, \varphi)$. Since α preserves φ, it follows that $\alpha\sigma_t^\varphi = \sigma_t^\varphi \alpha$ for all $t \in \mathbb{R}$ [**St**, p. 69] (see also [**HeTa**]). Further, α preserves each of $\mathfrak{F}_\varphi$, $\mathfrak{N}_\varphi$, $\mathfrak{M}_\varphi$ and $\mathfrak{A}_\varphi$ and is consequently implemented by an unique unitary V_α on $\mathcal{H}_\varphi$ such that $V_\alpha x_\varphi = (\alpha(x))_\varphi$, for $x_\varphi \in \mathfrak{A}_\varphi$, that is, $\mathrm{Ad}(V_\alpha) = \alpha$.

Let $\mathcal{G}(\cdot)$ denote the graph of an (unbounded) operator. Parts of the next proposition are known and for some parts we could not locate a reference. So we prove it entirely for the sake of convenience.

PROPOSITION 3.1. *The following hold.*

(1) $S_\varphi V_\alpha = V_\alpha S_\varphi$, $S_\varphi^* V_\alpha = V_\alpha S_\varphi^*$, $J_\varphi V_\alpha = V_\alpha J_\varphi$ *and* $\Delta_\varphi^z V_\alpha = V_\alpha \Delta_\varphi^z$ *for all* $z \in \mathbb{C}$ *(as closed operators).*
(2) V_α *preserves* $\mathfrak{A}_\varphi$, $\mathfrak{A}_\varphi'$.
(3) *For* $\xi \in \mathfrak{D}(S_\varphi)$ *and* $\eta \in \mathfrak{D}(S_\varphi^*)$, $V_\alpha L_\xi V_\alpha^* = L_{V_\alpha \xi}$ *and* $V_\alpha R_\eta V_\alpha^* = R_{V_\alpha \eta}$.
(4) V_α *preserves* $\mathfrak{P}_{S_\varphi}$ *and* $\mathfrak{P}_{S_\varphi^*}$.
(4) V_α *preserves* $\mathfrak{P}_\varphi$ *and* $\mathfrak{T}_\varphi$.

Proof. By the discussion above, it follows that $V_\alpha(\mathfrak{A}_\varphi) = \mathfrak{A}_\varphi$. Note that $V_\alpha S_\varphi$, $V_\alpha S_\varphi^*$, and $V_\alpha \Delta_\varphi^z$, $z \in \mathbb{C}$, are closed operators, as V_α is a unitary.

(1) From the above discussion, first note that $\Delta_\varphi^{it} V_\alpha = V_\alpha \Delta_\varphi^{it}$ for all $t \in \mathbb{R}$. Also note that it is enough to show that $\Delta_\varphi^s V_\alpha = V_\alpha \Delta_\varphi^s$ for all $s \in \mathbb{R}$ to conclude that $\Delta_\varphi^z V_\alpha = V_\alpha \Delta_\varphi^z$ for all $z \in \mathbb{C}$. Further, it is enough to establish the same when $s > 0$; the case for $s < 0$ follows by a symmetric argument, and for $s = 0$ the commutation is a tautology. Fix $0 < s \in \mathbb{R}$.

https://doi.org/10.1017/etds.2017.15 Published online by Cambridge University Press

Note that for $\xi \in \mathcal{H}_\varphi$, the function $i\mathbb{R} \ni it \mapsto \Delta_\varphi^{it}(\xi) \in \mathcal{H}_\varphi$ has an analytic extension denoted by F_ξ to the interior of $\mathcal{D}_s = \{z \in \mathbb{C} : 0 \le \Re(z) \le s\}$ which is continuous up to the boundary if and only if the function $i\mathbb{R} \ni it \mapsto \Delta_\varphi^{it}(V_\alpha \xi) \in \mathcal{H}_\varphi$ has the same property with analytic extension denoted by $F_{V_\alpha \xi}$. By uniqueness of extension, note that $F_{V_\alpha \xi} = V_\alpha F_\xi$. It readily follows that $\mathfrak{D}(\Delta_\varphi^s V_\alpha) = \mathfrak{D}(V_\alpha \Delta_\varphi^s)$ and $\Delta_\varphi^s V_\alpha = V_\alpha \Delta_\varphi^s$.

As $\mathfrak{D}(S_\varphi) = \mathfrak{D}(\Delta_\varphi^{1/2})$, the above argument shows that $\mathfrak{D}(S_\varphi V_\alpha) = \mathfrak{D}(V_\alpha S_\varphi) = \mathfrak{D}(S_\varphi)$. For $x_\varphi \in \mathfrak{A}_\varphi$, one has $S_\varphi V_\alpha x_\varphi = S_\varphi(\alpha(x))_\varphi = V_\alpha S_\varphi x_\varphi$. Consequently,

$$\mathcal{G}(S_\varphi V_{\alpha|\mathfrak{A}_\varphi}) = \mathcal{G}(V_\alpha S_{\varphi|\mathfrak{A}_\varphi}) \subseteq \overline{\mathcal{G}(V_\alpha S_{\varphi|\mathfrak{A}_\varphi})} = \mathcal{G}(V_\alpha S_\varphi),$$

as $\mathfrak{A}_\varphi$ is a core for $V_\alpha S_\varphi$, and the last statement is a direct application of the Hahn–Banach theorem. But $\overline{\mathcal{G}(S_\varphi V_{\alpha|\mathfrak{A}_\varphi})} = \overline{\mathcal{G}(V_\alpha S_{\varphi|\mathfrak{A}_\varphi})}$, thus $\overline{\mathcal{G}(S_\varphi V_{\alpha|\mathfrak{A}_\varphi})} = \mathcal{G}(V_\alpha S_\varphi)$. Comparing domains, it follows that $V_\alpha S_\varphi = S_\varphi V_\alpha$.

Again, since $S_\varphi = J_\varphi \Delta_\varphi^{1/2}$, it follows that $S_\varphi = V_\alpha J_\varphi \Delta_\varphi^{1/2} V_\alpha^* = V_\alpha J_\varphi V_\alpha^* \Delta_\varphi^{1/2}$ is the polar decomposition of S_φ as well. Thus, $J_\varphi V_\alpha = V_\alpha J_\varphi$ by uniqueness of polar decomposition. Since $S_\varphi^* = J_\varphi \Delta_\varphi^{-1/2}$, so from what we have proved so far it readily follows that $S_\varphi^* V_\alpha = V_\alpha S_\varphi^*$.

The proof shows that all operators prescribed in (1) of the statement are closed.

(2) Since $J_\varphi \mathfrak{A}_\varphi = \mathfrak{A}'_\varphi$ and since V_α commutes with J_φ, we have $V_\alpha \mathfrak{A}'_\varphi = V_\alpha J_\varphi \mathfrak{A}_\varphi = J_\varphi V_\alpha \mathfrak{A}_\varphi = \mathfrak{A}'_\varphi$.

(3) We only prove the statement for R_η. The proof for L_ξ is similar. From (1), we have that V_α keeps $\mathfrak{D}(S_\varphi^*)$ invariant. Fix $\eta \in \mathfrak{D}(S_\varphi^*)$. From the discussion in §2, $R_{V_\alpha \eta}$ is closed. We claim that $V_\alpha R_\eta V_\alpha^*$ is closed as well. Let $\xi_n \in \mathfrak{D}(V_\alpha R_\eta V_\alpha^*)$ and $\xi_0, \zeta_0 \in \mathcal{H}_\varphi$ be such that $(\xi_n, (V_\alpha R_\eta V_\alpha^*)\xi_n) \to (\xi_0, \zeta_0)$ as $n \to \infty$ in $\mathcal{H}_\varphi \oplus \mathcal{H}_\varphi$. Note that $V_\alpha^* \xi_n \in \mathfrak{D}(R_\eta)$ for all n and $V_\alpha^* \xi_n \to V_\alpha^* \xi_0$ and $(R_\eta V_\alpha^*)\xi_n \to V_\alpha^* \zeta_0$ as $n \to \infty$. As R_η is closed, it follows that $V_\alpha^* \xi_0 \in \mathfrak{D}(R_\eta)$ and $(R_\eta V_\alpha^*)\xi_0 = V_\alpha^* \zeta_0$, that is, $(V_\alpha R_\eta V_\alpha^*)\xi_0 = \zeta_0$. This establishes the claim. Now note that if $x_\varphi \in \mathfrak{A}_\varphi$, then by an easy approximation argument it follows that

$$\begin{aligned}(V_\alpha R_\eta V_\alpha^*)(x_\varphi) &= (V_\alpha R_\eta)((\alpha^{-1}(x))_\varphi = V_\alpha(R_\eta^0((\alpha^{-1}(x))_\varphi) = V_\alpha(L_{(\alpha^{-1}(x))_\varphi}(\eta)) \\ &= V_\alpha \alpha^{-1}(x)\eta = x(V_\alpha \eta) = L_{x_\varphi}(V_\alpha \eta) \\ &= R_{V_\alpha \eta}(x_\varphi).\end{aligned}$$

Now use the fact that $\mathfrak{A}_\varphi$ is a core for both $V_\alpha R_\eta V_\alpha^*$ and $R_{V_\alpha \eta}$ to conclude that $V_\alpha R_\eta V_\alpha^* = L_{V_\alpha \eta}$.

