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Solvability of the Bézout equation for Banach algebra-valued H^∞ functions on the polydisk

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Abstract. In connection with the still unsolved multidimensional corona problem for algebras of bounded holomorphic functions on convex domains, we study the solvability of the Bézout equation for the algebra of bounded holomorphic functions on the polydisk with values in a complex Banach algebra. Assuming local solvability of the Bézout equation on a special open cover of the maximal ideal space of the algebra, we combine a dimension-induction scheme with a careful analysis of the topological structure of this space to glue local solutions into a global one. As an application, we obtain the solvability of the Bézout equation for a broader class of subalgebras containing the slice algebra of bounded holomorphic functions, the case of the latter having been previously proved by the first author.

1 Introduction

Let A be a unital commutative complex Banach algebra. Its maximal ideal space $M(A) \subset A^*$ is the set of nonzero homomorphisms $A \to \mathbb{C}$. Every element of $M(A)$ is a continuous linear functional on A of norm at most one, so $M(A)$ lies in the closed unit ball of A^*. It is a compact Hausdorff space in the weak- $*$ topology of A^*. The Gelfand transform

$$\hat{a}(\xi) := \xi(a), \quad a \in A, \quad \xi \in M(A),$$

is a norm nonincreasing morphism of Banach algebras $A \to C(M(A))$. If $A \subset C_b(X)$, the algebra of bounded continuous functions on a Hausdorff space X with the supremum norm, and A separates points of X, then the Gelfand transform is an isometric embedding. In this case, the evaluation map

$$\iota : X \hookrightarrow M(A), \quad \iota(x)(f) = f(x), \quad f \in A,$$

is continuous, and the complement of the closure of $\iota(X)$ in $M(A)$ is called the *corona* of A.

The *corona problem* asks whether this complement is empty. Equivalently (cf. Garnett [19, Chapter V, Theorem 1.8]), it asks whether for every finite collection

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$f_1, \dots, f_k \in A$ satisfying the corona condition

$$\inf_{x\in X} \max_{1\le j\le k} |f_j(x)| > 0, \tag{1.1}$$

the corresponding Bézout equation is solvable, that is, whether there exist $g_1, \dots, g_k \in A$ such that

$$f_1 g_1 + \cdots + f_k g_k = 1 \qquad \text{on } X. \tag{1.2}$$

For $A = H^\infty(\mathbb{D})$, the algebra of bounded holomorphic functions on the unit disk, the problem was raised by Kakutani in 1941. Newman [33] introduced the term *corona* and showed that the problem is equivalent to an interpolation question later solved by Carleson [11]. Carleson proved that the corona is empty, equivalently that every collection of bounded holomorphic functions on $\mathbb{D}$ satisfying (1.1) admits a solution to (1.2). Hörmander [26] introduced $\bar\partial$-methods together with the Koszul complex to simplify the proof, and Wolff [41] later gave the shortest argument by directly obtaining uniform estimates on the boundary circle of the solution of the relevant $\bar\partial$-equation (see also Garnett [19]).

Following Carleson's theorem, the corona problem was settled for many classes of Riemann surfaces, including finite bordered surfaces [1, 2, 14, 16, 23, 34–37], certain infinitely connected domains ("roadrunner" domains) treated by Behrens [4, 5], with subsequent extensions in [12, 13, 31], and finitely sheeted coverings treated by Nakai [30]. It was later shown, starting with the work of Cole (see Gamelin [18]), that there exist Riemann surfaces with nontrivial corona (see also Barrett–Diller [3] and Lárusson [27]).

For planar domains, the general corona problem is still unsolved. In this direction, Gamelin [17] has shown that the corona problem for planar domains is local in the sense that it depends only on the behavior of the domain locally about each boundary point. Moore [29] proved that the corona is empty for domains whose boundary lies on the graph of a $C^{1+\varepsilon}$ function, extending the earlier result of Garnett and Jones [20] for Denjoy domains. Further extensions include Handy's theorem for complements of certain Cantor sets [22] and NewDelman's result for subsets of Lipschitz graphs [32].

The problem also remains open in several variables for the ball and the polydisk. In fact, no examples are known of domains in $\mathbb{C}^n$, $n \ge 2$, whose algebra H^∞ has a nontrivial corona.

2 Main results

In this article, we study the corona problem for the Banach algebra $H^\infty(\mathbb{D}^n)$ of bounded holomorphic functions on the polydisk $\mathbb{D}^n \subset \mathbb{C}^n$. As before, $M(H^\infty(\mathbb{D}^n))$ denotes its maximal ideal space, equipped with the weak-* topology.

Throughout this section, we use the notation

$$\widehat{U} \subset \big[M(H^\infty(\mathbb{D}))\big]^n, \qquad U := \widehat{U} \cap \mathbb{D}^n,$$

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for an open set $\widehat{U}$ in the product maximal ideal space and its intersection U with the polydisk.

Given a unital commutative complex Banach algebra $(A, \|\cdot\|_A)$, we write $H^\infty(\mathbb{D}^n, A)$ for the Banach algebra of bounded holomorphic functions from $\mathbb{D}^n$ into A, equipped with the norm

$$\|f\|_\infty = \sup_{z\in\mathbb{D}^n} \|f(z)\|_A. \tag{2.1}$$

One of our main results is the following.

Theorem 2.1 *Let A be a unital commutative complex Banach algebra, and let $f_1, \dots, f_k \in H^\infty(\mathbb{D}^n, A)$. Suppose there exist open sets $\widehat{U}_1, \dots, \widehat{U}_k \subset [M(H^\infty(\mathbb{D}))]^n$ such that*

$$[M(H^\infty(\mathbb{D}))]^n = \bigcup_{i=1}^k \widehat{U}_i, \quad \text{and} \quad U_i := \mathbb{D}^n \cap \widehat{U}_i.$$

Assume that for each i, there exist $g_{i1}, \dots, g_{ik} \in H^\infty(U_i, A)$ such that

$$\sum_{j=1}^k f_j g_{ij} = 1 \quad \text{on } U_i.$$

Then there exist $g_1, \dots, g_k \in H^\infty(\mathbb{D}^n, A)$ such that

$$\sum_{j=1}^k f_j g_j = 1 \quad \text{on } \mathbb{D}^n.$$

Next, we study the structure of the maximal ideal space $M(H^\infty(\mathbb{D}^n))$. To this end, we consider the natural continuous projection

$$\pi_n : M(H^\infty(\mathbb{D}^n)) \longrightarrow \left[M(H^\infty(\mathbb{D}))\right]^n, \tag{2.2}$$

defined as the transpose of the canonical isometric embedding

$$\widehat{\otimes}_\varepsilon^n H^\infty(\mathbb{D}) \hookrightarrow H^\infty(\mathbb{D}^n),$$

where $\widehat{\otimes}_\varepsilon^n H^\infty(\mathbb{D})$ denotes the n-fold injective (or ε-) tensor product of $H^\infty(\mathbb{D})$, regarded as the closed subalgebra of $H^\infty(\mathbb{D}^n)$ generated by bounded holomorphic functions in the individual coordinate variables. In particular, π_n restricts to the identity on $\mathbb{D}^n \subset M(H^\infty(\mathbb{D}^n))$.

For each point $\xi \in \left[M(H^\infty(\mathbb{D}))\right]^n$, define the corresponding fiber

$$F_\xi := \pi_n^{-1}(\xi) \cap \mathrm{cl}(\mathbb{D}^n) \subset M(H^\infty(\mathbb{D}^n)), \tag{2.3}$$

where $\mathrm{cl}(\mathbb{D}^n)$ denotes the closure of $\mathbb{D}^n$ in $M(H^\infty(\mathbb{D}^n))$.

We equip $\mathrm{cl}(\mathbb{D}^n)$ with the topology induced from $M(H^\infty(\mathbb{D}^n))$. Given an open subset $U \subset \mathrm{cl}(\mathbb{D}^n)$, we write $\mathcal{O}(U) \subset C(U)$ for the algebra of continuous functions on U that are holomorphic on $U \cap \mathbb{D}^n$ in the usual sense.

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For each fiber F_ξ, we define the associated uniform algebra $A(F_\xi) \subset C(F_\xi)$ as the uniform closure of the algebras $\mathcal{O}(U)|_{F_\xi}$, where U ranges over all open neighborhoods of F_ξ in $\mathrm{cl}(\mathbb{D}^n)$.

Since F_ξ is compact, the family of sets of the form

$$\pi_n^{-1}(V) \cap \mathrm{cl}(\mathbb{D}^n),$$

with V ranging over open neighborhoods of $\xi \in \big[M(H^\infty(\mathbb{D}))\big]^n$, forms a basis of neighborhoods of F_ξ in $\mathrm{cl}(\mathbb{D}^n)$; that is, every open neighborhood of F_ξ contains a set of this form. Moreover, each such set V contains a compact subset of the form $W = W_1 \times \cdots \times W_n$, where, for $\xi = (\xi_1, \ldots, \xi_n) \in \big[M(H^\infty(\mathbb{D}))\big]^n$, each component $W_i \subset M(H^\infty(\mathbb{D}))$ is given by

$$W_i := \big\{\eta \in M(H^\infty(\mathbb{D})) : |\hat{f}(\eta)| \le 1\big\},$$

for some function $f \in H^\infty(\mathbb{D})$ satisfying $\hat{f}(\xi_i) = 0$. Such a choice is guaranteed by the classical result of Suárez [38, Theorem 2.4], which asserts that the algebra $H^\infty(\mathbb{D})$ is *separating*.

In particular, the set

$$W \cap \mathbb{D}^n = (W_1 \cap \mathbb{D}) \times \cdots \times (W_n \cap \mathbb{D})$$

satisfies the hypotheses of the Runge-type approximation theorem established in [10]. That result asserts that every function in $H^\infty\big(\mathbb{D}^n \cap \pi_n^{-1}(V)\big)$ can be uniformly approximated on $W \cap \mathbb{D}^n$ by functions from $H^\infty(\mathbb{D}^n)$.

