HYDRODYNAMICAL IMAGES

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Consider irrotational two-dimensional flow of imcompressible inviscid fluid in the z-plane. Suppose that there are no rigid boundaries and let the complex potential of the motion be

$$f(z) = \phi + i\psi.$$

The singularities of $f(z)$ are determined by the sources, vortices, etc., to which the motion may be due. In particular a stream entails a simple pole at infinity.

If we introduce into the field a circular cylinder typified by its cross-section the circle C, $|z| = a$, the flow is thereby disturbed and the complex potential becomes

$$w = f(z) + f_1(z).$$

It is of course assumed that no singularity of the original flow is within C. The perturbing term $f_1(z)$ must then necessarily have all its singularities within C, and these singularities determine the image system in C of the external flow given by $f(z)$. Since the circle is a streamline, the stream function has a constant value on C, which without loss of generality may be taken to be zero, and then for points on C the complex potential $f(z) + f_1(z)$ must be purely real.

These conditions are all satisfied by the complex potential

$$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right),$$

where the bar denotes the complex conjugate*. For, if z is exterior to C, then a^2/z is within C. Since by hypothesis the singularities of $f(z)$ are all exterior to C, those of $\bar{f}(a^2/z)$ are all within C. Moreover, when z is on C, $a^2/z = \bar{z}$ and w is then purely real.

Thus the image problem is completely solved for a circular cylinder.

For example, if the cylinder is placed in the stream whose complex potential is $Ve^{i\alpha}z$, we have

$$w = Ve^{i\alpha}z + Ve^{-i\alpha}\left(\frac{a^2}{z}\right),$$

* If $f(x)$ is a function which takes complex values for real values of x, $\bar{f}(x)$ is the function which takes the corresponding conjugate complex values for the same real values of x, and $\bar{f}(z)$ is then obtained by writing z in place of x. See, for example, Milne-Thomson, *Theoretical hydrodynamics* (London, 1938), p. 138.

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so that the image is a doublet $a^2Ve^{-i\alpha}$ at the centre. Similarly for a vortex of strength κ at the point z_0

$$w = i\kappa \log (z-z_0) - i\kappa \log \left(\frac{a^2}{z} - \bar{z}_0\right) = i\kappa \log (z-z_0) - i\kappa \log \left(\frac{a^2}{z} - \frac{a^2}{z_0'}\right),$$

where z_0' is the inverse of z_0 with respect to C, so that the image system is a vortex of strength $-\kappa$ at the inverse point, and a vortex of strength κ at the centre.

It is now clear that to find the disturbance due to placing a cylinder of any form of section in a field of two-dimensional irrotational flow it is only necessary to know the conformal transformation which maps the region exterior to the contour of the section on the region exterior to C. The complex potential can then be written down by the above method.

Another application is to the two-dimensional aerofoil; for example in the biplane problem the effect of, say, the lower plane upon the upper can be obtained to a first approximation by replacing the lower plane by a suitably placed vortex* or by a vortex and a doublet†.

It is then necessary to find the relation between the circulation $2\pi\kappa$ about the upper plane and the strength of the vortex which replaces the lower plane. This amounts to finding κ such that the point z_0, on the circle C, which transforms into the trailing edge of the upper plane may be a stagnation point. The complex potential of the whole motion is

$$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right) + i\kappa \log z,$$

and the condition to be satisfied is $dw/dz = 0$ when $z = z_0$. But $a^2/z_0 = \bar{z}_0$. Therefore

$$\begin{aligned} \kappa &= iz_0 f'(z_0) - i\bar{z}_0 \bar{f}'(\bar{z}_0) \\ &= \text{real part of } -2iz_0(u_0 - iv_0), \end{aligned}$$

where $u_0 - iv_0 = -f'(z_0)$ is the complex velocity at the point z_0, *when the circle is absent.*

Thus $\kappa = 2av$, where v is the component, in the opposite sense to κ, of the velocity $u_0 - iv_0$ along the tangent at the point z_0 to the circle C.

* E. Pistolesi "Alcune considerazioni sul problema del biplano indefinito", *Atti della Pontificia Acc. delle Scienze* (1928).

† C. B. Millikan "An extended theory of thin airfoils and its application to the biplane problem", *U.S. Nat. Advisory Cttee for Aeronautics Report, Washington*, no. (362) 1930.

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