J. Plasma Physics (1998), *vol.* 59, *part* 4, *pp.* 719–725. Printed in the United Kingdom
© 1998 Cambridge University Press

Turbulence in incompressible fluids and magnetic fields

E. PIÑA

Departamento de Física, Universidad Autónoma Metropolitana–Iztapalapa, AP 55-534, CP 09340, México, DF Mexico

This paper is dedicated to Professor Radu Balescu on the occasion of his 65th birthday

(Received 2 September 1997 and in revised form 3 October 1997)

The Hamiltonian dynamics of a two-degree-of-freedom system is shown to mimic the structure of an incompressible fluid if the fluid streamlines are considered instead of the pathlines used by many authors. This approach is formally equivalent to the determination of magnetic field lines, which satisfy the same mathematical equations. Turbulence in this case is seen as Hamiltonian chaotic dynamics perturbed by time, which plays the role of the codimension parameter of such chaos. Any spatial Lie symmetry leads to a laminar flow, whereas stationary flow could be considered chaotic but not turbulent. The transition from laminar to turbulent flow is associated with the destruction of KAM trajectories produced by time evolution.

1. Introduction

Understanding the turbulent behaviour of a fluid is one of the most important current areas of work in physics, and for many years has presented a continuous challenge to a number of researchers.

In recent times, many papers have been published in which the meaning of turbulence in a fluid is illustrated by the chaotic behaviour of the pathlines of the hydrodynamic velocity field. In order to have a model of turbulence, the pathlines are shown to obey dynamic equations of the same type as those found in the chaotic dynamics of Hamiltonian systems. See Arnold (1965), Hénon (1966), Aref (1984), Aref and Balanchander (1986), Dombre *et al.* (1986), Chien *et al.* (1986), Chaiken *et al.* (1987), Feingold *et al.* (1988), Ziglin (1988), McLaughlin (1988), Zaslavski *et al.* (1988), Aref and Jones (1989), Polvani and Wisdom (1990), Wang *et al.* (1990, 1991), Zaslavski and Tippett (1991) and Lau and Finn (1992).

However, the velocity fields considered in these papers are stationary fields independent of time, or time-dependent fields of one- or two-dimensional flows. Also, the independent variable in the pathlines is the time, and this choice forces the independent variable to have the same physical meaning both in the hydrodynamic problem and in the differential equations of chaotic classical mechanics associated with the problem.

In this paper, a different philosophy is assumed. The idea is to use the streamlines, the tangent lines to the velocity field, in order to associate a

https://doi.org/10.1017/S0022377898006643 Published online by Cambridge University Press

Hamiltonian dynamics to any incompressible turbulent fluid. The streamlines are defined for each value of the time, and their coordinates are expressed in general in terms of a variable that is not equal to the time. These streamlines will coincide with the pathlines used by other authors when the velocity field is time-independent, but this case could be considered as a particular case of this more general approach. The absence of an explicit time in the velocity is here interpreted as the possibility of existence of dynamic chaos, with no turbulence.

The main change with respect to the work of other authors is evident when one introduces an explicit time dependence in the velocity field. In this philosophy, the time plays the role of the codimension of the space of parameters in Hamiltonian dynamics. The possibility of associating a chaotic behaviour with the set of streamlines for each time produces a time-dependent chaotic behaviour as the set of streamlines changes with time. In this case, turbulence is chaos very similar to mechanical chaos, but moving, since each value of time can give a different image of that chaotic structure. The definitions and the differential equations satisfied by the pathlines and the streamlines are the same as in many textbooks (e.g. Currie 1974).

The hydrodynamic velocity $\mathbf{u}$ of the incompressible fluid satisfies the continuity equation

$$\nabla \cdot \mathbf{u} = 0. \tag{1}$$

In addition to this equation, the velocity $\mathbf{u}$ satisfies a nonlinear and time-dependent equation of motion such as the Euler or Navier–Stokes hydrodynamic equation:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla \mathbf{p} + \mathbf{F} + \ldots \quad . \tag{2}$$

The velocity should also obey some boundary conditions expressing adherence to fixed or moving walls and the time dependence of incoming and outgoing fluid into the system.

