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Multifractal formalism for Benedicks–Carleson quadratic maps

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Abstract. For a positive measure set of non-uniformly expanding quadratic maps on the interval we effect a multifractal formalism, i.e., decompose the phase space into level sets of time averages of a given continuous function and consider the associated *Birkhoff spectrum* which encodes this decomposition. We derive a formula which relates the Hausdorff dimension of level sets to entropies and Lyapunov exponents of invariant probability measures, and then use this formula to show that the spectrum is continuous. In order to estimate the Hausdorff dimension from above, one has to 'see' sufficiently many points. To this end, we construct a family of towers. Using these towers we establish a large deviation principle of empirical distributions, with Lebesgue as a reference measure.

1. *Introduction*

Let $X = [-1, 1]$, and let $f_a \colon X \circlearrowleft$ be the quadratic map given by $f_a x = 1 - ax^2$, where $0 < a \leq 2$. It is well known [**2, 3, 18**] that there exists a set of a-values near 2 with positive Lebesgue measure for which the corresponding $f = f_a$ admits an invariant probability measure μ that is absolutely continuous with respect to Lebesgue. In this paper we develop a theory of multifractal formalism for a positive measure set of these quadratic maps.

Given a function $\varphi \colon X \to \mathbb{R}$ we consider sets of the form

$$K_\varphi(\alpha) = \left\{ x \in X \colon \lim_{n\to\infty} \frac{1}{n} S_n\varphi(x) = \alpha \right\}, \quad \alpha \in \mathbb{R},$$

where $S_n\varphi = \sum_{i=0}^{n-1} \varphi \circ f^i$. The following characteristic of the sets $K_\varphi(\alpha)$ has been studied in the literature:

$$B_\varphi(\alpha) = \dim_H K_\varphi(\alpha),$$

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where $\dim_H$ denotes the Hausdorff dimension. This function of α is called a *Birkhoff spectrum* of φ. In the case $\varphi = \log|Df|$ it is called a *Lyapunov spectrum.* Multifractal formalism aims to relate these spectra to other characteristics of the system, and to study the regularity of the spectra as functions of α, for instance, continuity, smoothness and convexity. With this study one tries to get more refined descriptions of the dynamics than purely stochastic considerations.

In the creation of the theory of multifractal formalism, uniform hyperbolicity or the absence of critical points have been assumed to obtain good descriptions of the spectra (see, e.g., [**9, 26, 28–30, 35**]). Our aim here is to incorporate into the theory certain non-uniformly expanding quadratic maps on the interval with critical points. We provide a simple set of conditions satisfied on a positive measure set in the parameter space of the quadratic maps, and give a partial description of the Birkhoff spectrum when these conditions are met.

We formulate our conditions as follows:

(A1) $f = f_a$ where a is sufficiently near 2;

(A2) $|Df^n(f0)| \geq e^{\lambda n}$ for every $n \geq 0$, where $\lambda = \frac{9}{10}\log 2$;

(A3) $|f^n 0| \geq e^{-(1/100)\sqrt{n}}$ for every $n \geq 1$;

(A4) f is topologically mixing on $[f^2 0,\ f0]$.

Benedicks and Carleson [**3**] proved the abundance of parameters near 2 for which (A2) holds. For these parameters, there exists a unique absolutely continuous invariant probability measure μ (acip for short). The abundance of parameters for which (A3) holds was proved by Benedicks and Young [**4**], and previously by Benedicks and Carleson [**2**] under slightly different hypotheses. For their parameters, (A4) holds (see [**36**, Lemma 2.1]). The parameter sets they constructed have 2 as a full Lebesgue density point. Hence, given $a_0 < 2$ arbitrarily near 2, there is a set $A \subset [a_0, 2]$ with positive Lebesgue measure such that (A2)–(A4) hold for all $a \in A$.

Let $C(X)$ denote the space of continuous functions on X, and $\mathcal{M}_f$ the space of f-invariant probability measures endowed with the topology of weak convergence. For $\varphi \in C(X)$ define

$$c_\varphi = \inf_{x\in X} \underline{\lim}_{n\to\infty} \frac{1}{n} S_n\varphi(x) \quad \text{and} \quad d_\varphi = \sup_{x\in X} \overline{\lim}_{n\to\infty} \frac{1}{n} S_n\varphi(x).$$

Since $\mathcal{M}_f$ is compact and φ is continuous, one has $c_\varphi = \min\{\nu(\varphi)\colon \nu \in \mathcal{M}_f\}$ and $d_\varphi = \max\{\nu(\varphi)\colon \nu \in \mathcal{M}_f\}$, where $\nu(\varphi) = \int \varphi\, d\nu$. Define sets $K_\varphi(\alpha)$ as above, and consider the decomposition

$$X = \left(\bigcup_{\alpha\in[c_\varphi, d_\varphi]} K_\varphi(\alpha) \right) \cup \hat{K}_\varphi.$$

Here, $\hat{K}_\varphi$ is the set of points in X for which $(1/n)S_n\varphi$ does not converge. This decomposition has extremely complicated topological structures. Indeed, by (A4), $K_\varphi(\alpha)$ and $\hat{K}_\varphi$ are dense in X unless they are empty. If $c_\varphi < d_\varphi$, then $\hat{K}_\varphi$ is non-empty and carries the full Hausdorff dimension [**1, 6**]. Since c_φ and d_φ are attained by ergodic measures, both $K_\varphi(c_\varphi)$ and $K_\varphi(d_\varphi)$ are non-empty. Using (A4) one can construct points with time averages converging to any number $\alpha \in (c_\varphi, d_\varphi)$. Hence, any $K_\varphi(\alpha)$ in the decomposition is non-empty.

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Let $h(\nu)$ denote the entropy of $\nu \in \mathcal{M}_f$ and define $\lambda(\nu) = \int \log |Df| \, d\nu$, which we call the Lyapunov exponent of ν. This value is well defined [**5**], and by a result of [**25**],

$$\lambda_{\inf} = \inf\{\lambda(\mu) \colon \mu \in \mathcal{M}_f\} > 0.$$

Relationships between entropies, Lyapunov exponents, and dimensions of invariant probability measures were studied in the literature [**16, 23, 37**]. The next theorem relates the Birkhoff spectrum to entropies and Lyapunov exponents of invariant probability measures.

THEOREM A. *If $f = f_a$ satisfies* (A1)–(A4)*, then for any $\varphi \in C(X)$ and $\alpha \in [c_\varphi, d_\varphi]$,*

$$B_\varphi(\alpha) = \lim_{\varepsilon \to 0} \sup \left\{ \frac{h(\nu)}{\lambda(\nu)} \colon \nu \in \mathcal{M}_f, \, |\nu(\varphi) - \alpha| < \varepsilon \right\}.$$

In addition, the Birkhoff spectrum $\alpha \mapsto B_\varphi(\alpha)$ is continuous.

In [**6**], the first-named author derived the same formula as in Theorem A for a class of one-dimensional maps. This class includes maps whose critical points are non-recurrent and with no neutral or stable periodic point (the so-called Misiurewicz maps). Theorem A allows the recurrence of the critical point at a sub-exponential rate by condition (A3). Although φ is required to be continuous, an extension of the formula to cover the Lyapunov spectrum will be given in our forthcoming work.

The multifractal formalism for one-dimensional maps with critical points is a rapidly expanding area of research, and quite a few results have been obtained lately. For multimodal maps satisfying growth conditions of derivatives along the orbits of critical points, Iommi and Todd [**17**] obtained a formula which relates the Lyapunov spectra to thermodynamic pressures. See Gelfert *et al* [**13**] and Przytycki and Rivera-Letelier [**31**] for results on the Lyapunov spectra of rational maps on the Riemann sphere. A key idea common to these recent works is to construct a sequence of nice induced systems that 'exhausts' the original system. Although a proof of Theorem A relies on the same idea, our induced systems are equipped with a special recurrence property. This requires a new construction.

The formula in Theorem A yields several properties of the Birkhoff spectrum. For instance, it is easy to show that B_φ is monotone increasing on the interval $[c_\varphi, \mu(\varphi)]$, while it is monotone decreasing on $[\mu(\varphi), d_\varphi]$ as a function of α. From the formula it readily follows that B_φ is upper semi-continuous. We are able to show that B_φ is lower semi-continuous, and so it is continuous. This phenomenon illustrates what is sometimes called the *multifractal miracle*—even though the decomposition of the phase space into the level sets is intricate and extremely complicated, the function B_φ which encodes this decomposition is continuous.

It is an interesting problem to study better regularities of the spectrum. If the dynamics is uniformly hyperbolic and the function φ is Hölder continuous, then the spectrum is real analytic and concave [**30**]. For one-dimensional maps with parabolic fixed points, the non-analyticity of the Lyapunov spectra implies the finiteness of absolutely continuous invariant measures [**24**]. For quadratic maps, only numerical results are known (see, e.g., [**14, 15**]).

Our strategy for the lower estimate of $B_\varphi(\alpha)$ is to construct certain Cantor sets in $K_\varphi(\alpha)$, and then put probability measures on them for which the Mass Distribution Principle holds

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(see [**37**, Proposition 2.1]). The presence of the critical point does not matter because small derivatives tend to improve lower estimates of the Hausdorff dimension.

For the upper estimate, we approximate $B_\varphi(\alpha)$ from above by the dimensions of ergodic measures (cf. Proposition 3.1). To construct such an ergodic sequence we construct a family of uniformly hyperbolic induced systems with finitely many branches. We pick the corresponding family of equilibrium states for weighted geometric potentials, and then spread them out to produce a sequence of ergodic measures with the desired property.

The biggest difficulty is to construct such a family of induced systems. We do this in two steps. We first construct a family of towers, with a special property that a *positive definite fraction of points in each partition element quickly falls down to the ground floor*. For this construction we make an important use of condition (A3). We then construct the desired family of induced systems by choosing a subsystem from each tower dynamical system.

Using the family of towers used in the proof of Theorem A we establish a large deviation principle for the Lebesgue measure. Let $\mathcal{M}$ denote the space of probability measures on X endowed with the topology of weak convergence. Define a free energy function $F\colon \mathcal{M} \to \mathbb{R} \cup \{-\infty\}$ by

$$F(\nu) = \begin{cases} h(\nu) - \lambda(\nu) & \text{if } \nu \in \mathcal{M}_f, \\ -\infty & \text{otherwise.} \end{cases}$$

By Ruelle's inequality [**33**], $F(\nu) \le 0$, and the equality holds only if $\nu = \mu$ [**22**].

It is known [**5**] that the Lyapunov exponent is not lower semi-continuous, and so $-F$ may not be lower semi-continuous. Hence, we introduce its lower-semi-continuous regularization $I\colon \mathcal{M} \to [0, \infty]$ by

$$I(\nu) = -\inf_{\mathcal{G}} \sup\{F(\xi)\colon \xi \in \mathcal{G}\},$$

where the infimum is taken over all neighborhoods $\mathcal{G}$ of ν in $\mathcal{M}$. Denote by $|\cdot|$ the Lebesgue measure on X and let $\log 0 = -\infty$. Let $\delta_x^n = (1/n)\sum_{i=0}^{n-1} \delta_{f^i x}$ where $\delta_{f^i x}$ is the Dirac measure at $f^i x$.

THEOREM B. *Let $f = f_a$ satisfy* (A1)–(A4). *Then the large deviation principle holds for $(f, |\cdot|)$ with I the rate function, namely, for any open set $\mathcal{G} \subset \mathcal{M}$,*

$$\varliminf_{n\to\infty} \frac{1}{n} \log |\{x \in X\colon \delta_x^n \in \mathcal{G}\}| \ge -\inf\{I(\nu)\colon \mu \in \mathcal{G}\},$$

and for any closed set $\mathcal{K} \subset \mathcal{M}$,

$$\varlimsup_{n\to\infty} \frac{1}{n} \log |\{x \in X\colon \delta_x^n \in \mathcal{K}\}| \le -\inf\{I(\nu)\colon \mu \in \mathcal{K}\}.$$

The large deviation principle has been proved in different settings, for different reference measures and with different assumptions on the hyperbolicity of the systems [**10, 21, 27, 31**]. For the positive measure set of quadratic maps that we treat here, the large deviation principle for the acips was proved in [**8**]. Theorem B is not a consequence of this, because the density of the acip is unbounded.

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The Contraction Principle in large deviations [**12**] allows us to obtain a formula for fluctuations of time averages of continuous functions. Let $\varphi \in C(X)$. We assume $c_\varphi < d_\varphi$, for otherwise it is meaningless to consider φ. Define a function $F_\varphi : [c_\varphi, d_\varphi] \to \mathbb{R}$ by

$$F_\varphi(\alpha) = \sup\{F(\nu) : \nu \in \mathcal{M}_f,\ \nu(\varphi) = \alpha\}.$$

COROLLARY 1. *If $c_\varphi \le \alpha < \beta \le d_\varphi$, then*

$$\lim_{n\to\infty} \frac{1}{n} \log \left| \left\{ \alpha \le \frac{1}{n} S_n\varphi \le \beta \right\} \right| = \max_{\alpha \le t \le \beta} F_\varphi(t).$$

Keller and Nowicki [**20**] obtained a *local* result which claims the existence of the limit provided φ is Hölder continuous and α, β are sufficiently near the mean $\mu(\varphi)$. Corollary 1 is a *full* result with no restriction on α or β.

The next corollary follows from Varadhan's integral lemma [**12**, p. 137] and the convex duality of Fenchel–Legendre transforms [**12**, p. 152].