Summarizing the above, we obtain that for every open neighborhood $U \subset \mathrm{cl}(\mathbb{D}^n)$ of F_ξ, the algebra $\mathcal{O}(U)|_{F_\xi}$ is contained in the uniform closure of the algebra $\mathcal{O}\big(M(H^\infty(\mathbb{D}^n))\big)|_{F_\xi}$, where

$$\mathcal{O}\big(M(H^\infty(\mathbb{D}^n))\big) := \widehat{H^\infty(\mathbb{D}^n)} = \big\{\hat{f} : f \in H^\infty(\mathbb{D}^n)\big\}.$$

In view of the definition of the algebra $A(F_\xi)$, it follows that $A(F_\xi)$ coincides with the uniform closure of the algebra $\widehat{H^\infty(\mathbb{D}^n)}|_{F_\xi}$. It then follows from a standard result in the theory of uniform algebras that the maximal ideal space of $A(F_\xi)$ coincides with the H^∞-hull of F_ξ, namely,

$$M(A(F_\xi)) = \left\{\zeta \in M(H^\infty(\mathbb{D}^n)) : |\hat{f}(\zeta)| \le \sup_{F_\xi} |\hat{f}| \text{ for all } f \in H^\infty(\mathbb{D}^n)\right\}. \tag{2.4}$$

Theorem 2.2 *For each $\xi \in [M(H^\infty(\mathbb{D}))]^n$, we have*

$$\pi_n^{-1}(\xi) = M(A(F_\xi)).$$

The corona problem for $H^\infty(\mathbb{D}^n)$ is solvable (i.e., $\mathrm{cl}(\mathbb{D}^n) = M(H^\infty(\mathbb{D}^n))$) if and only if

$$M(A(F_\xi)) = F_\xi \quad \textit{for all } \xi \in \big[M(H^\infty(\mathbb{D}))\big]^n.$$

To describe certain fibers F_ξ more explicitly, we begin by recalling classical results of Hoffman on the structure of the maximal ideal space $M(H^\infty(\mathbb{D}))$.

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The pseudohyperbolic metric on the unit disk $\mathbb{D}$ is given by

$$\rho(z,w) := \left|\frac{z-w}{1-\bar{w}z}\right|, \quad z,w \in \mathbb{D}, \tag{2.5}$$

and admits a natural extension to $M(H^\infty(\mathbb{D}))$ via the formula

$$\rho(\xi,\eta) := \sup\left\{|\hat{f}(\eta)| : f \in H^\infty(\mathbb{D}),\ \hat{f}(\xi) = 0,\ \|f\|_\infty \le 1\right\}, \quad \xi,\eta \in M(H^\infty(\mathbb{D})). \tag{2.6}$$

The *Gleason part* of a point $x \in M(H^\infty(\mathbb{D}))$ is

$$\pi(\xi) := \{\eta \in M(H^\infty(\mathbb{D})) : \rho(\xi,\eta) < 1\}. \tag{2.7}$$

These parts form a partition of $M(H^\infty(\mathbb{D}))$: for any ξ, η, either $\pi(\xi) = \pi(\eta)$ or $\pi(\xi) \cap \pi(\eta) = \varnothing$.

A fundamental result of Hoffman [25] asserts that each Gleason part in $M(H^\infty(\mathbb{D}))$ is either a singleton or an analytic disk (i.e., the image of a continuous injective map $\tau\colon \mathbb{D} \to M(H^\infty(\mathbb{D}))$ such that $\hat{f} \circ \tau \in H^\infty(\mathbb{D})$ for all $f \in H^\infty(\mathbb{D})$). A point $\xi \in M(H^\infty(\mathbb{D}))$ lies in a nontrivial part if and only if it belongs to the closure of an interpolating sequence. The union M_a of all such parts is open in the maximal ideal space, and its complement $M_s := M(H^\infty(\mathbb{D}))\backslash M_a$ is closed and totally disconnected by a theorem of Suárez [39].

The space $[M(H^\infty(\mathbb{D}))]^n$ admits a natural stratification according to the number of coordinates lying in the open subset M_a of nontrivial Gleason parts. Each stratum is determined by the subset of indices $j \in \{1,\dots,n\}$ for which the jth coordinate belongs to M_a (and hence the remaining coordinates lie in M_s).

We now describe the structure of fibers of the map

$$\pi_n : M(H^\infty(\mathbb{D}^n)) \to [M(H^\infty(\mathbb{D}))]^n$$

over the first two strata: (i) the open stratum M_a^n and (ii) the codimension-one stratum consisting of those $\xi \in [M(H^\infty(\mathbb{D}))]^n$ with exactly one coordinate in M_s and the remaining $n-1$ in M_a.

Theorem 2.3 *Let $\xi \in \left[M\left(H^\infty(\mathbb{D})\right)\right]^n$. Then:*

(i) *If $\xi \in M_a^n$, then the fiber $\pi_n^{-1}(\xi)$ coincides with*

$$F_\xi := \pi_n^{-1}(\xi) \cap \mathrm{cl}(\mathbb{D}^n),$$

and F_ξ lies in the closure of an interpolating sequence for $H^\infty(\mathbb{D}^n)$.[1] *In particular, F_ξ is totally disconnected.*

(ii) *If exactly one coordinate of ξ lies in M_s and the remaining $n-1$ in M_a, then the fiber $\pi_n^{-1}(\xi)$ has covering dimension ≤ 2 and decomposes as the disjoint union of a compact totally disconnected subset and a (possibly empty) open subset consisting of pairwise disjoint analytic disks.*

In both cases, $M\left(A(F_\xi)\right) = F_\xi$.

[1] A sequence $\{z_k\} \subset \mathbb{D}^n$ is interpolating for $H^\infty(\mathbb{D}^n)$ if, for every bounded sequence $\{a_k\} \in \ell^\infty$, there exists $f \in H^\infty(\mathbb{D}^n)$ such that $f(z_k) = a_k$ for all k.

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In case (ii), the fiber F_ξ, via the embedding R^* constructed in Lemma 5.1, is a disjoint union of Gleason parts for the algebra $H^\infty(\mathbb{D}\times\mathbb{N})$ of bounded holomorphic functions on the disjoint countable union of open disks. Each such part is either an analytic disk or a single point. For a detailed study of the structure of $M\big(H^\infty(\mathbb{D}\times\mathbb{N})\big)$, see, for example, [8].

Recall that the *covering dimension* of a normal space X, denoted $\dim X$, is defined as follows:

- $\dim X < n$ if every finite open cover of X has a finite open refinement of order $\le n$ (i.e., no point lies in more than $n+1$ sets of the refinement);
- $\dim X = n$ if $\dim X < n+1$ and $\dim X \not< n$;
- $\dim X = \infty$ if no such integer n exists.

We mention that every compact Hausdorff totally disconnected space has covering dimension 0 (see [15, Chapter 1, Section 6.7]).

Remark 2.4 For strata of $[M(H^\infty(\mathbb{D}))]^n$ of codimension between 2 and $n-1$, which arise only for $n\ge 3$, the structure of fibers F_ξ is more subtle. Nevertheless, one can still show that $M(A(F_\xi)) = F_\xi$ provided that the corona theorem holds for the algebras $H^\infty(\mathbb{D}^k)$ with $k = 2,3,\dots,n-1$.

The main difficulty arises for the lowest stratum M_s^n. In this case, the fiber F_ξ may contain the image of an injective continuous map $\tau\colon\mathbb{D}^n\to \mathrm{cl}(\mathbb{D}^n)$ such that $\hat f\circ\tau\in H^\infty(\mathbb{D}^n)$ for all $f\in H^\infty(\mathbb{D}^n)$. The existence of such "embedded" n-dimensional analytic structures obstructs an inductive approach to proving the corona theorem, since no analogous slice reduction is available in this stratum. We will explore this phenomenon in more detail in a forthcoming paper.

We now illustrate the scope of Theorems 2.1 and 2.2 by extending the class of subalgebras of $H^\infty(\mathbb{D}^n)$ over which the corresponding Bézout equations admit global solutions, and by showing that such solvability is preserved under all perturbations in $H^\infty(\mathbb{D}^n)$ up to the maximal size allowed by the corona condition.

Let $S_n\big(H^\infty(\mathbb{D})\big)\subset H^\infty(\mathbb{D}^n)$ denote the slice algebra, that is, the closed subalgebra consisting of those holomorphic functions whose boundary values on

$$\mathbb{T}^n := \{z=(z_1,\dots,z_n)\in\mathbb{C}^n : |z_i|=1,\ \text{for all } i\}$$

belong to

$$\widehat{\otimes}_\varepsilon^n L^\infty(\mathbb{T})\subset L^\infty(\mathbb{T}^n).$$

Equivalently, these are the holomorphic functions on $\mathbb{D}^n$ that can be uniformly approximated by bounded functions, which are *separately harmonic* in each variable.

It is not known in general whether $S_n\big(H^\infty(\mathbb{D})\big)$ coincides with the uniform algebra generated by $H^\infty(\mathbb{D})$ in each coordinate, that is, whether

$$S_n\big(H^\infty(\mathbb{D})\big) = \widehat{\otimes}_\varepsilon^n H^\infty(\mathbb{D}),$$

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a question related to the longstanding open problem of whether $H^\infty(\mathbb{D})$ has the approximation property (see Lindenstrauss [28, Question 7]; see also Bourgain and Reinov [6] for related results).

Let $H_1, \dots, H_m \in H^\infty(\mathbb{D}^n) \backslash S_n\big(H^\infty(\mathbb{D})\big)$, and set

$$H := (H_1, \dots, H_m) : \mathbb{D}^n \longrightarrow \mathbb{C}^m.$$

We denote by

$$A(S_n; H) := \langle\, S_n\big(H^\infty(\mathbb{D})\big),\ H_1, \dots, H_m \,\rangle \subset H^\infty(\mathbb{D}^n)$$

the uniform closure of the algebra generated by the slice algebra and the functions $H_1, \dots, H_m$.

Theorem 2.5 *Assume that, for every point $x \in \mathbb{D}^n$, there exists a neighborhood $U_x \subset \mathbb{D}^n$ of x and a one-dimensional complex analytic subset $\Sigma_x \subset \mathbb{C}^m$ such that $H(U_x) \subset \Sigma_x$.*

Fix $\delta > 0$ and suppose $g_1, \dots, g_k \in A(S_n; H)$ satisfy the corona condition

$$\inf_{z \in \mathbb{D}^n} \max_{1 \le i \le k} |g_i(z)| \ \ge\ \delta > 0.$$

Then for every $h_1, \dots, h_k \in H^\infty(\mathbb{D}^n)$ such that

$$\max_{1 \le i \le k} \|h_i\|_\infty < \delta,$$

the Bézout equation

$$(g_1 + h_1) f_1 + \cdots + (g_k + h_k) f_k = 1 \quad \text{on } \mathbb{D}^n$$

admits a solution $f_1, \dots, f_k \in H^\infty(\mathbb{D}^n)$.