(4) This follows directly from the definitions and (1) and (3).

(5) Note that $V_\alpha \Delta_\varphi^z = \Delta_\varphi^z V_\alpha$ for all $z \in \mathbb{C}$ and $V_\alpha(\mathfrak{A}_\varphi \cap \mathfrak{A}'_\varphi) = \mathfrak{A}_\varphi \cap \mathfrak{A}'_\varphi$ from (1) and (2). Fix $z \in \mathbb{C}$. Let $x_\varphi \in \mathfrak{T}_\varphi$; then $\Delta_\varphi^z V_\alpha(x_\varphi) = V_\alpha(\Delta_\varphi^z x_\varphi) \in \mathfrak{A}_\varphi \cap \mathfrak{A}'_\varphi$. Secondly, by what has been proved so far, we have

$$\begin{aligned}\Delta_\varphi^z L_{V_\alpha x_\varphi} \Delta_\varphi^{-z} &= \Delta_\varphi^z V_\alpha L_{x_\varphi} V_\alpha^* \Delta_\varphi^{-z} \\ &= V_\alpha(\Delta_\varphi^z L_{x_\varphi} \Delta_\varphi^{-z}) V_\alpha^* \subset V_\alpha L_{\Delta_\varphi^z x_\varphi} V_\alpha^* \\ &= L_{V_\alpha \Delta_\varphi^z x_\varphi} \\ &= L_{\Delta_\varphi^z V_\alpha x_\varphi} \quad \text{(from equation (2))}. \end{aligned} \tag{3}$$

https://doi.org/10.1017/etds.2017.15 Published online by Cambridge University Press

From the first three equalities in equation (3), it follows that

$$\xi \in \mathfrak{D}(\Delta_\varphi^z L_{V_\alpha x_\varphi} \Delta_\varphi^{-z}) \Leftrightarrow \xi \in \mathfrak{D}(V_\alpha \Delta_\varphi^z L_{x_\varphi} \Delta_\varphi^{-z} V_\alpha^*) \Leftrightarrow V_\alpha^* \xi \in \mathfrak{D}(\Delta_\varphi^z L_{x_\varphi} \Delta_\varphi^{-z})$$
$$\Leftrightarrow V_\alpha^* \xi \in \mathfrak{D}(\Delta_\varphi^{-z}) \Leftrightarrow \xi \in \mathfrak{D}(\Delta_\varphi^{-z}) \quad (\text{as } x_\varphi \in \mathfrak{T}_\varphi).$$

Similarly, it follows that $\Delta_\varphi^z R_{V_\alpha x_\varphi} \Delta_\varphi^{-z} \subset R_{\Delta_\varphi^z V_\alpha x_\varphi}$ and $\mathfrak{D}(\Delta_\varphi^z R_{V_\alpha x_\varphi} \Delta_\varphi^{-z}) = \mathfrak{D}(\Delta_\varphi^{-z})$. As $z \in \mathbb{C}$ was arbitrary, it follows that $V_\alpha \mathfrak{T}_\varphi = \mathfrak{T}_\varphi$.

Since $\mathfrak{P}_\varphi = \overline{\{x_\varphi J_\varphi x_\varphi : x_\varphi \in \mathfrak{T}_\varphi\}}$ and J_φ commutes with V_α, it follows that $V_\alpha \mathfrak{P}_\varphi = \mathfrak{P}_\varphi$. This completes the proof. □

4. *Polar decomposition*

In this section, we investigate the polar decomposition of left and right multiplication operators affiliated to von Neumann algebras. The set-up of this section is the same as that of §3.

LEMMA 4.1. *The following hold.*

(1) *For $\xi \in \mathfrak{D}(S_\varphi)$, let $L_\xi = v|L_\xi|$ denote the polar decomposition of L_ξ. Then $|L_\xi| = L_{v^*\xi}$ with $v^*\xi \in \mathfrak{P}_{S_\varphi}$ and $S_\varphi v^* \xi = v^* \xi$.*

(2) *For $\eta \in \mathfrak{D}(S_\varphi^*)$, let $R_\eta = v|R_\eta|$ denote the polar decomposition of R_η. Then $|R_\eta| = R_{v^*\eta}$ with $v^*\eta \in \mathfrak{P}_{S_\varphi^*}$ and $S_\varphi^* v^* \eta = v^* \eta$.*

Proof. We will only prove (1); the proof of (2) is similar.

(1) As before, note that, for $\xi \in \mathfrak{D}(S_\varphi)$, the operator L_ξ^0 with $\mathfrak{D}(L_\xi^0) = \mathfrak{A}_\varphi'$ and (densely) defined by $L_\xi^0(\zeta) = R_\zeta(\xi)$ for all $\zeta \in \mathfrak{A}_\varphi'$ is closable and affiliated to M, and L_ξ is the closure of L_ξ^0. Thus, $|L_\xi|$ is positive, self-adjoint, affiliated to M and $v \in M$.

Following the discussion in §2, let $L_{v^*\xi}^0 : \mathfrak{A}_\varphi' \subseteq \mathcal{H}_\varphi \to \mathcal{H}_\varphi$ be defined by $L_{v^*\xi}^0(\zeta) = R_\zeta(v^*\xi) = v^* R_\zeta(\xi) = v^* L_\xi(\zeta)$, $\zeta \in \mathfrak{A}_\varphi'$. As v is a partial isometry, it follows that $L_{v^*\xi}^0$ is closable and its closure $L_{v^*\xi}$ is equal to $v^* L_\xi = |L_\xi|$ which is positive and self-adjoint. From the alternative characterization of $\mathfrak{P}_{S_\varphi}$ (see [**StZs**, pp. 260–261]), it follows that $v^*\xi \in \mathfrak{P}_{S_\varphi}$. Consequently, $S_\varphi v^* \xi = v^* \xi$. □

The following lemma is crucial for our purposes.

LEMMA 4.2. *Let M be a von Neumann algebra equipped with an f.n.s. weight φ. Let $\alpha \in \mathrm{Aut}(M, \varphi)$. Let $V_\alpha : \mathcal{H}_\varphi \to \mathcal{H}_\varphi$ denote the unique unitary defined by $V_\alpha x_\varphi = (\alpha(x))_\varphi$, $x_\varphi \in \mathfrak{A}_\varphi$. Then the following hold.*

(1) *Let $\xi \in \mathfrak{D}(S_\varphi)$. Then $V_\alpha \xi \in \mathfrak{D}(S_\varphi)$, and if $\zeta \in \mathfrak{P}_{S_\varphi}$ is such that $L_\xi = v L_\zeta$ denotes the polar decomposition of L_ξ, then*

$$L_{V_\alpha \xi} = \alpha(v) L_{V_\alpha \zeta}$$

is the polar decomposition of $L_{V_\alpha \xi}$.

(2) *Let $\eta \in \mathfrak{D}(S_\varphi^*)$. Then $V_\alpha \eta \in \mathfrak{D}(S_\varphi^*)$, and if $\zeta \in \mathfrak{P}_{S_\varphi^*}$ is such that $R_\eta = v R_\zeta$ denotes the polar decomposition of R_η, then*

$$R_{V_\alpha \eta} = \alpha(v) R_{V_\alpha \zeta}$$

is the polar decomposition of $R_{V_\alpha \eta}$.

https://doi.org/10.1017/etds.2017.15 Published online by Cambridge University Press

Proof. As for Lemma 4.1, we will only prove (1); the proof of (2) is similar.

(1) First note that from Lemma 4.1, the polar decomposition of L_ξ is of the form as it appears in the statement. Secondly, if $L_{\xi_1} = L_{\xi_2}$ for $\xi_1, \xi_2 \in \mathfrak{D}(S_\varphi)$, then $\xi_1 = \xi_2$. Indeed, for all $\eta \in \mathfrak{A}'_\varphi$ we have $R_\eta(\xi_1) = R_\eta(\xi_2)$. Thus $R_\eta(\xi_1 - \xi_2) = 0$ for all $\eta \in \mathfrak{A}'_\varphi$, and hence $M'(\xi_1 - \xi_2) = 0$. Then the cyclic projection associated to the vector $\xi_1 - \xi_2$ swept by M' is the zero projection in M. Consequently, $\xi_1 = \xi_2$. Thus, ζ is uniquely determined and $\zeta = v^*\xi \in \mathfrak{P}_{S_\varphi}$ from Lemma 4.1.

