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ON MAXIMAL CYCLIC SUBGROUPS OF FINITE GROUPS

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Abstract

Let $\lambda(G)$ be the maximum number of subgroups in an irredundant cover of the finite group G. We establish bounds on the order, exponent and derived length of the group in terms of this invariant.

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1. Introduction

Throughout this paper, all groups are finite. Let G be a noncyclic group. A cover of G is a family $\{X_i\}_{i=1}^n$ of proper subgroups such that $G = \bigcup_{i=1}^n X_i$. A cover is said to be *irredundant* if none of the subgroups can be omitted. A cover is called a partition of G if $X_i \cap X_j = \{1\}$ whenever $i \neq j$. It is natural to ask what information about G can be deduced from properties of the subgroups X_i and their number n.

Interest in covers of groups by proper subgroups began with the classical work of Scorza [12], which showed that a group G admits an irredundant cover by exactly three subgroups when $G/N \cong C_2 \times C_2$ for some normal subgroup N. Later, Cohn [4] introduced the invariant $\sigma(G)$, defined as the minimal number of subgroups in an irredundant cover of a noncyclic group, and classified the groups G with $\sigma(G) = 3, 4, 5$. He also noted that no example with $\sigma(G) = 7$ seemed to exist and conjectured that every noncyclic solvable group satisfies $\sigma(G) = p^a + 1$. Both statements were confirmed by Tomkinson [15].

A cyclic subgroup H of G is called a maximal cyclic subgroup if it is not properly contained in any other cyclic subgroup of G. If G is noncyclic, then its maximal cyclic subgroups form an irredundant cover. In [11], Rogério introduced another cover invariant defined as follows: $\lambda(G)$ denotes the maximum size of an irredundant cover of a group G. Rogério observed that the number of maximal cyclic subgroups of G

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TABLE 1. Examples of groups with derived length at most 6.

Groups/GAP ID	$\lambda(G)$	$\mathsf{dl}(G)$	Theorem A
S_3, [6, 1]	4	2	2
$\mathsf{SL}(2,3)$, [24, 3]	7	3	5
[48, 28]	13	4	6
$\mathsf{AGL}(2,3)$, [432, 734]	123	5	11
[1296, 2890]	276	6	13

coincides with $\lambda(G)$. Groups G with $\lambda(G) \leqslant 6$ are classified in [1, 11]. In addition, [1] shows that if $\lambda(G) \leqslant 30$, then G is solvable. See also [3, 7] for related work.

This work is motivated by several recent contributions. Qureshi and Reis [9] introduced flower groups, while Gao and Garonzi [6] and Zarrin [17] obtained structural bounds in terms of certain special covers. We refine these ideas, incorporating results of Cohn [4] and Rogério [11], and obtain new bounds for the derived length and order of a group in terms of its maximal cyclic subgroups. Regarding the derived length, we follow the method used in [6] and improve the estimate by at least one.

THEOREM 1.1. *Let G be a noncyclic solvable group with $\lambda(G) = \lambda$. Then, the derived length is at most*

$$\begin{cases} 2 & \text{if } \lambda \leqslant 6, \\ \frac{5}{2} \cdot \log_3(\lambda - 1) + 1 & \text{if } \lambda \geqslant 7. \end{cases}$$

Note that the bound is exact for $\lambda(G) \leqslant 6$; $\mathsf{SL}(2,3)$ has derived length 3 and $\lambda(\mathsf{SL}(2,3)) = 7$. In Table 1, we use GAP [13] to list some solvable groups, illustrating how far the bound appears to be from the precise value. We also establish a similar bound for another type of cover (see Theorem 3.4).

Let G be a finite noncyclic group with $\lambda(G) = \lambda$ and let $S = \{C_1, \ldots, C_\lambda\}$ be the set of its maximal cyclic subgroups. The group G is called a flower group if there exists a subgroup $C_0 \leqslant G$ such that

$$C_i \cap C_j = C_0 \quad (1 \leqslant i < j \leqslant \lambda).$$

Here, C_0 is the pistil and the subgroups in S are the petals [9]. Note that if G is a flower group with trivial pistil, then the family $\{C_1, \ldots, C_\lambda\}$ is a partition of G. For a deeper discussion of partitionable groups, we refer the reader to [16].

A group G is said to be an M_C-group if every proper maximal cyclic subgroup of G is a maximal subgroup and G has at least one maximal cyclic subgroup (see [8]). In the next result, we establish a connection between M_C-groups and flower groups.