COROLLARY 2. *For any $\varphi \in C(X)$, the limit*

$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \int e^{S_n\varphi}\, dx$$

exists. In addition, (P, I) form a Legendre pair, namely the following holds:

$$P(\varphi) = \max\{\nu(\varphi) - I(\nu) : \nu \in \mathcal{M}_f\} \quad \text{for all } \varphi \in C(X),$$
$$I(\nu) = \max\{\nu(\varphi) - P(\varphi) : \varphi \in C(X)\} \quad \text{for all } \nu \in \mathcal{M}_f.$$

The rest of this paper consists of four sections. In §2, we construct a family of towers, collecting materials in [**8**] as far as needed. In §3, using this family of towers we estimate $B_\varphi(\alpha)$ from above. In §4, we estimate $B_\varphi(\alpha)$ from below and complete the proof of the formula in Theorem A. We then use this formula to prove the continuity of the Birkhoff spectrum. In §5, we prove Theorem B.

2. *Construction of a family of towers*

In this section, for a map f satisfying (A1)–(A4) we first introduce the machinery in [**8**] for recovering small derivatives near the critical point. We then construct a family of induced maps and associated towers. Important constants are $0 < \varepsilon \ll 1$ and $N \gg 1$, chosen in this order. In this section we suppose they are given. In §§3 and 5 we let ε approach 0.

We use the following standard notation: for a set $A \subset X$, $d(0, A) = \inf\{|x| : x \in A\}$; given a partition $\mathcal{P}$ of $A \subset X$ and $B \subset A$, $\mathcal{P}|B = \{\omega \cap B : \omega \in \mathcal{P}\}$.

2.1. *Recovering expansion.*

The next lemma states that the dynamics outside of a small neighborhood of the critical point is uniformly expanding with an exponent independent of the size of the neighborhood.

LEMMA 2.1. (Cf. [**8**, Lemma 2.5]) *The following holds for any $\hat\delta > 0$: if $x \in X$, $n \ge 1$ are such that $|f^i x| \ge \hat\delta$ for every $0 \le i \le n-1$, then $|Df^n(x)| \ge \hat\delta e^{(\lambda/3)n}$. Moreover, if $|f^n x| < \hat\delta$, then $|Df^n(x)| \ge e^{(\lambda/3)n}$.*

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A proof of this lemma is almost identical to that of [**8**, Lemma 2.5], and hence is omitted. Although particular values of $\hat{\delta}$ were chosen there, this choice is not essential.

To deal with the loss of expansion due to returns to the critical region, we mimic the binding argument of Benedicks and Carleson [**2, 3**]: subdivide the interval into pieces, and deal with them independently. For $p > 0$ let

$$\delta_p = \sqrt{\frac{e^{-\varepsilon p}}{10}\left[\sum_{i=0}^{p-1}\frac{|Df^i(f0)|}{|f^{i+1}0|}\right]^{-1}}. \tag{1}$$

If $\delta_p \le |x| < \delta_{p-1}$, then we regard the orbit of x as bound to the orbit of 0 up to time p.

LEMMA 2.2. *For any $\varepsilon > 0$ there exists $N > 0$ such that if $p \ge N$ and $\delta_p \le |x| < \delta_{p-1}$, then:*

(a) $|Df^p(x)| \ge e^{(\lambda/3)p}$;

(b) $\log|x|^{-2/\log 5} \le p \le \log|x|^{-2/\lambda}$.

Proof. (a) and the second inequality in (b) are due to [**8**, Lemma 2.3]. Rearranging $|x|^2 \ge \delta_p^2 \ge 5^{-p}$, which follows from the definition of δ_p in (1), and then using $|Df| \le 4$, (A3) yields the first inequality in (b). □

2.2. *Construction of a partition with slowly recurrent points.* We construct a partition of a small neighborhood of the critical point which is well-adapted to later constructions.

To start, for each $p > N$ cut the interval $[\delta_p, \delta_{p-1})$ into $\lfloor e^{3\varepsilon p}\rfloor$ intervals of equal length and denote them by $\hat{I}_{p,j}$ ($j = 1, 2, \ldots, [e^{3\varepsilon p}]$), from right to left. This defines a partition of the interval $(0, \delta_N)$, but it is not satisfactory for our construction, because there is no control over the iterates of the boundary points of the partition elements. To rectify this, we show in the next lemma the existence of a point in each $\hat{I}_{p,j}$ that is slowly recurrent to the critical point 0. We then use these points as partition points.

LEMMA 2.3. *For each (p, j) there exists $x \in \hat{I}_{p,j}$ such that $|f^n x| \ge \delta_N e^{-\varepsilon n}$ for every $n \ge \varepsilon^{-1}$.*

Proof. Set $t_0 = 0$, $\omega_0 = \hat{I}_{p,j}$ and $p_0 = p$. For every $\varepsilon^{-1} \le n \le p_0$ we have

$$d(0, f^n\omega_0) \ge |f^n 0| - |f^n\omega_0| \ge (1/2)|f^n 0| \ge (1/2)e^{-(\sqrt{n}/100)} \ge e^{-\varepsilon n}, \tag{2}$$

where we have used the bounded distortion of f^{n-1} on $f\omega_0$ from [**8**, Lemma 2.1] for the second inequality. The third one follows from (A3). The last one holds for sufficiently small ε.

By induction we choose a sequence $n_0 < n_1 < \cdots$ of integers and a sequence $\omega_0 \supsetneq \omega_1 \supsetneq \cdots$ of closed intervals such that, for every $k \ge 0$,

$$\begin{aligned} f^{n_k}\omega_k = \hat{I}_{p_k,j_k} \text{ for some } p_k,\ j_k \text{ and } d(0, f^n\omega_k) \ge \delta_N e^{-\varepsilon n} \\ \text{for every } n_k \le n \le n_k + p_k - 1. \end{aligned} \tag{3}$$

From (2) and (3), the point in the singleton $\bigcap_{k\ge 0}\omega_k$ satisfies the desired property.

For the rest of the proof, we assume (3) holds for some $k = l$, and then indicate how to choose t_{l+1} and ω_{l+1} for which (3) holds for $k = l + 1$. An argument to show (3) for $k = 0$ is included in the general step of the induction below.

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Given n_l, ω_l such that $f^{n_l}\omega_l = \hat{I}_{p_l,j_l}$, define $n_l + p_l \le t_1 < t_2 < \cdots$ inductively as follows: t_1 is the smallest $t \ge n_l + p_l$ with $d(0,\ f^t\omega_l) < \delta_N$. Suppose t_i has been defined. If $f^{t_i}\omega_l$ intersects no more than two $\hat{I}_{p,j}$-intervals, say $\hat{I}_{p,j}$ and $\hat{I}_{p',j'}$, $p \le p'$ (possibly $\hat{I}_{p,j} = \hat{I}_{p',j'}$), then define t_{i+1} to be the smallest $t \ge t_i + q_i$ with $d(0,\ f^t\omega_l) < \delta_N$, where $q_i = p$ if $f^{t_i}\omega_l \subset (-\delta_N, \delta_N)$, and $q_i = 1$ otherwise. If $f^{t_i}\omega_l$ intersects more than three $\hat{I}_{p,j}$-intervals, then t_{i+1} is undefined.

The expansion estimates in Lemmas 2.1 and 2.2 imply that one finally reaches t_s such that $f^{t_s}\omega_l$ intersects more than three $\hat{I}_{p,j}$-intervals. For all $\theta \in f^{n_l+p_l}\omega_l$ we have

$$|Df^{n_{l+1}-n_l-p_l}(\theta)| \ge \delta \exp\left(\frac{\lambda}{3} \sum_{\substack{1\le i\le s-1\\ q_i\ne 1}} q_i\right) \ge \delta_N. \tag{4}$$

This and $|f^{n_l+p_l}\omega_l| \ge e^{-5\varepsilon p_l}$, which follows from [**8**, Lemma 2.6(a)], yield $|f^{n_{l+1}}\omega_l| \ge \delta_N e^{-5\varepsilon p_l}$. By Lemma 2.2(b), $p_l \le (2/\lambda)(-\log\delta_N + \varepsilon n_l) \le (3/\lambda)\varepsilon n_l$, and thus $|f^{n_{l+1}}\omega_l| \ge \delta_N e^{-5\varepsilon p_l} \ge \delta_N e^{-(15/\lambda)\varepsilon^2 n_{l+1}}$. From this and the upper estimate of the length of $\hat{I}_{p,j}$ in [**8**, Lemma 2.6(b)], one can choose $\omega_{l+1} \subset \omega_l$ such that $f^{n_{l+1}}\omega_{l+1} = \hat{I}_{p_{l+1},j_{l+1}}$ and $d(0,\ f^{n_{l+1}}\omega_{l+1}) \ge \delta_N e^{-\varepsilon n_{l+1}}$.

It remains only to estimate the distance of the forward iterates of $f^{n_l+p_l-1}\omega_{l+1}$ to the critical point. We first consider the case $n = t_i$ with $q_i \ne 1$. Equation (4) implies that for some $\theta \in f^{n_l+p_l}\omega_l$ we have

$$2 \ge |f^{n_{l+1}}\omega_l| = |Df^{n_{l+1}-n_l-p_l}(\theta)| \cdot |f^{n_l+p_l}\omega_l| \ge \delta_N e^{(\lambda/3)q_i} e^{-5\varepsilon p_l}.$$

Taking logs and then rearranging the result we have

$$q_i \le -(4/\lambda)\log\delta_N + (15/\lambda)\varepsilon p_l \le (16/\lambda)\varepsilon p_l \le (48/\lambda^2)\varepsilon^2 n_l. \tag{5}$$

Hence, for $n = t_i$ we obtain

$$d(0,\ f^n\omega_l) \ge \delta_{q_i} \ge 5^{-q_i} \ge e^{-\varepsilon n_l} \ge \delta_N e^{-\varepsilon n}.$$

Next we consider the case $n \in (t_i, t_i + q_i]$ with $q_i \ne 1$. Let J denote the minimal interval containing $f^{t_i}\omega_l$ and 0. Then $|fJ| \le 2\delta_{q_i}^2$, and the distortion of f^{n-t_i-1} on fJ is bounded by 2 by [**8**, Lemma 2.1]. Hence

$$|f^{n-t_i}J| \le 2|Df^{n-t_i-1}(f0)||fJ| \le 4|Df^{n-t_i-1}(f0)|\delta_{q_i}^2 \le (9/10)|f^{n-t_i+1}0|,$$

and thus

$$\begin{aligned} d(0,\ f^n\omega_l) &\ge |f^{n-t_i+1}0| - |f^{n-t_i}J| \ge (1/10)|f^{n-t_i+1}0| \ge (1/10)e^{-(1/100)\sqrt{q_i}} \\ &\ge e^{-\varepsilon t_i} \ge \delta e^{-\varepsilon n}. \end{aligned}$$

The second inequality follows from (A3) and the third from (5).

Note that the above argument may be extended to the case $n \in [n_{l+1}+1, n_{l+1}+p_{l+1}-1]$. For all $n \in [n_l + p_l - 1, n_{l+1} - 1]$ other than those treated so far, the desired estimate holds because $f^n\omega_l$ is not contained in $(-\delta_N, \delta_N)$ and intersects at most one $\hat{I}_{p,j}$. The assumption of the induction has been recovered. □

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In view of Lemma 2.3, for each $\hat{I}_{p,j}$ fix once and for all a point $x_{p,j} \in \hat{I}_{p,j}$ such that $|f^n x_{p,j}| \geq \delta_N e^{-\varepsilon n}$ holds for every $n \geq \varepsilon^{-1}$. Set $\delta = x_{N,1}$. Note that $\delta < \delta_N$, and $\delta \to 0$ as $N \to \infty$. Using the points $x_{p,j}$ as partition points we construct a countable partition of the interval $(0, \delta)$ in such a way† that:

(i) each element of the partition contains exactly one element of $\{\hat{I}_{p,j}\}$;

(ii) each element of the partition is contained in three contiguous elements of $\{\hat{I}_{p,j}\}$.

The construction is straightforward. The boundary points of partition elements belong to $\{x_{p,j}\}$. Let $I_{p,j}$ denote the element of the partition containing $\hat{I}_{p,j}$. Let $I_{p,-j} = -I_{p,j}$, the mirror image of $I_{p,j}$ with respect to 0.

2.3. *Construction of dynamical partitions.* Let $\Lambda^+ = I_{N,1}$, which is the right extremal $I_{p,j}$-interval. Let $\Lambda^- = -\Lambda^+$ and $\Lambda = \Lambda^- \cup \Lambda^+$. Let $\hat{x}$ denote the orientation reversing fixed point of f in X and set $\hat{X} = [-\hat{x}, \hat{x}]$. By induction on the number of iterations we construct a 'decreasing' sequence $\{\tilde{\mathcal{P}}_n\}_{n\geq 0}$ of partitions of $\hat{X}$ into intervals, and introduce the notion of bound/free states.

Start with $\tilde{\mathcal{P}}_0 = \{[-\hat{x}, -\delta], [\delta, \hat{x}]\} \cup \{I_{p,j}\}_{p,j}$. We refer to the intervals $f[-\hat{x}, -\delta]$, $f[\delta, \hat{x}]$, $fI_{p,j}$ and $f^p I_{p,j}$ as *free*, and to $f^i I_{p,j}$ $(1 \leq i \leq p-1)$ as *bound*. Call p a *bound period of* $I_{p,j}$ *at time* 0.

Let $n \geq 1$. The f^n-images of elements of $\tilde{\mathcal{P}}_{n-1}$ are in two phases: either *bound* or *free*. If $\omega \in \tilde{\mathcal{P}}_{n-1}$, $f^n\omega$ is free and $d(0, f^n\omega) < \delta$, then $\tilde{\mathcal{P}}_n$ subdivides ω. For each resulting element $\omega' \in \tilde{\mathcal{P}}_n|\omega$ with $d(0, f^n\omega') < \delta$, an integer $p_n(\omega')$ is attached; this integer is called a *bound period of* ω' *at time* n.