By Proposition 3.1, it follows that $V_\alpha\xi$ and $V_\alpha\zeta$ are in $\mathfrak{D}(S_\varphi)$. Therefore, $L^0_{V_\alpha\xi}$, $L^0_{V_\alpha\zeta}$ are closable and $L_{V_\alpha\xi}$, $L_{V_\alpha\zeta}$ respectively denote their closures. We have $\xi = v\zeta$. Thus, $V_\alpha\xi = V_\alpha(v\zeta)$. Approximating ζ by elements of $\mathfrak{A}_\varphi$ in $\|\cdot\|_\varphi$ and using the continuity of $\alpha(v)$, it is easy to check that $V_\alpha\xi = \alpha(v)V_\alpha\zeta$. Thus, $L_{\alpha(v)V_\alpha\zeta}$ is closed. For $\eta \in \mathfrak{A}'_\varphi$, note that

$$L_{\alpha(v)V_\alpha\zeta}(\eta) = R_\eta(\alpha(v)V_\alpha\zeta) = \alpha(v)R_\eta(V_\alpha\zeta) = \alpha(v)L_{V_\alpha\zeta}(\eta).$$

Since $\mathfrak{A}'_\varphi$ is a core for both $L_{\alpha(v)V_\alpha\zeta}$ and $\alpha(v)L_{V_\alpha\zeta}$ and both operators are closed, so

$$L_{V_\alpha\xi} = \alpha(v)L_{V_\alpha\zeta}. \tag{4}$$

We claim that equation (4) defines the polar decomposition of $L_{V_\alpha\xi}$. Indeed, $V_\alpha\zeta \in \mathfrak{P}_{S_\varphi}$ by Proposition 3.1, so $L_{V_\alpha\zeta}$ is positive, self-adjoint and affiliated to M (again by Proposition 3.1) and $\alpha(v) \in M$ is a partial isometry. The left support of $L_{V_\alpha\zeta}$ (which is its right support as well), is the projection onto the subspace $\overline{L_{V_\alpha\zeta}\mathfrak{A}'_\varphi}^{\|\cdot\|_\varphi}$, as $\mathfrak{A}'_\varphi$ is a core for $L_{V_\alpha\zeta}$. Again, by Proposition 3.1, we have $L_{V_\alpha\zeta} = V_\alpha L_\zeta V_\alpha^*$.

From Lemma 3.1, $V_\alpha\mathfrak{A}'_\varphi = \mathfrak{A}'_\varphi$ and hence

$$\overline{L_{V_\alpha\zeta}\mathfrak{A}'_\varphi}^{\|\cdot\|_\varphi} = \overline{V_\alpha L_\zeta V_\alpha^*\mathfrak{A}'_\varphi}^{\|\cdot\|_\varphi} = \overline{V_\alpha L_\zeta\mathfrak{A}'_\varphi}^{\|\cdot\|_\varphi} = V_\alpha\overline{L_\zeta\mathfrak{A}'_\varphi}^{\|\cdot\|_\varphi}.$$

But the projection onto $\overline{L_\zeta\mathfrak{A}'_\varphi}^{\|\cdot\|_\varphi}$ is the left (as well as the right) support of L_ζ, which by definition of polar decomposition is v^*v. It follows that each of the left and the right support of $L_{V_\alpha\zeta}$ is $V_\alpha v^*vV_\alpha^* = \alpha(v^*v)$. The argument is then complete upon invoking the uniqueness of the polar decomposition. □

5. *Amplification*

In the next section we will work with amplification of W^*-dynamical systems where the amplification is obtained by tensoring the original von Neumann algebra by a matrix algebra of appropriate size. Thus, we give a short account of the standard form associated to such amplification. For details the reader is referred to [**St**].

It is a general fact that if $\mathfrak{A}_i \subseteq \mathcal{H}_i$ is a left Hilbert algebra with associated quadruple $(S_i, S_i^*, \Delta_i, J_i)$ and $\overline{\mathfrak{A}_i} = \mathcal{H}_i$ (as a Hilbert space) for $i = 1, 2$, then $\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2 \subseteq \mathcal{H}_1 \otimes \mathcal{H}_2$ equipped with the tensor product involutive algebra structure and scalar product of $\mathcal{H}_1 \otimes \mathcal{H}_2$ is a left Hilbert algebra with associated quadruple $(S_1 \otimes S_2, S_1^* \otimes S_2^*, \Delta_1 \otimes \Delta_2, J_1 \otimes J_2)$. Further, $L_{\xi_1\otimes\xi_2} = L_{\xi_1} \otimes L_{\xi_2}$ for all $\xi_1 \in \mathfrak{D}(S_1)$ and $\xi_2 \in \mathfrak{D}(S_2)$ and $R_{\eta_1\otimes\eta_2} = R_{\eta_1} \otimes R_{\eta_2}$ for all $\eta_1 \in \mathfrak{D}(S_1^*)$ and $\eta_2 \in \mathfrak{D}(S_2^*)$. Moreover, $\mathfrak{L}(\mathfrak{A}) = \mathfrak{L}(\mathfrak{A}_1) \overline{\otimes} \mathfrak{L}(\mathfrak{A}_2)$ and $\mathfrak{R}(\mathfrak{A}') = \mathfrak{R}(\mathfrak{A}'_1) \overline{\otimes} \mathfrak{R}(\mathfrak{A}'_2)$.

https://doi.org/10.1017/etds.2017.15 Published online by Cambridge University Press

Again, if M_i is a von Neumann algebra with an f.n.s. weight φ_i for $i = 1, 2$, then there exists a unique f.n.s. weight ψ on $M_1 \overline{\otimes} M_2$ such that

$$x_1 \in \mathfrak{M}_{\varphi_1} \quad \text{and} \quad x_2 \in \mathfrak{M}_{\varphi_2} \Rightarrow x_1 \otimes x_2 \in \mathfrak{M}_{\psi} \text{ and } \psi(x_1 \otimes x_2) = \varphi_1(x_1)\varphi_2(x_2). \tag{5}$$

Moreover, $\sigma_t^{\psi} = \sigma_t^{\varphi_1} \otimes \sigma_t^{\varphi_2}$ for all $t \in \mathbb{R}$. This fact is extremely delicate. The f.n.s. weight ψ is denoted by $\varphi_1 \otimes \varphi_2$.

Thus, in accordance with tensor products of left Hilbert algebras, note that if $\varphi_{\mathfrak{A}_i}$ is the f.n.s. weight associated to a left Hilbert algebra $\mathfrak{A}_i \subseteq \mathcal{H}_i$ for $i = 1, 2$, as above, then $\varphi_{\mathfrak{A}_1} \otimes \varphi_{\mathfrak{A}_2}$ is the f.n.s. weight associated to the left Hilbert algebra $\mathfrak{A}_1 \otimes \mathfrak{A}_2$.

As in §3, let M be a von Neumann algebra and let φ be an f.n.s. weight on M. Let $n \in \mathbb{N}$, and let $\widetilde{M} = M \otimes M_n(\mathbb{C})$. Consider the f.n.s. weight $\widetilde{\varphi} = \varphi \otimes tr$ on $\widetilde{M}$, where tr denotes the normalized trace on $M_n(\mathbb{C})$. The GNS Hilbert space $\mathcal{H}_{\widetilde{\varphi}} = L^2(\widetilde{M}, \widetilde{\varphi})$ of $\widetilde{M}$ with respect to $\widetilde{\varphi}$ is $\mathcal{H}_{\varphi} \otimes M_n(\mathbb{C})$. Let 1_n be the identity operator on the Hilbert space $M_n(\mathbb{C})$.

Vectors in $\mathcal{H}_{\widetilde{\varphi}}$ are viewed as $n \times n$ matrices with elements from $\mathcal{H}_{\varphi}$ and a generic element in $\mathcal{H}_{\widetilde{\varphi}}$ will be written as $[\xi_{ij}]$ with $\xi_{ij} \in \mathcal{H}_{\varphi}$ for $1 \leq i, j \leq n$. The obvious analogous notational convention is adopted for elements in $\widetilde{M}$.

From the discussion in the beginning of this section, the associated operators in the modular theory of $\widetilde{M}$ with respect to $\widetilde{\varphi}$ are given by $S_{\widetilde{\varphi}} = S_{\varphi} \otimes J_{tr}$, $J_{\widetilde{\varphi}} = J_{\varphi} \otimes J_{tr}$, $\Delta_{\widetilde{\varphi}} = \Delta_{\varphi} \otimes 1_n$ and the modular automorphisms are given by $\sigma_t^{\widetilde{\varphi}} = \sigma_t^{\varphi} \otimes 1$, $t \in \mathbb{R}$. Thus, $\Delta_{\widetilde{\varphi}}^{it}[\xi_{ij}] = [\Delta_{\varphi}^{it}\xi_{ij}]$, for all $t \in \mathbb{R}$ and all $\xi_{ij} \in \mathcal{H}_{\varphi}$ for $1 \leq i, j \leq n$. Also note that, since $S_{\widetilde{\varphi}} = S_{\varphi} \otimes J_{tr}$, it follows that $[\xi_{ij}] \in \mathfrak{D}(S_{\widetilde{\varphi}})$ if and only if $\xi_{ij} \in \mathfrak{D}(S_{\varphi})$ for all $1 \leq i, j \leq n$.

