A series expansion for standard normal probabilities

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The goal of this paper is to derive a series expansion for standard normal probabilities $F(b)$, where $b \geqslant 0$ and the upper half of the cumulative distribution function (or CDF) of the standard normal is given by

$$F(b) = \frac{1}{\sqrt{2\pi}} \int_0^b e^{-\frac{1}{2}z^2} dz. \tag{1}$$

See [1, Chapter 4] for information and results about the normal distribution. It is known that this integral is not elementary, and the usual approach is to use the Taylor series expansion for e^z in order to evaluate the integral. We know that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

and therefore that

$$e^{-\frac{1}{2}z^2} = \sum_{n=0}^{\infty} \frac{(-\frac{1}{2}z^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{2^n n!}. \tag{2}$$

Using (2) in (1) and integrating the Taylor series term by term, we obtain:

$$F(b) = \frac{1}{\sqrt{2\pi}} \int_0^b \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{2^n n!} dz = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \int_0^b (-1)^n \frac{z^{2n}}{2^n n!} dz$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} (-1)^n \left[\frac{z^{2n+1}}{(2n+1)2^n n!} \right]_0^b = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{b^{2n+1}}{(2n+1)2^n n!}, \tag{3}$$

an alternating series which converges for all values of b. See also the Wikipedia page [2] for more information on the normal distribution. From (3) we derive a different (but equivalent) series for $F(b)$, and the starting point for our investigations is to consider the following function of t, where $t \geqslant 0$:

$$f(t) = \int_0^1 e^{-\frac{1}{2}tx^2} dx. \tag{4}$$

We now integrate (4), using integration by parts repeatedly, as follows:

$$f(t) = \int_0^1 e^{-\frac{1}{2}tx^2} dx = \int_0^1 e^{-\frac{1}{2}tx^2} \cdot 1 \, dx = \left[x e^{-\frac{1}{2}tx^2} \right]_0^1 - \int_0^1 x e^{-\frac{1}{2}tx^2} (-tx) \, dx$$

$$= e^{-\frac{1}{2}t} + t \int_0^1 e^{-\frac{1}{2}tx^2} x^2 dx = e^{-\frac{1}{2}t} + t \left(\left[\frac{1}{3} x^3 e^{-\frac{1}{2}tx^2} \right]_0^1 - \int_0^1 \frac{1}{3} x^3 e^{-\frac{1}{2}tx^2} (-tx) \, dx \right)$$

$$= e^{-\frac{1}{2}t} + e^{-\frac{1}{2}t} \frac{t}{3} + t^2 \int_0^1 e^{-\frac{1}{2}tx^2} \cdot \frac{1}{3} x^4 dx$$

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$$= e^{-\frac{1}{2}t} + e^{-\frac{1}{2}t}\frac{t}{3} + t^2\left(\left[\frac{1}{3}\cdot\frac{1}{5}x^5e^{-\frac{1}{2}tx^2}\right]_0^1 - \int_0^1 \frac{1}{3}\cdot\frac{1}{5}x^5e^{-\frac{1}{2}tx^2}(-tx)\,dx\right)$$

$$= e^{-\frac{1}{2}t} + e^{-\frac{1}{2}t}\frac{t}{3} + e^{-\frac{1}{2}t}\frac{t^2}{3\cdot5} + t^3\int_0^1 e^{-\frac{1}{2}tx^2}\cdot\frac{1}{3}\cdot\frac{1}{5}x^6dx$$

$$= e^{-\frac{1}{2}t} + e^{-\frac{1}{2}t}\frac{t}{3} + e^{-\frac{1}{2}t}\frac{t^2}{3\cdot5} + t^3\left(\left[\frac{1}{3\cdot5\cdot7}x^7e^{-\frac{1}{2}tx^2}\right]_0^1 - \int_0^1 \frac{1}{3\cdot5\cdot7}x^7e^{-\frac{1}{2}tx^2}(-tx)\,dx\right)$$

$$= e^{-\frac{1}{2}t} + e^{-\frac{1}{2}t}\frac{t}{3} + e^{-\frac{1}{2}t}\frac{t^2}{3\cdot5} + e^{-\frac{1}{2}t}\frac{t^2}{3\cdot5\cdot7} + t^4\int_0^1 e^{-\frac{1}{2}tx^2}\cdot\frac{1}{3\cdot5\cdot7}x^8dx. \tag{5}$$