2. The hydrodynamics–mechanics equivalence

The problem of finding the streamlines tangent to an incompressible velocity is equivalent to a two-degree-of-freedom Hamiltonian dynamical system. This theorem is found in the literature associated with the field lines of a magnetic field, which obey the differential equations

$$\nabla \cdot \mathbf{u} = 0, \qquad \mathrm{d}\mathbf{r} \times \mathbf{u} = 0. \tag{3}$$

The hydrodynamic velocity replaces the magnetic field in these equations. This has been expressed as a variational principle by Morozov and Solovev (1966):

$$\delta \int dr^i \, \mathbf{A}_i(\mathbf{r}) = 0, \tag{4}$$

where $\mathbf{A}_i$ are the components of the vector potential for the hydrodynamic velocity:

$$\mathbf{u} = \nabla \times \mathbf{A}. \tag{5}$$

These lines can be found using a Hamiltonian formalism given by several authors. In particular, Cary and Littlejohn (1983) have developed a

https://doi.org/10.1017/S0022377898006643 Published online by Cambridge University Press

Hamiltonian formalism in non-canonical variables to find these lines; and Janaki and Ghosh (1987) have given a Hamiltonian formalism with the equations

$$p = \int dy\, u_3(x, y,), \tag{6a}$$

$$H = \int dy\, u_1(x, y, z), \tag{6b}$$

where u_1 and u_3 are two components of the velocity, and (x, y, z) are spatial coordinates. The Hamiltonian H is a function of the canonical variables x, p and the 'time' z. Piña and Ortiz (1988) have shown the equivalence of the formulations of Cary and Littlejohn (1983) and Janaki and Ghosh (1987).

In (6) the two components of the velocity are written only as functions of the position, not the time, to emphasize that in (6) the coordinate z plays the role of the mechanical 'time' whereas the real time of hydrodynamics is only a constant parameter in this formalism.

The converse theorem is also true. Birkhoff (1935), in considering the Hamiltonian dynamics of the restricted three-body problem (a two-degree-of-freedom problem), observed that one can associate the set of trajectories with the stationary movement of an incompressible fluid.

This idea can be generalized to any two-degree-of-freedom Hamiltonian dynamics. One can associate with it an incompressible flow as follows. Assume as starting point a Lagrangian of any two-degree-of-freedom dynamical system:

$$\tfrac{1}{2}[a_{11}(z_1, z_2)\,\dot{z}_1^2 + 2a_{12}(z_1, z_2)\,\dot{z}_1\,\dot{z}_2 + a_{22}(z_1, z_2)\,\dot{z}_2^2] + b_1(z_1, z_2)\,\dot{z}_1 + b_2(z_1, z_2)\,\dot{z}_2 - \phi(z_1, z_2), \tag{7}$$

where the dot denotes the derivative with respect to a real time and z_1 and z_2 are the two coordinates, and the functions a_{11}, a_{12}, a_{22}, b_1, b_2 and ϕ are functions of the coordinates (not explicitly written in the following).

By using Maupertius' principle, the orbits are solutions to the variational principle, which does not depend explicitly on the real time,

$$\delta \int \left[\left(2(E-\phi)[a_{11}(dz_1)^2 + 2a_{12}\,dz_1\,dz_2 + a_{22}(dz_2)^2] \right)^{1/2} + b_1\,dz_1 + b_2\,dz_2 \right] = 0, \tag{8}$$

where E is the constant of motion

$$E = \tfrac{1}{2} \sum_{i,j} a_{ij}\,\dot{z}_i\,\dot{z}_j + \phi. \tag{9}$$

Using the coordinate z_2 as a new independent variable, renamed s,

$$s = z_2, \tag{10}$$

an explicit s-dependent Lagrangian function L results from Maupertius' principle as