THEOREM 1.2. *Let G be a noncyclic M_C-group. Then, G is a flower group.*

Our next result gives a bound for $|G|$ in terms of $\lambda(G)$ and $\exp(G)$. These parameters are independent (see Remark 3.12(1)).

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THEOREM 1.3. *Let G be a noncyclic group with $\lambda(G) = \lambda$.*

(1) *The order $|G|$ is at most $\exp(G) \cdot (\lambda - 1)$.*
(2) *Let G be a flower group with pistil C_0. Then, each petal C_i has order at most $|C_0| \cdot (\lambda - 1)$. Moreover, $|G| \leqslant |C_0| \cdot (\lambda - 1)^2$.*
(3) *Let G be a flower p-group with pistil C_0. Then, $\exp(G) \leqslant |C_0| \cdot (\lambda - 1)$.*

The bounds obtained above are sharp. For instance, the quaternion group $G = Q_8$ is a flower group with $\exp(G) = 4$, $\lambda(G) = 3$ and $|C_0| = 2$. See Remarks 3.11 and 3.12 for further details on Theorem 1.3.

We say that a family of groups is maximal cyclic bounded MCB if for every natural number n, there are only finitely many groups G in the family with $\lambda(G) \leqslant n$ (see [6, Section 1] for more details). Flower groups are not MCB (see Example 3.9). However, by restricting the size of the pistil, we obtain MCB families. For each positive integer k, let $\mathfrak{X}_k$ denote the set of all flower groups G with $|C_0| \leqslant k$. The following result is immediate from Theorem 1.3(2).

COROLLARY 1.4. *For every k, the family $\mathfrak{X}_k$ is* MCB.

2. Preliminaries

Throughout, we will use the usual notation, for example, S_n and A_n denote respectively the symmetric and alternating groups on n letters, and D_n stands for the dihedral group of order $2n$. Let H be a subgroup of a group G. The subgroup $\bigcap_{g \in G} H^g$ is called the *core* of H in G and is denoted by H_G. The derived length of a solvable group G will be denoted by $\mathsf{dl}(G)$.

We begin by recalling a few known results that will be used throughout the paper.

PROPOSITION 2.1 (Rogério [11, Propositions 4 and 5]). *Let G be a group.*

(a) *If $H_1, \ldots, H_n$ are the maximal cyclic subgroups of G, then $\{H_i\}_{i=1}^n$ is an irredundant cover of G and $\lambda(G) = n$. Moreover, $\{H_i\}_{i=1}^n$ is the unique covering by proper subgroups of G if and only if H_i is a maximal subgroup of G for $i = 1, \ldots, n$.*
(b) *If $M \trianglelefteq G$, then $\lambda(G/M) \leqslant \lambda(G)$. If $N = \bigcap_{i=1}^{\lambda} H_i$, where $\lambda = \lambda(G)$ and $H_1, H_2, \ldots, H_\lambda$ are the maximal cyclic subgroups of G, then N is a central subgroup of G and $\lambda(G/N) = \lambda(G)$.*

LEMMA 2.2 (Cohn [4, Theorem 1]). *Let G be a group. If $\{X_i\}_{i=1}^n$ is an irredundant cover of G, then $|G| \leqslant \sum_{i=2}^n |X_i|$.*

The following result is a direct consequence of [14, Lemma 3.3].

LEMMA 2.3. *Let G be a noncyclic group with $\lambda(G) = \lambda$ and let $H_1, \ldots, H_\lambda$ be its maximal cyclic subgroups ordered so that $|H_i| \geqslant |H_{i+1}|$. Then, $|G : H_j| \leq \lambda - 1$ for $j = 1, 2$.*

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3. Proofs

3.1. Proof of Theorem 1.1. In [1, Theorems A and B] and [11, Theorem 1], the authors give explicit presentations for all groups with $\lambda(G) \in \{3, 4, 5, 6\}$. By direct inspection of these presentations, we obtain the following result.