Remark 2.6 (a) The hypothesis of Theorem 2.5 on the map $H = (H_1, \dots, H_m)$ is equivalent to the existence of holomorphic maps

$$h : \mathbb{D}^n \to \mathbb{D}, \qquad \widetilde{H} : \mathbb{D} \to \mathbb{C}^m$$

with $H = \widetilde{H} \circ h$ (see Lemma 6.1). In particular, Theorem 2.5 applies whenever the components of H factor through a holomorphic map from $\mathbb{D}^n$ to $\mathbb{D}$.

(b) If $g_1, \dots, g_k \in S_n\big(H^\infty(\mathbb{D})\big)$ satisfy the corona condition

$$\inf_{z \in \mathbb{D}^n} \max_{1 \le i \le k} |g_i(z)| > 0,$$

then the Bézout equation

$$g_1 f_1 + \cdots + g_k f_k = 1 \quad \text{on } \mathbb{D}^n$$

admits a solution $f_1, \dots, f_k \in S_n\big(H^\infty(\mathbb{D})\big)$. This was established in [7, Corollary 1.16] as a consequence of the more general result [7, Theorem 1.12]. We note that Theorem 2.5 is significantly stronger: it applies already when H consists of a single function from $H^\infty(\mathbb{D}^n) \backslash S_n\big(H^\infty(\mathbb{D})\big)$.

(c) The perturbation condition in Theorem 2.5 is optimal. Indeed, assume

$$\inf_{z \in \mathbb{D}^n} \max_{1 \le i \le k} |g_i(z)| = \delta > 0.$$

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Choose a sequence $(z_\nu) \subset \mathbb{D}^n$ such that $\max_{1\le i\le k} |g_i(z_\nu)| \to \delta$. Set

$$v_\nu := \big(g_1(z_\nu), \dots, g_k(z_\nu)\big) \in \mathbb{C}^k.$$

The sequence (v_ν) is bounded, so a subsequence converges to a vector $c = (c_1, \dots, c_k) \in \mathbb{C}^k$. For this limit,

$$\max_{1\le i\le k} |c_i| = \delta$$

and, along the same subsequence,

$$\inf_{z\in\mathbb{D}^n} \max_{1\le i\le k} |g_i(z) - c_i| = 0.$$

Thus, the corresponding Bézout equation

$$(g_1 - c_1)f_1 + \cdots + (g_k - c_k)f_k = 1$$

has no solution in $H^\infty(\mathbb{D}^n)$. Indeed, if such a solution existed, then along the chosen subsequence

$$1 \le \sum_{i=1}^{k} \|f_i\|_\infty |g_i(z_\nu) - c_i| \longrightarrow 0,$$

a contradiction.

3 Proof of Theorem 2.1

Proof We argue by induction on the number of variables n.

For $n = 1$, the statement is Theorem 8.3 of [9]. Assume the result holds for dimension n, and consider $H^\infty(\mathbb{D}^{n+1}, A)$.

Let $(\widehat{U}_i)_{1\le i\le k}$ be an open cover of $[M(H^\infty(\mathbb{D}))]^{n+1}$. By the product topology, there exist a finite open cover $(\widehat{V}_j)_{1\le j\le l}$ of $[M(H^\infty(\mathbb{D}))]^n$ and, for each j, a finite open cover $(\widehat{W}_{jm})_{1\le m\le l_j}$ of $M(H^\infty(\mathbb{D}))$ such that the cover $(\widehat{V}_j \times \widehat{W}_{jm})_{j,m}$ of $[M(H^\infty(\mathbb{D}))]^{n+1}$ refines $(\widehat{U}_i)_{1\le i\le k}$. Set

$$U_{jm} := V_j \times W_{jm}, \qquad V_j = \mathbb{D}^n \cap \widehat{V}_j, \quad W_{jm} = \mathbb{D} \cap \widehat{W}_{jm}.$$

Then $(U_{jm})_{j,m}$ is a finite cover of $\mathbb{D}^{n+1}$ refining $(U_i)_{1\le i\le k}$.

Suppose the Bézout equation

$$\sum_{i=1}^{k} f_i g_i = 1 \tag{3.1}$$

admits local solutions $(g_{1;jm}, \dots, g_{k;jm}) \in [H^\infty(U_{jm}, A)]^k$ on the cover $(U_{jm})_{j,m}$. For each $i \in \{1, \dots, k\}$, define slice functions on W_{jm},

$$g^s_{i;jm}(z)(z_1, \dots, z_n) := g_{i;jm}(z_1, \dots, z_n, z), \qquad (z_1, \dots, z_n) \in V_j,\ z \in W_{jm}.$$

These belong to $H^\infty(W_{jm}, H^\infty(V_j, A))$. Likewise, each f_i defines

$$f^s_{i;j} \in H^\infty(\mathbb{D}, H^\infty(V_j, A)).$$

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Passing to slice functions in (3.1) yields that the $H^\infty(V_j, A)$-valued Bézout equation on $\mathbb{D}$,

$$\sum_{i=1}^{k} f_{i;j}^s g_i = 1, \tag{3.2}$$

has local solutions $(g_{1;jm}^s, \dots, g_{k;jm}^s) \in [H^\infty(W_{jm}, H^\infty(V_j, A))]^k$ on the cover (W_{jm}) of $\mathbb{D}$.

Since $W_{jm} = \mathbb{D} \cap \widehat{W}_{jm}$ and $\mathcal{W}_j = (\widehat{W}_{jm})_{1\le m\le l_j}$ is an open cover of $M(H^\infty(\mathbb{D}))$, applying [9, Theorem 8.3], we obtain global solutions $(h_{1;j}, \dots, h_{k;j}) \in [H^\infty(\mathbb{D}, H^\infty(V_j, A))]^k, 1 \le j \le l$, of (3.2).

Define

$$g_{i;j}(z_1, \dots, z_{n+1}) := (h_{i;j}(z_{n+1}))(z_1, \dots, z_n), \qquad (z_1, \dots, z_{n+1}) \in V_j \times \mathbb{D}.$$

Then $g_{i;j} \in H^\infty(V_j \times \mathbb{D}, A)$ and

$$\sum_{i=1}^{k} f_i g_{i;j} = 1 \quad \text{on } V_j \times \mathbb{D}.$$

Hence, we obtain local bounded holomorphic solutions of the original Bézout equation (3.1) on the cover $(V_j \times \mathbb{D})_{1\le j\le l}$ of $\mathbb{D}^{n+1}$.

Now view each $g_{i;j}$ as a slice function

$$g_{i;j}^s \in H^\infty(V_j, H^\infty(\mathbb{D}, A)).$$

Similarly, f_i induce slice functions

$$f_i^s \in H^\infty(\mathbb{D}^n, H^\infty(\mathbb{D}, A)).$$

Then the solvability of the Bézout equation (3.1) on the cover $(V_j \times \mathbb{D})$ transforms into the identities

$$\sum_{i=1}^{k} f_i^s g_{i;j}^s = 1 \quad \text{in } H^\infty(\mathbb{D}, A) \text{ for each } V_j,$$

that is, we obtain, with respect to the cover $(V_j)_{1\le j\le l}$, local bounded holomorphic solutions of the Bézout equation on $\mathbb{D}^n$,

$$\sum_{i=1}^{k} f_i^s h_i = 1 \tag{3.3}$$

with values in the algebra $H^\infty(\mathbb{D}, A)$, over the cover $(V_j)_{1\le j\le l}$ of $\mathbb{D}^n$, where $V_j = \mathbb{D}^n \cap \widehat{V}_j$ and $\mathcal{V} = (\widehat{V}_j)_{1\le j\le l}$ is a finite open cover of $[M(H^\infty(\mathbb{D}))]^n$.

By the induction hypothesis, there exists a solution $(h_1, \dots, h_k) \in [H^\infty(\mathbb{D}^n, H^\infty(\mathbb{D}, A))]^k$ of (3.3).

Finally, set

$$g_i(z_1, \dots, z_{n+1}) := (h_i(z_1, \dots, z_n))(z_{n+1}), \quad (z_1, \dots, z_{n+1}) \in \mathbb{D}^{n+1}.$$

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Then $g_i \in H^\infty(\mathbb{D}^{n+1}, A)$ and

$$\sum_{i=1}^{k} f_i g_i = 1.$$

This completes the induction. ∎

4 Proof of Theorem 2.2

Proof Recall that $A(F_\xi)$ is the uniform closure of $\widehat{H^\infty(\mathbb{D}^n)}|_{F_\xi}$, hence $M\big(A(F_\xi)\big)$ is exactly the H^∞-hull of F_ξ. Also recall that

$$\pi_n \colon M(H^\infty(\mathbb{D}^n)) \longrightarrow \big[M(H^\infty(\mathbb{D}))\big]^n$$

is the transpose of the canonical embedding

$$\widehat{\otimes}_\varepsilon^n H^\infty(\mathbb{D}) \hookrightarrow H^\infty(\mathbb{D}^n),$$

and that the embedded algebra separates the fibers of π_n.

Since the H^∞-hull of $\pi_n^{-1}(\xi)$ coincides with $\pi_n^{-1}(\xi)$, we have $M\big(A(F_\xi)\big) \subset \pi_n^{-1}(\xi)$.

For the reverse inclusion, suppose $\zeta \in \pi_n^{-1}(\xi) \backslash M\big(A(F_\xi)\big)$. By the definition of the H^∞-hull, there exists $f \in H^\infty(\mathbb{D}^n)$ with $\hat{f}(\zeta) = 1$ and

$$m := \max_{F_\xi} |\hat{f}| < 1.$$

Set $g := 1 - f$. Then $\hat{g}(\zeta) = 0$ and $|\hat{g}(x)| \geq 1 - m > 0$ for all $x \in F_\xi$.