Given $\omega \in \tilde{\mathcal{P}}_{n-1}$, $\tilde{\mathcal{P}}_n|\omega$ is defined as follows. If $f^n\omega$ is free and contains more than two $I_{p,j}$-intervals, then let $\tilde{\mathcal{P}}_n$ subdivide ω according to the (p, j)-locations of its f^n-image. In all other cases, let $\tilde{\mathcal{P}}_n|\omega = \{\omega\}$. Partition points are inserted only to ensure that the f^n-images of $\tilde{\mathcal{P}}_n$-elements intersecting $(-\delta, \delta)$ contain exactly one $I_{p,j}$. The f^n-images out of $(-\delta, \delta)$ are treated as follows. Let $\omega' \subset \omega$ be such that $f^n\omega'$ is a component of $f^n\omega \backslash (-\delta, \delta)$. We let $\omega' \in \tilde{\mathcal{P}}_n$ if $|f^n\omega'| \geq |\Lambda^+|$. Otherwise, we glue ω' to the adjacent element whose f^n-image is contained in $\Lambda^\pm$.

The bound periods at time n of the elements of $\tilde{\mathcal{P}}_n|\omega$ are determined by the p-locations of their f^n-images. Namely, if $\tilde{\mathcal{P}}_n$ subdivides ω and $\omega' \in \tilde{\mathcal{P}}_n|\omega$, then $p_n(\omega') = p$ where p is such that $f^n\omega' \supset I_{p,j}$ holds for some j. If $\tilde{\mathcal{P}}_n|\omega = \{\omega\}$, then $p_n(\omega) = \min\{p\colon I_{p,j} \cap f^n\omega \neq \emptyset \text{ for some } j\}$.

To proceed, for $\omega' \in \tilde{\mathcal{P}}_n$ we say $f^{n+1}\omega'$ is *bound* if there exists $k \leq n$ such that $\omega' \in \tilde{\mathcal{P}}_k$, $p_k(\omega')$ makes sense and satisfies $n + 1 < k + p_k(\omega')$. Otherwise we say $f^{n+1}\omega'$ is *free*. This completes the construction of $\tilde{\mathcal{P}}_n$ $(n = 0, 1, \ldots)$.

The following bounded distortion can be proved similarly‡ to [**8**, Lemma 2.7]. Set $C_0 = \exp(-\delta^3)$. Let $\omega \in \tilde{\mathcal{P}}_{n-1}$ and suppose that $f^n\omega$ is free. Then

$$\frac{|Df^n(x)|}{|Df^n(y)|} \leq C_0 \quad \text{for all } x, y \in \omega. \tag{6}$$

† Note that not all $x_{p,j}$ are used in this construction, because of the requirement (ii).

‡ Although the value of δ is slightly different from the one used in [**8**], the technical adjustment is minimal.

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2.4. *Inducing time estimates.* We define inductively a partition $\mathcal{Q}$ of Λ into intervals and an associated inducing time $R\colon \mathcal{Q} \to \mathbb{N}$ as follows. Let $\omega \in \tilde{\mathcal{P}}_{n-1}|\Lambda$. If $f^n\omega$ is free and $f^n\omega \supset 3\Lambda^+$ or $3\Lambda^-$, then set $\omega \cap f^{-n}\Lambda^+ \in \mathcal{Q}$ or $\omega \cap f^{-n}\Lambda^- \in \mathcal{Q}$, and $R(\omega) = n$. We iterate the remaining parts $f^n\omega \backslash \Lambda^+$ or $f^n\omega \backslash \Lambda^-$, which is the union of elements of $\tilde{\mathcal{P}}_n$, and repeat the same procedure. By definition, for each $\omega \in \mathcal{Q}$, $f^{R(\omega)}$ sends ω diffeomorphically onto Λ^+ or Λ^-.

Set $\hat{\zeta} = |\Lambda^+|/(2C_0) \in (0, 1)$ and $C_1 = 1 + \hat{\zeta}^{-1}$. Note that $\hat{\zeta} \to 0$ and $C_1 \to \infty$ as $\delta \to 0$. Let $\theta = 10^{-10\,000}$ and set $\zeta = \max\{e^{-(\lambda/14)}, (1 - \hat{\zeta})^\theta\}$. Lemma 2.4 below applied to $\Lambda^\pm$ implies that the measure of the tail set

$$\{R > n\} = \bigcup_{\omega \in \mathcal{Q}\colon\, R(\omega) > n} \omega$$

decays exponentially fast. In particular, $\mathcal{Q}$ is a partition of a full measure subset of Λ.

LEMMA 2.4. *There exists $k_0 = k_0(\delta)$ such that the following holds for every $k \geq k_0$: let $\omega \in \tilde{\mathcal{P}}_{k-1}$ and suppose that $\omega \subset \{R > k\}$ and $f^k\omega$ is free. Then*

$$|\{R > k + l\} \cap \omega| \leq C_1\zeta^l|\omega| \quad \textit{for every } l \geq (16\varepsilon/\lambda)k.$$

Proof. Let $\mathcal{Q}'$ denote the set of all $\omega' \in \mathcal{Q}|\{R > k + l\} \cap \omega$ for which there exists $n \in [k, k + l]$ such that $d(0, f^n\omega_n) < \delta$ holds for the element $\omega_n \in \tilde{\mathcal{P}}_n$ containing ω'. Let $\mathcal{Q}''$ denote the collection of elements of $\mathcal{Q}|\{R > k + l\} \cap \omega$ which do not belong to $\mathcal{Q}'$.

Each $\eta \in \mathcal{Q}'$ has an *itinerary* $(n_1, p_1, j_1), \ldots, (n_s, p_s, j_s)$ $(s \leq [l/N])$ that is defined as follows: $k \leq n_1 < \cdots < n_s \leq k + l$ is a sequence of integers, associated with a sequence $\omega_0 \supset \omega_{n_1} \supset \cdots \supset \omega_{n_s} \supset \eta$ of intervals such that ω_{n_i} $(i = 1, \ldots, s)$ is the element of $\tilde{\mathcal{P}}_{n_i}$ containing ω that arises out of the subdivision at time n_i, with $I_{p_i,j_i} \subset f^{n_i}\omega_{n_i}$; $\omega_{n_s} \in \tilde{\mathcal{P}}_{k+l-1}$. Since $|f^{n_s+p_s}\omega_{n_s}| \leq 2$, for some $x \in \omega_{n_s}$ we have $|\omega_{n_s}| \leq 2|Df^{n_s+p_s}(x)|^{-1}$. By Lemmas 2.1 and 2.2(a) we have $|Df^{n_s+p_s}(x)| \geq \delta e^{(\lambda/3)\sum_{i=1}^s p_i}|Df^k(x)|$, and by (6), $|Df^k(x)| \geq (1/C_0)|f^k\omega|/|\omega|$. Combining these three inequalities we obtain

$$|\eta| \leq |\omega_{n_s}| \leq 2C_0\delta^{-1}e^{-(\lambda/3)\sum_{i=1}^s p_i}\frac{|\omega|}{|f^k\omega|},$$

and therefore

$$\sum_{\eta\in\mathcal{Q}'}|\eta| \leq 2C_0\delta^{-1}\sum_s\sum_P e^{-(\lambda/3)P}\#\left\{\{(n_i, p_i, j_i)\}_{i=1}^s\colon \sum_{i=1}^s p_i = P\right\}\frac{|\omega|}{|f^k\omega|}.$$

From the proof of [**8**, Lemma 2.8] the cardinality is $\leq e^{\varepsilon n}e^{4\varepsilon P}$, and from the proof of [**8**, Sublemma 2.9] $n_{i+1} - n_i \leq 2p_i$ holds for every $1 \leq i \leq s$, where $n_{s+1} > k + l$ is such that $\tilde{\mathcal{P}}_{n_{s+1}}$ partitions ω_{n_s}. It follows that $l < n_{s+1} \leq n_1 + 2\sum_{i=1}^s p_i$. If $n_1 \leq k + l/2$ then $\sum_{i=1}^s p_i \geq l/4$, and therefore

$$\sum_{\substack{\eta\in\mathcal{Q}'\\ n_1\leq k+l/2}}|\eta| \leq C_0\delta^{-1}\frac{l}{N}\sum_{P\geq l/4}e^{(8\varepsilon-(\lambda/3))P}\frac{|\omega|}{|f^k\omega|} \leq e^{-(\lambda/13)l}\frac{|\omega|}{|f^k\omega|}, \tag{7}$$

where the last inequality holds provided k is sufficiently large because $l \geq \sqrt{\varepsilon}k$.

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For those $\eta \in \mathcal{Q}'$ with $n_1 > k + l/2$, a similar reasoning shows

$$|\eta| \le |\omega_{n_1}| \le C_0 \delta^{-1} e^{-\lambda(n_1-k)} \frac{|\omega|}{|f^k\omega|} \le C_0 \delta^{-1} e^{-(\lambda l/2)} \frac{|\omega|}{|f^k\omega|} \le e^{-(\lambda l/3)} \frac{|\omega|}{|f^k\omega|},$$

and therefore

$$\sum_{\substack{\eta\in\mathcal{Q}' \\ n_1>k+l/2}} |\eta| \le C_0\delta^{-1} \frac{l}{N} \sum_{P\le l} e^{4\varepsilon P+\varepsilon l-(\lambda/3)l} \frac{|\omega|}{|f^k\omega|} \le e^{-(\lambda l/4)} \frac{|\omega|}{|f^k\omega|}. \tag{8}$$

We now treat elements of $\mathcal{Q}''$. Let $t_1 \ge k$ be such that ω is subdivided at time t_1. Since $I_{p,j} \supset f^n\omega$ holds for some $n < k$ we have $|f^k\omega| \ge \delta e^{-5\varepsilon k}$. If $t_1 - k \ge 16(\varepsilon/\lambda)k$, then $|f^{t_1}\omega| \ge \delta e^{-5\varepsilon k} e^{(\lambda/3)(t_1-k)} \ge \delta e^{\varepsilon k} > 2 = |X|$, which is a contradiction. Hence $t_1 - k < (16\varepsilon/\lambda)k$, and so $t_1 < k + l$.

Let $t \ge k$. We say $\tilde{\omega} \in \tilde{\mathcal{P}}_t|\omega$ is an *escaping component at time t* if $\tilde{\omega}$ arises out of subdivision at time t and satisfies $d(0,\ f^t\tilde{\omega}) = \delta$. Let $\mathcal{E}_1$ denote the collection of escaping components at time $k + r$. If $\mathcal{E}_1 = \emptyset$, then $\mathcal{Q}'' = \emptyset$. Hence we assume $\mathcal{E}_1 \ne \emptyset$.

Each $\eta \in \mathcal{Q}''$ has an *itinerary* $(t_1, \epsilon_1), \ldots, (t_q, \epsilon_q)$ that is defined as follows: $k \le t_1 < \cdots < t_q < k + l$ is a sequence of integers, associated with a nested sequence $\omega \supset \omega_{t_1} \supset \cdots \supset \omega_{t_q} \supset \eta$ of intervals such that for each i, ω_{t_i} is an escaping component at time t_i and $\epsilon_i = +$ (respectively $\varepsilon_i = -$) if $f^{t_i}\omega_{t_i}$ is right (respectively left) of the critical point; ω_{t_q} is the smallest escaping component containing η. Call q the *length of the itinerary of* η.

For $\theta > 0$ let $\mathcal{Q}''_{\le\theta l} = \{\eta \in \mathcal{Q}'':$ the length of the itinerary is $\le \theta l\}$. The number of all itineraries of length q is $\le 2^q \binom{l}{q}$, and so the Stirling formula implies one can choose small θ such that $\#\mathcal{Q}''_{\le\theta l} \le e^{\lambda l/100}$. Then

$$\sum_{\eta\in\mathcal{Q}''_{\le\theta l}} |\eta| \le \#\mathcal{Q}''_{\theta l} e^{-\lambda l/2} \frac{|\omega|}{|f^k\omega|} \le e^{-(\lambda/3)l} \frac{|\omega|}{|f^k\omega|}. \tag{9}$$

To treat elements in $\mathcal{Q}''_{>\theta l} = \{\eta \in \mathcal{Q}'':$ the length of the itinerary is $> \theta l\}$, define for each $q \ge 1$ a collection $\mathcal{E}_q$ of escaping components (at variable times) inductively as follows: each $\omega \in \mathcal{E}_q$ is an escaping component at some time, say $t = t(\omega)$. Let $t' > t$ denote the time at which ω is subdivided. Then ω contains no or at most two escaping components at time t'. We let them be in $\mathcal{E}_{q+1}$. Let $E_q = \bigcup_{\omega\in\mathcal{E}_q} \omega$. If $\omega \in \mathcal{E}_q$ and $\omega \cap E_{q+1} \ne \emptyset$, then from the bounded distortion (6),

$$\frac{|\omega\backslash E_{q+1}|}{|\omega|} \ge C_0^{-1} \frac{|f^{t'}(\omega\backslash E_{q+1})|}{|f^{t'}\omega|} \ge C_0^{-1} \frac{|\Lambda^+|}{|X|} = \hat{\zeta},$$

and so $|\omega \cap E_{q+1}| \le |\omega| - |\omega\backslash E_{q+1}| \le (1-\hat{\zeta})|\omega|$. Hence, $|E_{q+1}| \le (1-\hat{\zeta})|E_q|$, and thus $|E_q| \le (1-\hat{\zeta})^q|\omega|$. By definition, if the itinerary of $\eta \in \mathcal{Q}''_{>\theta l}$ is of length q, then η is contained in an element of $\mathcal{E}_q$. Hence,

$$\sum_{\eta\in\mathcal{Q}''_{>\theta l}} |\eta| \le \sum_{\theta l\le q\le l} |E_q| \le \sum_{q\ge\theta l} (1-\hat{\zeta})^q \le \hat{\zeta}^{-1}(1-\hat{\zeta})^{\theta l}. \tag{10}$$

Since θ is independent of δ and $\hat{\zeta} \to 0$ as $\delta \to 0$, we have $e^{-(\lambda/14)} \le \zeta$. Equations (7)–(10) yield $|\{R > k + l\} \cap \omega| \le e^{-(\lambda/14)l} + \hat{\zeta}^{-1}(1-\hat{\zeta})^{\theta l} \le C_1\zeta^l$. □

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2.5. *Bounded distortion.* We prove a statement on distortions. Let J be an interval. A differentiable map $g\colon J \to \mathbb{R}$ without a critical point has *distortion bounded by* $\kappa \geq 1$ if

$$\sup_{x,y\in J} \frac{|Dg(x)|}{|Dg(y)|} \leq \kappa.$$

Let $J \subset T$ be two intervals and $n > 0$ such that $f^n|T$ is strictly monotone. We say f^nT contains a ξ-scaled neighborhood of f^nJ if the lengths of both components of $f^n(T\setminus J)$ are $\geq \xi|f^nJ|$. The following is known as the Koebe Principle [**11**, Ch. IV.1].