We now adopt the double factorial notation, where $(2n-1)!!$ denotes the product of the first n odd natural numbers. Thus

$$(2n-1)!! = (2n-1)(2n-3)(2n-5)\cdots5\cdot3\cdot1.$$

Continuing from (5), we see that the repeated application of integration by parts results in the following formula:

$$\int_0^1 e^{-\frac{1}{2}tx^2}dx$$

$$= e^{-\frac{1}{2}t}\left(1 + \frac{t}{3} + \frac{t^2}{3\cdot5} + \frac{t^3}{3\cdot5\cdot7} + \cdots + \frac{t^{n-1}}{(2n-1)!!}\right) + t^n\int_0^1 e^{-\frac{1}{2}tx^2}\frac{x^{2n}}{(2n-1)!!}dx. \tag{6}$$

Notice that an upper bound for the size of the error term in (6) may be found very easily. Thus for any $t > 0$ we have

$$\left|t^n\int_0^1 e^{-\frac{1}{2}tx^2}\frac{x^{2n}}{(2n-1)!!}\right| \leqslant t^n\int_0^1\left|e^{-\frac{1}{2}tx^2}\right|\frac{x^{2n}}{(2n-1)!!}dx$$

$$\leqslant t^n\int_0^1 1\frac{x^{2n}}{(2n-1)!!}dx = \left[t^n\frac{x^{2n+1}}{(2n+1)!!}\right]_0^1 = \frac{t^n}{(2n+1)!!}. \tag{7}$$

And notice in (7) that

$$\lim_{n\to\infty}\frac{t^n}{(2n+1)!!} = 0$$

for any $t > 0$. Thus we conclude from (6) and (7) that

$$\int_0^1 e^{-\frac{1}{2}tx^2}dx = e^{-\frac{1}{2}t}\sum_{n=0}^{\infty}\frac{t^n}{(2n+1)!!}. \tag{8}$$

Now recall that

$$f(t) = \int_0^1 e^{-\frac{1}{2}tx^2}dx. \tag{9}$$

We now make the following change of variables in (9). Let

$$x = \frac{1}{\sqrt{t}}z \Rightarrow dx = \frac{1}{\sqrt{t}}dz.$$

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Then (9) becomes

$$f(t) = \int_0^{\sqrt{t}} e^{-\frac{1}{2}z^2} \cdot \frac{1}{\sqrt{t}}\, dz = \frac{1}{\sqrt{t}} \int_0^{\sqrt{t}} e^{-\frac{1}{2}z^2} dz. \tag{10}$$

Substituting (10) into (8), we obtain:

$$\frac{1}{\sqrt{t}} \int_0^{\sqrt{t}} e^{-\frac{1}{2}z^2} dz = e^{-\frac{1}{2}t} \sum_{n=0}^{\infty} \frac{t^n}{(2n+1)!!}$$

$$\int_0^{\sqrt{t}} e^{-\frac{1}{2}z^2} dz = \sqrt{t} \cdot e^{-\frac{1}{2}t} \cdot \sum_{n=0}^{\infty} \frac{t^n}{(2n+1)!!}. \tag{11}$$

We make the substitution $\sqrt{t} = b \Rightarrow t = b^2$ in (11) to obtain

$$\int_0^{b} e^{-\frac{1}{2}z^2} dz = b \cdot e^{-\frac{1}{2}b^2} \cdot \sum_{n=0}^{\infty} \frac{b^{2n}}{(2n+1)!!}. \tag{12}$$

Finally, multiplying through (12) by the constant $\frac{1}{\sqrt{2\pi}}$, we obtain the following series expansion for standard normal probabilities, for $b \geqslant 0$:

$$F(b) = \frac{1}{\sqrt{2\pi}} \int_0^{b} e^{-\frac{1}{2}z^2} dz = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}b^2} \sum_{n=0}^{\infty} \frac{b^{2n+1}}{(2n+1)!!}. \tag{13}$$

Clearly (13), which is not alternating, is a rapidly converging series for small values of b, but for very large values of b the series

$$\sum_{n=0}^{\infty} \frac{b^{2n+1}}{(2n+1)!!}$$

will start to dominate. As $b \to \infty$ in (13), it is clear that the left-hand side of (13) approaches $\frac{1}{2}$ since the standard normal distribution integrates to 1 and is symmetrical. Thus the right-hand side of (13) must also approach $\frac{1}{2}$ as $b \to \infty$. Thus we must have

$$\lim_{b \to \infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}b^2} \sum_{n=0}^{\infty} \frac{b^{2n+1}}{(2n+1)!!} = \frac{1}{2}. \tag{14}$$