LEMMA 3.1. *Let G be a nonabelian group with $\lambda = \lambda(G) \leqslant 6$. Then, G is metabelian.*

PROOF OF THEOREM 1.1. By Lemma 3.1, it suffices to consider $\lambda \geqslant 7$. Let $H_1, \ldots, H_\lambda$ be the maximal cyclic subgroups of G, ordered so that $|H_i| \geqslant |H_{i+1}|$ for each i. By Proposition 2.1(a), their union forms an irredundant cover of G. By Lemma 2.3, the index of H_1 in G is at most $\lambda - 1$ and the quotient $G/(H_1)_G$ embeds into the symmetric group $\mathsf{S}_{\lambda-1}$. Dixon's theorem [5, Theorem 1] then shows that the derived length of this quotient is at most $(5/2)\log_3(\lambda - 1)$. As the core of H_1 is cyclic, the stated estimate follows immediately. □

REMARK 3.2. Here, $[m, n]$ denotes the `SmallGroup`(m,n) in the GAP library [13]. In Table 1, we have used GAP to select a solvable group for each derived length at most 6, computed $\lambda(G)$ and listed the bound obtained from Theorem 1.1 for comparison (see also [6, Theorem 1.1]).

REMARK 3.3. Our methods do not yield good bounds for the derived length $\mathsf{dl}(G)$ in terms of $\lambda(G)$. Roughly speaking, the approach fails because it relies on Tomkinson's result [14] which gives very large bounds whose precise values are unknown. A different strategy is needed.

3.2. Derived length and centralisers. The main result of this subsection lies slightly outside the main scope of the paper. However, its application is natural and arises from the same ideas employed in the proof of Theorem 1.1.

Let G be a group. Here, $\mathsf{cent}(G)$ denotes the set $\{C_G(g) \mid g \in G\}$, where $C_G(g)$ is the centraliser of the element g in G. For a positive integer, the group G is called an n-centraliser group if $|\mathsf{cent}(G)| = n$. Zarrin [17] proves that if G is a solvable n-centraliser group, then $\mathsf{dl}(G) \leqslant n$. Using the same ideas, together with Dixon's theorem [5, Theorem 1], we obtain the following bound.

THEOREM 3.4. *Let G be a solvable n-centraliser group. Then,*

$$\mathsf{dl}(G) \leqslant \begin{cases} 2 & \text{if } n \leqslant 5, \\ \frac{5}{2} \cdot \log_3(n-1) + 2 & \text{if } n \geqslant 6. \end{cases}$$

Before giving the proof, we present two lemmas about n-centraliser groups.

LEMMA 3.5 (Belcastro and Sherman [2, Theorems 2 and 4, page 367]).

(a) *There are no 2-centraliser and no 3-centraliser groups.*
(b) *A group G is a 4-centraliser group if and only if $G/Z(G) = C_2 \times C_2$.*
(c) *A group G is a 5-centraliser group if and only if $G/Z(G) = C_3 \times C_3$ or S_3.*

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LEMMA 3.6 (Zarrin [17, Lemma 2.2]). *Let G be an n-centraliser group. Let*

$$N := \bigcap_{a \in G} N_G(C_G(a)).$$

Then, G/N is isomorphic to a subgroup of S_{n-1}.

PROOF OF THEOREM 3.4. Assume that $n \leqslant 5$. By Lemma 3.5, the group $G/Z(G)$ is abelian or metacyclic. Note that if $G/Z(G)$ is metacyclic, then G is metabelian, since a central-by-cyclic group is abelian.

Now, suppose $n \geqslant 6$. By Lemma 3.6, the quotient G/N embeds in S_{n-1}. By Dixon's result [5, Theorem 1], the derived length of G/N is at most $(5/2)\log_3(n-1)$. Hence, it suffices only to prove that N is metabelian.

First, we prove that N is 2-Engel. Choose arbitrarily $a, b \in N$. We claim that $[b, a, a] = 1$. In fact, by definition,

$$(a^{-1})^b \in C_G(a)^b = C_G(a).$$

Hence, $(a^{-1})^b a = [b, a] \in C_G(a)$ and so $[b, a, a] = 1$. This shows that N is 2-Engel. According to Levi's theorem [10, Theorem 12.3.6](iv), we deduce that N is nilpotent of class at most 3. In particular, N is metabelian, which completes the proof. □

REMARK 3.7. There is no general relation between $\lambda(G)$ and $|\mathsf{cent}(G)|$. Indeed, for the dihedral group D_n, we have $\lambda(D_n) = n + 1$, while

$$|\mathsf{cent}(D_n)| = \begin{cases} n+2 & \text{if } n \text{ is odd,} \\ \frac{1}{2}n + 2 & \text{if } n \text{ is even.} \end{cases}$$

3.3. Proof of Theorem 1.2. First, we state the classification of finite M_C-groups obtained by Juriaans and Rogério [8].

LEMMA 3.8 (Juriaans and Rogério [8, Theorem 2.5]). *Let G be a finite group. Then, G is an M_C-group if and only if one of the following holds:*

(a) *G is cyclic or $G \cong C_p \times C_p \times C_n$, where p is a prime not dividing n;*
(b) *$G \cong C_n \times Q_8$ with n odd, where $Q_8 = \langle x, y \mid x^4 = 1, x^2 = y^2, x^y = x^{-1} \rangle$ is the quaternion group of order 8;*
(c) *there exist primes $q < p$, an exact sequence*

$$1 \to Z(G) \to G \to C_p \rtimes C_q \to 1,$$

and $Z(G) \leqslant \langle x \rangle$ for any maximal cyclic group $\langle x \rangle \leqslant G$.