$$L = \left\{ 2(E-\phi) \left[a_{11} \left(\frac{dz_1}{ds} \right)^2 + 2a_{12} \frac{dz_1}{ds} + a_{22} \right] \right\}^{1/2} + b_1 \frac{dz_1}{ds} + b_2, \tag{11}$$

https://doi.org/10.1017/S0022377898006643 Published online by Cambridge University Press

and from it a Hamiltonian formalism follows:

$$p_1 = \frac{\partial L}{\partial (dz_1/ds)}$$

$$= b_1 + [2(E-\phi)]^{1/2} \frac{a_{11}\, dz_1/ds + a_{12}}{[a_{11}(dz_1/ds)^2 + 2a_{12}\, dz_1/ds + a_{22}]^{1/2}}, \tag{12a}$$

$$H = p_1 \frac{dz_1}{ds} - L$$

$$= -b_2 + [2(E-\phi)]^{1/2} \frac{-a_{12}\, dz_1/ds - a_{22}}{[a_{11}(dz_1/ds)^2 + 2a_{12}\, dz_1/ds + a_{22}]^{1/2}}. \tag{12b}$$

This can be recognized as a particular case of the previous formalism associated with the streamlines of the incompressible fluid by Janaki and Ghosh (1987) if one identifies

$$x = z_1, \qquad y = f(dz_1/ds), \qquad z = s = z_2, \tag{13}$$

where f is an arbitrary function.

The components of the hydrodynamic velocity, which has zero divergence, can be found if an equivalence is made between the corresponding terms of (6) and (12). Birkhoff's example in his treatment of the restricted three-body problem is a particular case of this calculation.

Note that in this case the 'real time' of the mechanical system does not appear in (12). The 'time' associated with the Hamiltonian system in (12) is the coordinate z_2.

3. Discussion

A consequence of these results is the possibility of relating all the experience obtained recently in the study of two-degree-of-freedom Hamiltonian systems (Lichtenberg and Leiberman 1983; Ozorio de Almeida 1988; Roberts and Quispel 1992; Arnold *et al.* 1993) with this particular way of regarding the turbulent flow of an incompressible fluid.

The first aspect to take into account is the chaotic dynamics of those systems that are highly unstable with respect to initial conditions (Lichtenberg and Leiberman 1983; Ozorio de Almeida 1988; Roberts and Quispel 1992; Arnold *et al.* 1993). The periodic orbits of these systems are isolated, and do not form parts of foliated structures. These periodic orbits are, in general, classified into two types: stable and unstable. The stable orbits have an infinite number of other longer-periodic orbits turning around them an integer number of windings. These turning orbits come in pairs of one stable and another unstable orbit of similar length. These orbits around a closed orbit are ordered from nearest to farthest by the Farey sequence, the denominator in that sequence corresponding to the winding number of the orbit. Between any two winding numbers, there exists another winding number.

After the identification made in this paper, we find that the hydrodynamic velocity field is tangent to streamlines that behave in a very similar way to the chaotic structures of Hamiltonian systems, including the same complex organization (Lichtenberg and Leiberman 1983; Ozorio de Almeida 1988;

https://doi.org/10.1017/S0022377898006643 Published online by Cambridge University Press

Roberts and Quispel 1992; Arnold *et al.* 1993). Around any closed streamline, a cable of closed streamlines is formed, winding around the shortest streamlines in ordered series of turnings according to the Farey sequence, in pairs of stable and unstable closed streamlines. These streamlines are located around the shorter streamline, separated by tubes around that streamline. These tubes are associated with KAM surfaces, which are densely covered by streamlines of infinite length.