It is possible to establish (14) by using the properties of Gauss's error function, written as $erf(x)$. The Gauss error function is defined by the equation:

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-t^2} dt. \tag{15}$$

By making the change of variable $t^2 = \frac{1}{2}z^2$ in (15) one may show the following two facts about $erf(x)$:

$$erf(\infty) = 1$$

and

$$\frac{1}{2} erf\left(\frac{b}{\sqrt{2}}\right) = \frac{1}{\sqrt{2\pi}} \int_0^{b} e^{-\frac{1}{2}z^2} dz = F(b).$$

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From (13) and the properties of $erf(x)$, we see that

$$\lim_{b\to\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}b^2}\sum_{n=0}^{\infty}\frac{b^{2n+1}}{(2n+1)!!}=\lim_{b\to\infty}F(b)=\lim_{b\to\infty}\frac{1}{2}erf\left(\frac{b}{\sqrt{2}}\right)=\frac{1}{2}erf(\infty)=\frac{1}{2}\cdot 1=\frac{1}{2}.$$

This establishes (14). For small values of b, one may use (13) as an alternative to (3) for numerical work with standard normal probabilities.

We conclude the paper with the following observation. From (3) and (13) we have established that

$$\frac{1}{\sqrt{2\pi}}\int_0^b e^{-\frac{1}{2}z^2}dz=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}b^2}\sum_{n=0}^{\infty}\frac{b^{2n+1}}{(2n+1)!!}=\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{b^{2n+1}}{(2n+1)2^n\cdot n!}. \quad (16)$$

From (2) we write the Taylor series expansion for $e^{-\frac{1}{2}b^2}$ as

$$e^{-\frac{1}{2}b^2}=\sum_{n=0}^{\infty}(-1)^n\frac{b^{2n}}{2^n n!}.$$

Thus we have established by virtue of (16) the following power series identity:

$$\left(\sum_{n=0}^{\infty}(-1)^n\frac{b^{2n}}{2^n n!}\right)\left(\sum_{n=0}^{\infty}\frac{b^{2n+1}}{(2n+1)!!}\right)=\sum_{n=0}^{\infty}(-1)^n\frac{b^{2n+1}}{(2n+1)2^n n!}, \quad (17)$$

where we have cancelled the constant of $\frac{1}{\sqrt{2\pi}}$ from (16). We point out that it is also possible to verify (17) using two series expansions for the error function. From Zwillinger [3 Section 6.13] we may expand $erf(x)$ as given by (15) as follows:

$$erf(x)=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)n!}x^{2n+1}=\frac{2}{\sqrt{\pi}}\left(x-\frac{x^3}{3}+\frac{1}{2!}\frac{x^5}{5}-\frac{1}{3!}\frac{x^7}{7}+\ldots\right). \quad (18)$$

Another known expansion also from Zwillinger [3] for $erf(x)$ is given by

$$erf(x)=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{3}{2})e^{-x^2}}{\Gamma(n+\frac{3}{2})}x^{2n+1}=\frac{2}{\sqrt{\pi}}e^{-x^2}\left(x+\frac{2x^3}{3}+\frac{4x^5}{15}+\ldots\right), \quad (19)$$

where $\Gamma(n)$ in (19) denotes the gamma function. Equating (18) to (19) and cancelling the factor of $\frac{2}{\sqrt{\pi}}$ that appears in both, we obtain:

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)n!}x^{2n+1}=\sum_{n=0}^{\infty}\frac{\Gamma(\frac{3}{2})e^{-x^2}}{\Gamma(n+\frac{3}{2})}x^{2n+1}. \quad (20)$$

Using the properties of the gamma function, we note that

$$\Gamma\left(n+\frac{3}{2}\right)=\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)=\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right)$$

$$=\ldots$$

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$$= \left(n + \frac{1}{2}\right)\left(n - \frac{1}{2}\right)\ldots\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right).$$

Thus

$$\frac{\Gamma\left(n + \frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} = \left(n + \frac{1}{2}\right)\left(n - \frac{1}{2}\right)\ldots\left(\frac{3}{2}\right) = \left(\frac{2n+1}{2}\right)\left(\frac{2n-1}{2}\right)\ldots\left(\frac{3}{2}\right)$$
$$= \frac{(2n+1)(2n-1)\ldots(3)}{2^n} = \frac{(2n+1)!!}{2^n}.$$