LEMMA 2.5. *Let $J \subset T$ be two intervals and $n > 0$ such that $f^n|T$ is strictly monotone and f^nT contains a ξ-scaled neighborhood of f^nJ. Then $f^n|J$ has distortion bounded by $((1+\xi)/\xi)^2$.*

Let $J \subset \hat{X}$ be an interval and $n > 0$. We say f^nJ is a *free segment* (respectively *bound segment*) if it is the union of elements of $\tilde{\mathcal{P}}_n$, and for any $\omega \in \tilde{\mathcal{P}}_n|J$, $f^n\omega$ is free (respectively bound). A free segment f^nJ is *maximal* if there is no interval $I \subset \hat{X}$ containing J such that f^nI is a free segment.

LEMMA 2.6. *If $n \geq N$ and f^nJ is a maximal free segment not containing $\{\hat{x}, -\hat{x}\}$, then there is an interval $T \supset J$ such that $f^n|T$ is strictly monotone and f^nT contains an $e^{-6\varepsilon n}$-scaled neighborhood of f^nJ. In particular, $f^n|J$ has distortion bounded by $e^{13\varepsilon n}$.*

Proof. By the assumption, to each side of J is attached an interval $\omega \in \tilde{\mathcal{P}}_n$ such that $f^n\omega$ is bound. Let k denote the maximal $i < n$ such that $f^i\omega$ is free and set $p = p_i(\omega)$. Then $k < n < k + p$. If $k + p = n + 1$, then using $|Df| \leq 4$ and [**8**, Lemma 2.6(a)] we have

$$|f^n\omega| \geq (1/4)|f^{n+1}\omega| = (1/4)|f^{k+p}\omega| \geq (1/4)e^{-5\varepsilon p} \geq e^{-6\varepsilon n}.$$

If $k + p > n + 1$, then $p = N$ and so

$$|f^n\omega| \geq (1/4)^{k+p-n}|f^{k+p}\omega| \geq (1/4)^N|f^{k+p}\omega| \geq (1/4)^N e^{-5\varepsilon N} \geq e^{-6\varepsilon n}. \qquad \square$$

2.6. *Construction of finite partitions.* The partition $\tilde{\mathcal{P}}_n$ restricted to $\{R > n\}$ is actually too fine to be used for an upper estimate of the Hausdorff dimension. Hence, we construct a finite partition $\mathcal{P}_n$ by gluing some elements of $\tilde{\mathcal{P}}_n$.

Start with $\mathcal{P}_0 = \{\Lambda^-, \Lambda^+\}$. Assume inductively that $\mathcal{P}_{n-1}$ has been constructed with the following properties:

(P1)$_{n-1}$ it is a partition of the set $\{R > n-1\}$ into a finite number of intervals each of which is the union of a countable number of elements of $\tilde{\mathcal{P}}_{n-1}|\{R > n-1\}$;

(P2)$_{n-1}$ for any $\omega \in \mathcal{P}_{n-1}$ let

$$\mathrm{fr}(\omega) = \bigcup\{\omega' \in \tilde{\mathcal{P}}_{n-1}|\omega\colon f^n\omega' \text{ is free}\}$$

and

$$\mathrm{bo}(\omega) = \bigcup\{\omega' \in \tilde{\mathcal{P}}_{n-1}|\omega\colon f^n\omega' \text{ is bound}\}.$$

These two sets are intervals unless empty. In addition, $\mathrm{fr}(\omega)$ is the union of at most $[e^{3\varepsilon(n-1)}]$ elements of $\tilde{\mathcal{P}}_{n-1}$.

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Let $\omega \in \mathcal{P}_{n-1}$ and write $\{R = n\} = \{R > n-1\} \backslash \{R > n\}$. The partition $\mathcal{P}_n$ on $\omega \backslash \{R = n\}$ is defined as follows. Let $\mathrm{bo}(\omega) \in \mathcal{P}_n$ unless empty. If $\mathrm{fr}(\omega) \neq \emptyset$ then there are two cases:

- if $\mathrm{fr}(\omega) \cap \{R = n\} = \emptyset$, then define $\mathcal{P}_n | \mathrm{fr}(\omega)$ by dividing $f^n \mathrm{fr}(\omega)$ into at most two intervals, one of which is to the right of 0 and the other to the left of 0;
- if $\mathrm{fr}(\omega) \cap \{R = n\} \neq \emptyset$, then $\mathrm{fr}(\omega) \backslash \{R = n\}$ consists of at most three intervals ω^-, ω^+, ω^0, where the corresponding f^n-images are: to the left of Λ^-; to the right of Λ^+; in between Λ^- and Λ^+. Let $\omega^\pm \in \mathcal{P}_n$ unless empty. Finally, define $\mathcal{P}_n | \omega^0$ by dividing $f^n \omega^0$ into at most two intervals, one of which is to the right of 0 and the other to the left of 0.

This completes the definition of $\mathcal{P}_n$. $(\mathrm{P1})_n$ holds by construction. To see $(\mathrm{P2})_n$, let $\omega \in \mathcal{P}_n$. The subdivision algorithm described in §2.3 and the 'monotonicity' of the bound periods with respect to the distance to the critical point imply that $\mathrm{fr}(\omega)$, $\mathrm{bo}(\omega)$ are intervals or empty sets. By construction, $\mathrm{fr}(\omega) \in \tilde{\mathcal{P}}_n$, or else it is made up of elements of $\tilde{\mathcal{P}}_n$ with the same latest bound period at the same time k, $k < n+1$. Hence $\mathrm{fr}(\omega)$ is the union of at most $e^{3\varepsilon n}$ elements of $\tilde{\mathcal{P}}_n$.

2.7. *Abundance of long free segments.* The next lemma allows us to find long free segments in generic partition elements.

LEMMA 2.7. *There exists $k_1 = k_1(\delta) > 0$ such that if $k \geq k_1$ and $\omega \in \mathcal{P}_k$, then there exist $q \in [k+1, (1+3\varepsilon/\lambda)k]$ and $\omega' \in \tilde{\mathcal{P}}_q | \omega$ such that:*

(a) *$f^q \omega'$ is free;*

(b) *$|\omega'| \geq e^{-18\varepsilon k} |\omega|$;*

(c) *$\omega' \subset \{R > q-1\}$.*

Proof. We first consider the case $|\mathrm{fr}(\omega)| \geq (1/2)|\omega|$. By $(\mathrm{P2})_k$, $\mathrm{fr}(\omega)$ is the union of at most $[e^{3\varepsilon k}]$ elements of $\tilde{\mathcal{P}}_k$. Hence it is possible to choose $\omega' \in \tilde{\mathcal{P}}_k | \mathrm{fr}(\omega)$ such that $|\omega'| \geq (1/2) e^{-3\varepsilon k} |\omega|$. Set $q = k+1$. Then (a) and (b) hold. (c) holds because $\omega \subset \{R > k\} = \{R > q-1\}$.

We now consider the case $|\mathrm{bo}(\omega)| \geq (1/2)|\omega|$. In this case, we shall choose ω' to be a certain subinterval of $\mathrm{bo}(\omega)$. Let i denote the maximal $j \leq k$ such that $f^j \mathrm{bo}(\omega)$ is a free segment. Let r denote the minimum of the bound period $p_i : \tilde{\mathcal{P}}_i | \mathrm{bo}(\omega) \to \mathbb{N}$ at time i. Set $q = i + r$.

SUBLEMMA 2.8. *$r \leq k$ and $k+1 < q \leq (1+3/\lambda)k$.*

Proof. Since $f^{k+1} \mathrm{bo}(\omega)$ is a bound segment, $k+1 < q$. To show the rest, for $i-1 \leq j \leq k+1$ let ω_j denote the element of $\mathcal{P}_j$ containing $\mathrm{bo}(\omega)$. We have $\omega_{i-1} \supset \omega_i \supset \cdots \supset \omega_k \supset \omega_{k+1}$, $\omega_k = \omega$ and $\omega_{k+1} = \mathrm{bo}(\omega)$. Note that $\tilde{\mathcal{P}}_j | \omega_j = \{\omega' \in \tilde{\mathcal{P}}_i | \omega_i : p_i(\omega') > j - i\}$. We treat two cases separately.

Case I: $\omega_i = \mathrm{bo}(\omega)$. Since $f^i \mathrm{bo}(\omega)$ is a free segment, $\omega_i \subset \mathrm{fr}(\omega_{i-1})$. Since $\mathrm{fr}(\omega_{i-1})$ is the union of elements of $\tilde{\mathcal{P}}_{i-1}$, for any point x in the boundary of $\mathrm{fr}(\omega_{i-1})$ we have $|f^i x| \geq \delta e^{-\varepsilon i}$. In other words, $f^i \mathrm{fr}(\omega_{i-1})$ is not contained in $(-\delta e^{-\varepsilon i}, \delta e^{-\varepsilon i})$, and the same holds for $f^i \omega_i$. By Lemma 2.2(b) we have $r \leq (3\varepsilon/\lambda) i < k$, and so $q \leq k + r \leq (1 + 3\varepsilon/\lambda)k$ for sufficiently large k.

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Case II: $\omega_i \supsetneq \mathrm{bo}(\omega)$. Let r' denote the mimimum of the bound period $p_i : \tilde{\mathcal{P}}_i|\omega_i \to \mathbb{N}$ at time i. If $i + r' \le k + 1$, then the monotonicity of the bound period implies $\omega_i = \cdots = \omega_{i+r'-1} \supsetneq \omega_{i+r'} \supsetneq \cdots \supsetneq \omega_{k+1}$, and $\omega_{k+1} \notin \mathcal{P}_{k+2}$. This implies $r = k + 2 - i$, and so $q = k + 2 \le (1 + 3\varepsilon/\lambda)k$. If $i + r' > k + 1$, then $\mathrm{fr}(\omega) \cap \omega_i = \emptyset$, and thus all $\mathrm{fr}(\omega)$, $\mathrm{bo}(\omega)$, ω_i share exactly one boundary point. Since f^i sends ω diffeomorphically onto its image, $f^i\mathrm{bo}(\omega)$ must come close to the boundary of $(-\delta, \delta)$ so that $r = N$. Hence, $q \le (1 + 3\varepsilon/\lambda)k$. □

Choose $\omega' \in \tilde{\mathcal{P}}_i|\mathrm{bo}(\omega)$ such that $r = p_i(\omega')$. Then $f^q\omega'$ is free, and $\omega' \subset \{R > q - 1\}$. As for (b), since $f^i\omega'$ contains some $I_{r,j}$ we have $|f^i\omega'| \ge (\delta_{r-1} - \delta_r)e^{-3\varepsilon r}$. Since $|f^i\mathrm{bo}(\omega)| \le \delta_{r-1}$ and $\delta_r \le e^{-(\varepsilon/2)}\delta_{r-1}$,

$$\frac{|f^i\omega'|}{|f^i\mathrm{bo}(\omega)|} \ge \frac{(\delta_{r-1} - \delta_r)e^{-3\varepsilon r}}{\delta_{r-1}} \ge (1 - e^{-(\varepsilon/2)})e^{-3\varepsilon r}.$$

Suppose that $\mathrm{bo}(\omega)$ is contained in an interval which does not contain $\{\pm\hat{x}\}$ and whose f^i-image is a maximal free segment. By Lemma 2.6 and $r \le k$ in Sublemma 2.8,

$$|\omega'| \ge e^{-13\varepsilon i}(1 - e^{-(\varepsilon/2)})e^{-3\varepsilon r}|\mathrm{bo}(\omega)| \ge e^{-17\varepsilon k}|\mathrm{bo}(\omega)| \ge (1/2)e^{-17\varepsilon k}|\omega| \ge e^{-18\varepsilon k}|\omega|.$$

Even if the above is not the case, the proof of Lemma 2.6 implies essentially the same distortion bounds, and so the same lower estimate of $|\omega'|$ holds. □

2.8. *Special property of the partition.* The next lemma asserts that a positive definite fraction of points in each element of $\mathcal{P}_k$ quickly return to the base Λ.

LEMMA 2.9. *There exists $k_2 \ge \max\{k_0, k_1\}$ such that, if $k \ge k_2$ and $\omega \in \mathcal{P}_k$, then there exists $\tilde{\omega} \in \mathcal{Q}$ such that:*

(a) $\tilde{\omega} \subset \omega$ *and* $|\tilde{\omega}| \ge e^{-\sqrt{\varepsilon}k}|\omega|$;

(b) $k < R(\tilde{\omega}) \le (1 + 19\varepsilon/\lambda)k$.