Consequently, if $[\xi_{ij}] \in \mathfrak{D}(S_{\widetilde{\varphi}})$, then the densely defined operator $L^0_{[\xi_{ij}]}$, defined as before, is closable and its closure $L_{[\xi_{ij}]}$ admits a polar decomposition $L_{[\xi_{ij}]} = [u_{ij}]|L_{[\xi_{ij}]}|$, where $[u_{ij}] \in \widetilde{M}$ is a partial isometry and $|L_{[\xi_{ij}]}|$ is positive, self-adjoint and affiliated to $\widetilde{M}$. From Proposition 4.1, $[\zeta_{ij}] = [u_{ij}]^*[\xi_{ij}] \in \mathfrak{P}_{S_{\widetilde{\varphi}}}$ and

$$L_{[\xi_{ij}]} = [u_{ij}]L_{[\zeta_{ij}]}. \tag{6}$$

6. *Weight preserving dynamics*

For the remainder of this paper, the terminology 'f.n.s. weight' will mean an *infinite* weight which is faithful, normal and semifinite unless otherwise stated. A W^*-dynamical system denoted by $\mathfrak{M} = (M, \varphi, \beta, G)$ is a quadruple, such that M is a von Neumann algebra, φ is an f.n.s. weight on M and β is a strongly continuous action of a separable locally compact group G on M by φ-preserving automorphisms. This assumption will be in force throughout the rest of the paper (though more general groups can also be allowed).

Since G acts by φ-preserving automorphisms, we have $\beta_g \sigma_t^{\varphi} = \sigma_t^{\varphi} \beta_g$ for all $t \in \mathbb{R}$ and $g \in G$ (as noted in §3); thus $\beta(G)$ is a symmetry (sub)group of the associated Hamiltonian. As noted before, β_g is implemented by an unique unitary V_g on $\mathcal{H}_{\varphi}$ such that $V_g x_{\varphi} = (\beta_g(x))_{\varphi}$, for $x \in \mathfrak{A}_{\varphi}$, that is, $\mathrm{Ad}(V_g) = \beta_g$ for all $g \in G$. It is easy to check that $\pi_V : G \ni g \mapsto V_g \in \mathcal{U}(\mathcal{H}_{\varphi})$ is a strongly continuous unitary representation of G on $\mathcal{H}_{\varphi}$. Recall that $\mathfrak{A}_{\varphi} = \mathfrak{N}_{\varphi} \cap \mathfrak{N}_{\varphi}^*$.

Definition 6.1. We make the following definitions.

(1) The action β is called ergodic if $x \in M$ and $\beta_g(x) = x$ for all $g \in G$ implies $x \in \mathbb{C}1$.
(2) The representation π_V is called ergodic if $\xi \in \mathcal{H}_{\varphi}$ and $V_g \xi = \xi$ for all $g \in G$ implies $\xi = 0$.

https://doi.org/10.1017/etds.2017.15 Published online by Cambridge University Press

(3) The action β is called compact relative to $\mathfrak{A}_\varphi$ if the orbit $\{(\beta_g(x))_\varphi : g \in G\}$ is pre-compact in $\mathcal{H}_\varphi$ in $\|\cdot\|_\varphi$ for all $x \in \mathfrak{A}_\varphi$.

(4) The representation π_V is called compact if the orbit $\{V_g\xi : g \in G\}$ of ξ is pre-compact in $\mathcal{H}_\varphi$ in $\|\cdot\|_\varphi$ for all $\xi \in \mathcal{H}_\varphi$.

(5) The action β is called weak mixing relative to $\mathfrak{A}_\varphi$ if for each $x, y \in \mathfrak{A}_\varphi$ and $\epsilon > 0$ there exists $g \in G$ such that $|\varphi(y^*\alpha_g(x))| < \epsilon$.

(6) The representation π_V is called weak mixing if for each $\xi, \eta \in \mathcal{H}_\varphi$ and $\epsilon > 0$ there exists $g \in G$ such that $|\langle V_g\xi, \eta\rangle_\varphi| < \epsilon$.

(7) The action β is called (strong) mixing if $\varphi(y^*\alpha_g(x)) \to 0$ as $g \to \infty$ for each $x, y \in \mathfrak{A}_\varphi$ [**KrSu**].

(8) The representation π_V is called (strong) mixing if $\langle V_g\xi, \eta\rangle_\varphi \to 0$ as $g \to \infty$ for each $\xi, \eta \in \mathcal{H}_\varphi$.

Note that by a simple approximation argument it follows that (5) and (6) in Definition 6.1 are equivalent and they are further equivalent to the fact that π_V has no non-trivial finite-dimensional invariant subspace. The last statement about finite-dimensional invariant subspaces is a standard fact in ergodic theory concerning unitary representations (see [**KeLi**]). Similarly, (7) and (8) are also equivalent.

It is trivial that (4) $\Rightarrow$ (3). It is also true that (3) $\Rightarrow$ (4). For the sake of completeness, we sketch a proof. Fix $\xi \in \mathcal{H}_\varphi$. Assuming (3), we will show that $\overline{\{V_g\xi : g \in G\}}^{\|\cdot\|_\varphi}$ is sequentially compact, which will establish that $\overline{\{V_g\xi : g \in G\}}^{\|\cdot\|_\varphi}$ is compact. So let $\xi_n \in \overline{\{V_g\xi : g \in G\}}^{\|\cdot\|_\varphi}$. Given $\epsilon > 0$, for each n find $g_n \in G$ such that $\|\xi_n - V_{g_n}\xi\|_\varphi < \epsilon/8$ and also choose $x_\varphi \in \mathfrak{A}_\varphi$ such that $\|\xi - x_\varphi\|_\varphi < \epsilon/4$. Thus, $\|V_{g_n}\xi - V_{g_n}x_\varphi\|_\varphi < (\epsilon/4)$ for all n. There exists a subsequence g_{n_k} of g_n such that $V_{g_{n_k}}x_\varphi$ is Cauchy. Consequently,

$$\begin{aligned}\|\xi_{n_k} - \xi_{n_l}\|_\varphi &\le \|\xi_{n_k} - V_{g_{n_k}}\xi\|_\varphi + \|\xi_{n_l} - V_{g_{n_l}}\xi\|_\varphi + \|V_{g_{n_k}}\xi - V_{g_{n_l}}\xi\|_\varphi \\ &< \frac{\epsilon}{4} + \|V_{g_{n_k}}\xi - V_{g_{n_k}}x_\varphi\|_\varphi + \|V_{g_{n_l}}\xi - V_{g_{n_l}}x_\varphi\|_\varphi + \|V_{g_{n_k}}x_\varphi - V_{g_{n_l}}x_\varphi\|_\varphi \\ &< \frac{3\epsilon}{4} + \|V_{g_{n_k}}x_\varphi - V_{g_{n_l}}x_\varphi\|_\varphi, \quad \text{for all } k, l.\end{aligned}$$

It follows that $\{\xi_{n_k}\}$ is Cauchy and hence converges in $\overline{\{V_g\xi : g \in G\}}^{\|\cdot\|_\varphi}$. This shows that (3) and (4) are equivalent.

However, (1) and (2) of Definition 6.1 are *not equivalent* unless $\varphi(1) < \infty$. For a proof of the equivalence of (1) and (2) when φ is a state, see [**BCM**]. Nevertheless, (1) implies (2) for f.n.s. weights, and its proof requires delicate and rigorous argument as we show in the next result.

It is a standard result (see, for example, [**StZs**, 10.16]) that for $\xi \in \mathcal{H}_\varphi$ and $r > 0$,

$$\xi_r = \sqrt{\frac{r}{\pi}} \int_{-\infty}^{\infty} e^{-rt^2} \Delta_\varphi^{it}\xi \, dt \to \xi, \quad \text{as } r \to \infty \text{ in } \|\cdot\|_\varphi. \tag{7}$$

Note that each ξ_r is analytic vector. We will refer to such a net of vectors $(\xi_r)_{r>0}$ as an analytic approximation of ξ from the Gaussian core or simply analytic approximation of ξ. Consequently, $\xi_r \in \mathfrak{D}(S_\varphi)$ for all $r > 0$.

https://doi.org/10.1017/etds.2017.15 Published online by Cambridge University Press

THEOREM 6.2. *Let G act ergodically on M preserving φ. Let $\xi \in \mathcal{H}_\varphi$ be a vector such that $V_g\xi = \langle g, \chi\rangle\xi$ for all $g \in G$, where χ is a (continuous) character of G. Then $\xi = 0$. In particular, π_V is ergodic.*

Proof. Assume $\xi \neq 0$. Let $\{\xi_r : r > 0\}$, with ξ_r defined in equation (7), be an analytic approximation of ξ. Fix $r_0 > 0$ such that $\xi_r \neq 0$ for all $r \geq r_0$. Note that $V_g\xi_r = \langle g, \chi\rangle\xi_r$ for all $g \in G$ and $r \geq r_0$, as $V_g\Delta_\varphi^{it} = \Delta_\varphi^{it}V_g$ for all $g \in G$ and for all $t \in \mathbb{R}$.