PROOF OF THEOREM 1.2. The result follows by inspecting the three cases in Lemma 3.8.

Case (a): $G = C_p \times C_p \times C_n \cong C_p \times C_{pn}$ *with* $p \nmid n$. By Proposition 2.1(b), $\lambda(G) = p + 1$. We can write G as $\langle a, b \mid a^p = b^{pn} = 1,\ [a, b] = 1 \rangle$. The maximal cyclic subgroups are $H_i = \langle a^i b \rangle$ for $i \in \{1, \ldots, p\}$ and $H_{p+1} = \langle ab^p \rangle$. Moreover,

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$H_i \cap H_j = \langle b^p \rangle$ for all $i \neq j$. Since $\langle b^p \rangle$ has order n and is the common intersection of all distinct maximal cyclic subgroup, G is a flower group, which gives item (a).

Case (*b*): $G = Q_8 \times C_n$, *where n is odd.* We can rewrite G as

$$\langle a, x, y \mid a^n = x^4 = [a, x] = [a, y] = 1, x^2 = y^2, x^y = x^{-1} \rangle.$$

By [11], $\lambda(G) = 3$. The maximal cyclic subgroups of G are $H_1 = \langle ay \rangle$, $H_2 = \langle ax \rangle$ and $H_3 = \langle axy \rangle$, and $H_i \cap H_j = \langle x^2 a \rangle$ whenever $i \neq j$. Hence, G is a flower group with pistil $\langle x^2 a \rangle$, which gives item (b).

Case (c): there exist primes p and q such that G fits into the short exact sequence

$$1 \to Z(G) \to G \to C_p \rtimes C_q \to 1,$$

and every maximal cyclic subgroup $\langle x \rangle$ *of G contains the centre.* Let $H_1, \ldots, H_\lambda$ be the maximal cyclic subgroups of G. By Proposition 2.1(b),

$$H_1 \cap \cdots \cap H_\lambda \leqslant Z(G).$$

However, by definition, $Z(G) \leqslant H_i \cap H_j$ for all i, j. Consequently, G is a flower group with pistil $Z(G)$, which completes the proof. □

EXAMPLE 3.9. Note that the family of M_C-groups is not MCB (and the same holds for flower groups, since all noncyclic M_C-groups are flower groups). As an example, let $G = G_k = C_3 \rtimes C_{2^k}$ with the action given by inversion. Each G_k is an M_C-group and there is a short exact sequence $1 \to Z(G_k) \to G_k \to \mathsf{S}_3 \to 1$, where $Z(G_k)$ is a cyclic group of order 2^{k-1}. By Proposition 2.1(b), $\lambda(G_k) = 4$ for every k. In particular, this family is not MCB.

3.4. Proof of Theorem 1.3. The next lemma gives a classification of flower groups and expresses the order of these groups in terms of the orders of their petals.

PROPOSITION 3.10 (Qureshi and Reis [9, Proposition 3.3]). *For a finite noncyclic group G and a cyclic subgroup* C_0 *of G, the following are equivalent:*

(a) *G is a flower group with pistil* C_0*;*
(b) *for any* $g \in G \setminus C_0$*, there exists a unique maximal cyclic group C such that* $g \in C$ *and* $C_0 \subseteq C$*;*
(c) *there exist* $k \geqslant 1$ *and distinct maximal cyclic subgroups* $C_1, \ldots, C_k$ *of G such that* $C_i \cap C_j = C_0$ *for* $1 \leqslant i < j \leqslant k$ *and*

$$\sum_{i=1}^{k} |C_i| - (k-1)|C_0| = |G|.$$

PROOF OF THEOREM 1.3. (1) Let G be a noncyclic group and let $H_1, \ldots, H_\lambda$ be the maximal cyclic subgroups of G, where $\lambda = \lambda(G)$. By Proposition 2.1(a), the union

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$\bigcup_{i=1}^{\lambda} H_i$ forms an irredundant cover of G. By Lemma 2.2,

$$|G| \leqslant \sum_{i=1}^{\lambda-1} |H_i|.$$

Since each H_i is cyclic, its order divides $\exp(G)$. Therefore,

$$|G| \leqslant \sum_{i=1}^{\lambda-1} |H_i| \leqslant (\lambda - 1) \cdot \exp(G),$$

which is the claimed bound.