Proof. Choose $q \in [k + 1, (1 + 3\varepsilon/\lambda)k]$ and $\omega' \in \tilde{\mathcal{P}}_q|\omega$ for which the conclusions of Lemma 2.7 hold. By Lemma 2.4,

$$|\omega' \cap \{R < q + (16\varepsilon/\lambda)k\}| \ge (1 - C_1\zeta^k)|\omega'| \ge (1/2)|\omega'|.$$

Since the f^r-image of $f^q\omega'$ is folded at most 2^r times, $\#\{\omega \in \mathcal{Q}|\omega' : R(\omega) = q + r\} \le 2^{r+1}$ and so

$$\begin{aligned}\#\{\omega \in \mathcal{Q}|\omega' : R(\omega) < q + (16\varepsilon/\lambda)k\} &= \sum_{r=0}^{[(16\varepsilon/\lambda)k]} \#\{\omega \in \mathcal{Q}|\omega' : R(\omega) = q + r\} \\ &\le \sum_{r=0}^{[(16\varepsilon/\lambda)k]} 2^{r+1} \le e^{(17\varepsilon/\lambda)k},\end{aligned}$$

where the last inequality holds for sufficiently large k. Then it is possible to choose $r \le (16\varepsilon/\lambda)k$ and $\tilde{\omega} \in \mathcal{Q}|\omega'$ such that $R(\tilde{\omega}) = q + r \le (1 + 19\varepsilon/\lambda)k$ and $|\tilde{\omega}| \ge (1/2)e^{-(17\varepsilon/\lambda)k}|\omega'|$. From this and Lemma 2.7(b) we obtain (a). □

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2.9. *Towers.* We now translate Lemma 2.9 into the language of towers. By Lemma 2.4, we may think of $R\colon \mathcal{Q} \to \mathbb{N}$ as a function on a full measure subset of Λ in the obvious way. Let

$$\Delta = \{(x, \ell)\colon x \in \Lambda,\ \ell = 0, 1, \ldots, R(x) - 1\},$$

which we call a *tower*, and define a *tower map* $\hat{f}\colon \Delta \circlearrowleft$ by

$$\hat{f}(x, \ell) = \begin{cases} (x, \ell + 1) & \text{if } \ell + 1 < R(x), \\ (f^{R(x)}x, 0) & \text{if } \ell + 1 = R(x). \end{cases}$$

The point (x, ℓ) is considered to be climbing the tower in the first case, and falling down from the tower in the second case. Define $\Delta_\ell = \{(x, \ell) \in \Delta\colon R(x) > \ell\}$. With the canonical identification $\{R > \ell\} \ni x \mapsto (x, \ell) \in \Delta_\ell$ we transplant the partition $\mathcal{P}_\ell$ of $\{R > \ell\}$ to the partition of Δ_ℓ and also denote it by $\mathcal{P}_\ell$. Let $\mathcal{D} = \bigcup_{\ell \geq 0} \mathcal{P}_\ell$. This is a partition of Δ with a Markov property: for any $\omega \in \mathcal{D}$, $\hat{f}\omega$ is a finite union of elements of $\mathcal{D}$. We identify $\Delta_0 = \{(x, 0)\colon x \in \Lambda\}$ with Λ under the action of the map $\pi\colon \Delta \to \Lambda$ given by $\pi(x, \ell) = x$.

LEMMA 2.10. *The following holds for sufficiently large n: for any $A \in \bigvee_{i=0}^{n-1} \hat{f}^{-i}\mathcal{D}$ with $A \subset \Delta_0$ there exist an interval $\tilde{A} \subset \Delta_0$ and $t \in [(1 - \varepsilon)n, (1 + 20\varepsilon/\lambda)n]$ such that:*

(a) $\tilde{A} \subset A$, *and* $\hat{f}^t\tilde{A} = \Lambda^+$ *or* $= \Lambda^-$;

(b) $|\tilde{A}| \geq e^{-2\sqrt{\varepsilon}n}|A|$.

Proof. In the first n iterates under $\hat{f}$, the interval A continues climbing the tower, or else falls down from the tower several times. Let $j = \max\{i \geq 0\colon \hat{f}^i A \subset \Delta_0\}$. Since $f^j A \subset \Lambda^\pm$ and $f^j|A$ is extended to a diffeomorphism onto $3\Lambda^\pm$, $f^j|A$ has distortion bounded by four. Set $\omega = f^j A$ and $k = n - j - 1$. Since $A \subset \Delta_0$ we have $\omega \in \mathcal{P}_k$. If $k \geq k_2$, then take a subinterval $\tilde{\omega} \subset \omega$ for which the conclusions of Lemma 2.9 hold, and define $t = j + R(\tilde{\omega})$. The bounds on t follow from Lemma 2.9(b). Define $\tilde{A}$ to be the subinterval of A such that $\tilde{\omega} = f^j\tilde{A}$. Then

$$\frac{|\tilde{A}|}{|A|} \geq \frac{1}{4}\frac{|\tilde{\omega}|}{|\omega|} \geq \frac{1}{4}e^{-\sqrt{\varepsilon}k} \geq e^{-2\sqrt{\varepsilon}n}. \tag{11}$$

The second inequality follows from Lemma 2.9(a). If $k < k_2$, then set $t = j$ and define $\tilde{A}$ to be the subinterval of A such that $f^t\tilde{A} = \Lambda^\pm$. □

3. *Upper estimate of Birkhoff spectrum*

We put together the constructions and the results in §2 to obtain an upper estimate of the Birkhoff spectrum. For $\varphi \in C(X)$, $k \geq 0$, $\alpha \in [c_\varphi, d_\varphi]$ and $\varepsilon > 0$ consider the set

$$\Gamma_k = \Gamma_k(\varphi; \alpha, \varepsilon) = \left\{x \in \Lambda\colon \left|\frac{1}{n}S_n\varphi(x) - \alpha\right| < \varepsilon \text{ for every } n \geq k\right\}.$$

Note that Γ_k is increasing in k. Since $K_\varphi(\alpha)$ is dense in X and Λ contains open sets, $\Gamma_k \neq \emptyset$ holds for sufficiently large k. Define $\sigma = \sigma(\varphi; \alpha, \varepsilon)$ by

$$\sigma = \sup\left\{\frac{h(\mu)}{\lambda(\mu)}\colon \mu \in \mathcal{M}_f,\ |\mu(\varphi) - \alpha| \leq \sqrt{\varepsilon}\right\} + \varepsilon^{1/3}. \tag{12}$$

Since $\lambda_{\inf} > 0$, σ stays bounded from above as $\varepsilon \to 0$.

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PROPOSITION 3.1. *If $\varphi \in C(X)$ is Lipschitz, then for any $\alpha \in [c_\varphi, d_\varphi]$ and $\varepsilon > 0$,*

$$\dim_H \Gamma_k(\varphi; \alpha, \varepsilon) \le \sigma(\varphi; \alpha, \varepsilon) \quad \textit{for every } k \ge 0.$$

We finish the upper estimate of $B_\varphi(\alpha)$ assuming the conclusion of Proposition 3.1. Set $Y = [f^2 0, f 0]$. Points in $X \setminus Y$ are mapped to Y by some positive iterates. The countable stability and the invariance of the Hausdorff dimension under the action of Lipschitz continuous homeomorphisms yields $B_\varphi(\alpha) = \dim_H(K_\varphi(\alpha) \cap Y)$. We estimate the right-hand side.

By (A4) there exists $M > 0$ such that $f^M \Lambda = Y$. Then

$$\dim_H(K_\varphi(\alpha) \cap Y) = \dim_H f^M(K_\varphi(\alpha) \cap \Lambda) \le \dim_H(K_\varphi(\alpha) \cap \Lambda) \le \lim_{k\to\infty} \dim_H \Gamma_k,$$

where the last inequality is because $K_\varphi(\alpha) \cap \Lambda \subset \bigcup_{k\ge n} \Gamma_k$ for every $n \ge 0$. If φ is Lipschitz continuous, then by Proposition 3.1,

$$B_\varphi(\alpha) \le \lim_{k\to\infty} \dim_H \Gamma_k \le \sigma.$$

Letting $\varepsilon \to 0$ we get

$$B_\varphi(\alpha) \le \lim_{\varepsilon\to 0} \sup\left\{ \frac{h(\mu)}{\lambda(\mu)} : \mu \in \mathcal{M}_f,\ |\mu(\varphi) - \alpha| < \varepsilon \right\}.$$

If φ is merely continuous, then take a Lipschitz continuous $\tilde{\varphi}$ such that $\|\varphi - \tilde{\varphi}\| < \varepsilon/2$, $c_{\tilde{\varphi}} = c_\varphi$ and $d_{\tilde{\varphi}} = d_\varphi$. Then for any $\alpha \in [c_\varphi, d_\varphi]$ and small $\varepsilon > 0$, $\Gamma_k(\varphi; \alpha, \varepsilon) \subset \Gamma_k(\tilde{\varphi}; \alpha, 2\varepsilon)$ holds. By Proposition 3.1 there exists $\xi \in \mathcal{M}_f$ such that

$$\dim_H \Gamma_k(\varphi; \alpha, \varepsilon) \le \dim_H \Gamma_k(\tilde{\varphi}; \alpha, 2\varepsilon) \le \frac{h(\xi)}{\lambda(\xi)} + (3\varepsilon)^{1/3},$$

and

$$|\xi(\varphi) - \alpha| \le |\xi(\varphi) - \xi(\tilde{\varphi})| + |\xi(\tilde{\varphi}) - \alpha| < \varepsilon/2 + \sqrt{2\varepsilon} < 2\sqrt{\varepsilon}.$$

The rest of the argument is identical to the previous case.

The rest of this section is entirely devoted to the proof of Proposition 3.1. In §3.1 we extract from the towers uniformly hyperbolic invariant sets (horseshoes). In §3.2 we construct invariant measures on the horseshoes, and use them to complete the proof of the proposition.

3.1. *Construction of a horseshoe.* Define

$$\mathcal{A}_n = \left\{ A \in \bigvee_{i=0}^{n-1} \hat{f}^{-i}\mathcal{D} : A \subset \Delta_0,\ \left| \frac{1}{n} S_n\varphi(x) - \alpha \right| < \varepsilon \text{ for some } x \in A \right\}.$$

If $\Gamma_k \ne \emptyset$ then for every $n \ge k$ we have $\mathcal{A}_n \ne \emptyset$, and $\Gamma_k \subset \bigcup_{A\in\mathcal{A}_n} A$. We use this family of coverings for the upper estimate of the Hausdorff dimension.

Let Ω be a finite collection of pairwise disjoint closed intervals in Λ and r a positive integer. We say Ω *generates a horseshoe for* f^r if f^r sends each element of Ω diffeomorphically onto $\hat{X}$. By a *horseshoe* we mean the set

$$H_r(\Omega) = \bigcap_{j=0}^{\infty} (f^r)^{-j} \left(\bigcup_{I\in\Omega} I \right).$$

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LEMMA 3.2. *For any $\varepsilon > 0$ there exists $n' > 0$ such that if $n \geq n'$ then $\mathcal{A}_n \neq \emptyset$ and there exists a finite collection $\mathcal{K}$ of closed intervals in Λ and an integer $q \in [(1-\varepsilon)n, (1+21\varepsilon/\lambda)n]$ such that:*

(a) *$\mathcal{K}$ generates a horseshoe for f^q;*
(b) *$\sum_{K\in\mathcal{K}} |K|^\sigma \geq e^{-3\sqrt{\varepsilon}\sigma n} \sum_{A\in\mathcal{A}_n} |A|^\sigma$;*
(c) *for all $x \in H_q(\mathcal{K})$, $|(1/q)S_q\varphi(x) - \alpha| \leq \sqrt{\varepsilon}$.*

Proof. For each $A \in \mathcal{A}_n$, fix once and for all an interval $\tilde{A}$ and an integer $t = t_A$ for which the conclusions of Lemma 2.10 hold. Let $\mathcal{A}_n(t) = \{A \in \mathcal{A}_n : t_A = t\}$. Then $t_A \in [(1-\varepsilon)n, (1+19\varepsilon/\lambda)n]$. Let t_0 be a value of t which maximizes $\sum_{A\in\mathcal{A}_n(t)} |\tilde{A}|^\sigma$. Then

$$\sum_{A\in\mathcal{A}_n(t_0)} |\tilde{A}|^\sigma \geq \frac{1}{(1+20/\lambda)\varepsilon n} \sum_{A\in\mathcal{A}_n} |\tilde{A}|^\sigma. \tag{13}$$

By (A4) it is possible to choose a constant $\tau > 0$, an integer $u > 0$ and a closed interval $I^+ \subset \Lambda^+$ such that Λ^+ contains the τ-scaled neighborhood of I^+, and f^u sends I^+ diffeomorphically onto $\hat{X}$. Define $q = t_0 + u$. The bounds on q hold for sufficiently large n.