The existence of whirlpools, densely distributed in the field, is explained by the fact that the neighbourhood of a closed stable streamline rotates with an angular velocity related to the Poincaré exponents of that line. These exponents determine the linear behaviour near a periodic orbit.

This rotation (and these exponents), change as functions of any parameter in the Hamiltonian (Lichtenberg and Leiberman 1983; Ozorio de Almeida 1988; Roberts and Quispel 1992; Arnold *et al.* 1993). For a time-dependent compressible fluid, such a parameter is the time, producing rotation of fluid around any closed stable streamline in the fluid.

If the associated Hamiltonian system is non-integrable then the periodic closed streamlines are densely distributed in the fluid in non-foliated structures, and the generation of whirlpools densely fills the regions of closed streamlines.

On changing a parameter in the Hamiltonian, one finds bifurcation of one stable orbit, which becomes unstable and produces two stable orbits, separating from the unstable one (Lichtenberg and Leiberman 1983; Ozorio de Almeida 1988; Roberts and Quispel 1992; Arnold *et al.* 1993). This phenomena, as time evolves, will cause in many places of the fluid the formation of two vortex rings from one, or the reverse phenomenon. These bifurcations have been studied in great detail by several authors (e.g. Greene *et al.* 1981).

The hyperbolic closed streamlines of the fluid are at the centre of extremely chaotic structures of homoclinic orbits approaching asymptotically to that unstable closed streamlines (Lichtenberg and Leiberman 1983; Ozorio de Almeida 1998; Roberts and Quispel 1992; Arnold *et al.* 1993).

A physical consequence implicit in the previous discussion is the fact that the extreme complexity of the turbulent fluid comes about from the property of mass conservation in the continuity equation. The nonlinearities of the equation of motion are responsible for the time change of the chaotic structure, but not necessarily for the existence of that structure.

On the other hand, laminar flow can be associated with integrable cases of the dynamics where the periodic orbits are parabolic and non-isolated, having neighbouring orbits of the same period, all of them on a surface that is a constant of motion. The constants of motion foliate the space, giving the ordered structure of the laminar fluid.

Because the chaotic structure shows its full complexity at small distances, after considering the molecular structure and macroscopic experiments, one needs to take account of this chaotic structure only for length scales that are not too small.

Therefore laminar flow should arise not only for exactly integrable Hamiltonians producing foliated structures, but also for those perturbed Hamiltonians that are nearly integrable (Lichtenberg and Leiberman 1983; Ozorio de Almeida 1988; Roberts and Quispel 1992; Arnold *et al.* 1993), with a set of parallel quasifoliated KAM surfaces, producing experimentally the

https://doi.org/10.1017/S0022377898006643 Published online by Cambridge University Press

appearance of an integrable Hamiltonian. The onset of turbulence should be associated with the moment at which a large number of the quasifoliated KAM surfaces disappear and the mixing of streamlines from all regions of the fluid is complete.

Let us now compare this theory with other work. One notes that the existence of a constant of motion in the Hamiltonian dynamics (12) gives the laminar integrability of the motion. The laminar behaviour of the two-dimensional stationary fluid dynamics, found by Aref and co-workers (Aref 1984; Aref and Balanchander 1986; Aref and Jones 1989) is a corollary. The chaotic structure that results when time dependence is included by these authors can be well understood from the particular point of view presented here.

The chaotic cases with Eulerian time-independent velocities considered by many authors (see the references in the first paragraph of Section 1) are particular cases of the present theory. Our streamlines coincide in such cases with pathlines that could be parameterized in terms of the real time, but do not move globally. These authors have previously found a chaotic dynamical structure called Lagrangian turbulence, but with less complexity than our chaotic moving structure, which exists only when the velocity field is time-dependent with no symmetry. Note also that the onset of turbulence has not been described by those authors, since an explicit time dependence in the velocity is avoided.