Fix $r > r_0$. Since $\xi_r \in \mathfrak{D}(S_\varphi)$, from Proposition 3.1 it follows that $V_g\xi_r \in \mathfrak{D}(S_\varphi)$ as well. Let $\zeta_r \in \mathfrak{P}_{S_\varphi}$ be such that $L_{\xi_r} = v_r L_{\zeta_r}$ is the polar decomposition of L_{ξ_r} (see Lemma 4.1). By Lemma 4.2, $L_{V_g\xi_r} = \beta_g(v_r)L_{V_g\zeta_r}$ is the polar decomposition of $L_{V_g\xi_r}$, for each $g \in G$. On the other hand, $L_{V_g\xi_r} = \langle g, \chi\rangle L_{\xi_r} = \langle g, \chi\rangle v_r L_{\zeta_r}$ for all $g \in G$. Therefore, for all $g \in G$,

$$L_{V_g\xi_r} = \langle g, \chi\rangle v_r L_{\zeta_r} = \beta_g(v_r)L_{V_g\zeta_r}.$$

Hence, by uniqueness of the polar decomposition, it follows that for all $g \in G$,

$$\beta_g(v_r) = \langle g, \chi\rangle v_r \quad \text{and} \quad V_g\zeta_r = \zeta_r.$$

It follows that $\beta_g(v_r^*v_r) = v_r^*v_r$ and $\beta_g(v_rv_r^*) = v_rv_r^*$ for all $g \in G$, and thus the hypothesis implies that $v_r^*v_r = 1 = v_rv_r^*$ (as $\xi_r \neq 0$), as v_r is supposed to be a non-zero partial isometry. Consequently, v_r is a unitary.

Note that L_{ζ_r} is positive, self-adjoint and affiliated to M. Thus, by the spectral theorem, write $L_{\zeta_r} = \int_0^\infty \lambda\, de_\lambda$. Note that $e_\lambda \in M$ for all $\lambda \geq 0$. Since $V_g\zeta_r = \zeta_r$, we have that $L_{\zeta_r} = \int_0^\infty \lambda\, d\beta_g(e_\lambda)$ for all $g \in G$. By uniqueness of resolution of the identity in the spectral theorem, it follows that $\beta_g(e_\lambda) = e_\lambda$ (cf. [**KRI**, p. 350]) for all $g \in G$. Ergodicity of β forces that e_λ is trivial for all λ. Since the support of L_{ζ_r} is 1, there exists $\lambda_0 \geq 0$ such that $e_\lambda = 1$ for all $\lambda > \lambda_0$ and $e_\lambda = 0$ for $\lambda < \lambda_0$. Consequently, $L_{\zeta_r} = c1$, where $c \in (0, \infty]$.

If $c \in (0, \infty)$, then L_{ζ_r} is bounded and thus $\zeta_r \in \mathfrak{A}_\varphi'' = \mathfrak{A}_\varphi$. From equation (1), it follows that $\varphi(1) = (1/c^2)\|\zeta_r\|_\varphi^2$, whence φ is a finite weight. Since this is impossible by the hypothesis, we have $c = \infty$. Since $L_\varsigma \neq 0$ for any $0 \neq \varsigma \in \mathfrak{D}(S_\varphi)$, it is only possible that $\zeta_r = 0$.

Thus, we have a contradiction unless $\xi = 0$. This completes the proof. □

Remark 6.3. Note that ergodicity of π_V only entails absence of fixed points of β inside $\mathfrak{A}_\varphi$ but beyond this not much can be said. In fact, ergodicity of π_V does not guarantee ergodicity of the action β. We provide an example to justify this point. Let $\mathcal{K} = L^2([0, 1], \lambda)$, where λ is the normalized Lebesgue measure on $[0, 1]$. Let $M = \mathbf{B}(\mathcal{K})$ and let Tr be the f.n.s. trace on M scaled so that it assigns the value 1 to each minimal projection. Then $\mathcal{H}_{\mathrm{Tr}} = \mathcal{HS}(\mathcal{K})$ (upon identification), where $\mathcal{HS}(\mathcal{K})$ denotes the collection of Hilbert–Schmidt operators on $\mathcal{K}$. Let $A = L^\infty([0, 1], \lambda)$ and let $G \subseteq \mathcal{U}(A)$ be a discrete group of unitaries that generates A as a von Neumann algebra. Let G act on M via unitary conjugation and let the action be denoted by β. Since the trace Tr of an operator in M is independent of the choice of orthonormal basis, β preserves Tr. Let $\mathfrak{M} = (M, \mathrm{Tr}, \beta, G)$. In this case, π_V is ergodic. Indeed, if $s \in \mathcal{HS}(\mathcal{K})$ is such that $usu^* = s$ for all $u \in G$, then s commutes with A. But A is a maximal abelian self-adjoint

https://doi.org/10.1017/etds.2017.15 Published online by Cambridge University Press

(masa) in M, as it is an abelian algebra having a cyclic vector in $\mathcal{K}$. Consequently, $s \in A$. But this is impossible unless $s = 0$, as the spectral projections of s^*s have to lie in A and no projection in A is finite with respect to Tr. This proves π_V is ergodic. Clearly, β is not ergodic. In fact, $M^G = A$, as A is a masa.

The next theorem, the main theorem of this paper, is a generalization of [**BCM**, Theorem 6.10]. The idea in its proof is to perform an appropriate 'amplification' of the set-up in Theorem 6.2. In its proof, we invoke the set-up laid out in §5.

THEOREM 6.4. *Let $\mathfrak{M} = (M, \varphi, \beta, G)$ be an ergodic dynamical system. Then π_V cannot have any non-zero finite-dimensional invariant subspace, that is, $\mathfrak{M}$ is weakly mixing relative to $\mathfrak{A}_\varphi$.*

Proof. Suppose $0 \neq \mathcal{K}$ is a finite-dimensional invariant subspace of π_V. We claim that there exists a non-zero finite-dimensional invariant subspace of π_V of analytic vectors. To see this, let $\dim(\mathcal{K}) = m$ and let $(\xi_1, \xi_2, \ldots, \xi_m)$ be an orthonormal basis of $\mathcal{K}$. Let $\xi_{i,r}$, $r > 0$, be analytic approximations of ξ_i for all $1 \leq i \leq m$ obtained from equation (7). Choose $r_0 > 0$ such that $\xi_{i,r} \neq 0$ for all $r > r_0$ and for all $1 \leq i \leq m$. Fix $r > r_0$. V_g commutes with Δ_φ^{it} for all $t \in \mathbb{R}$ and all $g \in G$, so $\mathcal{K}_r = \operatorname{span}\{\xi_{i,r} : 1 \leq i \leq m\}$ is an invariant subspace of π_V. Let $(\zeta_1, \zeta_2, \ldots, \zeta_n)$ be an orthonormal basis of $\mathcal{K}_r$. Clearly, $n \leq m$. From equation (7), it follows that $\zeta_i \in \mathfrak{D}(\Delta_\varphi^z)$ for all $z \in \mathbb{C}$, so $\zeta_i \in \mathfrak{D}(S_\varphi)$ for all $1 \leq i \leq n$.

Following §5, consider $\widetilde{M} = M \otimes M_n(\mathbb{C})$, equipped with the f.n.s. weight $\widetilde{\varphi} = \varphi \otimes tr$, where tr is the normalized trace on $M_n(\mathbb{C})$. Note that $\widetilde{M}$ acts via left multiplication on $\mathcal{H}_{\widetilde{\varphi}}$ in the GNS representation. Let G act on $\widetilde{M}$ via $\widetilde{\beta}_g = \beta_g \otimes 1$, $g \in G$. By ergodicity of the G-action on M, it follows that the fixed point algebra $\widetilde{M}^G$ of the G-action on $\widetilde{M}$ is given by $\widetilde{M}^G = 1 \otimes M_n(\mathbb{C})$.

Note that $\widetilde{\beta}_g$ preserves $\widetilde{\varphi}$ for all $g \in G$. Indeed, $[x_{ij}] \in \widetilde{M}_+$ is such that $\widetilde{\varphi}([x_{ij}]) < \infty$ if and only if $\varphi(x_{ii}) < \infty$ for all $1 \leq i \leq n$ (note that $x_{ii} \geq 0$). Now use the discussion preceding Proposition 3.1 to see that $\widetilde{\beta}_g$ preserves $\widetilde{\varphi}$. Moreover, from equation (5) it follows that $\widetilde{\beta}_g$ is implemented by the unitary $\widetilde{V}_g = V_g \otimes 1_n$ (see §5) for each $g \in G$.