Let $I^- = -I^+$. For each $A \in \mathcal{A}_n(t_0)$ define $K(A)$ to be the preimage of I^+ or I^- under $f^{t_0}|\tilde{A}$, according to whether $f^{t_0}\tilde{A} = \Lambda^+$ or $= \Lambda^-$. Set $\mathcal{K} = \{K(A) : A \in \mathcal{A}_n(t_0)\}$. Then $\mathcal{K}$ is a finite collection of pairwise disjoint closed intervals in Λ, and f^q sends each element of $\mathcal{K}$ diffeomorphically onto $\hat{X}$. Set $c = (\tau/(1+\tau))^2|I^+|/|\Lambda^+|$. Then

$$\begin{aligned}\sum_{K\in\mathcal{K}} |K|^\sigma &\geq c^\sigma \sum_{A\in\mathcal{A}_n(t_0)} |\tilde{A}|^\sigma \geq \frac{c^\sigma}{(1+20/\lambda)\varepsilon n} \sum_{A\in\mathcal{A}_n} |\tilde{A}|^\sigma \\ &\geq \frac{1}{(1+20/\lambda)\varepsilon n}(ce^{-2\sqrt{\varepsilon}n})^\sigma \sum_{A\in\mathcal{A}_n} |A|^\sigma \geq e^{-3\sqrt{\varepsilon}\sigma n} \sum_{A\in\mathcal{A}_n} |A|^\sigma.\end{aligned}$$

SUBLEMMA 3.3. *There exists a constant $C > 0$ such that if $n \geq N$ and $\omega \in \mathcal{P}_{n-1}$, then for all $x, y \in \omega$,*

$$|S_n\varphi(x) - S_n\varphi(y)| \leq \mathrm{Lip}(\varphi) \cdot C\delta^{-1},$$

where $\mathrm{Lip}(\varphi)$ *denotes the Lipschitz constant of φ.*

Proof. Let $0 \leq i \leq n-1$. We call $f^i\omega$ free if there exists an interval $J \subset \hat{X}$ containing ω such that $f^i J$ is a free segment. Let i_0 denote the maximal $i \leq n-1$ such that $f^i\omega$ is free. From the construction in §2 one can find integers $0 \leq r_1 < \cdots < r_s = i_0$, $p_1, \ldots, p_s$ such that: r_1 is the smallest $i \geq 0$ with $f^i\omega \cap (-\delta, \delta) \neq \emptyset$; $\delta_{p_k} \leq d(0, f^{r_k}\omega) \leq \delta_{p_k-2}$ and r_{k+1} is the smallest $i \geq r_k + p_k$ with $f^i\omega \cap (-\delta, \delta) \neq \emptyset$ $(k = 1, \ldots, s-1)$; $d(0, f^{r_s}\omega) \leq \delta_{p_s-2}$ and $n \leq r_s + p_s$. Then, similarly to the proof of [**8**, Sublemma 3.16] one can show that

$$\sum_{i=0}^{n-1} |f^i\omega| \leq C\delta^{-1}.$$

This implies the desired inequality since φ is Lipschitz continuous. □

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To prove (c), for each $A \in \mathcal{A}_n(t_0)$ pick $x_A \in A$ such that $|(1/n)S_n\varphi(x_A) - \alpha| \le \varepsilon$. We have

$$S_q\varphi(x_A) \ge S_n\varphi(x_A) - \sup|\varphi| \cdot |q-n| \ge \alpha n - \sup|\varphi| \cdot (21\varepsilon/\lambda)n \ge (\alpha - \sqrt{\varepsilon}/2)q.$$

In the same way we have $S_q\varphi(x_A) \le (\alpha + \sqrt{\varepsilon}/2)q$. Then

$$|S_q\varphi(x_A) - q\alpha| \le \sqrt{\varepsilon} \cdot q/2. \tag{14}$$

By Sublemma 3.3, for any $x \in A$ we have $|S_{t_0}\varphi(x_A) - S_{t_0}\varphi(x)| \le \mathrm{Lip}(\varphi) \cdot C\delta^{-1}$. Since $q - t_0 = u$ and $q \ge (1-\varepsilon)n$, for sufficiently large n we have

$$|S_q\varphi(x_A) - S_q\varphi(x)| \le \mathrm{Lip}(\varphi) \cdot C\delta^{-1} + 2\sup|\varphi| \cdot (q - t_0) \le \sqrt{\varepsilon} \cdot q/2. \tag{15}$$

Equations (14) and (15) yield $|S_q\varphi(x) - q\alpha| \le \sqrt{\varepsilon} \cdot q$. □

3.2. *Construction of a measure on the horseshoe.* For sufficiently large n, choose a finite collection $\mathcal{K}$ of closed intervals in Λ and a positive integer q for which the conclusions of Lemma 3.2 hold. Set $F = f^q$. By construction, F is uniformly expanding on each element of $\mathcal{K}$. Hence, $F|H_q(\mathcal{K})$ is topologically conjugate to the one-sided full shift on $\#\mathcal{K}$ symbols. Write $\mathcal{K} = \{K_1, \ldots, K_{\#\mathcal{K}}\}$. For $\ell > 0$ and an $(\ell+1)$-string $(a_0, \ldots, a_\ell)$ of integers in $[1, \#\mathcal{K}]$, define an interval

$$K_{a_0 \cdots a_\ell} = K_{a_0} \cap F^{-1}K_{a_1} \cap \cdots \cap F^{-\ell}K_{a_\ell}.$$

Set $\kappa = 2C_0 \sup_{x,y \in I^+}(|Df^u(x)|/|Df^u(y)|)$. We have

$$\frac{|K_{a_0\cdots a_\ell}|}{|K_{a_0\cdots a_{\ell-1}}|} \ge \frac{1}{2}\frac{|F^{\ell-1}K_{a_0\cdots a_\ell}|}{|F^{\ell-1}K_{a_0\cdots a_{\ell-1}}|} = \frac{1}{2}\frac{|\{x \in K_{a_{\ell-1}} : Fx \in K_{a_\ell}\}|}{|K_{a_{\ell-1}}|} \ge \kappa^{-1}|K_{a_\ell}|.$$

The first inequality follows from the Koebe Principle, and the second one from (6) and the definition of I^+, u. Then

$$\begin{aligned}\sum_{(a_0,\ldots,a_\ell)} |K_{a_0\cdots a_\ell}|^\sigma &= \sum_{(a_0,\ldots,a_{\ell-1})} |K_{a_0\cdots a_{\ell-1}}|^\sigma \sum_{a_\ell} \frac{|K_{a_0\cdots a_\ell}|^\sigma}{|K_{a_0\cdots a_{\ell-1}}|^\sigma} \\ &\ge \kappa^{-\sigma} \sum_{i=1}^{\#\mathcal{K}} |K_i|^\sigma \sum_{(a_0,\ldots,a_{\ell-1})} |K_{a_0\cdots a_{\ell-1}}|^\sigma \\ &\ge \cdots \ge \left(\kappa^{-\sigma} \sum_{i=1}^{\#\mathcal{K}} |K_i|^\sigma\right)^{\ell+1}.\end{aligned}$$

This yields

$$\varliminf_{\ell\to\infty} \frac{1}{\ell} \log \sum_{(a_0,\ldots,a_\ell)} |K_{a_0\cdots a_\ell}|^\sigma \ge \log \sum_{i=1}^{\#\mathcal{K}} |K_i|^\sigma - \sigma \log \kappa. \tag{16}$$

Let $\nu_{a_0\cdots a_\ell}$ denote the uniform distribution on the orbit of the $(\ell+1)$-periodic point of F in $K_{a_0\cdots a_\ell}$. Define an F-invariant probability measure ν_ℓ supported on $H_q(\mathcal{K})$ by

$$\nu_\ell = \rho_\ell \sum_{(a_0,\ldots,a_\ell)} |K_{a_0\cdots a_\ell}|^\sigma \nu_{a_0\cdots a_\ell},$$

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where $\rho_\ell = 1/\sum_{(a_0,\dots,a_\ell)} |K_{a_0\cdots a_\ell}|^\sigma$ is the normalizing constant. Pick an accumulation point of the sequence $\{\nu_\ell\}$ and denote it by ν_0. Taking a subsequence if necessary we may assume this convergence takes place for the entire sequence. Using the relation $\nu_\ell(K_{a_0\cdots a_\ell}) = \rho_\ell |K_{a_0\cdots a_\ell}|^\sigma$ and $|K_{a_0\cdots a_\ell}| \le \kappa |\hat{X}| \exp\{-(\ell+1)\int \log |DF|\, d\nu_{a_0\cdots a_\ell}\}$, which follows from the bounded distortion, we have

$$\begin{aligned}\log \sum_{(a_0,\dots,a_\ell)} |K_{a_0\cdots a_\ell}|^\sigma &= \sum_{(a_0,\dots,a_\ell)} \nu_\ell(K_{a_0\cdots a_\ell})(-\log \nu_\ell(K_{a_0\cdots a_\ell}) + \sigma \log |K_{a_0\cdots a_\ell}|) \\ &\le -\sum_{(a_0,\dots,a_\ell)} \nu_\ell(K_{a_0\cdots a_\ell}) \log \nu_\ell(K_{a_0\cdots a_\ell}) \\ &\quad - \sigma(\ell+1)\int \log |DF|\, d\nu_\ell + \sigma \log \kappa |\hat{X}|.\end{aligned}$$

A slight modification of the argument in the proof of the Variational Principle [**34**, Theorem 9.10] shows that

$$\overline{\lim_{\ell\to\infty}} \frac{1}{\ell} \log \sum_{(a_0,\dots,a_\ell)} |K_{a_0\cdots a_\ell}|^\sigma \le h_F(\nu_0) - \sigma \int \log |DF|\, d\nu_0, \tag{17}$$

where $h_F(\nu_0)$ denotes the entropy of (F, ν_0). Let $\xi = (1/q)\sum_{i=0}^{q-1} (f^i)_*\nu_0$, which is f-invariant. It follows from Lemma 3.2(c) that $|\xi(\varphi) - \alpha| \le \sqrt{\varepsilon}$ since ν_0 is supported on $H_q(\mathcal{K})$. Then by the definition of σ in (12), (16) and (17) yield

$$\begin{aligned}\log \sum_{i=1}^{\#\mathcal{K}} |K_i|^\sigma &\le h_F(\nu_0) - \sigma \int \log|DF|\, d\nu_0 + \sigma \log \kappa \\ &= q(h(\xi) - \sigma\lambda(\xi)) + \sigma \log \kappa \le -q\varepsilon^{1/3}\lambda(\xi) + \sigma \log \kappa.\end{aligned}$$

Lemma 3.2(b) gives

$$\log \sum_{A\in\mathcal{A}_n} |A|^\sigma \le 4\sqrt{\varepsilon}\sigma n + \log \sum_{i=1}^{\#\mathcal{K}} |K_i|^\sigma.$$

Since $q \ge (1-\varepsilon)n$ and $\lambda(\xi) \ge \lambda_{\inf} > 0$, we have

$$\begin{aligned}\log \sum_{A\in\mathcal{A}_n} |A|^\sigma &\le 4\sqrt{\varepsilon}\sigma n - q\varepsilon^{1/3}\lambda(\xi) + \sigma \log \kappa \\ &\le (4\sqrt{\varepsilon}\sigma - (1-\varepsilon)\varepsilon^{1/3}\lambda_{\inf})n + \sigma \log \kappa \le -\varepsilon^{1/3}\lambda_{\inf} n/2,\end{aligned}$$

where the last inequality holds for sufficiently large n. It follows that $\sum_{A\in\mathcal{A}_n} |A|^\sigma$ has a negative growth rate as n increases. In addition, the above inequality implies that the diameters of the elements of $\mathcal{A}_n$ decrease uniformly as n increases. Therefore, the Hausdorff σ-measure of Γ_k is zero and so $\dim_H \Gamma_k \le \sigma$. This completes the proof of Proposition 3.1.

4. *Lower estimate and continuity of Birkhoff spectrum*

In this section we estimate $B_\varphi(\alpha)$ from below, and finish the proof of the formula in Theorem A. We then use this formula to prove the continuity of the Birkhoff spectrum.

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4.1. *Lower estimate of the Birkhoff spectrum.* To estimate $B_\varphi(\alpha)$ from below we will construct a sufficiently large set of points for which the time averages of φ are precisely equal to α. Let $\mathcal{M}_f^e$ denote the set of ergodic elements of $\mathcal{M}_f$.

PROPOSITION 4.1. *Let $\varphi \in C(X)$ and $\alpha \in [c_\varphi, d_\varphi]$. Let $\{\mu_i\}_i$ be a sequence in $\mathcal{M}_f^e$ such that $|\mu_i(\varphi) - \alpha| < 1/i$ and $h(\mu_i)/\lambda(\mu_i)$ converges as $i \to \infty$. There exists a closed set $\Gamma \subset K_\varphi(\alpha)$ such that*

$$\dim_H(\Gamma) \geq \lim_{i\to\infty} \frac{h(\mu_i)}{\lambda(\mu_i)}.$$

It then follows that

$$B_\varphi(\alpha) \geq \lim_{\varepsilon\to 0} \sup\left\{\frac{h(\mu)}{\lambda(\mu)} : \mu \in \mathcal{M}_f^e, |\mu(\varphi) - \alpha| < \varepsilon\right\}. \tag{18}$$

To finish, it only remains to show that the supremum of the right-hand side of (18) may be taken over all invariant probability measures that are not necessarily ergodic.