By continuity of $\hat{g}$ and compactness of F_ξ, there exists an open neighborhood $\mathcal{U}$ of F_ξ in $\mathrm{cl}(\mathbb{D}^n)$ on which $|\hat{g}| \geq \frac{1}{2}(1-m)$. Since the sets

$$\pi_n^{-1}(\widehat{U}) \cap \mathrm{cl}(\mathbb{D}^n), \quad \widehat{U} \subset \big[M(H^\infty(\mathbb{D}))\big]^n$$

with $\widehat{U}$ an open neighborhood of ξ, form a neighborhood basis of F_ξ, we may choose $\widehat{U}$ so that

$$\pi_n^{-1}(\widehat{U}) \cap \mathrm{cl}(\mathbb{D}^n) \subset \mathcal{U}.$$

Set $U := \widehat{U} \cap \mathbb{D}^n$. Then

$$|g(z)| \geq \tfrac{1}{2}(1-m) \quad \text{for all } z \in U. \tag{4.1}$$

For each $x = (x_1, \ldots, x_n) \in \big[M(H^\infty(\mathbb{D}))\big]^n \backslash \widehat{U}$, choose an index $r = r(x)$ such that $x_r \neq \xi_r$. Since $H^\infty(\mathbb{D})$ separates points of its maximal ideal space [38, Theorem 2.4], choose $\psi_x \in H^\infty(\mathbb{D})$ with $\widehat{\psi_x}(x_r) \neq \widehat{\psi_x}(\xi_r)$, and set

$$\varphi_x := \psi_x - \widehat{\psi_x}(\xi_r).$$

Then

$$\widehat{\varphi_x}(\xi_r) = 0, \quad \widehat{\varphi_x}(x_r) \neq 0.$$

Define

$$g_x(z_1, \ldots, z_n) := \varphi_x(z_r).$$

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Then $g_x \in \widehat{\otimes}_\varepsilon^n H^\infty(\mathbb{D})$, and if

$$G_x(\eta_1, \dots, \eta_n) := \widehat{\varphi_x}(\eta_r),$$

then $\widehat{g_x} = G_x \circ \pi_n$. In particular, $\widehat{g_x}$ vanishes on $\pi_n^{-1}(\xi)$. By continuity of G_x, there exist an open neighborhood $\widehat{W}_x$ of x and $\delta_x > 0$ such that

$$|\widehat{g_x}(y)| \geq \delta_x \quad \text{for all } y \in \pi_n^{-1}(\widehat{W}_x).$$

The compact set $\left[M(H^\infty(\mathbb{D}))\right]^n \backslash \widehat{U}$ is covered by finitely many such neighborhoods $\widehat{W}_{x_1}, \dots, \widehat{W}_{x_q}$. Write

$$g_i := g_{x_i}, \quad \widehat{W}_i := \widehat{W}_{x_i}, \quad W_i := \widehat{W}_i \cap \mathbb{D}^n, \quad \delta_i := \delta_{x_i}.$$

Then

$$\widehat{g_i}\big|_{\pi_n^{-1}(\xi)} = 0, \quad |g_i(z)| \geq \delta_i \quad \text{for all } z \in W_i, \quad 1 \leq i \leq q. \tag{4.2}$$

The set $\widehat{U}$, together with the sets $\widehat{W}_i$, $1 \leq i \leq q$, forms a finite open cover of $\left[M(H^\infty(\mathbb{D}))\right]^n$. On U, the tuple $(g, g_1, \dots, g_q)$ has the bounded holomorphic local Bézout solution $(g^{-1}, 0, \dots, 0)$, while on W_i, it has the local solution whose g_i-coordinate is g_i^{-1} and whose other coordinates are zero. Therefore, Theorem 2.1, with $A = \mathbb{C}$, yields $f_0, f_1, \dots, f_q \in H^\infty(\mathbb{D}^n)$ such that

$$g f_0 + g_1 f_1 + \cdots + g_q f_q = 1 \quad \text{on } \mathbb{D}^n.$$

Taking Gelfand transforms and restricting to $\pi_n^{-1}(\xi)$, equation (4.2) gives

$$\widehat{f_0}(x)\, \widehat{g}(x) = 1 \qquad \text{for all } x \in \pi_n^{-1}(\xi).$$

But $\hat{g}(\zeta) = 0$, a contradiction. Hence, $\pi_n^{-1}(\xi) \subset M\big(A(F_\xi)\big)$, and therefore

$$\pi_n^{-1}(\xi) = M\big(A(F_\xi)\big).$$

Finally,

$$\mathrm{cl}(\mathbb{D}^n) = M(H^\infty(\mathbb{D}^n))$$

(i.e., the corona theorem holds for $H^\infty(\mathbb{D}^n)$) if and only if

$$M\big(A(F_\xi)\big) = F_\xi \qquad \text{for all } \xi \in \left[M(H^\infty(\mathbb{D}))\right]^n,$$

equivalently, the H^∞-hull of each fiber F_ξ coincides with F_ξ. ∎

5 Proof of Theorem 2.3

Proof of (i) Let $\xi = (\xi_1, \dots, \xi_n) \in M_a^n$. Since each coordinate $\xi_j \in M(H^\infty(\mathbb{D}))$ lies in a nontrivial Gleason part, a classical result of Hoffman (see, e.g., [25, Theorem 5.5]) guarantees the existence of an interpolating sequence

$$\zeta_j = \{z_{j,k}\}_{k=1}^\infty \subset \mathbb{D}$$

whose closure in $M(H^\infty(\mathbb{D}))$ contains ξ_j, for $j = 1, \dots, n$.

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Define the product sequence

$$\zeta := \zeta_1 \times \cdots \times \zeta_n = \{z_k := (z_{1,k}, \dots, z_{n,k})\}_{k=1}^{\infty} \subset \mathbb{D}^n,$$

whose closure in the product topology on $[M(H^\infty(\mathbb{D}))]^n$ contains ξ.

Claim 1 The sequence ζ is interpolating for $H^\infty(\mathbb{D}^n)$.

Indeed, for each j and k, let $\varphi_{j,k} \in H^\infty(\mathbb{D})$ be bounded holomorphic functions satisfying

$$\varphi_{j,k}(z_{j,\ell}) = \delta_{k,\ell}, \quad \sum_{k=1}^{\infty} |\varphi_{j,k}(z)| \le C_j, \quad \text{for all } z \in \mathbb{D},$$

for some constants $C_j = C_j(\zeta_j) > 0$ (see [19, Chapter VII, Theorem 2.1]). Define

$$\psi_k(z) := \prod_{j=1}^{n} \varphi_{j,k}(z_j), \quad z = (z_1, \dots, z_n) \in \mathbb{D}^n, \quad k \in \mathbb{N}.$$

Then $\psi_k \in H^\infty(\mathbb{D}^n)$, and

$$\psi_k(z_\ell) = \delta_{k,\ell}, \quad \forall\, k, \ell \in \mathbb{N}, \qquad \sum_{k=1}^{\infty} |\psi_k(z)| \le C := \prod_{j=1}^{n} C_j, \quad \forall\, z \in \mathbb{D}^n.$$

Thus, the map

$$L\colon \ell^\infty(\zeta) \to H^\infty(\mathbb{D}^n), \qquad L\big((c_k)_{k=1}^{\infty}\big) := \sum_{k=1}^{\infty} c_k \psi_k,$$

is a bounded linear operator of norm $\le C$, with $L((c_k))|_\zeta = (c_k)$ for all $(c_k) \in \ell^\infty(\zeta)$. Hence, ζ is interpolating for $H^\infty(\mathbb{D}^n)$, as claimed.

It follows from *Claim 1* that the restriction map

$$H^\infty(\mathbb{D}^n) \to \ell^\infty(\zeta), \qquad f \mapsto f|_\zeta,$$

induces, by transposition, a continuous map of maximal ideal spaces

$$i_\zeta\colon M\big(\ell^\infty(\zeta)\big) \longrightarrow M\big(H^\infty(\mathbb{D}^n)\big).$$

Because the restriction map is surjective, i_ζ is injective; since its domain is compact and its codomain is Hausdorff, i_ζ is a topological embedding. Since $\ell^\infty(\zeta)$ is self-adjoint, its maximal ideal space $M(\ell^\infty(\zeta))$ is homeomorphic to the Stone–Čech compactification $\beta(\mathbb{N})$, and

$$i_\zeta\big(M(\ell^\infty(\zeta))\big) = \mathrm{cl}(\zeta) \subset \mathrm{cl}(\mathbb{D}^n)$$

is totally disconnected.

Let $B_{\zeta_j} \in H^\infty(\mathbb{D})$ be the Blaschke product vanishing precisely on ζ_j, and denote by $\hat{B}_{\zeta_j} \in C\big(M(H^\infty(\mathbb{D}))\big)$ its Gelfand transform. Let $p_j\colon [M(H^\infty(\mathbb{D}))]^n \to M(H^\infty(\mathbb{D}))$ be the projection onto the jth coordinate, and define

$$\widetilde{B}_{\zeta_j} := \hat{B}_{\zeta_j} \circ p_j \in C\big([M(H^\infty(\mathbb{D}))]^n\big).$$

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By the product topology, the joint zero set of the functions $\widetilde{B}_{\zeta_j}$, $1 \le j \le n$, coincides with the closure of the interpolating sequence ζ in $[M(H^\infty(\mathbb{D}))]^n$ and hence contains ξ.

By definition, the pullbacks $\pi_n^*\widetilde{B}_{\zeta_j} \in C\big(M(H^\infty(\mathbb{D}^n))\big)$ are the Gelfand transforms of the functions $(p_j|_{\mathbb{D}^n})^*(B_{\zeta_j}) \in H^\infty(\mathbb{D}^n)$, given by

$$(p_j|_{\mathbb{D}^n})^*(B_{\zeta_j})(z_1,\dots,z_n) = B_{\zeta_j}(z_j), \qquad 1 \le j \le n,$$

and they vanish on F_ξ.

We next identify the joint zero set of these pullbacks.

Claim 2 The joint zero set of the functions $\pi_n^*\widetilde{B}_{\zeta_j}$ on $\mathrm{cl}(\mathbb{D}^n) \subset M(H^\infty(\mathbb{D}^n))$, $1 \le j \le n$, is precisely $\mathrm{cl}(\zeta)$.

To prove this, suppose, to the contrary, that there exists a point $\eta \in \mathrm{cl}(\mathbb{D}^n)\backslash\mathrm{cl}(\zeta)$ such that

$$\pi_n^*\widetilde{B}_{\zeta_j}(\eta) = 0 \quad \text{for all } j = 1,\dots,n.$$

Let $U \subset M(H^\infty(\mathbb{D}^n))$ be an open neighborhood of η whose closure is disjoint from $\mathrm{cl}(\zeta)$. For each $\varepsilon \in (0,1)$, consider the open set

$$U_\varepsilon := U \cap \big\{\theta \in M(H^\infty(\mathbb{D}^n)) : |\pi_n^*\widetilde{B}_{\zeta_j}(\theta)| < \varepsilon \text{ for all } 1 \le j \le n\big\}.$$

Define the pseudohyperbolic metric on $\mathbb{D}^n$ by

$$\rho_n(z,w) := \max_{1\le j\le n} \rho(z_j,w_j), \qquad z,w \in \mathbb{D}^n,$$

where ρ is the pseudohyperbolic metric on $\mathbb{D}$ (see (2.5)). Since $\mathbb{D}^n$ is dense in $\mathrm{cl}(\mathbb{D}^n)$, we may select a point

$$z_\varepsilon \in U_\varepsilon \cap \mathbb{D}^n.$$

By a result of Hoffman (see, e.g., [19, Chapter X, Lemma 1.4]), we have

$$\lim_{\varepsilon\to 0} \mathrm{dist}(z_\varepsilon,\zeta) = 0, \tag{5.1}$$

where, for $z \in \mathbb{D}^n$ and $Z \subset \mathbb{D}^n$,

$$\mathrm{dist}(z,Z) := \inf_{y\in Z} \rho_n(z,y).$$

Let $\nu \in \mathrm{cl}(\mathbb{D}^n)$ be a limit point of the sequence $\{z_{1/k}\}_{k\in\mathbb{N}}$. By the Schwarz–Pick lemma (see, e.g., [19, Chapter I, Lemma 1.2]) and (5.1), we obtain $\nu \in \mathrm{cl}(\zeta)$. On the other hand, since $\nu \in \mathrm{cl}(U)$ and $\mathrm{cl}(U) \cap \mathrm{cl}(\zeta) = \varnothing$, a contradiction, proving *Claim 2*.