In conclusion, the set of streamlines of any incompressible fluid has, for each value of the time, the standard structure of the orbits of a Hamiltonian with two degrees of freedom. As time evolves, this set moves, as do the orbits of the dynamical system, which are modified by changes in the parameter of a parameter-dependent Hamiltonian. The onset of turbulence occurs when the KAM trajectories disappear.

References

Aref, H. 1984 *J. Fluid Mech.* **143**, 1.

Aref, H. and Balanchander, S. 1986 *Phys. Fluids* **29**, 3515.

Aref, H. and Jones, S. W. 1989 *Phys. Fluids* **A1**, 470.

Arnold, V. I. 1965 *C. R. Acad. Sci. Paris* **261**, 17.

Arnold, V. I., Kozlov, V. V. and Neishstadt, A. I. 1993 Mathematical aspects of classical and celestial mechanics. *Encyclopedia of Mathematical Sciences*, Vol. 3: *Dynamical Systems III* (ed. V. I. Arnold). Springer-Verlag, Berlin.

Birkhoff, G. D. 1935 *Ann. Sc. Norm. Sup. di Pisa, S. 2*, **4**, 267.

Cary, J. R. and Littlejohn, R. G. 1983 *Ann. Phys (NY)* **151**, 1.

Chaiken, J., Chu, C. K., Tabor, M. and Tan, M. Q. 1987 *Phys. Fluids* **30**, 687.

Chien, W. L. *et al.* 1986 *J. Fluid Mech.* **170**, 355.

Currie, I. G. 1974 *Fundamental Mechanics of Fluids*, pp. 38 and 40. McGraw-Hill, New York.

Dombre, T., Frisch, U., Greene, J. M., Hénon, M., Mehr, A. and Soward, A. M. 1986 *J. Fluid Mech.* **167**, 353.

Feingold, M., Kadanoff, L. P. and Piro, O. 1988 *J. Statist. Phys.* **50**, 529.

Greene, J. M., MacKay, R. S., Vivaldi, F. and Feigenbaum, M. J. 1981 *Physica* **D3**, 468.

Hénon, M., 1966 *C. R. Acad. Sci. Paris* **262**, 312.

Janaki, M. S. and Ghosh, G. 1987 *J. Phys. A: Math. Gen.* **20**, 3679.

Lau, Y.-T. and Finn, J. M. 1992 *Physica* **D57**, 283.

https://doi.org/10.1017/S0022377898006643 Published online by Cambridge University Press

Lichtenberg, A. J. and Leiberman, M. A. 1983 *Regular and Stochastic Motion*. Springer-Verlag, New York.

McLaughlin, J. B. 1988 *Phys. Fluids* **31**, 2544.

Morozov, A. I. and Solovev, L. S. 1966 *Reviews of Plasma Physics*, Vol. 2 (ed. M. A. Leontovich), p. 229. Consultants Bureau, New York.

Ozorio de Almeida, A. M. 1988 *Hamiltonian Systems*. Cambridge University Press.

Piña, E. and Ortiz, T. 1988 *J. Phys. A: Math. Gen.* **21**, 1293.

Polvani, L. M. and Wisdom, J. 1990 *Phys. Fluids* **A2**, 123.

Roberts, J. A. G. and Quispel, G. R. W. 1992 *Phys. Rep.* **216**, 63.

Wang, L.-P., Burton, T. D. and Stock, D. E. 1990 *Phys. Fluids* **A2**, 1305.

Wang, L.-P., Burton, T. D. and Stock, D. E. 1991 *Phys. Fluids* **A3** 1073.

Zaslavsky, G. M. and Tippett, M. K. 1991 *Phys. Rev. Lett.* **67**, 3251.

Zaslavskii, G. M., Sagedev, R. G. and Chernikov, A. A. 1988 *Soviet Phys. JETP* **67**, 270.

Ziglin, S. L. 1988 *Math. USSR Isv.* **31**, 407.

https://doi.org/10.1017/S0022377898006643 Published online by Cambridge University Press