Thus (20) can be rewritten as

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\,n!}x^{2n+1} = e^{-x^2}\sum_{n=0}^{\infty}\frac{2^n}{(2n+1)!!}x^{2n+1}. \tag{21}$$

Reversing the sides of (21) and replacing e^{-x^2} with its series expansion, we obtain

$$\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{2^n x^{2n+1}}{(2n+1)!!}\right) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)\,n!}. \tag{22}$$

We now replace x in (22) by $\dfrac{b}{\sqrt{2}}$, and then multiply through the equation by the constant $\sqrt{2}$, to obtain the following result:

$$\left(\sum_{n=0}^{\infty}(-1)^n\frac{b^{2n}}{2^n n!}\right)\left(\sum_{n=0}^{\infty}\frac{b^{2n+1}}{(2n+1)!!}\right) = \sum_{n=0}^{\infty}(-1)^n\frac{b^{2n+1}}{(2n+1)\,2^n n!}.$$

Thus we have again arrived at (17) by equating the two known series expansions for the error function. We also mention that *Mathematica* recognises the first series in (17) as $e^{-\frac{1}{2}b^2}$, the second series as $e^{\frac{1}{2}b^2}\sqrt{\frac{\pi}{2}}erf\left(\frac{b}{\sqrt{2}}\right)$ and the third series on the right-hand side of (17) as $\sqrt{\frac{\pi}{2}}erf\left(\frac{b}{\sqrt{2}}\right)$. We may confirm this by noting that the right-hand side of (17) is simply $\sqrt{2\pi}F(b)$, and we had established after (15) that $F(b) = \frac{1}{2}erf\left(\frac{b}{\sqrt{2}}\right)$. Thus the series on the right-hand side of (17) is exactly

$$\sqrt{2\pi}F(b) = \sqrt{2\pi}\cdot\frac{1}{2}erf\left(\frac{b}{\sqrt{2}}\right) = \sqrt{\frac{\pi}{2}}erf\left(\frac{b}{\sqrt{2}}\right),$$

as found by *Mathematica*.

Now (17) is an interesting power series identity in its own right. Another approach to analysing it is to look at the coefficients of the powers of b on each side of the equation. If we cancel a factor of b from both sides and equate the coefficients of b^{2n}, we get

$$\sum_{k=0}^{n}\frac{(-1)^k}{2^k k!}\cdot\frac{1}{[2(n-k)+1]!!} = \frac{(-1)^n}{(2n+1)\,2^n n!}. \tag{23}$$

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Multiplying through (23) by $2^n \cdot n!$, we obtain

$$\sum_{k=0}^{n} (-1)^k 2^{n-k} \frac{n!}{k!\,[2(n-k)+1]!!} = \frac{(-1)^n}{(2n+1)}. \tag{24}$$

Dividing (24) through by $(-1)^n$, and noting that $(-1)^{k-n} = (-1)^{n-k}$, we obtain

$$\sum_{k=0}^{n} (-1)^{n-k} 2^{n-k} \frac{n!}{k!\,[2(n-k)+1]!!} = \frac{1}{2n+1}. \tag{25}$$

Re-indexing the sum in (25) by changing k to $n-k$, we obtain

$$\sum_{k=0}^{n} (-1)^k 2^k \frac{n!}{(n-k)!\,(2k+1)!!} = \frac{1}{2n+1}. \tag{26}$$

In order to simplify (26), we first note that

$$\begin{aligned}(2k+1)!! &= (2k+1)(2k-1)\ldots 5\cdot 3\cdot 1\\ &= (2k+1)((2k-1)\ldots 5\cdot 3\cdot 1)\cdot\frac{(2k)(2k-2)\ldots(2)}{(2k)(2k-2)\ldots(2)}\\ &= \frac{(2k+1)(2k)(2k-1)\ldots 3\cdot 2\cdot 1}{2^k k!}\\ &= \frac{(2k+1)!}{2^k k!}.\end{aligned} \tag{27}$$

Using (27) to simplify (26), we obtain

$$\sum_{k=0}^{n} (-1)^k 2^{2k} \frac{n!\,k!}{(n-k)!\,(2k+1)!} = \frac{1}{2n+1}. \tag{28}$$

Multiplying the fraction on the left-hand side of (28) by $\dfrac{k!}{k!}$, we finally obtain

$$\sum_{k=0}^{n} \frac{(-1)^k 2^{2k}}{2k+1} \frac{\binom{n}{k}}{\binom{2k}{k}} = \frac{1}{2n+1}. \tag{29}$$