Claim 2 implies that

$$F_\xi := \pi_n^{-1}(\xi) \cap \mathrm{cl}(\mathbb{D}^n) \subset \big\{\nu \in \mathrm{cl}(\mathbb{D}^n) : \pi_n^*\widetilde{B}_{\zeta_j}(\nu) = 0,\ 1 \le j \le n\big\} \subset \mathrm{cl}(\zeta).$$

Since ζ is an interpolating sequence for $H^\infty(\mathbb{D}^n)$, the restriction of the algebra $\mathcal{O}\big(M(H^\infty(\mathbb{D}^n))\big)$ to $\mathrm{cl}(\zeta)$ coincides with the full algebra $C\big(\mathrm{cl}(\zeta)\big)$. By the Tietze–Urysohn extension theorem, the same property holds for any closed subset of $\mathrm{cl}(\zeta)$, and in particular for the compact set $F_\xi \subset \mathrm{cl}(\zeta)$. Hence, $A(F_\xi) = C(F_\xi)$, and therefore

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the maximal ideal space $M(A(F_\xi))$ coincides with F_ξ. By Theorem 2.2, we conclude that

$$\pi_n^{-1}(\xi) = F_\xi \subset \mathrm{cl}(\zeta).$$

As $\mathrm{cl}(\zeta)$ is totally disconnected, the same is true for F_ξ.

This completes the proof of part (i) of the theorem.

Proof of (ii) If exactly one coordinate of ξ lies in M_s and the remaining $n-1$ coordinates lie in M_a, then the natural action of the symmetric group on coordinates of $\mathbb{D}^n$ preserves both algebras $\widehat{\otimes}_\varepsilon^n H^\infty(\mathbb{D})$ and $H^\infty(\mathbb{D}^n)$. Consequently, it extends to actions on their maximal ideal spaces $\big[M\big(H^\infty(\mathbb{D})\big)\big]^n$ and $M\big(H^\infty(\mathbb{D}^n)\big)$, and these actions commute with the projection

$$\pi_n\colon M\big(H^\infty(\mathbb{D}^n)\big) \to \big[M\big(H^\infty(\mathbb{D})\big)\big]^n.$$

In particular, the fibers of π_n corresponding to points in the same orbit are homeomorphic and have identical analytic structure. Hence, without loss of generality, we may and will assume that

$$\xi \in M_a^{n-1} \times M_s.$$

We first show that

$$\pi_n^{-1}(\xi) = F_\xi := \pi_n^{-1}(\xi) \cap \mathrm{cl}(\mathbb{D}^n)$$

(see Lemma 5.1 below).

Let $\xi = (\xi^{n-1}, \xi_n)$, where $\xi^{n-1} \in M_a^{n-1}$ and $\xi_n \in M_s$. By *Claim 1* in the proof of part (i) of the theorem, there exists an interpolating sequence $\zeta = \{z_k\} \subset \mathbb{D}^{n-1}$ for $H^\infty(\mathbb{D}^{n-1})$, which is the Cartesian product of interpolating sequences for $H^\infty(\mathbb{D})$ in the coordinate disks, such that the closure of ζ in $\big[M(H^\infty(\mathbb{D}))\big]^{n-1}$ contains the point ξ^{n-1}. Moreover, there exist functions $\psi_k \in H^\infty(\mathbb{D}^{n-1})$, $k \in \mathbb{N}$, such that

$$\psi_k(z_i) = \delta_{k,i}, \quad \forall\, k, i \in \mathbb{N}, \qquad \sum_{k=1}^{\infty} |\psi_k(z)| \le C = C(\zeta), \quad \forall\, z \in \mathbb{D}^{n-1}.$$

Then for every uniformly bounded sequence of functions $\{f_k\} \subset H^\infty(\mathbb{D})$, there exists $f \in H^\infty(\mathbb{D}^n)$ such that

$$f(z_k, z) = f_k(z) \quad \forall\, z \in \mathbb{D},\ \forall\, k \in \mathbb{N}.$$

Indeed, set

$$f(w, z) := \sum_{k=1}^{\infty} \psi_k(w)\, f_k(z), \qquad (w, z) \in \mathbb{D}^{n-1} \times \mathbb{D}.$$

Since $\sum_{k=1}^{\infty} |\psi_k(w)| \le C$ for all $w \in \mathbb{D}^{n-1}$ and $\sup_k \|f_k\|_\infty < \infty$, the series defining f converges uniformly on compact subsets of $\mathbb{D}^{n-1} \times \mathbb{D}$, and

$$\|f\|_\infty \le C \sup_k \|f_k\|_\infty.$$

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Uniform convergence on compacta shows that $f \in H^\infty(\mathbb{D}^n)$. Moreover, for each $i \in \mathbb{N}$ and every $z \in \mathbb{D}$,

$$f(z_i, z) = \sum_{k=1}^{\infty} \psi_k(z_i)\, f_k(z) = \sum_{k=1}^{\infty} \delta_{k,i}\, f_k(z) = f_i(z),$$

as required.

Thus, the restriction map

$$R\colon H^\infty(\mathbb{D}^n) \to H^\infty(\mathbb{D} \times \mathbb{N}), \quad R(f)(k) := f\big|_{\{z_k\}\times\mathbb{D}}, \tag{5.2}$$

is a surjective homomorphism of Banach algebras, where $H^\infty(\mathbb{D} \times \mathbb{N})$ is equipped with the norm

$$\|(g_k)_{k\in\mathbb{N}}\| := \sup_{k\in\mathbb{N}} \|g_k\|_\infty.$$

In particular, the induced transpose map on maximal ideal spaces

$$R^*\colon M\big(H^\infty(\mathbb{D} \times \mathbb{N})\big) \longrightarrow M\big(H^\infty(\mathbb{D}^n)\big), \qquad R^*(\varphi) := \varphi \circ R,$$

is a continuous embedding, and its image is H^∞-convex; that is, for every character $\eta \in M\big(H^\infty(\mathbb{D}^n)\big)\backslash R^*\big(M(H^\infty(\mathbb{D} \times \mathbb{N}))\big)$, there exists $f \in H^\infty(\mathbb{D}^n)$ such that

$$|\hat{f}(\eta)| > \sup_{R^*(M(H^\infty(\mathbb{D}\times\mathbb{N})))} |\hat{f}|.$$

Lemma 5.1 *We have*

$$R^*\big(M(H^\infty(\mathbb{D} \times \mathbb{N}))\big) \supset \pi_n^{-1}(\xi).$$

Moreover,

$$\pi_n^{-1}(\xi) = F_\xi.$$

Proof Let us first prove that $\pi_n^{-1}(\xi) \subset \operatorname{im} R^*$. Set

$$I := \ker R = \big\{f \in H^\infty(\mathbb{D}^n) : f(z_k, z) = 0 \ \forall\, k \in \mathbb{N},\ \forall z \in \mathbb{D}\big\}.$$

Let U be an open neighborhood of

$$F_\xi := \pi_n^{-1}(\xi) \cap \operatorname{cl}(\mathbb{D}^n)$$

in $M\big(H^\infty(\mathbb{D}^n)\big)$. As observed above, we may assume, without loss of generality, that

$$U = \pi_n^{-1}(V) \cap \operatorname{cl}(\mathbb{D}^n)$$

for some open neighborhood V of ξ in $\big[M(H^\infty(\mathbb{D}))\big]^n$. Since the closure of $\zeta \times \mathbb{D}$ in $\big[M(H^\infty(\mathbb{D}))\big]^n$ contains ξ, there exists a subsequence $\zeta_U \subset \zeta$ such that $\zeta_U \times \mathbb{D} \subset V$. Varying U over a basis of open neighborhoods of F_ξ, we conclude that

$$F_\xi \subset \operatorname{cl}(\zeta \times \mathbb{D}) \qquad (\text{recall that } \operatorname{cl} = \operatorname{cl}_{M(H^\infty(\mathbb{D}^n))}).$$

Thus, the Gelfand transform of every $f \in I$ vanishes on F_ξ, and hence on its H^∞-hull. By Theorem 2.2, this hull equals $\pi_n^{-1}(\xi)$. Therefore, every $\eta \in \pi_n^{-1}(\xi)$ annihilates

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I and thus factors through

$$H^\infty(\mathbb{D}^n)/I \simeq H^\infty(\mathbb{D}\times\mathbb{N}),$$

that is,

$$\eta = R^*(\varphi) \qquad \text{for some } \varphi \in M\big(H^\infty(\mathbb{D}\times\mathbb{N})\big).$$

This proves $\pi_n^{-1}(\xi) \subset \operatorname{im} R^*$. Using this fact, let us prove that $\pi_n^{-1}(\xi) = F_\xi$.

Indeed, the corona theorem for $H^\infty(\mathbb{D}\times\mathbb{N})$ (which follows from uniform bounds in the proof of Carleson's original theorem, see, e.g., [19]) implies that $\mathbb{D}\times\mathbb{N}$ is an open dense subset of $M(H^\infty(\mathbb{D}\times\mathbb{N}))$. By construction of R^*,

$$R^*(\mathbb{D}\times\mathbb{N}) = \zeta\times\mathbb{D}.$$

Since R^* is continuous and injective, it follows that

$$R^*\big(M(H^\infty(\mathbb{D}\times\mathbb{N}))\big) = \operatorname{cl}\big(\zeta\times\mathbb{D}\big) \subset \operatorname{cl}(\mathbb{D}^n).$$

Hence,

$$\pi_n^{-1}(\xi) = \pi_n^{-1}(\xi)\cap R^*\big(M(H^\infty(\mathbb{D}\times\mathbb{N}))\big) \subset \pi_n^{-1}(\xi)\cap \operatorname{cl}(\mathbb{D}^n) = F_\xi.$$

Thus, $\pi_n^{-1}(\xi) = F_\xi$, completing the proof of the lemma. ∎

By Lemma 5.1 and Theorem 2.2, we conclude that $M(A(F_\xi)) = F_\xi$.