(2) Assume G is a flower group. As in item (1), the maximal cyclic subgroups form an irredundant cover and these subgroups are precisely the petals $C_1, \ldots, C_\lambda$. Order the C_i so that $|C_i| \geqslant |C_{i+1}|$ for each i. With this ordering, it suffices to prove that

$$|C_1| \leqslant (\lambda - 1) \cdot |C_0|.$$

Combining Lemma 2.2 and Proposition 3.10, we obtain

$$\sum_{i=1}^{\lambda} |C_i| - (\lambda - 1)|C_0| = |G| \leqslant \sum_{i=2}^{\lambda} |C_i|,$$

and the desired inequality for $|C_1|$ follows.

It remains to bound the order $|G|$. By definition, $C_0 = C_1 \cap C_2$. By Lemma 2.3, $|G : C_i| \leqslant \lambda - 1$ for $i \in \{1, 2\}$. By the Poincaré lemma [10, 1.3.11], the index

$$|G : C_0| = |G : C_1 \cap C_2| \leqslant |G\colon C_1| \cdot |G\colon C_2| \leqslant (\lambda - 1)^2.$$

In particular, the order $|G|$ is at most $(\lambda - 1)^2 \cdot |C_0|$, as required.

(3) Let G be a flower p-group with pistil C_0. Set $m = \mathsf{lcm}(|C_1|, |C_2|, \ldots, |C_\lambda|)$. Since G is a p-group, each C_i has order a power of p, so that every element order divides m. Thus, $m = \exp(G)$. By item (2), the exponent $\exp(G)$ divides $|C_0| \cdot (\lambda - 1)$. This completes the proof. □

We finish the paper with two remarks that provide concrete examples illustrating the scope and limitations of Theorem 1.3.

REMARK 3.11. The bound in Theorem 1.3(1) is attained for the quaternion group, but for many groups, it is much larger than $|G|$. We propose the following refinement. Define

$$M := \max\{|H| \mid H \text{ is a maximal cyclic in } G\}.$$

Since the bound in Theorem 1.3(1) involves the order of a maximal cyclic subgroup, the same conclusion remains valid if we replace $\exp(G)$ with M. In fact, using M gives a sharper estimate in general: for p-groups, we have $M = \exp(G)$ and for many groups of mixed order, the bound becomes strictly smaller. For example, if $G = \mathsf{SL}(2, 3)$, then

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$|G| = 24$, $\exp(G) = 12$, $|M| = 6$ and $\lambda(G) = 7$. In particular, the resulting bound is

$$|G| \leqslant M \cdot (\lambda - 1) = 36 < \exp(G) \cdot (\lambda - 1) = 72.$$

REMARK 3.12. The parameters $\lambda(G)$, $\exp(G)$ and $|C_0|$ behave independently in general.

(1) The parameters $\lambda(G)$ and $\exp(G)$ are independent. If G is an elementary abelian p-group of rank r, then $\exp(G) = p$ and

$$\lambda(G) = \frac{p^r - 1}{p - 1}.$$

In contrast, if p is an odd prime and $G = C_2 \times C_2 \times C_p$, then $\lambda(G) = 3$, while $\exp(G) = 2p$.

(2) Let $n \geqslant 2$. If $G = D_{2^n}$, then G is a flower group with trivial pistil. Moreover, $\lambda(D_{2^n}) = 2^n + 1$ and $\exp(D_{2^n}) = 2^n$. Thus, the bounds in Theorem 1.3(2) and (3) are attained for this family.

(3) Let p be an odd prime. If $G = D_p$, then $\lambda(G) = p + 1$ and $\exp(G) = 2p$. This shows that $\exp(G)$ cannot be bound by $|C_0|(\lambda(G) - 1)$ without imposing additional assumptions.

(4) To see the independence of $\lambda(G)$ and the size of the pistil, take $n \geqslant 2$ and consider the group $G = \langle a, b \mid a^3 = 1 = b^{2^n},\ a^b = a^{-1} \rangle$. This is a flower group with $|C_0| = 2^{n-1}$ and $\lambda(G) = 4$.

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