Now the left-hand side of (29) is a very interesting combinatorial expression that *Mathematica* identifies immediately as the hypergeometric function ${}_2F_1\left(1, -n; \tfrac{3}{2}; 1\right)$, and it is able to use the properties of these functions to simplify it to the right-hand side, which is $\frac{1}{2n+1}$. One can experiment with *Mathematica* to see if (29) generalises, and it turns out that if we remove the $2k+1$ from the denominator on the left-hand side of (29), we obtain a similar identity corresponding to the hypergeometric function ${}_2F_1\left(1, -n; \tfrac{1}{2}; 1\right)$:

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$$\sum_{k=0}^{n} (-1)^k 2^{2k} \frac{n!\,k!}{(n-k)!\,(2k)!} = \sum_{k=0}^{n} (-1)^k 2^{2k} \frac{\binom{n}{k}}{\binom{2k}{k}} = \frac{1}{1-2n}. \quad (30)$$

Note that we have again multiplied and divided by $k!$ to derive (30). We prove the combinatorial identities (29) and (30) in the Appendix by elementary methods, without appealing to the properties of hypergeometric functions.

Appendix

Here we prove the combinatorial identities (29) and (30) by elementary methods. We start by showing (30), namely that

$$\sum_{k=0}^{n} (-1)^k 2^{2k} \frac{\binom{n}{k}}{\binom{2k}{k}} = \frac{1}{1-2n},$$

where n is a non-negative integer. Let

$$S_n = \sum_{k=0}^{n} (-1)^k 2^{2k} \frac{\binom{n}{k}}{\binom{2k}{k}},$$

and we will assume inductively that $S_n = \frac{1}{1-2n}$. Then

$$\sum_{k=0}^{n} (-1)^{k+1} 2^{2k} (2n-1) \frac{\binom{n}{k}}{\binom{2k}{k}} = 1.$$

Note that $2n-1 = 2(n+1)-3$ and that $(n+1)\binom{n}{k} = (k+1)\binom{n+1}{k+1}$. Thus

$$3 \sum_{k=0}^{n} (-1)^k 2^{2k} \frac{\binom{n}{k}}{\binom{2k}{k}} + \sum_{k=0}^{n} (-1)^{k+1} 2^{2k} \cdot 2 \cdot (k+1) \frac{\binom{n+1}{k+1}}{\binom{2k}{k}} = 1.$$

Thus

$$3\left(\frac{1}{1-2n}\right) + \sum_{k=0}^{n} (-1)^{k+1} 2^{2k} \cdot 2 \cdot (k+1) \frac{\binom{n+1}{k+1} \cdot \frac{(2k+2)(2k+1)}{(k+1)(k+1)}}{\binom{2k}{k} \cdot \frac{(2k+2)(2k+1)}{(k+1)(k+1)}} = 1.$$

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Note the cancellation of $(k + 1)$; we obtain:

$$3\left(\frac{1}{1 - 2n}\right) + \sum_{k=0}^{n} (-1)^{k+1} 2^{2k} \cdot 2^2 \cdot (2k + 1) \frac{\binom{n+1}{k+1}}{\binom{2k+2}{k+1}} = 1.$$

Note again that $2k + 1 = 2(k + 1) - 1$ and that $(k+1)\binom{n+1}{k+1} = (n+1)\binom{n}{k}$. Thus

$$3\left(\frac{1}{1-2n}\right) - \sum_{k=0}^{n} (-1)^{k+1} 2^{2(k+1)} \frac{\binom{n+1}{k+1}}{\binom{2k+2}{k+1}} + \sum_{k=0}^{n} (-1)^{k+1} 2^{2k} \cdot 2^3 \cdot (n+1) \frac{\binom{n}{k}}{\binom{2k+2}{k+1}} = 1. \quad (31)$$

$$3\left(\frac{1}{1-2n}\right) - (S_{n+1} - 1) + 2(n+1) \sum_{k=0}^{n} (-1)^{k+1} 2^{2(k+1)} \frac{\binom{n+1}{k+1} - \binom{n}{k+1}}{\binom{2k+2}{k+1}} = 1, \quad (32)$$

using Pascal's identity and recognising the first sum in (31) as being $S_{n+1} - 1$. Thus (32) becomes