We now use the known structure of $M\big(H^\infty(\mathbb{D}\times\mathbb{N})\big)$ to analyze the internal geometry of F_ξ. As shown in [8], every Gleason part in $M\big(H^\infty(\mathbb{D}\times\mathbb{N})\big)$ is either a singleton or an analytic disk; the set of nontrivial parts is open and dense, the set of trivial parts is totally disconnected, and

$$\dim M\big(H^\infty(\mathbb{D}\times\mathbb{N})\big) = 2.$$

We relate this structure to the fibers of the projection T^* introduced below.

The Banach algebra homomorphism

$$T\colon \ell^\infty \longrightarrow H^\infty(\mathbb{D}\times\mathbb{N}), \qquad T\big((c_k)_{k\in\mathbb{N}}\big) := (g_k)_{k\in\mathbb{N}}, \quad g_k(z)\equiv c_k,\ z\in\mathbb{D}, \tag{5.3}$$

is an isometric embedding. Identifying $M(\ell^\infty)$ with the Stone–Čech compactification $\beta\mathbb{N}$, its transpose

$$T^*\colon M\big(H^\infty(\mathbb{D}\times\mathbb{N})\big) \longrightarrow \beta\mathbb{N}$$

is surjective. For characters corresponding to points $(z,n)\in\mathbb{D}\times\mathbb{N}$, the map T^* assigns evaluation at the index n; in particular,

$$T^*\big(\mathbb{D}\times\mathbb{N}\big) = \mathbb{N}\subset\beta\mathbb{N}.$$

Fix $\chi\in\beta\mathbb{N}\backslash\mathbb{N}$. If $G\subset M\big(H^\infty(\mathbb{D}\times\mathbb{N})\big)$ is an analytic disk with $T^*(G)$ containing χ, then, since T^* is continuous and $\beta\mathbb{N}$ is totally disconnected, the connected set $T^*(G)$ must be a singleton. Hence, $T^*(G) = \{\chi\}$, so $G\subset (T^*)^{-1}(\chi)$. Consequently, for every $\chi\in\beta\mathbb{N}\backslash\mathbb{N}$, the fiber $(T^*)^{-1}(\chi)$ is a disjoint union of analytic disks and points belonging to trivial Gleason parts of $M\big(H^\infty(\mathbb{D}\times\mathbb{N})\big)$ (each such part is a singleton).

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Lemma 5.2 *Let G be an analytic disk in $M\big(H^\infty(\mathbb{D}\times\mathbb{N})\big)$ such that $R^*(G)$ contains a point $\eta\in F_\xi=\pi_n^{-1}(\xi)$, where $\xi=(\xi^{n-1},\xi_n)\in M_a^{n-1}\times M_s\subset\big[M(H^\infty(\mathbb{D}))\big]^n$. Then*

$$R^*(G)\subset F_\xi.$$

Proof Let

$$p^{n-1}\colon\mathbb{D}^n\to\mathbb{D}^{n-1}$$

be the projection onto the first $n-1$ coordinates. It induces the following algebra homomorphisms. First, the pullback

$$(p^{n-1})^*\colon H^\infty(\mathbb{D}^{n-1})\longrightarrow H^\infty(\mathbb{D}^n),\qquad (p^{n-1})^*(f)=f\circ p^{n-1},\tag{5.4}$$

whose transpose is the map

$$\hat{p}^{n-1}\colon M\big(H^\infty(\mathbb{D}^n)\big)\longrightarrow M\big(H^\infty(\mathbb{D}^{n-1})\big),\qquad \hat{p}^{n-1}(\varphi)=\varphi\circ(p^{n-1})^*,$$

and $\hat{p}^{n-1}|_{\mathbb{D}^n}=p^{n-1}$.

Second, the restriction

$$(p^{n-1})^*\big|_{\widehat{\otimes}_\varepsilon^{n-1}H^\infty(\mathbb{D})}:\widehat{\otimes}_\varepsilon^{n-1}H^\infty(\mathbb{D})\longrightarrow\widehat{\otimes}_\varepsilon^{n}H^\infty(\mathbb{D})$$

whose transpose

$$\tilde{p}^{n-1}\colon\big[M\big(H^\infty(\mathbb{D})\big)\big]^n\longrightarrow\big[M\big(H^\infty(\mathbb{D})\big)\big]^{n-1},$$

coincides with the projection onto the first $n-1$ coordinates.

The embeddings $\widehat{\otimes}_\varepsilon^k H^\infty(\mathbb{D})\hookrightarrow H^\infty(\mathbb{D}^k)$ are compatible with the pullback $(p^{n-1})^*$. Passing to transposes (see (2.2)) yields

$$\tilde{p}^{n-1}\circ\pi_n\;=\;\pi_{n-1}\circ\hat{p}^{n-1}.\tag{5.5}$$

Let $\zeta=\{z_k\}\subset\mathbb{D}^{n-1}$ be the interpolating sequence fixed at the beginning of the proof, whose closure in $\big[M(H^\infty(\mathbb{D}))\big]^{n-1}$ contains ξ^{n-1}. Define the restriction homomorphism

$$R_\zeta\colon H^\infty(\mathbb{D}^{n-1})\longrightarrow\ell^\infty(\mathbb{N}),\qquad R_\zeta(f)=\big(f(z_k)\big)_{k\in\mathbb{N}}.$$

By the definitions of R, T, and $(p^{n-1})^*$, see (5.2)–(5.4), we have the commutative relation

$$T\circ R_\zeta\;=\;R\circ(p^{n-1})^*.$$

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Passing to transposes and combining with (5.5) gives the commutative diagram:

$$
\begin{array}{ccc}
M\big(H^\infty(\mathbb{D}\times\mathbb{N})\big) & \xrightarrow{\quad T^*\quad} & \beta\mathbb{N} \\
\Big\downarrow{\scriptstyle R^*} & & \Big\downarrow{\scriptstyle (R_\zeta)^*} \\
M\big(H^\infty(\mathbb{D}^n)\big) & \xrightarrow[\quad \hat{p}^{\,n-1}\quad]{} & M\big(H^\infty(\mathbb{D}^{\,n-1})\big) \\
\Big\downarrow{\scriptstyle \pi_n} & & \Big\downarrow{\scriptstyle \pi_{n-1}} \\
\big[M\big(H^\infty(\mathbb{D})\big)\big]^n & \xrightarrow[\quad \tilde{p}^{\,n-1}\quad]{} & \big[M\big(H^\infty(\mathbb{D})\big)\big]^{n-1}.
\end{array}
\tag{5.6}
$$

Since $R^*(G)\subset M\big(H^\infty(\mathbb{D}^n)\big)$ meets the fiber $F_\xi=\pi_n^{-1}(\xi)$ and T^* is constant on G, the diagram (5.6) gives, for $\eta\in R^*(G)\cap F_\xi$,

$$
(\pi_{n-1}\circ\hat{p}^{\,n-1})(\eta)=\xi^{n-1}. \tag{5.7}
$$

Let $T^*(G)=\{\omega\}\subset\beta\mathbb{N}$. Then the top square in (5.6) yields

$$
\hat{p}^{\,n-1}\big(R^*(G)\big)=\{(R_\zeta)^*(\omega)\}.
$$

Applying π_{n-1} and using (5.7), we obtain

$$
(\pi_{n-1}\circ\hat{p}^{\,n-1})\big(R^*(G)\big)=\{\xi^{n-1}\}.
$$

For the last coordinate, let

$$
p_n:\big[M(H^\infty(\mathbb{D}))\big]^n\longrightarrow M(H^\infty(\mathbb{D}))
$$

be the projection onto the nth factor. Then $(p_n\circ\pi_n)\big(R^*(G)\big)$ is an analytic disk in $M(H^\infty(\mathbb{D}))$ passing through ξ_n. Since $\xi_n\in M_s$ is a trivial Gleason part, Hoffman's results [25, Theorems 4.3, 5.5, and 5.6] imply that this image is a singleton:

$$
(p_n\circ\pi_n)\big(R^*(G)\big)=\{\xi_n\}.
$$

Combining the two displays gives

$$
\pi_n\big(R^*(G)\big)=\{(\xi^{n-1},\xi_n)\}=\{\xi\},
$$

hence $R^*(G)\subset\pi_n^{-1}(\xi)=F_\xi$, as claimed. ∎

Combining Lemmas 5.1 and 5.2 with the structure of $M\big(H^\infty(\mathbb{D}\times\mathbb{N})\big)$ established in [8], we obtain that F_ξ is a disjoint union of its Gleason parts, each of which is either a single point or an analytic disk. Moreover, since F_ξ embeds into $M\big(H^\infty(\mathbb{D}\times\mathbb{N})\big)$, its covering dimension satisfies

$$
\dim F_\xi\le 2.
$$

To complete the proof of part (ii), it remains to show that the subset of trivial Gleason parts in F_ξ is nonempty. Equivalently, if the fiber F_ξ contains a nontrivial

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Gleason part (an analytic disk), then it must also contain at least one trivial Gleason part.

Lemma 5.3 *The closure in $M\big(H^\infty(\mathbb{D}\times\mathbb{N})\big)$ of any nontrivial Gleason part of $H^\infty(\mathbb{D}\times\mathbb{N})$ contains a trivial Gleason part of this algebra.*

Proof It was shown in [4, Lemma 2] that the algebra $H^\infty(\mathbb{D}\times\mathbb{N})$ is logmodular on $\mathbb{D}\times\mathbb{N}$. This, together with the general result of Hoffman [24], implies that the Gleason parts of $H^\infty(\mathbb{D}\times\mathbb{N})$ are either points or analytic disks. The latter means that there exists a continuous injective map

$$\pi : \mathbb{D} \longrightarrow M\big(H^\infty(\mathbb{D}\times\mathbb{N})\big)$$

onto this Gleason part such that $\hat{f}\circ\pi \in H^\infty(\mathbb{D})$ for all $f \in H^\infty(\mathbb{D}\times\mathbb{N})$.