Let

$$\zeta = \begin{pmatrix} \zeta_1 & 0 & 0 & \cdots & 0 \\ \zeta_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \zeta_n & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathcal{H}_{\widetilde{\varphi}}.$$

Note that $\zeta \in \mathfrak{D}(S_{\widetilde{\varphi}})$, since $\zeta_j \in \mathfrak{D}(S_\varphi)$ for all $1 \leq j \leq n$. Thus from the discussion in §§2 and 3, L_ζ is closed and affiliated to $\widetilde{M}$. Let $L_\zeta = [u_{ij}]L_{[\eta_{ij}]}$ denote the polar decomposition of L_ζ, where $[\eta_{ij}] \in \mathfrak{P}_{S_{\widetilde{\varphi}}}$ as in Lemma 4.1. Then by Lemma 4.2, it follows that for all $g \in G$,

$$L_{\widetilde{V}_g\zeta} = \widetilde{\beta}_g([u_{ij}])L_{\widetilde{V}_g[\eta_{ij}]} \tag{8}$$

is the polar decomposition of $L_{\widetilde{V}_g\zeta}$.

Since $\mathcal{K}_r$ is an invariant subspace of the G-action, for every $g \in G$ we have that $(V_g\zeta_i)_{i=1}^n$ is an orthonormal basis of $\mathcal{K}_r$. Thus, for $g \in G$, there exists a unitary $v_g \in$

https://doi.org/10.1017/etds.2017.15 Published online by Cambridge University Press

$\mathcal{U}(M_n(\mathbb{C}))$ such that $(V_g\zeta_1, V_g\zeta_2, \ldots, V_g\zeta_n)^T = (1 \otimes v_g)(\zeta_1, \zeta_2, \ldots, \zeta_n)^T$, where the superscript T denotes the transpose.

Hence, denoting by $O_{n,n-1}$ the $n \times (n-1)$ zero matrix, we have

$$\widetilde{V}_g\zeta = \begin{pmatrix} V_g\zeta_1 & 0 & \cdots & 0 \\ V_g\zeta_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ V_g\zeta_n & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} (1 \otimes v_g) & \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_n \end{pmatrix} & O_{n,n-1} \end{pmatrix}$$

$$= (1 \otimes v_g) \begin{pmatrix} \zeta_1 & 0 & \cdots & 0 \\ \zeta_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \zeta_n & 0 & \cdots & 0 \end{pmatrix}$$

$$= (1 \otimes v_g)[u_{ij}][\eta_{ij}], \quad g \in G. \tag{9}$$

Let $w_g = (1 \otimes v_g)[u_{ij}]$, $g \in G$. Since $w_g^* w_g = [u_{ij}]^*[u_{ij}]$, it follows from equation (9) that $L_{\widetilde{V}_g\zeta} = (1 \otimes v_g)[u_{ij}]L_{[\eta_{ij}]}$ is also the polar decomposition of $L_{\widetilde{V}_g\zeta}$ for all $g \in G$. Therefore, by uniqueness of the polar decomposition it follows that for all $g \in G$,

$$\widetilde{\beta}_g([u_{ij}]) = (1 \otimes v_g)[u_{ij}] \quad \text{and} \quad L_{\widetilde{V}_g[\eta_{ij}]} = L_{[\eta_{ij}]}. \tag{10}$$

Denote by $\mathfrak{A}_{\widetilde{\varphi}}$ the full left Hilbert algebra associated to $\widetilde{M}$ (that is, $\mathfrak{A}_{\widetilde{\varphi}} = \mathfrak{A}''_{\widetilde{\varphi}}$). Equation (10) forces $R_{[\varsigma_{ij}]}\widetilde{V}_g[\eta_{ij}] = R_{[\varsigma_{ij}]}[\eta_{ij}]$ for all $[\varsigma_{ij}] \in \mathfrak{A}'_{\widetilde{\varphi}}$ and all $g \in G$. Consequently, $\widetilde{V}_g[\eta_{ij}] = [\eta_{ij}]$ for all $g \in G$.

Consider the spectral decomposition $L_{[\eta_{ij}]} = \int_0^\infty \lambda \, de_\lambda$, where $e_\lambda \in \widetilde{M} = M \otimes M_n(\mathbb{C})$ for all $\lambda \geq 0$. Since $\widetilde{V}_g[\eta_{ij}] = [\eta_{ij}]$, arguing as in the proof of Theorem 6.2, it follows that $\widetilde{\beta}_g(e_\lambda) = e_\lambda$ for all $\lambda \geq 0$ and for all $g \in G$. Thus, $e_\lambda \in 1 \otimes M_n(\mathbb{C})$ for all $\lambda \geq 0$.

For $\lambda \geq 0$, write $e_\lambda = 1 \otimes f_\lambda$, where $f_\lambda \in M_n(\mathbb{C})$. Clearly, f_λ, $\lambda \geq 0$, is a commuting family of projections in $M_n(\mathbb{C})$. Let $p_N = \chi_{L_{[\eta_{ij}]}}[0, N]$. Then $p_N \in 1 \otimes M_n(\mathbb{C})$ for all $N \geq 1$. Note that $L_{p_N[\eta_{ij}]} = p_N L_{[\eta_{ij}]} = \int_0^N \lambda \, de_\lambda$, and thus $L_{p_N[\eta_{ij}]}$ is bounded for all N. Thus $L_{p_N[\eta_{ij}]} \in \widetilde{M}$, and because its spectrum is contained inside $[0, N]$, it is positive for all $N \geq 1$. But at the same time $L_{p_N[\eta_{ij}]} \in 1 \otimes M_n(\mathbb{C})$ for all N. Hence, for $N \geq 1$, there exist (bounded) linear operators $L_N : (M_n(\mathbb{C}), tr) \to (M_n(\mathbb{C}), tr)$ such that $L_{p_N[\eta_{ij}]} = 1 \otimes L_N$ and $L_N = \int_0^N \lambda \, df_\lambda$. Note that $L = \int_0^\infty \lambda \, df_\lambda$ is a densely defined, closed, positive, self-adjoint operator on $(M_n(\mathbb{C}), tr)$ and if $q_N = \chi_L([0, N])$, then $q_N L = L_N$ for all $N \geq 1$. But, as $M_n(\mathbb{C})$ is finite-dimensional, L is everywhere defined and bounded. Consequently, if $L = [a_{ij}]$, then it follows that

$$L_{[\eta_{ij}]} = 1 \otimes L = [a_{ij}1] \in \widetilde{M}_+.$$

It follows that $[\eta_{ij}] \in \mathfrak{A}_{\widetilde{\varphi}}$, and, using equations (1) and (5), it follows that $\|[\eta_{ij}]\|_{\widetilde{\varphi}}^2 = \widetilde{\varphi}(1 \otimes L^2) = \infty$. This is a contradiction.

Consequently, π_V cannot admit any non-trivial finite-dimensional invariant subspace and hence is weakly mixing from [**KeLi**]. □

https://doi.org/10.1017/etds.2017.15 Published online by Cambridge University Press

COROLLARY 6.5. *Let β be an action of a separable locally compact group G on a von Neumann algebra M preserving an f.n.s. weight φ, and having a non-zero compact component (relative to $\mathfrak{A}_\varphi$). Then $M^G \neq \mathbb{C}1$.*

The next example shows that ergodic hierarchy in the context of faithful normal state preserving actions does not generalize in the sense of representation theory to the context of f.n.s. weights. There are restrictions and the structure of these dynamical systems seems far more complex than that of infinite measure preserving transformations.

Example 6.6. Consider the Tr-preserving dynamical system $\mathfrak{M}$ discussed in Remark 6.3. We will make free use of all the notation discussed in Remark 6.3. We claim that for an appropriate choice of the group G, the induced representation π_V is in fact mixing and has countable Lebesgue spectrum; thus π_V shares spectral properties similar to Bernoulli shifts. But β is not even ergodic. Thus, semifiniteness of the fixed point subalgebra with respect to the weight seems to play a role in the ergodic hierarchy.