$$3\left(\frac{1}{1-2n}\right) - (S_{n+1} - 1) + 2(n+1)\left[\sum_{k=0}^{n} (-1)^{k+1} 2^{2(k+1)} \frac{\binom{n+1}{k+1}}{\binom{2k+2}{k+1}} - \sum_{k=0}^{n} (-1)^{k+1} 2^{2(k+1)} \frac{\binom{n}{k+1}}{\binom{2k+2}{k+1}}\right] = 1. \quad (33)$$

$$3\left(\frac{1}{1 - 2n}\right) - (S_{n+1} - 1) + 2(n + 1)[S_{n+1} - 1 - (S_n - 1)] = 1,$$

as we recognize the sums $S_{n+1} - 1$ and $S_n - 1$ appearing in (33). Thus

$$3\left(\frac{1}{1 - 2n}\right) + (2(n + 1) - 1)S_{n+1} + 1 - 2(n + 1)\left(\frac{1}{1 - 2n}\right) = 1,$$

since $S_n = \dfrac{1}{1 - 2n}$. Thus

$$1 + (2(n + 1) - 1)S_{n+1} = 0.$$

$$S_{n+1} = \frac{1}{1 - 2(n + 1)},$$

completing the proof of (30) by induction.

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Next we show (29), namely that

$$\sum_{k=0}^{n} \frac{(-1)^k 2^{2k}}{2k+1} \frac{\binom{n}{k}}{\binom{2k}{k}} = \frac{1}{2n+1},$$

where n is a non-negative integer. The strategy of the proof is to suppose that

$$\sum_{k=0}^{n} \frac{(-1)^k 2^{2k}}{2k+1} (2n+1) \frac{\binom{n}{k}}{\binom{2k}{k}} = \alpha_n,$$

where α_n represents a function depending on n. We will then show that $\alpha_n \equiv 1$ for all n, and this will suffice to establish (29). Similar to the proof of (30), we start by noting that $2n+1 = 2(n+1) - 1$, and that

$$(n+1)\binom{n}{k} = (k+1)\binom{n+1}{k+1}.$$

Thus we have

$$\sum_{k=0}^{n} \frac{(-1)^k 2^{2k}}{2k+1} (-1) \frac{\binom{n}{k}}{\binom{2k}{k}} + \sum_{k=0}^{n} \frac{(-1)^k 2^{2k}}{2k+1} 2(k+1) \frac{\binom{n+1}{k+1}}{\binom{2k}{k}} = \alpha_n.$$

$$\frac{-\alpha_n}{2n+1} + \sum_{k=0}^{n} \frac{(-1)^k 2^{2k}}{2k+1} 2(k+1) \frac{\binom{n+1}{k+1} \cdot \frac{(2k+2)(2k+1)}{(k+1)(k+1)}}{\binom{2k}{k} \cdot \frac{(2k+2)(2k+1)}{(k+1)(k+1)}} = \alpha_n.$$

Notice the cancellation of the factors of $(k+1)$ and $(2k+1)$ in the above; we obtain

$$\frac{-\alpha_n}{2n+1} + \sum_{k=0}^{n} (-1)^k 2^{2k} \cdot 2 \frac{\binom{n+1}{k+1} \cdot 2}{\binom{2k+2}{k+1}} = \alpha_n.$$

$$\frac{-\alpha_n}{2n+1} - \sum_{k=0}^{n} (-1)^{k+1} 2^{2(k+1)} \frac{\binom{n+1}{k+1}}{\binom{2k+2}{k+1}} = \alpha_n.$$

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We now re-index the sum to obtain

$$\frac{-\alpha_n}{2n+1} - \left[\sum_{k=0}^{n+1} (-1)^k 2^{2k} \frac{\binom{n+1}{k}}{\binom{2k}{k}} - 1 \right] = \alpha_n. \tag{34}$$

$$\frac{-\alpha_n}{2n+1} - \left[\frac{1}{1-2(n+1)} - 1 \right] = \alpha_n, \tag{35}$$

where we have replaced the sum appearing in (34) by $\frac{1}{1-2(n+1)}$, by virtue of the fact that we have already established (30). Thus (35) becomes

$$\frac{-\alpha_n}{2n+1} + \frac{1}{2(n+1)-1} + 1 = \alpha_n.$$

$$\frac{-\alpha_n}{2n+1} + \frac{1}{2n+1} + 1 = \alpha_n.$$

$$-\alpha_n + 1 + 2n + 1 = (2n+1)\alpha_n.$$

$$2n + 2 = (2n+2)\alpha_n.$$

$$\alpha_n = 1.$$

Thus $\alpha_n \equiv 1$ for all n, and this suffices to establish (29).

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