Suppose, to the contrary, that there exists a nontrivial Gleason part $G \subset M\big(H^\infty(\mathbb{D}\times\mathbb{N})\big)$ whose closure $\overline{G}$ consists entirely of nontrivial Gleason parts, so that $\overline{G}$ is a disjoint union of analytic disks.

For $f \in H^\infty(\mathbb{D}\times\mathbb{N})$, let $\mathcal{C}_{\hat{f}}$ denote the set of all analytic disks in $\overline{G}$ on which the Gelfand transform $\hat{f}$ is constant.

Claim Each $\mathcal{C}_{\hat{f}}$ is a nonempty compact subset of $\overline{G}$.

Indeed, the function $|\hat{f}|$, being continuous on the compact set $\overline{G}$, attains its maximum at some point η. Let G_η be the nontrivial Gleason part containing η, and let $\pi : \mathbb{D} \to G_\eta$ be its analytic parametrization (as in the above definition). Then the function $|\hat{f}\circ\pi|$ is subharmonic on $\mathbb{D}$ and attains its maximum at the interior point $\pi^{-1}(\eta)$. By the maximum modulus principle, it must be constant, and hence $\hat{f}\circ\pi$, and therefore $\hat{f}$, is constant on G_η. Thus, $G_\eta \subset \mathcal{C}_{\hat{f}}$, showing that $\mathcal{C}_{\hat{f}}$ is nonempty.

Next, we show that $\mathcal{C}_{\hat{f}}$ is compact. Let θ be a limit point of $\mathcal{C}_{\hat{f}}$. By the hypothesis on $\overline{G}$, the point θ belongs to some maximal analytic disk, that is, to the Gleason part of θ. By [8, Theorems 2.5(b) and (d)], there exists an open neighborhood U of θ in the set of nontrivial Gleason parts of $H^\infty(\mathbb{D}\times\mathbb{N})$, and a homeomorphism

$$h : S\times\mathbb{D} \longrightarrow U,$$

where S is a compact subset of the Stone–Čech compactification $\beta\mathbb{N}$, such that each slice $h(\{\omega\}\times\mathbb{D})$ lies in a nontrivial Gleason part of $H^\infty(\mathbb{D}\times\mathbb{N})$, and for every $f \in H^\infty(\mathbb{D}\times\mathbb{N})$, the pullback $\hat{f}\circ h(\omega,\cdot)$ belongs to $H^\infty(\mathbb{D})$.

Since θ is a limit point of $\mathcal{C}_{\hat{f}}$, there exists a net $(\theta_\alpha) \subset \mathcal{C}_{\hat{f}} \cap U$ converging to θ. If θ_α lies in the image under h of the slice corresponding to ω_α, then $\hat{f}$ is constant on this image. Passing to the limit within U shows that $\hat{f}$ is constant on the image under h of the slice containing θ. By analyticity, $\hat{f}$ is constant on the entire Gleason part containing θ; hence, $\theta \in \mathcal{C}_{\hat{f}}$.

Thus, $\mathcal{C}_{\hat{f}}$ is closed in $\overline{G}$, and compactness follows since $\overline{G}$ is compact.

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Further, given any finite collection $f_1, \dots, f_k \in H^\infty(\mathbb{D} \times \mathbb{N})$, consider the continuous nonnegative function

$$\widehat{F} := \sum_{j=1}^{k} |\widehat{f_j}|^2$$

on $\overline{G}$. It attains its maximum at some point η. Let G_η be the nontrivial Gleason part containing η, and let $\pi : \mathbb{D} \to G_\eta$ be its analytic parametrization. Then

$$\widehat{F} \circ \pi = \sum_{j=1}^{k} |\widehat{f_j} \circ \pi|^2$$

is a continuous nonnegative subharmonic function on $\mathbb{D}$ and attains its maximum at the interior point $\pi^{-1}(\eta)$. By the maximum principle for subharmonic functions, it is constant. Hence,

$$0 = \Delta(\widehat{F} \circ \pi) = 4 \sum_{j=1}^{k} \left|(\widehat{f_j} \circ \pi)'\right|^2,$$

so each $\widehat{f_j} \circ \pi$ is constant. Thus, each $\widehat{f_j}$ is constant on G_η, and consequently

$$\mathcal{C}_{\hat{f}_1} \cap \cdots \cap \mathcal{C}_{\hat{f}_k} \neq \varnothing.$$

Since each $\mathcal{C}_{\hat{f}}$ is compact and all finite intersections are nonempty, the finite intersection property implies that

$$\bigcap_{f \in H^\infty(\mathbb{D} \times \mathbb{N})} \mathcal{C}_{\hat{f}} \neq \varnothing.$$

Hence, there exists an analytic disk in $\overline{G}$ on which $\hat{f}$ is constant for every $f \in H^\infty(\mathbb{D} \times \mathbb{N})$. This yields a contradiction, since the Gelfand transforms of functions from the algebra $H^\infty(\mathbb{D} \times \mathbb{N})$ separate points of its maximal ideal space. Therefore, $\overline{G}$ cannot consist solely of analytic disks, that is, it must contain a trivial Gleason part, as claimed. ∎

Since F_ξ is contained in the homeomorphic image of $M\big(H^\infty(\mathbb{D} \times \mathbb{N})\big)$ under R^*, which preserves the analytic structure, the subset of trivial Gleason parts in F_ξ is nonempty. This completes the proof of part (ii) of Theorem 2.3. ∎

Remark 5.4 It is not difficult to construct points ξ as in part (ii) for which the corresponding fibers F_ξ contain analytic disks. Whether this holds for all such ξ remains an open question. We conjecture that every fiber of this type contains analytic disks.

6 Proof of Theorem 2.5

We begin with the following auxiliary result.

Lemma 6.1 *Let $H = (H_1, \dots, H_m) \in [H^\infty(\mathbb{D}^n)]^m$. The following are equivalent:*

(1) *For each $x \in \mathbb{D}^n$, there exist a neighborhood $U_x \subset \mathbb{D}^n$ of x and a one-dimensional complex analytic subset $\Sigma_x \subset \mathbb{C}^m$ such that $H(U_x) \subset \Sigma_x$.*

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(2) *There exist holomorphic maps*

$$h : \mathbb{D}^n \to \mathbb{D}, \qquad \widetilde{H} : \mathbb{D} \to \mathbb{C}^m$$

such that $H = \widetilde{H} \circ h$.

Proof We assume that H is nonconstant; otherwise, both statements are trivial.

(1) $\Rightarrow$ (2). Choose radii $0 < r_1 < r_2 < \cdots \nearrow 1$ and consider the exhaustion of $\mathbb{D}^n$ by open polydisks $\mathbb{D}^n_{r_j}$ of radii r_j. By compactness of the closure $\overline{\mathbb{D}}^n_{r_j}$, there exist $x_1, \ldots, x_N$ with

$$\overline{\mathbb{D}}^n_{r_j} \subset U_{x_1} \cup \cdots \cup U_{x_N},$$

hence

$$H(\mathbb{D}^n_{r_j}) \subset \Sigma_j := \Sigma_{x_1} \cup \cdots \cup \Sigma_{x_N},$$

a one-dimensional analytic set.

Let $\nu_j : \Sigma_j^\nu \to \Sigma_j$ be the normalization [21, Chapter 8]. By its universal property, the holomorphic map $H|_{\mathbb{D}^n_{r_j}}$ lifts to a holomorphic map

$$H_j^\nu : \mathbb{D}^n_{r_j} \longrightarrow \Sigma_j^\nu, \qquad H = \nu_j \circ H_j^\nu.$$

Since H_j^ν is nonconstant and $\mathbb{D}^n_{r_j}$ is connected, the open mapping theorem implies that

$$R_j := H_j^\nu(\mathbb{D}^n_{r_j})$$

is a connected open Riemann surface.

Set

$$h_j := H_j^\nu : \mathbb{D}^n_{r_j} \to R_j, \qquad \widetilde{H}_j := \nu_j|_{R_j} : R_j \to \mathbb{C}^m,$$

so $H|_{\mathbb{D}^n_{r_j}} = \widetilde{H}_j \circ h_j$. If $r_i < r_j$, uniqueness of normalization factorizations yields a unique holomorphic map $\varphi_{ij} : R_i \to R_j$ such that $h_j = \varphi_{ij} \circ h_i$ on $\mathbb{D}^n_{r_i}$. Thus, $\{R_j, \varphi_{ij}\}$ forms a direct system of Riemann surfaces. ∎

6.1 Formation of the limit surface

Define the inductive limit

$$R := \varinjlim R_j.$$

Each φ_{ij} is a local biholomorphism on the relevant component of R_i, so the complex structures are compatible under pullback. Hence, R becomes a Hausdorff one-dimensional complex manifold. Since each R_j is connected and the overlaps $\varphi_{ij}(R_i) \cap R_j$ are nonempty open subsets, the union $R = \bigcup_j R_j$ is connected, and hence R is a connected Riemann surface.

Since the factorizations are compatible, the h_j glue to a holomorphic map

$$h_0 : \mathbb{D}^n \to R,$$

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and the $\widetilde{H}_j$ glue to a holomorphic map

$$\widetilde{H}_0 : R \to \mathbb{C}^m,$$

yielding

$$H = \widetilde{H}_0 \circ h_0.$$

6.2 Uniformization

As $\widetilde{H}_0$ is bounded and nonconstant, R admits a nonconstant bounded holomorphic function, hence R is hyperbolic. Thus, its universal cover is $\mathbb{D}$. Let $p : \mathbb{D} \to R$ be the covering map. Since $\mathbb{D}^n$ is simply connected, the covering homotopy theorem implies the existence of a holomorphic map

$$h : \mathbb{D}^n \to \mathbb{D}, \qquad h_0 = p \circ h,$$

and setting $\widetilde{H} := \widetilde{H}_0 \circ p$ gives $H = \widetilde{H} \circ h$.

(2) $\Rightarrow$ (1). Assume $H = \widetilde{H} \circ h$ with $h : \mathbb{D}^n \to \mathbb{D}$ and $\widetilde{H} : \mathbb{D} \to \mathbb{C}^m$ nonconstant holomorphic maps. Fix $x \in \mathbb{D}^n$, set $\zeta_0 := h(x)$, $y_0 := H(x)$.

Since h is nonconstant, the open mapping theorem implies that $h(\mathbb{D}^n)$ is an open subset of $\mathbb{D}$. Hence, we can choose an open disk $W_0 \subset h(\mathbb{D}^n)$ centered at ζ_0 and set

$$W := W_0 - \zeta_0 = \{\zeta - \zeta_0 : \zeta \in W_0\},$$

so that W is an open disk centered at 0.