Using (A4) and a one-dimensional version of Katok's Theorem [**19**], for any $\mu \in \mathcal{M}_f$ and $\varepsilon > 0$ one can find $\nu \in \mathcal{M}_f^e$ such that: $|\mu(\varphi) - \nu(\varphi)| \leq \varepsilon$; $h(\nu) \geq h(\mu) - \varepsilon$; $\lambda(\nu) \leq \lambda(\mu) + \varepsilon$. Since $0 < \lambda_{\inf} \leq \lambda(\mu) \leq \log 4$ and $h(\nu) \leq \log 2$, we have

$$\frac{h(\nu)}{\lambda(\nu)} \geq \frac{h(\mu) - \varepsilon}{\lambda(\mu) + \varepsilon} = \frac{h(\mu)}{\lambda(\mu)} - \frac{\varepsilon(h(\mu) + \lambda(\mu))}{\lambda(\mu)(\lambda(\mu) + \varepsilon)} \geq \frac{h(\mu)}{\lambda(\mu)} - \frac{3\varepsilon \log 2}{\lambda_{\inf}^2},$$

and therefore

$$\begin{aligned}\sup\left\{\frac{h(\nu)}{\lambda(\nu)} : \nu \in \mathcal{M}_f^e, |\nu(\varphi) - \alpha| < 2\varepsilon\right\} \\ \geq \sup\left\{\frac{h(\mu)}{\lambda(\mu)} : \mu \in \mathcal{M}_f, |\mu(\varphi) - \alpha| < \varepsilon\right\} - \frac{3\varepsilon \log 2}{\lambda_{\inf}^2}.\end{aligned}$$

Letting $\varepsilon \to 0$ and then using (18) we obtain

$$B_\varphi(\alpha) \geq \lim_{\varepsilon\to 0} \sup\left\{\frac{h(\mu)}{\lambda(\mu)} : \mu \in \mathcal{M}_f,\ |\mu(\varphi) - \alpha| < \varepsilon\right\}.$$

From this and the upper estimate in §3 we obtain the formula in Theorem A.

The rest of this subsection is devoted to a proof of Proposition 4.1. If $h(\mu_i) \to 0$ then there is nothing to prove since $\lambda(\mu_i) \geq \lambda_{\inf} > 0$. So we may assume $h(\mu_i) > 0$ for each i. By a result of [**19**], for any ergodic measure with positive entropy one can construct a horseshoe and use it to approximate its entropy, Lyapunov exponent and the integral of a continuous function. Namely, for each i there exist $\beta_i > 0$, a closed interval L_i and a family Ω_i of pairwise disjoint closed intervals in the interior of L_i such that:

(i) for each $I \in \Omega_i$, $f^{\beta_i} I = L_i$;

(ii) for any $x \in \bigcup_{I\in\Omega_i} I$ and $\psi \in \{\varphi, \log|Df|\}$, $|(1/\beta_i) S_{\beta_i}\psi(x) - \mu_i(\psi)| \leq 1/i$;

(iii) $(1/\beta_i) \log \#\Omega_i \geq h(\mu_i) - 1/i$.

We construct a family of intervals at smaller and smaller scales which wander around different horseshoes. By (A4), for each i it is possible to choose $\gamma_i > 0$ and a closed interval $\tilde{L}_i \subset L_i$ such that f^{γ_i} sends $\tilde{L}_i$ homeomorphically onto Y. Choose a sequence $\{\kappa_i\}$ of positive integers inductively as follows.

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Start with $\kappa_1 = 1$. Given κ_{i-1}, choose κ_i to be a large integer which depends on $\beta_1, \beta_2, \ldots, \beta_{i+1}, \gamma_1, \gamma_2, \ldots, \gamma_{i-1}, \kappa_1, \kappa_2, \ldots, \kappa_{i-1}, i, \sup|\varphi|, \alpha$. Requirements among these constants will be made explicit at the end of the proof.

For each $k \geq 1$, let $n = n(k)$, $s = s(k)$ be integers such that

$$k = \kappa_1 + \kappa_2 + \cdots + \kappa_n + s \quad \text{and} \quad 0 \leq s < \kappa_{n+1}.$$

Let

$$\Omega^{(k)} = \underbrace{\Omega_1 \times \cdots \times \Omega_1}_{\kappa_1} \times \underbrace{\Omega_2 \times \cdots \times \Omega_2}_{\kappa_2} \times \cdots \times \underbrace{\Omega_n \times \cdots \times \Omega_n}_{\kappa_n} \times \underbrace{\Omega_{n+1} \times \cdots \times \Omega_{n+1}}_{s}.$$

Elements of $\Omega^{(k)}$ are denoted by $(I_1, \ldots, I_k)$, i.e., $I_1 \in \Omega_1, \ldots, I_{\kappa_1} \in \Omega_1, I_{\kappa_1+1} \in \Omega_2$, and so on.

For each $k \geq 1$ and $(I_1, \ldots, I_k) \in \Omega^{(k)}$ we associate a closed interval $[I_1, \ldots, I_k]$ inductively as follows. Observe that $\Omega^{(1)} = \Omega_1$. For each $I \in \Omega^{(1)}$, define $[I] = I$. Given $k \geq 1$, $(I_1, \ldots, I_k) \in \Omega^{(k)}$ and $[I_1, \ldots, I_k]$, $(I_1, \ldots, I_k, I_{k+1}) \in \Omega^{(k+1)}$, define $[I_1, \ldots, I_k, I_{k+1}] \subset [I_1, \ldots, I_k]$ by

$$[I_1, \ldots, I_k, I_{k+1}] = \begin{cases} (f^t|[I_1, \ldots, I_k])^{-1} I_{k+1} & \text{if } s < \kappa_{n+1} - 1, \\ (f^t|[I_1, \ldots, I_k])^{-1}([(f^{\beta_{n+1}}|I_{k+1})^{-1} \tilde{L}_{n+1}] & \text{if } s = \kappa_{n+1} - 1, \end{cases}$$

where $t = t(k)$ is defined by

$$t = \beta_1\kappa_1 + \gamma_1 + \beta_2\kappa_2 + \gamma_2 + \cdots + \beta_n\kappa_n + \gamma_n + \beta_{n+1}s.$$

Set

$$\mathcal{F}^{(k)} = \{[I_1, \ldots, I_k] \colon (I_1, \ldots, I_k) \in \Omega^{(k)}\}.$$

This is a collection of pairwise disjoint closed intervals with the following properties: if $s(k) > 0$, then $f^{t(k)}[I_1, \ldots, I_k] = L_{n(k)+1}$; if $s(k) = 0$, then $f^{t(k)}$ sends $[I_1, \ldots, I_k]$ homeomorphically onto Y. Observe that $\{\bigcup_{I \in \mathcal{F}^{(k)}} I\}_k$ is a nested sequence of closed sets. Define

$$\Gamma = \bigcap_{k=1}^{\infty} \bigcup_{I \in \mathcal{F}^{(k)}} I.$$

Points in Γ continue traveling from one horseshoe to the next generated by Ω_k, $k \geq 1$. For the choice of $\{\kappa_i\}$ we will request

$$\kappa_i \gg \max\{\beta_1, \beta_2, \ldots, \beta_{i+1}, \gamma_1, \gamma_2, \ldots, \gamma_{i-1}, \kappa_1, \kappa_2, \ldots, \kappa_{i-1}\}. \tag{19}$$

Then, generic finite orbits of Γ spend most of their time near the last or the second to last horseshoes, and gain time averages in this duration. As a result, the time averages along the finite orbits become nearly α. In fact, the following lemma holds.

LEMMA 4.2. $\Gamma \subset K_\varphi(\alpha)$.

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Proof. Let $x \in \Gamma$. For a large integer q let $k \geq 1$ be maximal such that $t(k) \leq q$. Then $q - t(k) \leq \beta_{n(k)+1}$. Splitting the time interval $[0, q-1]$ is a concatenation of the duration around horseshoes and the transition between horseshoes, and then applying (ii) to each of the corresponding orbit segments we have

$$\begin{aligned}
|S_q\varphi(x) - q\alpha| \leq{}& \sum_{j=0}^{\kappa_1-1} |S_{\beta_1}\varphi(f^{\beta_1 j}x) - \beta_1\alpha| \\
&+ |S_{\gamma_1}\varphi(f^{\kappa_1\beta_1}x) - \gamma_1\alpha| \\
&+ \sum_{j=0}^{\kappa_2-1} |S_{\beta_2}\varphi(f^{\beta_1\kappa_1+\gamma_1+\beta_2 j}x) - \beta_2\alpha| \\
&+ |S_{\gamma_2}\varphi(f^{\beta_1\kappa_1+\gamma_1+\beta_2\kappa_2}x) - \gamma_2\alpha| + \cdots \\
&+ \sum_{j=0}^{\kappa_n-1} |S_{\beta_n}\varphi(f^{\sum_{i=1}^{n-1}(\beta_i\kappa_i+\gamma_i)+\beta_n j}x) - \beta_n\alpha| \\
&+ |S_{\gamma_n}\varphi(f^{\sum_{i=1}^{n}\beta_i\kappa_i+\sum_{i=1}^{n-1}\gamma_i}x) - \gamma_n\alpha| \\
&+ \sum_{j=0}^{s-1} |S_{\beta_{n+1}}\varphi(f^{\sum_{i=1}^{n}(\beta_i\kappa_i+\gamma_i)+\beta_{n+1} j}x) - \beta_{n+1}\alpha| \\
&+ |S_{q-t}\varphi(f^t x) - (q-t)\alpha|.
\end{aligned}$$

Using (ii) and the fact that x is contained in an element of $\mathcal{F}^{(k)}$, for every $2 \leq \ell \leq n$ we have

$$\begin{aligned}
\sum_{j=0}^{\kappa_\ell-1} |S_{\beta_\ell}\varphi(f^{\sum_{i=1}^{\ell-1}(\beta_i\kappa_i+\gamma_i)+\beta_\ell j}x) - \beta_\ell\alpha| &\leq |S_{\beta_\ell}\varphi(f^{\sum_{i=1}^{\ell-1}(\beta_i\kappa_i+\gamma_i)+\beta_\ell j}x) - \beta_\ell\mu_\ell(\varphi)| \\
&\quad + |\beta_\ell\mu_\ell(\varphi) - \beta_\ell\alpha| \leq \frac{2\beta_\ell}{\ell}.
\end{aligned}$$

Summing these and other remainder terms we get

$$\begin{aligned}
|S_q\varphi(x) - q\alpha| &\leq \sum_{i=1}^{n} \gamma_i(\sup|\varphi| + \alpha) + \sum_{i=1}^{n} \frac{2\beta_i\kappa_i}{i} + \frac{2\beta_{n+1}s}{n+1} + (q-t)(\sup|\varphi| + \alpha) \\
&\leq \frac{3\beta_n\kappa_n}{n} + \frac{2\beta_{n+1}s}{n+1} \leq \frac{5q}{n},
\end{aligned}$$

where κ_n is chosen sufficiently large so that the second inequality holds. Since $n \to \infty$ as $q \to \infty$, $x \in K_\varphi(\alpha)$ follows. □

For each $I \in \mathcal{F}^{(k)}$ choose a point $x_I \in I \cap \Gamma$ and define an atomic probability measure ν_k uniformly distributed on the set $\{x_I : I \in \mathcal{F}^{(k)}\}$. Pick an accumulation point of the sequence $\{\nu_k\}$ and denote it by ν. Since Γ is closed we have $\nu(\Gamma) = 1$. For $x \in X$ and $\rho > 0$, let $D_\rho(x) = \{y : |x - y| \leq \rho\}$. The inequality in Proposition 4.1 follows from [**37**, Proposition 2.1] and the following lemma.

LEMMA 4.3. *For any $x \in \Gamma$ we have*

$$\underline{\lim}_{\rho\to 0} \frac{\log \nu D_\rho(x)}{\log \rho} \geq \lim_{i\to\infty} \frac{h(\mu_i)}{\lambda(\mu_i)}.$$

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Proof. Consider the set of pairs (n, s) of integers such that $n \geq 0$ and $0 \leq s < \kappa_{n+1}$. We introduce an order on this set as follows: $(n_1, s_1) < (n_2, s_2)$ if $n_1 < n_2$ or $n_1 = n_2$ and $s_1 < s_2$. For a pair (n, s) in this set, let

$$a_{n,s} = \exp\left[-\beta_n\kappa_n\left(\lambda(\mu_n) + \frac{2}{n}\right) - \beta_{n+1}s\left(\lambda(\mu_{n+1}) + \frac{1}{n+1}\right)\right].$$

Using (19) it is easy to show that $a_{n+1,0} < a_{n,\kappa_{n+1}-1}$. Hence the sequence $\{a_{n,s}\}$ is monotone decreasing. Then for given small $\rho > 0$ one can choose k such that $a_{n(k),s(k)} < \rho \leq a_{n(k-1),s(k-1)}$.