In all that follows, it is understood that A acts on $\mathcal{K}$ via multiplication operators. Note that $\mathcal{H}_{\mathrm{Tr}} = \mathcal{HS}(\mathcal{K})$. So let $T, S \in \mathcal{HS}(\mathcal{K})$. Then there exist square integrable kernels $K_T, K_S \in L^2(\lambda \otimes \lambda)$ such that $\mathrm{Int}(K_T) = T$ and $\mathrm{Int}(K_S) = S$, where $\mathrm{Int}(K)$ denotes the integral operator associated to a kernel $K \in L^2(\lambda \otimes \lambda)$, that is,

$$\mathrm{Int}(K) f(s) = \int_0^1 K(s, t) f(t)\, d\lambda(t) \quad \text{a.e. } s,\ f \in L^2(\lambda). \tag{11}$$

Note that if $u \in \mathcal{U}(A)$, then

$$(uTu^*) f(s) = \int_0^1 u(s) K_T(s, t) \overline{u(t)} f(t)\, d\lambda(t) \quad \text{a.e. } s,\ f \in L^2(\lambda).$$

By the standard theory of Hilbert–Schmidt operators we have $S^* = \mathrm{Int}(\overline{K_S \circ \theta})$, where $\theta : [0, 1] \times [0, 1] \to [0, 1] \times [0, 1]$ is the flip of coordinates. Again, from the standard theory there is a unique (up to null sets) square integrable (convolution) kernel K_0^u such that $\mathrm{Int}(K_0^u) = S^*(uTu^*) \in \mathcal{HS}(\mathcal{K})$, where K_0^u is given by

$$K_0^u(s, t) = \int_0^1 \overline{K_S(r, s)} u(r) K_T(r, t) \overline{u(t)}\, d\lambda(r) \quad \text{a.e. } d\lambda(s)\, d\lambda(t). \tag{12}$$

Via the GNS construction with respect to Tr, we have $\langle uTu^*, S\rangle_{\mathrm{Tr}} = \mathrm{Tr}(S^* uTu^*)$. Note that $S^* uTu^* \in \mathbf{B}(\mathcal{K})_*$ (identified as trace class operators), and so $S^* uTu^*$ is traceable (see [**Br**, p. 1186]). Consequently, by [**Br**, Theorems 3.1 and 3.5] and equation (12) it follows that $K_0^u(s, s)$ exists a.e. λ and

$$\begin{aligned}
\mathrm{Tr}(S^*(uTu^*)) &= \int_0^1 K_0^u(s, s)\, d\lambda(s) \\
&= \int_0^1 \left(\int_0^1 \overline{K_S(r, s)} u(r) K_T(r, s) \overline{u(s)}\, d\lambda(r) \right) d\lambda(s) \\
&= \int_0^1 \overline{u(s)} \left(\int_0^1 \overline{K_S(r, s)} K_T(r, s) u(r)\, d\lambda(r) \right) d\lambda(s), \quad u \in \mathcal{U}(A).
\end{aligned} \tag{13}$$

https://doi.org/10.1017/etds.2017.15 Published online by Cambridge University Press

Now assume $G = \mathbb{Z}$ by identifying $n \in \mathbb{Z}$ with the function $e_n : [0, 1] \ni s \mapsto e^{2\pi ins} \in \mathbb{C}$. Since $\mathbb{Z}$ acts on M by unitary conjugation preserving Tr, we have $M^{\mathbb{Z}} = A$. However, for $T, S \in \mathcal{HS}(\mathcal{K})$, we have

$$\begin{aligned}&\mathrm{Tr}(S^*\beta_n(T))\\ &= \int_0^1 e^{-2\pi ins}\left(\int_0^1 e^{2\pi inr}\overline{K_S(r,s)}K_T(r,s)d\lambda(r)\right)d\lambda(s) \to 0, \quad \text{as } |n| \to \infty.\end{aligned}$$

The last statement is an application of the Riemann–Lebesgue lemma, dominated convergence theorem and Hölder's inequality. Indeed,

$$\begin{aligned}\left|\int_0^1 \overline{K_S(r,s)}K_T(r,s)\,d\lambda(r)\right| &\le \int_0^1 |K_S(r,s)||K_T(r,s)|\,d\lambda(r)\\ &\le \|K_S(\cdot,s)\|_2\|K_T(\cdot,s)\|_2, \ \lambda \quad \text{a.e. } s,\end{aligned}$$

by Hölder's inequality. Note that $s \mapsto \|K_S(\cdot,s)\|_2\|K_T(\cdot,s)\|_2$ is measurable, and again by Hölder's inequality we have

$$\int_0^1 \|K_S(\cdot,s)\|_2\|K_T(\cdot,s)\|_2\,d\lambda(s) \le \|K_T\|_2\|K_S\|_2 < \infty.$$

Consequently, π_V is mixing.

Note that Hilbert–Schmidt operators on $L^2(\lambda)$ are in bijective correspondence with functions in $L^2(\lambda \otimes \lambda)$. Consider the continuous kernels given by $K_{m,l}(s,t) = e^{2\pi ims}e^{2\pi ilt}$, $s, t \in [0, 1]$, $m, l \in \mathbb{Z}$. Then from equation (11) it follows that

$$\mathrm{Int}(K_{m,l})f = \langle f, e_{-l}\rangle_\lambda e_m, \quad f \in L^2(\lambda).$$

Denote $T_{m,l} = \mathrm{Int}(K_{m,l})$, $m, l \in \mathbb{Z}$. Fix $m, l \in \mathbb{Z}$. From equation (13), the Fourier coefficients of the elementary spectral measure $\mu_{T_{m,l}}$ of π_V corresponding to the vector $T_{m,l}$ have the property that

$$\widehat{\mu}_{T_{m,l}}(n) = \mathrm{Tr}(T_{m,l}^*\beta_n(T_{m,l})) = \int_0^1 e^{-2\pi ins}\left(\int_0^1 e^{2\pi inr}\,d\lambda(r)\right)d\lambda(s) = 0, \quad \text{for all } n \ne 0.$$

Therefore, by uniqueness of Fourier coefficients, $\mu_{T_{m,l}} = \mathfrak{m}$ for all $m, l \in \mathbb{Z}$, where $\mathfrak{m}$ is the normalized Haar measure on $\mathbb{T}$. Consequently, the spectral measure of π_V is the Haar measure.

Furthermore, we claim that the orbit $\pi_V T_{m,l}$ is orthogonal to $T_{m',l'}$ with respect to Tr whenever $(m, l) \ne (m', l')$ and $m' - m \ne l - l'$. Indeed, from equation (13) we have

$$\begin{aligned}\mathrm{Tr}(T_{m',l'}^*\beta_n(T_{m,l})) &= \int_0^1 e^{-2\pi ins}\left(\int_0^1 e^{2\pi inr}e^{2\pi i(m-m')r}e^{2\pi i(l-l')s}\,d\lambda(r)\right)d\lambda(s)\\ &= \int_0^1 e^{-2\pi ins}e^{2\pi i(l-l')s}\,d\lambda(s)\int_0^1 e^{2\pi inr}e^{2\pi i(m-m')r}\,d\lambda(r)\\ &= \begin{cases}1, & \text{if } n = m' - m \text{ and } n = l - l',\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Thus, choosing $m' - m \ne l - l'$ forces $\pi_V T_{m,l}$ and $\pi_V T_{m',l'}$ to be orthogonal, and hence $\overline{\mathrm{span}\,\pi_V T_{m,l}}$ is orthogonal to $\overline{\mathrm{span}\,\pi_V T_{m',l'}}$. Consequently, the spectral multiplicity of the action is infinite.

https://doi.org/10.1017/etds.2017.15 Published online by Cambridge University Press

We record Example 6.6 in the form of a theorem below.

THEOREM 6.7. *There exists a non-ergodic f.n.s. weight preserving automorphism of a von Neumann algebra such that the induced representation on the GNS space has countable Lebesgue spectrum.*

It is not difficult to extend Example 6.6 to type II_∞ and type III factors. First, consider M to be a type II_∞ factor and write $M = N \overline{\otimes} \mathbf{B}(\mathcal{K})$, where N is a II_1 factor and $\mathcal{K} = L^2([0, 1], \lambda)$ as before. Let Tr denote the semifinite trace on $\mathbf{B}(\mathcal{K})$ as in Example 6.6 and let τ_N be the unique normal tracial state of N. Then $\mathrm{Tr}_M = \tau_N \otimes \mathrm{Tr}$ is an f.n.s. trace on M. Let A denote the continuous masa in $\mathbf{B}(\mathcal{K})$ as in Example 6.6 together with the action of $\mathbb{Z}$ considered there. Consider any diffuse masa $B \subseteq N$ and consider the τ_N-preserving action α of $\mathbb{Z}$ on N implemented by inner automorphisms resulting from a Haar unitary generator of B. Then $\alpha \otimes \beta$ (where β is as in Example 6.6) is a Tr_M-preserving action of $\mathbb{Z}^2$ on M. By Tomita's theorem on commutants, $B \overline{\otimes} A$ is a masa in M. Consequently, $M^{\mathbb{Z}^2} = B \overline{\otimes} A$ and the action $\alpha \otimes \beta$ is thus not ergodic. But the action is clearly mixing and the spectral multiplicity is infinite.

To find such examples in the context of type III factors consider the following. Let M be a type III factor equipped with a faithful normal state φ such that $(M^\varphi)' \cap M = \mathbb{C}1$, where $M^\varphi = \{x \in M : \sigma_t^\varphi(x) = x \ \forall t \in \mathbb{R}\}$ denotes the centralizer of φ. Then M^φ contains a diffuse masa B_0 of M; in fact one can choose B_0 to be singular in M [**Po**]. Note that the action of $\mathcal{U}(B_0)$ on M by inner automorphisms preserves φ as $B_0 \subseteq M^\varphi$. Now in the previous paragraph replace N by M, τ_N by φ and consider an analogous action of $\mathbb{Z}^2$ on $M \overline{\otimes} \mathbf{B}(\mathcal{K})$.