Define

$$\varphi(\zeta) := \widetilde{H}(\zeta + \zeta_0) - y_0, \qquad \zeta \in W,$$

so that $\varphi(0) = 0$. Write $\varphi = (\varphi_1, \dots, \varphi_m)$. Because $H = \widetilde{H} \circ h$ is nonconstant, $\widetilde{H}$ is nonconstant. Moreover, $h(\mathbb{D}^n)$ is open. If φ were constant on W, then $\widetilde{H}$ would be constant on the open disk W_0, and hence on $\mathbb{D}$ by the identity theorem, a contradiction. Thus, at least one component φ_j is not identically zero near 0. Set

$$k := \min_{1 \le j \le m} \operatorname{ord}_0 \varphi_j < \infty.$$

By permuting the coordinates in $\mathbb{C}^m$, if necessary, and precomposing φ with a biholomorphic change of variable on a smaller disk $W' \subset W$ fixing 0 (and then renaming W' by W), we may assume, without loss of generality, that

$$\varphi_1(\zeta) = \zeta^k, \qquad \varphi_j(\zeta) = \zeta^k \psi_j(\zeta), \quad j \ge 2, \ \ \zeta \in W,$$

with each ψ_j holomorphic near 0. Thus,

$$\varphi(\zeta) = \big(\zeta^k, \zeta^k \psi_2(\zeta), \dots, \zeta^k \psi_m(\zeta)\big), \qquad \zeta \in W.$$

Consider the projection onto the first coordinate,

$$\pi_1 : \mathbb{C}^m \to \mathbb{C}, \qquad \pi_1(z_1, \dots, z_m) = z_1.$$

Then, for all $\zeta \in W$,

$$\pi_1\big(\varphi(\zeta)\big) = \zeta^k.$$

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Hence, for some $r > 0$,

$$\pi_1 \circ \varphi : W\backslash\{0\} \longrightarrow \{w \in \mathbb{C} : 0 < |w| < r\}$$

is a finite unbranched covering of degree k. Consequently,

$$\widetilde{H}(W_0) = \varphi(W) + y_0$$

is a one-dimensional complex analytic subset of a neighborhood of $y_0 \in \mathbb{C}^m$.

Finally, let $U_x := h^{-1}(W_0)$. Then U_x is a neighborhood of x and

$$H(U_x) = \widetilde{H}(h(U_x)) = \widetilde{H}(W_0),$$

establishing (1). ∎

Proof of Theorem 2.5 We must show that whenever $g_1, \dots, g_k \in A(S_n; H)$, with H as in the hypotheses of the theorem, satisfy the corona condition

$$\inf_{z\in\mathbb{D}^n} \max_{1\le i\le k} |g_i(z)| \ge \delta > 0,$$

then for every $h_1, \dots, h_k \in H^\infty(\mathbb{D}^n)$ with $\max_i \|h_i\|_\infty < \delta$, the ideal generated by

$$F_i := g_i + h_i, \qquad i = 1, \dots, k,$$

is the whole algebra $H^\infty(\mathbb{D}^n)$.

Suppose, to the contrary, that the ideal generated by $F_1, \dots, F_k$ is proper. Then there exists a maximal ideal $\chi \in M\big(H^\infty(\mathbb{D}^n)\big)$ containing it; hence,

$$\widehat{F}_i(\chi) = 0, \qquad i = 1, \dots, k.$$

Let

$$\pi_n : M\big(H^\infty(\mathbb{D}^n)\big) \longrightarrow \big[M\big(H^\infty(\mathbb{D})\big)\big]^n$$

be the canonical projection introduced in (2.2), and set $\xi := \pi_n(\chi)$. The associated fiber is

$$F_\xi := \pi_n^{-1}(\xi) \cap \mathrm{cl}(\mathbb{D}^n).$$

Let $A(F_\xi)$ denote the uniform closure in $C(F_\xi)$ of the restrictions of $H^\infty(\mathbb{D}^n)$ to F_ξ. By the first assertion of Theorem 2.2, we have

$$\pi_n^{-1}(\xi) = M\big(A(F_\xi)\big),$$

so, in particular, $\chi \in M\big(A(F_\xi)\big)$. We therefore regard χ as a character of $A(F_\xi)$ and continue to denote it by χ.

By Lemma 6.1, there exist holomorphic maps $h : \mathbb{D}^n \to \mathbb{D}$ and $\widetilde{H} : \mathbb{D} \to \mathbb{C}^m$ such that $H = \widetilde{H} \circ h$. Since H is bounded on $\mathbb{D}^n$, it follows that $\widetilde{H}$ is bounded on $\mathbb{D}$.

The map h induces a homomorphism

$$h^* : H^\infty(\mathbb{D}) \longrightarrow H^\infty(\mathbb{D}^n), \qquad \varphi \longmapsto \varphi \circ h,$$

whose transpose determines a continuous map of maximal ideal spaces

$$\hat{h}^* : M\big(H^\infty(\mathbb{D}^n)\big) \longrightarrow M\big(H^\infty(\mathbb{D})\big).$$

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Set

$$K := \hat{h}^*(F_\xi) \subset M\big(H^\infty(\mathbb{D})\big).$$

Since the Gelfand transform of every function in $S_n(H^\infty(\mathbb{D}))$ is constant on the fibers of π_n, the restriction of each $\hat{g}_i$ to F_ξ is the uniform limit in $C(F_\xi)$ of the restrictions to F_ξ of Gelfand transforms of a sequence of holomorphic polynomials in the coordinate functions of H, that is, polynomials in $H_1, \dots, H_m$. Using the factorization $H = \widetilde{H} \circ h$, each such polynomial in $H_1, \dots, H_m$ can be written as a polynomial in the coordinate functions of $\widetilde{H}$ composed with h. Hence, its Gelfand transform descends, via $\hat{h}^*$, to the Gelfand transform of a holomorphic polynomial on $\mathbb{D}$ in the coordinates of $\widetilde{H}$.

Therefore, the restriction of each $\hat{g}_i$ to F_ξ factors through $\hat{h}^*$; that is, there exist continuous functions $\widetilde{g}_i \in C(K)$ such that

$$\hat{g}_i = \widetilde{g}_i \circ \hat{h}^* \quad \text{on } F_\xi, \quad i = 1, \dots, k.$$

Moreover, each $\widetilde{g}_i$ is the uniform limit on K of Gelfand transforms of functions from $H^\infty(\mathbb{D})$.

Let $\mathcal{A}(K)$ denote the uniform closure on K of the algebra of restrictions of continuous functions defined on open neighborhoods U of K whose restrictions to $U \cap \mathbb{D}$ are holomorphic. By results of Suárez ([40, Theorem 2.4 and Lemma 1.1] together with [38, Theorem 2.4]), the maximal ideal space of $\mathcal{A}(K)$ is canonically identified with K.

The restriction of $\hat{h}^*$ to F_ξ induces a pullback homomorphism

$$\Phi : \mathcal{A}(K) \longrightarrow A(F_\xi), \qquad f \longmapsto f \circ \hat{h}^*\big|_{F_\xi}.$$

Indeed, if U is an open neighborhood of K and f is continuous on U with $f|_{U\cap\mathbb{D}}$ holomorphic, then $(\hat{h}^*)^{-1}(U)$ is an open neighborhood of F_ξ in $M\big(H^\infty(\mathbb{D}^n)\big)$, and $f \circ \hat{h}^*$ is continuous on this neighborhood and holomorphic on its intersection with $\mathbb{D}^n$ (since $\hat{h}^*(z) = h(z)$ for $z \in \mathbb{D}^n$). Hence, the restriction of $f \circ \hat{h}^*$ to F_ξ lies in $A(F_\xi)$ (cf. the definition in Section 2 after (2.3)). Since $\mathcal{A}(K)$ is the uniform closure of such functions f on K, it follows that $\Phi(\mathcal{A}(K)) \subset A(F_\xi)$.

The transpose of Φ,

$$\Phi^* : M(A(F_\xi)) \longrightarrow M(\mathcal{A}(K)) \cong K,$$

is therefore well defined and continuous. Moreover, Φ^* coincides with the restriction of $\hat{h}^*$ to $M(A(F_\xi)) = \pi_n^{-1}(\xi)$. Indeed, for any $f \in H^\infty(\mathbb{D})$,

$$\Phi(\hat{f}|_K) = \hat{f} \circ \hat{h}^*\big|_{\pi_n^{-1}(\xi)} = \widehat{f \circ h}\big|_{\pi_n^{-1}(\xi)} = \widehat{h^*(f)}\big|_{\pi_n^{-1}(\xi)},$$

and evaluation at $\chi \in M(A(F_\xi))$ gives

$$\Phi^*(\chi)(\hat{f}|_K) = \chi\big(\widehat{h^*(f)}\big) = \hat{f}\big(\hat{h}^*(\chi)\big),$$

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which means precisely that $\Phi^*(\chi) = \hat{h}^*(\chi)$. Thus, we may identify Φ^* with the restriction of $\hat{h}^*$, and we continue to denote it by

$$\hat{h}^* : M(A(F_\xi)) \longrightarrow K.$$

In particular, for our chosen $\chi \in M(A(F_\xi))$,

$$\eta := \hat{h}^*(\chi) \in K, \qquad \hat{g}_i(\chi) = \widetilde{g}_i(\eta), \quad i = 1, \dots, k.$$

The corona condition now gives

$$\max_{1\le i\le k} |\widetilde{g}_i(\eta)| = \max_{1\le i\le k} |\hat{g}_i(\chi)| \ge \delta.$$

Hence, for some index i_0,

$$|\hat{g}_{i_0}(\chi)| \ge \delta.$$

On the other hand,

$$|\hat{h}_{i_0}(\chi)| \le \|h_{i_0}\|_\infty < \delta.$$

Therefore,

$$|\hat{F}_{i_0}(\chi)| = |\hat{g}_{i_0}(\chi) + \hat{h}_{i_0}(\chi)| \ge |\hat{g}_{i_0}(\chi)| - |\hat{h}_{i_0}(\chi)| > \delta - \delta = 0,$$

which contradicts $\hat{F}_{i_0}(\chi) = 0$.

Thus, no maximal ideal of $H^\infty(\mathbb{D}^n)$ can contain the ideal generated by $F_1, \dots, F_k$, and hence this ideal is all of $H^\infty(\mathbb{D}^n)$. The Bézout equation for $F_1, \dots, F_k$ therefore has a solution in $H^\infty(\mathbb{D}^n)$, completing the proof.

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