Let $I \in \mathcal{F}^{(k)}$. We have $\nu(\partial I) = 0$, and for every $q \geq k$,

$$\nu_q(I) = \frac{\#\{J \in \mathcal{F}^{(q)} \colon J \subset I\}}{\#\mathcal{F}^{(q)}} = \frac{1}{\#\mathcal{F}^{(k)}}.$$

Hence,

$$\nu(I) = \lim_{q\to\infty} \nu_q(I) = \frac{1}{\#\mathcal{F}^{(k)}}.$$

Using (ii) for $\psi = \log|Df|$ and (19), for all $x \in I$ we have

$$|Df^t(x)| \leq \exp\left[\beta_n\kappa_n\left(\lambda(\mu_n) + \frac{2}{n}\right) + \beta_{n+1}s\left(\lambda(\mu_{n+1}) + \frac{2}{n}\right)\right].$$

Since $f^t I \subset X$, the Mean Value Theorem gives

$$|I| \geq \frac{1}{2}\exp\left[-\beta_n\kappa_n\left(\lambda(\mu_n) + \frac{2}{n}\right) - \beta_{n+1}s\left(\lambda(\mu_{n+1}) + \frac{2}{n}\right)\right]. \tag{20}$$

Hence, for any $x \in \Gamma$, $D_\rho(x)$ intersects at most $2\exp\{\beta_{n+1}s[2/n - 1/(n+1)]\}$ elements of $\mathcal{F}^{(k)}$. Using (iii) we have

$$\#\mathcal{F}^{(k)} \geq (\#\Omega_n)^{\kappa_n} \cdot (\#\Omega_{n+1})^s \geq \exp\left[\beta_n\kappa_n\left(h(\mu_n) - \frac{1}{n}\right) + \beta_{n+1}s\left(h(\mu_{n+1}) - \frac{1}{n+1}\right)\right],$$

and therefore

$$\begin{aligned}\nu D_\rho(x) &\leq \frac{2}{\#\mathcal{F}^{(k)}}\exp\left[\beta_{n+1}s\left(\frac{2}{n} - \frac{1}{n+1}\right)\right]\\ &\leq 2\exp\left[-\beta_n\kappa_n\left(h(\mu_n) - \frac{1}{n}\right) - \beta_{n+1}s\left(h(\mu_{n+1}) - \frac{2}{n}\right)\right].\end{aligned}$$

This yields

$$\frac{\log \nu D_\rho(x)}{\log\rho} \geq \frac{\beta_n\kappa_n(h(\mu_n) - 1/n) + \beta_{n+1}s(h(\mu_{n+1}) - 2/n)}{\beta_n\kappa_n(\lambda(\mu_n) + 2/n) + \beta_{n+1}s(\lambda(\mu_{n+1}) + 1/(n+1))} + \frac{\log 2}{\log\rho}.$$

The desired inequality holds since $n \to \infty$ as $\rho \to 0$. This completes the proof of Lemma 4.3 and hence that of Proposition 4.1. □

4.2. *Continuity of the Birkhoff spectrum.* From the formula in Theorem A, the spectrum is upper semi-continuous. We argue by contradiction assuming that the spectrum is not lower semi-continuous at a point $\alpha_0 \in [c_\varphi, d_\varphi]$. Then it is possible to choose $\epsilon_0 > 0$ and a monotone sequence $\{\alpha_n\}$ such that $\alpha_n \to \alpha_0$ and

$$B_\varphi(\alpha_n) \leq B_\varphi(\alpha_0) - \epsilon_0. \tag{21}$$

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Let us suppose that $\{\alpha_n\}$ is monotone increasing. Take $\mu_c \in \mathcal{M}_f$ with $\mu_c(\varphi) = c_\varphi$. The formula in Theorem A allows us to choose a sequence $\{\mu_k\}$ in $\mathcal{M}_f$ such that $h(\mu_k)/\lambda(\mu_k) \geq B_\varphi(\alpha_0) - \epsilon_0/4$ and $\mu_k(\varphi) \to \alpha_0$. Choose a subsequence $\{\mu_{k(n)}\}$ such that $\alpha_n \leq \mu_{k(n)}(\varphi)$. For each n choose $0 \leq t_n \leq 1$ such that $(1-t_n)\mu_c(\varphi) + t_n\mu_{k(n)}(\varphi) = \alpha_n$, and define $\nu_n = (1-t_n)\mu_c + t_n\mu_{k(n)}$. For all large n we have

$$B_\varphi(\alpha_n) = B_\varphi[(1-t_n)\mu_c(\varphi) + t_n\mu_{k(n)}(\varphi)] \geq \frac{h(\nu_n)}{\lambda(\nu_n)} \geq B_\varphi(\alpha_0) - \epsilon_0/2.$$

The second inequality follows from the linearity of entropies and Lyapunov exponents on measures, $t_n \to 1$ and $\inf_n \lambda(\mu_{k(n)}) \geq \lambda_{\inf} > 0$. This yields a contradiction to (21). In the case where $\{\alpha_n\}$ is monotone decreasing, take $\mu_d \in \mathcal{M}_f$ with $\mu_d(\varphi) = d_\varphi$ and use it in place of μ_c.

5. *Large deviation principle*

In this last section we prove Theorem B. This amounts to proving the following proposition, which gives an upper bound of deviation probabilities in terms of the free energies of invariant measures.

PROPOSITION 5.1. *Let $d \geq 1$, $\varphi_1, \ldots, \varphi_d \in C(X)$ be Lipschitz, and let $\alpha_1, \ldots, \alpha_d \in \mathbb{R}$. For any $\varepsilon > 0$ there exists $n_0 > 0$ such that for every $n \geq n_0$ there exists $\eta \in \mathcal{M}_f$ such that:*

$$\frac{1}{n}\log\left|\left\{x \in \Lambda\colon \frac{1}{n}S_n\varphi_j(x) \geq \alpha_j,\ \ j = 1, \ldots, d\right\}\right| \leq (1-\varepsilon)F(\eta) + 4\sqrt{\varepsilon}; \tag{22}$$

$$\eta(\varphi_j) \geq \alpha_j - \sqrt{\varepsilon}, \quad j = 1, \ldots, d. \tag{23}$$

We finish the proof of Theorem B assuming the conclusion of the proposition. Recall that $M > 0$ is such that $f^M\Lambda = Y$. Let $\varepsilon_0 > 0$ be a small constant. For all large n we have

$$\left\{x \in Y\colon \frac{1}{n}S_n\varphi_j(x) \geq \alpha_j\right\} \subset f^M\left\{x \in \Lambda\colon \frac{1}{n}S_n\varphi_j(x) \geq \alpha_j - \varepsilon_0\right\},$$

where it is understood that j runs over $\{1, 2, \ldots, d\}$. By the Mean Value Theorem,

$$\left|\left\{x \in Y\colon \frac{1}{n}S_n\varphi_j(x) \geq \alpha_j\right\}\right| \leq 4^M \cdot \left|\left\{x \in \Lambda\colon \frac{1}{n}S_n\varphi_j(x) \geq \alpha_j - \varepsilon_0\right\}\right|.$$

From this inequality and Proposition 5.1 there exists $\eta \in \mathcal{M}_f$ such that $\eta(\varphi_j) \geq \alpha_j - \varepsilon_0 - \sqrt{\varepsilon}$ $(j = 1, \ldots, d)$ and

$$\frac{1}{n}\log\left|\left\{x \in Y\colon \frac{1}{n}S_n\varphi_j(x) \geq \alpha_j\right\}\right| \leq \frac{M}{n}\log 4 + (1 - C\varepsilon)F(\eta) + 4\sqrt{\varepsilon}.$$

Letting $n \to \infty$, and then $\varepsilon_0 \to 0$, $\varepsilon \to 0$ we get

$$\overline{\lim_{n\to\infty}}\ \frac{1}{n}\log\left|\left\{\frac{1}{n}S_n\varphi_j \geq \alpha_j\right\}\right| \leq \lim_{\varepsilon\to 0}\sup\{F(\nu)\colon \nu \in \mathcal{M}_f,\ \nu(\varphi_j) \geq \alpha_j - \sqrt{\varepsilon}\}. \tag{24}$$

The lower large deviations bound obtained in [**7**] gives

$$\underline{\lim_{n\to\infty}}\ \frac{1}{n}\log\left|\left\{\frac{1}{n}S_n\varphi_j > \alpha_j\right\}\right| \geq \sup\{F(\nu)\colon \nu \in \mathcal{M},\ \nu(\varphi_j) > \alpha_j\}, \tag{25}$$

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where $\sup \emptyset = -\infty$. Theorem B follows from (24) and (25) because the weak topology on $\mathcal{M}$ has a countable base generated by open sets of the form $\{\nu \in \mathcal{M}\colon \nu(\varphi_j) > \alpha_j,\ j = 1, \ldots, d\}$, where $d \geq 1$, each $\varphi_j \in C(X)$ is Lipschitz, and $\alpha_j \in \mathbb{R}$.

The rest of this section is devoted to the proof of Proposition 5.1. From the towers constructed in §2 we extract horseshoes, and construct invariant measures supported on them with the properties as in the statement of the proposition.

5.1. *Construction of a horseshoe.* Define

$$\mathcal{B}_n = \left\{ A \in \bigvee_{i=0}^{n-1} \hat{f}^{-i}\mathcal{D}\colon A \subset \Delta_0,\ \ \frac{1}{n} S_n\varphi_j(x) \geq \alpha_j,\ j = 1, \ldots, d \text{ for some } x \in A \right\}.$$

Observe that

$$\left| \left\{ x \in \Lambda\colon \frac{1}{n} S_n\varphi_j(x) \geq \alpha_j,\ j = 1, \ldots, d \right\} \right| \leq |\mathcal{B}_n|, \tag{26}$$

where $|\mathcal{B}_n| = \sum_{A \in \mathcal{B}_n} |A|$. To estimate this from above we use the following lemma, the proof of which closely follows that of Lemma 3.2 with σ replaced by 1.

LEMMA 5.2. *For any $\varepsilon > 0$ there exists $n'' > 0$ such that if $n \geq n''$ and $\mathcal{B}_n \neq \emptyset$ then there exist a finite collection $\mathcal{L}$ of pairwise disjoint closed intervals in Λ and an integer $r \in [(1-\varepsilon)n, (1 + 21\varepsilon/\lambda)n]$ such that:*

(a) *$\mathcal{L}$ generates a horseshoe for f^r;*

(b) *$\sum_{L \in \mathcal{L}} |L| \geq e^{-3\sqrt{\varepsilon}n} |\mathcal{B}_n|$;*

(c) *for all $x \in H_r(\mathcal{L})$, $(1/r) S_r\varphi_j(x) \geq \alpha_j - \sqrt{\varepsilon}$, $j = 1, \ldots, d$.*

Proof. For each $B \in \mathcal{B}_n$ fix once and for all an interval $\tilde{B}$ and an integer $t = t_B$ for which the conclusions of Lemma 2.10 hold. Let $\mathcal{B}_n(t) = \{B \in \mathcal{B}_n\colon t_B = t\}$. Then $t_B \in [(1-\varepsilon)n, (1 + 19\varepsilon/\lambda)n]$. Let t_1 be a value of t which maximizes $\sum_{B \in \mathcal{B}_n(t)} |\tilde{B}|$. Then

$$\sum_{B \in \mathcal{B}_n(t_1)} |\tilde{B}| \geq \frac{1}{(1 + 20/\lambda)\varepsilon n} \sum_{B \in \mathcal{B}_n} |\tilde{B}|. \tag{27}$$

For each $B \in \mathcal{B}_n(t_1)$ define $L(B)$ to be the preimage of I^+ or I^- under $f^{t_1}|\tilde{B}$, according to whether $f^{t_1}\tilde{B} = \Lambda^+$ or $= \Lambda^-$. Set $\mathcal{L} = \{L(B)\colon B \in \mathcal{B}_n(t_1)\}$ and $r = t_1 + u$. The bounds on r hold for sufficiently large n. Inequality (27) and Lemma 2.10 imply that

$$\sum_{L \in \mathcal{L}} |L| \geq e^{-3\sqrt{\varepsilon}n} |\mathcal{B}_n|.$$

To prove (c), for each $B \in \mathcal{B}_n(t_1)$ pick $x_B \in B$ such that $S_n\varphi(x_B) \geq \alpha_j n$ for $j = 1, \ldots, d$. We have

$$S_r\varphi_j(x_B) \geq S_n\varphi_j(x_B) - \sup|\varphi_j| \cdot |r - n| \geq \alpha_j n - (21\varepsilon/\lambda)n \geq (\alpha_j - \sqrt{\varepsilon}/2)r. \tag{28}$$

By Sublemma 3.3, for any $x \in B$ we have $|S_{t_1}\varphi_j(x_B) - S_{t_1}\varphi_j(x)| \leq \mathrm{Lip}(\varphi_j) \cdot C\delta^{-1}$, and thus

$$|S_r\varphi_j(x_B) - S_r\varphi_j(x)| \leq \mathrm{Lip}(\varphi_j) \cdot C\delta^{-1} + 2\sup|\varphi_j| \cdot (r - t_1) \leq \sqrt{\varepsilon}r/2. \tag{29}$$

Inequalities (28) and (29) yield $S_r\varphi_j(x) \geq (\alpha_j - \sqrt{\varepsilon})r$. □

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5.2. *Construction of a measure on the horseshoe.* We construct a measure η for which (22) and (23) hold. For sufficiently large n with $\mathcal{B}_n \neq \emptyset$, choose a finite collection $\mathcal{L}$ of pairwise disjoint closed intervals in Λ and a positive integer r for which the conclusions of Lemma 5.2 hold. Set $G = f^r$. The argument in §3.2 shows that there exists a G-invariant probability measure ν_∞ supported on $H_r(\mathcal{L})$ satisfying

$$h_G(\nu_\infty) - \int \log |DG| \, d\nu_\infty \geq \log \sum_{L\in\mathcal{L}} |L| - \log \kappa,$$

where $h_G(\nu_\infty)$ denotes the entropy of (G, ν_∞). Define $\eta \in \mathcal{M}_f$ by $\eta = (1/r) \sum_{i=0}^{r-1} (f^i)_* \nu_\infty$. From Lemma 5.2(c) it follows that $\eta(\varphi_j) \geq \alpha_j - \sqrt{\varepsilon}$, and (23) holds. Since $F(\eta) \leq 0$ and $r \geq (1-\varepsilon)n$, using Lemma 5.2(b) we have

$$\begin{aligned} n \cdot F(\eta) &\geq \frac{r}{1-\varepsilon} F(\eta) = \frac{1}{1-\varepsilon} (h_G(\nu_\infty) - \int \log |DG| \, d\nu_\infty) \\ &\geq \frac{1}{1-\varepsilon} \left(\log \sum_{L\in\mathcal{L}} |L| - \log \kappa \right) \geq \frac{1}{1-\varepsilon} (\log |\mathcal{B}_n| - 4\sqrt{\varepsilon} n). \end{aligned}$$

Rearranging this and using (26) yields

$$\frac{1}{n} \log \left| \left\{ x \in \Lambda : \frac{1}{n} S_n \varphi_j(x) \geq \alpha_j \quad j = 1, \ldots, d \right\} \right| \leq \frac{1}{n} \log |\mathcal{B}_n| \leq (1-\varepsilon) F(\eta) + 4\sqrt{\varepsilon}.$$

Hence, (22) holds.

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