Example 6.6 highlights that in order to understand the failure of the statement 'weak mixing of the representation $\pi_V \Rightarrow$ ergodicity of β' in the context of f.n.s. weight preserving actions, semifiniteness of φ with respect to the fixed point subalgebra $M^G = \{x \in M : \beta_g(x) = x \ \forall g \in G\}$ is involved (see [**Ta**]). In the next result, we show that Example 6.6 is prototypical in the failure of ergodicity of β in the presence of weak mixing of π_V.

THEOREM 6.8. *Let $\mathfrak{M} = (M, \varphi, \beta, G)$ be an f.n.s. weight preserving W^*-dynamical system. Assume that $\mathfrak{M}$ is weak mixing relative to $\mathfrak{A}_\varphi$. Then either $\mathfrak{M}$ is ergodic or $\varphi_{|(M^G)_+} = \infty$.*

Proof. If M^G is trivial, there is nothing to prove. So, let $M^G \neq \mathbb{C}1$. Suppose to the contrary that there exists $0 \neq x \in (M^G)_+$ such that $\varphi(x) < \infty$. Thus, M^G intersects $\mathfrak{F}_\varphi$ (defined in §3) non-trivially and the latter is contained in $\mathfrak{A}_\varphi$. But then π_V is weak mixing and yet contain non-zero fixed points. This is a contradiction. Therefore, $\varphi_{|(M^G)_+} = \infty$. □

Thus, we have the following corollary.

COROLLARY 6.9. *Let $\mathfrak{M} = (M, \varphi, \beta, G)$ be an f.n.s. weight preserving W^*-dynamical system such that π_V is weak mixing. If φ is semifinite on M^G, then φ is a finite weight.*

https://doi.org/10.1017/etds.2017.15 Published online by Cambridge University Press

In the next theorem, we allow finite weights as well. We now intend to show that up to scaling there can be at most one f.n.s. weight that is preserved by an ergodic action. Let $\mathcal{W}$ denote the collection of f.n.s. weights of a von Neumann algebra M.

THEOREM 6.10. *Let β be an ergodic action of a group G on a von Neumann algebra M. Suppose that there exists $\psi \in \mathcal{W}$ such that $\psi \circ \beta_g = \psi$ for all $g \in G$. If $\varphi \in \mathcal{W}$ is such that $\varphi \circ \beta_g = \varphi$ for all $g \in G$, then there exists $c > 0$ such that $\varphi = c\psi$. In particular, if both φ and ψ were states then $\varphi = \psi$.*

Moreover, if $G = \mathbb{R}$ and the action via modular automorphisms (σ_t^{ψ}) is ergodic, then φ commutes with ψ if and only if φ is a scalar multiple of ψ.

Proof. Note that $[D\varphi : D\psi]_t = u_t \in M$ is unitary for all $t \in \mathbb{R}$ as both $\varphi, \psi \in \mathcal{W}$. Then

$$\sigma_t^{\varphi}(x) = u_t \sigma_t^{\psi}(x) u_t^*, \quad x \in M.$$

Then, for all $g \in G$ and $t \in \mathbb{R}$, we have

$$\begin{aligned}
&\sigma_t^{\varphi}(\beta_g(x)) = u_t \sigma_t^{\psi}(\beta_g(x)) u_t^*, \quad \text{while} \\
&\sigma_t^{\varphi}(\beta_g(x)) = \beta_g(\sigma_t^{\varphi}(x)) \\
&\qquad = \beta_g(u_t)\beta_g(\sigma_t^{\psi}(x))\beta_g(u_t)^* = \beta_g(u_t)\sigma_t^{\psi}(\beta_g(x))\beta_g(u_t)^*, \quad x \in M.
\end{aligned}$$

Note that β_g preserves $\mathfrak{N}_{\varphi}$, $\mathfrak{N}_{\psi}$, $\mathfrak{N}_{\varphi}^*$ and $\mathfrak{N}_{\psi}^*$ for all $g \in G$. Therefore, it is routine to check that $t \mapsto \beta_g(u_t)$, $t \in \mathbb{R}$, satisfies all the properties of Connes' cocycle derivative (see [**St**, p. 46]) for all $g \in G$. Consequently, by the uniqueness of the cocycle derivative it follows that $\beta_g(u_t) = u_t$ for all $t \in \mathbb{R}$ and all $g \in G$. Hence, by ergodicity of the action, there exists a continuous function $h : \mathbb{R} \to \mathbb{T}$ such that $u_t = h(t)1$ for all $t \in \mathbb{R}$. The cocycle property of (u_t) forces $h(t + s) = h(t)h(s)$ for $t, s \in \mathbb{R}$. It follows that there exists $\theta \in \mathbb{R}$ such that $h(t) = e^{i\theta t}$, for all $t \in \mathbb{R}$. Thus, $u_{-i/2} = e^{\theta/2}1$. Consequently,

$$\varphi(x) = \psi(u_{-i/2}^* x u_{-i/2}) = e^{\theta}\psi(x), \quad x \in \mathfrak{M}_{\varphi},$$

forcing $\mathfrak{M}_{\varphi} = \mathfrak{M}_{\psi}$. Finally, put $c = e^{\theta}$.

The case when both $\varphi(1) = \psi(1) = 1$ entails $\theta = 0$. Thus, $\varphi = \psi$. The last conclusion is obvious. □

Acknowledgement. The authors thank Jon Bannon and Jan Cameron for numerous helpful discussions.

REFERENCES

[**Aa**] J. Aaronson. *An Introduction to Infinite Ergodic Theory (Mathematic Surveys and Monographs, 50)*. American Mathematical Society, Providence, RI, 1997.
[**BCM**] J. Bannon, J. Cameron and K. Mukherjee. On noncommutative joinings. *Preprint*, 2016.
[**Br**] C. Brislawn. Kernels of trace class operators. *Proc. Amer. Math. Soc.* **104**(4) (1988), 1181–1190.
[**Co**] A. Connes. Almost periodic states and factors of type III_1. *J. Funct. Anal.* **16** (1974), 415–445.
[**Ha**] A. Hajian. On ergodic measure-preserving transformations defined on an infinite measure space. *Proc. Amer. Math. Soc.* **16**(1) (1965), 45–48.

https://doi.org/10.1017/etds.2017.15 Published online by Cambridge University Press

[**HaKa**] A. Hajian and S. Kakutani. Weakly wandering sets and invariant measures. *Trans. Amer. Math. Soc.* **110**(1) (1964), 136–151.
[**Hal**] P. R. Halmos. On automorphisms of compact groups. *Bull. Amer. Math. Soc.* **49** (1943), 619–624.
[**HeTa**] R. H. Herman and M. Takesaki. States and automorphism groups of operator algebras. *Comm. Math. Phys.* **19** (1970), 142–160.
[**HLS**] R. Høegh-Krohn, M. B. Landstad and E. Størmer. Compact ergodic groups of automorphisms. *Ann. of Math. (2)* **114**(1) (1981), 75–86.
[**JaPi**] V. Jakšić and C. A. Pillet. A note on eigenvalues of Liouvilleans. *J. Statist. Phys.* **105**(5–6) (2001), 937–941.
[**Ka**] I. Kaplansky. Groups with representations of bounded degree. *Canad. J. Math.* **1** (1949), 105–112.
[**KeLi**] D. Kerr and Hanfeng Li. *Ergodic Theory: Independence and Dichotomies*, online book, 2014, http://www.math.tamu.edu/∼kerr/book/.
[**KRI**] R. Kadison and J. Ringrose. *Fundamentals of the Theory of Operator Algebras Vol. I: Elementary Theory*. Academic Press, New York, 1983.
[**KrSu**] U. Krengel and L. Sucheston. On mixing in infinite measure space. *Z. Wahrsch. verw. Geb.* **13** (1969), 150–164.
[**MSW**] R. Miles, M. Staines and T. Ward. Dynamical invariants for group automorphisms. *Recent Trends in Ergodic Theory and Dynamical Systems (Contemporary Mathematics, 631)*. American Mathematical Society, Providence, RI, 2015, pp. 231–258.
[**MuPa**] K. Mukherjee and I. Patri. Automorphisms of compact quantum groups. *Preprint*, 2016.
[**Po**] S. Popa. Singular maximal abelian *-subalgebras in continuous von Neumann algebras. *J. Funct. Anal.* **50**(2) (1983), 151–166.
[**Ta**] M. Takesaki. Conditional expectations in von Neumann algebras. *J. Funct. Anal.* **9** (1972), 306–321.
[**Ta2**] M. Takesaki. *Theory of Operator Algebras II (Encylopedia of Mathematical Sciences, 125)*. Springer, Berlin, 2003.
[**St**] S. Ştrătilă. *Modular Theory in Operator Algebras*. Abacus Press, Tonbridge Walls, 1981, Editura Academiei, Bucharest.
[**StZs**] S. Ştrătilă and L. Zsidó. *Lectures on von Neumann Algebras*. Abacus Press, Tunbridge Wells, 1979.

https://doi.org/10.1017/etds.2017.15 Published online by Cambridge University Press